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John Campbell

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q -analogues of π -formulas due to Ramanujan and Guillera

JOHN M. CAMPBELL

Abstract

The first known q -analogues for any of the 17 formulas for $\frac{1}{\pi}$ due to Ramanujan were introduced in 2018 by Guo and Liu (J Difference Equ Appl 29:505–513, 2018), via the q -Wilf–Zeilberger method. Through a “normalization” technique based on the q -polynomial coefficients involved in first-order difference equations obtained from the q -version of Zeilberger’s algorithm, we introduce q -WZ pairs that extend WZ pairs introduced by Guillera (Adv in Appl Math 29:599–603, 2002). We apply our q -WZ pairs to introduce and prove q -analogues of two of Ramanujan’s series for $\frac{1}{\pi}$ and q -analogues of Guillera’s first two series for $\frac{1}{\pi^2}$. We also apply our normalization method to further extend Guillera’s WZ pairs by introducing hypergeometric expansions for $\frac{1}{\pi^2}$.

Keywords: Wilf–Zeilberger method; Wilf–Zeilberger pair; difference equation; Ramanujan-type series, q -Pochhammer symbol, q -analogue.

2020 Mathematics Subject Classification: 05A10, 33F10.

1 Introduction

The Wilf–Zeilberger (WZ) method [20] is of great importance in combinatorics, special functions theory, and computer science. One of the most notable developments in the history of WZ theory is given by the discoveries due to Guillera, with a 2002 contribution of Guillera [13] having led to a breakthrough in the areas of mathematics related to Ramanujan’s series for $\frac{1}{\pi}$ [2, pp. 352–354] [21]. Guillera [13] applied the WZ method, through the use of Zeilberger’s EKHAD package for the Maple Computer Algebra System, to introduce and prove the first Ramanujan-type series for $\frac{1}{\pi^2}$. This motivates the development of techniques to build upon Guillera’s applications of the WZ method and the EKHAD package. This provides the main purpose of our paper.

Following the work of Guillera [13], a discrete function $A(n, k)$ is said to be *hypergeometric* if $\frac{A(n+1, k)}{A(n, k)}$ and $\frac{A(n, k+1)}{A(n, k)}$ are rational functions. A pair

(F, G) of bivariate, hypergeometric functions $F = F(n, k)$ and $G = G(n, k)$ is a *WZ pair* if the discrete difference equation

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k) \quad (1)$$

holds. For a WZ pair (F, G) , there exists a rational function $R(n, k)$ such that $G(n, k) = R(n, k)F(n, k)$ [27], and $R(n, k)$ is referred to as the *certificate* associated with (F, G) . Setting

$$H(n, k) := F(n+1, n+k) + G(n, n+k), \quad (2)$$

the relation

$$\sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} H(n, 0) \quad (3)$$

proved by Zeilberger provides a key in Guillera's derivations of remarkable formulas such as

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} (820n^2 + 180n + 13). \quad (4)$$

The WZ pair (F, G) employed by Guillera to prove (4) was determined through the EKHAD package and is such that

$$F(n, k) = 512 \frac{(-1)^k}{2^{16n} 2^{4k}} \frac{n^3}{4n - 2k - 1} \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k} \binom{n+k}{n}^2}. \quad (5)$$

The problem of determining q -analogues of the WZ pairs applied by Guillera [13] has led us to formulate an "EKHAD-type" method toward extending known WZ pairs, and we apply our method to obtain q -analogues of the π -formulas proved by Guillera [13].

2 Background and preliminaries

The Γ -function is defined for arguments with a positive real part so that $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ and its analytic continuation is obtained through the relation $\Gamma(x) = \frac{\Gamma(x+1)}{x}$. The Pochhammer symbol may then be defined so that $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$, and we write

$$\left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_k = \frac{(\alpha)_k (\beta)_k \cdots (\gamma)_k}{(A)_k (B)_k \cdots (C)_k}.$$

Similarly, the q -Pochhammer symbol may be defined so that

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

and we write

$$\left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \middle| q \right]_n = \frac{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}{(A; q)_n (B; q)_n \cdots (C; q)_n}.$$

We also let the q -bracket symbol be such that $[n]_q = \frac{1-q^n}{1-q}$.

A number of past research contributions have concerned q -analogues of Ramanujan's and Guillera's π -formulas, and our main results are inspired by many of these past research works. The WZ-based approach involved in our current paper is not equivalent to that involved in our past work [3] on a q -analogue of the Guillera formula

$$\frac{\pi^2}{4} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} 1, 1, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| q \right]_n (3n + 2)$$

introduced and proved via the WZ method in 2008 [11].

Our applications below based on the q -analogue of Zeilberger's algorithm are inspired by how Guo and Liu [16] applied the q -WZ method to prove the q -analogue

$$\sum_{n=0}^{\infty} q^{n^2} [q, q | q^2]_n \left[\begin{matrix} q^2 \\ q^4, q^4, q^4 \end{matrix} \middle| q^4 \right]_n [6n + 1]_q = (1 + q) \left[\begin{matrix} q^2, q^6 \\ q^4, q^4 \end{matrix} \middle| q^4 \right]_\infty$$

of Ramanujan's formula

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \middle| q \right]_n (6n + 1)$$

and the q -analogue

$$\sum_{n=0}^{\infty} (-1)^n q^{3n^2} \frac{[q, q, q | q^2]_n}{[q^4, q^4, q^4 | q^4]_n} [6n + 1]_q = \left[\begin{matrix} q^3, q^5 \\ q^4, q^4 \end{matrix} \middle| q^4 \right]_\infty$$

of Ramanujan's formula

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{8}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_n (6n + 1).$$

Zeilberger's algorithm and the "normalization" procedure that we apply are not involved in the work of Guo and Liu or in any other past references concerning q -analogues of Ramanujan's or Guillera's formulas.

As noted by Guo and Zudilin [17] the first known q -analogues for any of the 17 formulas for $\frac{1}{\pi}$ due to Ramanujan were introduced by Guo and Liu in 2018 [16]. Guo and Zudilin also proved the q -analogue

$$\sum_{n=0}^{\infty} (-1)^n q^{2n^2} \left[\begin{matrix} q^2, q^2 \\ q^4, q^4 \end{matrix} \middle| q^4 \right]_n \frac{(q; q^2)_{2n}}{(q^4; q^4)_{2n}} \times \\ \left([8n + 1]_q + [4n + 1]_q \frac{q^{4n+1}}{1 + q^{4n+2}} \right) = (1 + q) \left[\begin{matrix} q^2, q^6 \\ q^4, q^4 \end{matrix} \middle| q^4 \right]_{\infty}$$

of Ramanujan's formula

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1, 1 \end{matrix} \right]_n (20n + 3) \quad (6)$$

and the q -analogue

$$\sum_{n=0}^{\infty} q^{2n^2} \left[\begin{matrix} q \\ q^2 \end{matrix} \middle| q^2 \right]_{2n} \frac{[q, q | q^2]_n}{[q^6, q^6 | q^6]_n} [8n + 1]_q = \left[\begin{matrix} q^3 \\ q^2 \end{matrix} \middle| q^2 \right]_{\infty} \left[\begin{matrix} q^3 \\ q^6 \end{matrix} \middle| q^6 \right]_{\infty}$$

of Ramanujan's formula

$$\frac{2\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1, 1 \end{matrix} \right]_n (8n + 1).$$

The works of Guo and Liu and of Guo and Zudilin are related to many subsequent research works concerning q -analogues of π -formulas [4, 5, 6, 14, 15, 18, 19, 22, 23, 24, 25, 26] and motivate the new results proved in Section 3 below. Devising q -analogues for π -formulas is considered to be challenging [25], which motivates our EKHAD-normalization technique described in Section 3 below. Our approach toward the problem of finding q -analogues of Guillera's WZ pairs is not equivalent to Guillera's approach toward this problem [14], and we invite the interested reader to make the proper comparison.

3 q -analogues of Ramanujan's formulas

Out of the series expansions for $\frac{1}{\pi}$ discovered by Ramanujan, the first WZ proof for any such expansion is due to Zeilberger [9]. In 2002, Guillera [13] introduced WZ proofs of Ramanujan's formula in (6) and Ramanujan's formula such that

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{64}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_n (42n + 5). \quad (7)$$

The F -entry in the WZ pair (F, G) Guillera applied to prove (6) and (7) is such that

$$F(n, k) = 64 \frac{(-1)^n (-1)^k}{2^{10n} 2^{2k}} \frac{n^2}{4n - 2k - 1} \frac{\binom{2k}{k}^2 \binom{2n}{n}^2 \binom{4n-2k}{2n-k}}{\binom{2n}{k} \binom{n+k}{n}}. \quad (8)$$

This was obtained by Guillera through the EKHAD package. By rewriting (8) as

$$F(n, k) = -64 \left(-\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_n n^2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} - 2n, 1 + n \end{matrix} \right]_k, \quad (9)$$

this has led us toward a way of generalizing Guillera's WZ pair (F, G) , through the use of both Zeilberger's algorithm and its q -analogue.

Taking the combination of Pochhammer symbols in (9) indexed by k , we set

$$\mathcal{F}(n, k) := \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} - 2n, 1 + n \end{matrix} \right]_k. \quad (10)$$

Applying Zeilberger's algorithm to (10), we obtain the difference equation

$$p_1(n)\mathcal{F}(n+1, k) + p_2(n)\mathcal{F}(n, k) = \mathcal{G}(n, k+1) - \mathcal{G}(n, k) \quad (11)$$

for the polynomials $p_1(n) = (2n+1)(4n-1)(4n+1)$ and $p_2(n) = 128n^2(n+1)$, and for the companion $\mathcal{G}(n, k) = \mathcal{R}(n, k)\mathcal{F}(n, k)$, for the certificate $\mathcal{R}(n, k) = 2(n+1)(2k-4n+1)(2k+20n+3)$. By "normalizing" $\mathcal{F}(n, k)$ by setting

$$\overline{\mathcal{F}}(n, k) = \mathcal{F}(n, k) (-1)^n \prod_{m=1}^{n-1} \frac{p_1(m)}{p_2(m)}, \quad (12)$$

this normalized version of $\mathcal{F}(n, k)$ is none other, up to a constant scalar multiple, than the function $F(n, k)$ obtained by Guillera via the EKHAD package, i.e., so that the desired WZ difference equation in (1) is satisfied.

For a hypergeometric function $\mathcal{F}(n, k)$ that satisfies a first-order difference equation from Zeilberger's algorithm or its q -analogue, we define the **EKHAD-normalization** $\overline{\mathcal{F}}(n, k)$ of $\mathcal{F}(n, k)$ according to (12) and for the polynomial (or q -polynomial) coefficients $p_1(n)$ and $p_2(n)$ of $\mathcal{F}(n+1, k)$ and $\mathcal{F}(n, k)$ in the difference equation for \mathcal{F} given by Zeilberger's algorithm (or its q -version). By experimenting with variants and extensions of the input function in (10) using EKHAD-normalization, this has led us to extend Guillera's results [13], as below. This normalization approach is inspired by the *WZ seeds* recently introduced by Au [1], and we invite the interested reader to compare the respective approaches.

Experimentally, we have discovered that the q -analogue

$$q^k \left[\begin{matrix} q^{\frac{1}{2}}, q^{\frac{1}{2}} \\ q^{\frac{3}{2}-2n}, q^{1+n} \end{matrix} \middle| q \right]_k \quad (13)$$

of the input function $\mathcal{F}(n, k)$ in (10) satisfies a first-order recurrence via the q -version of Zeilberger's algorithm. The EKHAD-normalization of (13) has led us to construct a q -WZ pair to prove the following new result, which (as we later clarify) provides a q -analogue of Ramanujan's formula in (6).

Theorem 1. *For the q -polynomial*

$$\rho(n, q) = -2q^{2n+1} + q^{4n+1} - q^{4n+2} + q^{6n+1} + q^{8n+2} + q^{10n+3} - 1,$$

the equality

$$\sum_{n=0}^{\infty} (-1)^n q^{3(n-1)(n+1)} \left[\begin{matrix} q \\ -q^2, -q^2, q^2, q^2, q^2, -q^3, -q^3 \end{matrix} \middle| q^2 \right]_n \times \\ [q, q^3 \mid q^4]_n \rho(n, q) = \frac{(q-1)(q+1)^3}{q^3} \left[\begin{matrix} q, q \\ q^2, q^3 \end{matrix} \middle| q^2 \right]_{\frac{1}{2}}$$

holds for a free parameter q .

Proof. We set

$$F(n, k; q) := 4(-1)^n q^{\frac{3n^2}{2}+k-1} \frac{(1+q^{\frac{1}{2}})^3 (1-q^n)^2}{(1-q^{\frac{1}{2}})^2} [q^{-\frac{1}{2}}, q^{\frac{1}{2}} \mid q^2]_n \times$$

$$\left[\begin{array}{c} q^{\frac{1}{2}} \\ -1, -1, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q, q, q \end{array} \middle| q \right]_n \left[\begin{array}{c} q^{\frac{1}{2}}, q^{\frac{1}{2}} \\ q^{\frac{3}{2}-2n}, q^{1+n} \end{array} \middle| q \right]_k.$$

By then setting

$$R(n, k; q) := \frac{q^{2n} - q^{k+\frac{1}{2}}}{q^k (1 - q^n)^2 (q^n + 1)^2} \times \frac{q^{k+2n+\frac{1}{2}} - 2q^{n+\frac{1}{2}} - q^{2n+1} + q^{3n+\frac{1}{2}} + q^{4n+1} + q^{5n+\frac{3}{2}} - 1}{2q^{n+1} + q^{2n+\frac{3}{2}} + \sqrt{q}}$$

and $G(n, k; q) := R(n, k; q)F(n, k; q)$, we may verify that the difference equation

$$F(n + p + 1, k; q) - F(n + p, k; q) = G(n + p, k + 1; q) - G(n + p, k; q)$$

holds for a free parameter p . A telescoping phenomenon then gives us that

$$F(m + p + 1, k; q) - F(p, k; q) = \sum_{n=0}^m (G(n + p, k + 1; q) - G(n + p, k; q))$$

for positive integers m . Setting $m \rightarrow \infty$ and $p \rightarrow 0$, we obtain that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n + p, k; q) = \lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n + p, k + 1; q). \quad (14)$$

Carlson's theorem can then be applied to show that both sides of (14) are constant for every value of k , including non-integers, so that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n + p, 0; q) = \lim_{p \rightarrow 0} \left(G\left(p, \frac{1}{2}; q\right) + \sum_{n=1}^{\infty} G\left(n + p, \frac{1}{2}; q\right) \right). \quad (15)$$

We may verify that $\lim_{p \rightarrow 0} G(n + p, \frac{1}{2})$ vanishes for positive integers n , so that an application of the dominated convergence theorem using (15) gives us that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n + p, 0; q) = \lim_{p \rightarrow 0} G\left(p, \frac{1}{2}; q\right). \quad (16)$$

We again apply the dominated convergence theorem to interchange the limiting operations on the left of (16). After applying the substitution $q \mapsto q^2$, we obtain an equivalent version of the desired result. \square

Multiplying both sides of the equality in Theorem 3 by $\frac{1}{q-1}$ and setting $q \rightarrow 1$, the summand, for a nonnegative integer n , of the left-hand side reduces to the summand of the Ramanujan series in (6) for the same argument n , and the right-hand side reduces to $\frac{8}{\pi}$ according to the known property of the q -Gamma function

$$\Gamma_q(x) := (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+x-1}}$$

whereby

$$\lim_{q \rightarrow 1} \Gamma_{q^2} \left(\frac{1}{2} \right) = \sqrt{\pi}. \quad (17)$$

As clarified below, Theorem 3 provides a q -analogue of Ramanujan's formula in (7).

Theorem 2. *For the q -polynomials*

$$\rho_1(n, q) = (1 - q^{2n-1})(3q^{10n+3} + 3q^{8n+2} - q^{6n+3} + 2q^{6n+1} - 3q^{4n+2} - 3q^{2n+1} - 1)$$

and $\rho_2(n, q) = (1 + q^{2n+1})^2$, the equality

$$\sum_{n=0}^{\infty} (-1)^n q^{(n-1)(3n+5)} \left[\begin{matrix} q, q, q, q, q^3 \\ -q^2, -q^2, q^2, q^2, q^2, q^{3-4n}, q^{2n+2} \end{matrix} \middle| q^2 \right]_n \times \\ \left[\begin{matrix} q^{-1}, q \\ q^2, q^6 \end{matrix} \middle| q^4 \right]_n \frac{\rho_1(n, q)}{\rho_2(n, q)} = \frac{(q-1)^2(q+1)^2}{q^6} \left[\begin{matrix} q, q \\ q^2, q^3 \end{matrix} \middle| q^2 \right]_{\frac{1}{2}}$$

holds for a free parameter q .

Proof. For the same WZ pair (F, G) involved in our proof of Theorem 3, by setting $H(n, k; q)$ as in (2) so that

$$H(n, k; q) := F(n+1, n+k; q) + G(n, n+k; q), \quad (18)$$

an equivalent version of Zeilberger's identity in (3) gives us that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, 0; q) = \lim_{p \rightarrow 0} \sum_{n=0}^{\infty} H(n+p, 0; q). \quad (19)$$

From the same application of Carlson's theorem as in the proof of Theorem 3, we obtain from (19) that

$$\lim_{p \rightarrow 0} G\left(p, \frac{1}{2}; q\right) = \lim_{p \rightarrow 0} \sum_{n=0}^{\infty} H(n+p, 0; q). \quad (20)$$

Applying the dominated convergence theorem to interchange the order of limiting operations on the right of (20), and applying the substitution $q \mapsto q^2$, we obtain an equivalent version of the desired result. \square

Multiplying both sides of the q -identity in Theorem 3 by $\frac{q(q+1)^2}{(1-q)^2}$ and setting $q \rightarrow 1$, the summand of the infinite series for a given natural number n reduces to the summand of Ramanujan's series in (7) for the same integer n , and we may again apply the q -Gamma relation in (17) to obtain the desired multiple of $\frac{1}{\pi}$.

4 q -analogues of Guillera's formulas

By rewriting Guillera's F -function in (5) so that

$$F(n, k) = -512 \left(-\frac{1}{16}\right)^n \left[\begin{matrix} -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_n n^3 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} - 2n, 1 + n, 1 + n \end{matrix} \right]_k,$$

this leads us to apply EKHAD-normalization using a q -analogue of the factor

$$\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} - 2n, 1 + n, 1 + n \end{matrix} \right]_k, \quad (21)$$

namely

$$q^k \left[\begin{matrix} q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}} \\ q^{\frac{3}{2}-2n}, q^{1+n}, q^{1+n} \end{matrix} \middle| q \right]_k. \quad (22)$$

Using the q -version of Zeilberger's algorithm, we find that (22) satisfies a first-order recurrence, and this has led us to construct a q -WZ pair to prove the following q -analogue of the formula

$$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_n (120n^2 + 34n + 3) \quad (23)$$

due to Guillera [13].

Theorem 3. For the q -polynomial

$$\rho(n, q) = 3q^{2n+1} - 3q^{4n+1} + 3q^{4n+2} - 2q^{6n+1} - 3q^{6n+2} + q^{6n+3} + q^{8n+1} - 3q^{8n+2} - 2q^{8n+3} + 3q^{10n+2} - 3q^{10n+3} + 3q^{12n+3} + q^{14n+4} + 1.$$

the relation

$$\sum_{n=0}^{\infty} q^{2n^2} \left[\begin{matrix} q, q, q \\ -q^2, -q^2, -q^2, q^2, q^2, q^2, q^2, q^2, -q^3, -q^3, -q^3 \end{matrix} \middle| q^2 \right]_n \times \\ [q, q^3 \mid q^4]_n \rho(n, q) = (q-1)^2 (q+1)^4 \left[\begin{matrix} q, q, q \\ q^2, q^2, q^3 \end{matrix} \middle| q^2 \right]_{\frac{1}{2}}$$

holds for a free parameter q .

Proof. By setting $F(n, k; q)$ as

$$8q^{k+n^2} \frac{(\sqrt{q}+1)^5 (1-q^n)^3}{(1-\sqrt{q})^3 \sqrt{q}} \left[\begin{matrix} q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}} \\ -1, -1, -1, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q, q, q, q, q \end{matrix} \middle| q \right]_n \times \\ \left[\begin{matrix} q^{-\frac{1}{2}}, q^{\frac{1}{2}} \\ q^2 \end{matrix} \right]_n \left[\begin{matrix} q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}} \\ q^{\frac{3}{2}-2n}, q^{1+n}, q^{1+n} \end{matrix} \middle| q \right]_k,$$

we obtain the rational certificate $R(n, k; q)$ equal to

$$\frac{q^{\frac{1}{2}-k} (q^{k+1} - q^{2n+\frac{1}{2}})}{q^{3/2} (1 - q^{2n})^3 (1 + q^{\frac{2n+1}{2}})^3} \tau(n, k)$$

for the q -polynomial

$$\tau(n, k) := -3q^{k+2n+\frac{1}{2}} - 3q^{k+3n+1} + q^{k+4n+\frac{1}{2}} - 2q^{k+4n+\frac{3}{2}} + 3q^{k+5n+1} + \\ 3q^{k+6n+\frac{3}{2}} + q^{k+7n+2} + 3q^{n+\frac{1}{2}} + 3q^{2n+1} - 2q^{3n+\frac{1}{2}} + q^{3n+\frac{3}{2}} - 3q^{4n+1} - 3q^{5n+\frac{3}{2}} + 1.$$

By setting $G(n, k; q) := R(n, k; q)F(n, k; q)$, we find that the q -difference equation

$$F(n+p+1, k; q) - F(n+p, k; q) = G(n+p, k+1; q) - G(n+p, k; q)$$

holds for a free parameter p . We again apply a telescoping argument to obtain that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, k; q) = \lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, k+1; q), \quad (24)$$

and we again apply Carlson's theorem to obtain that (24) holds for all k , including non-integer values. We again obtain that

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, 0; q) = \lim_{p \rightarrow 0} \left(G\left(p, \frac{1}{2}; q\right) + \sum_{n=1}^{\infty} G\left(n+p, \frac{1}{2}; q\right) \right),$$

and we again find that this reduces to

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, 0; q) = \lim_{p \rightarrow 0} G\left(p, \frac{1}{2}; q\right),$$

and we again reverse the order of limiting operations on the left-hand side, and we again apply the substitution $q \mapsto q^2$. This gives us an equivalent version of the desired result. \square

Multiplying both sides of the identity in Theorem 3 by $\frac{(1+q)^2}{4(1-q)^2q^2}$ and setting $q \rightarrow 1$, the summand for the series in Theorem 3 for a given natural number n approaches the summand of the Guillera series in (23) for the same index n , and we may apply the q -Gamma relation in (17) to obtain a multiple of $\frac{1}{\pi^2}$.

For the same q -WZ pair (F, G) involved in our proof of Theorem 3, according to the relation

$$\lim_{p \rightarrow 0} \sum_{n=0}^{\infty} G(n+p, 0; q) = \lim_{p \rightarrow 0} \sum_{n=0}^{\infty} H(n+p, 0; q).$$

for H as in (18), this can be used to obtain a q -analogue of the Guillera formula [13] such that

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \left(-\frac{1}{1024} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_n (820n^2 + 180n + 13),$$

according to a similar approach as in our proof of Theorem 3. Explicitly, for the q -polynomials

$$\begin{aligned}
\rho_1(n, q) = & q^{2(n+1)^2} + 5q^{2(n+1)(n+3)} - q^{2n^2+2n+3} - 3q^{2n^2+4n+4} + q^{2n^2+6n+2} + \\
& 3q^{2n^2+6n+3} - q^{2n^2+6n+5} - q^{2n^2+8n+1} + 3q^{2n^2+8n+3} + 6q^{2n^2+8n+4} - \\
& 3q^{2n^2+10n+2} - 5q^{2n^2+10n+3} + q^{2n^2+10n+4} + 5q^{2n^2+10n+7} - 6q^{2n^2+12n+3} - \\
& 6q^{2n^2+12n+4} - 5q^{2n^2+12n+5} - 6q^{2n^2+12n+6} - q^{2n^2+12n+8} + 5q^{2n^2+14n+2} + \\
& q^{2n^2+14n+3} - 2q^{2n^2+14n+5} - 5q^{2n^2+14n+6} - 17q^{2n^2+14n+7} - 3q^{2n^2+14n+9} + \\
& 6q^{2n^2+16n+3} + 3q^{2n^2+16n+4} + 6q^{2n^2+16n+5} + 17q^{2n^2+16n+6} + q^{2n^2+16n+7} + \\
& 2q^{2n^2+16n+8} - q^{2n^2+16n+10} - q^{2n^2+18n+2} + 2q^{2n^2+18n+4} + q^{2n^2+18n+5} + \\
& 17q^{2n^2+18n+6} + 6q^{2n^2+18n+7} + 3q^{2n^2+18n+8} + 6q^{2n^2+18n+9} - 3q^{2n^2+20n+3} - \\
& 17q^{2n^2+20n+5} - 5q^{2n^2+20n+6} - 2q^{2n^2+20n+7} + q^{2n^2+20n+9} + 5q^{2n^2+20n+10} - \\
& q^{2n^2+22n+4} - 6q^{2n^2+22n+6} - 5q^{2n^2+22n+7} - 6q^{2n^2+22n+8} - 6q^{2n^2+22n+9} + \\
& 5q^{2n^2+24n+5} + q^{2n^2+24n+8} - 5q^{2n^2+24n+9} - 3q^{2n^2+24n+10} + 5q^{2n^2+26n+6} + \\
& 6q^{2n^2+26n+8} + 3q^{2n^2+26n+9} - q^{2n^2+26n+11} - q^{2n^2+28n+7} + 3q^{2n^2+28n+9} + \\
& q^{2n^2+28n+10} - 3q^{2n^2+30n+8} + q^{2n^2+30n+10} - q^{2n^2+32n+9}
\end{aligned}$$

and $\rho_2(n, q) = 1 - q^{4n-1}$, the relation

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left[-q^2, -q^2, -q^2, q^2, q^2, q^2, -q^3, -q^3, -q^3, q^4, q^4, q^{3-4n}, q^{4+2n}, q^{4+2n} \mid q^2 \right]_n \\
& \times [q^{-1}, q \mid q^4]_n \frac{\rho_1(n, q)}{\rho_2(n, q)} = (1 - q)^3 q^2 (q + 1)^4 (1 - q^2)^2 \left[\begin{matrix} q, q, q \\ q^2, q^2, q^3 \end{matrix} \mid q^2 \right]_{\frac{1}{2}}
\end{aligned}$$

holds for a free parameter q .

5 Conclusion

Our EKHAD-normalization technique can be used to produce q -analogues of many further series for π , and this includes q -analogues of further Ramanujan-type series for $\frac{1}{\pi}$ proved via the WZ method by Guillera [10]. For the sake of brevity, we leave this to a separate project, and we briefly conclude by considering how our EKHAD-normalization technique can be applied to obtain hypergeometric evaluations for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$, using variants and extensions of the hypergeometric F -functions applied by Guillera [13].

As a generalization of the hypergeometric function in (21), we define

$$\mathcal{F}_\alpha(n, k) := \left[\begin{matrix} \alpha, \frac{1}{2}, 1 - \alpha \\ \frac{3}{2} - 2n, n + 1, n + 1 \end{matrix} \right]_{k+\beta n}. \quad (25)$$

By applying EKHAD-normalization, according to (12), for special cases of the parameters and variables involved in (25), and applying WZ relations as in (14) and (19), this leads us to expansions for $\frac{1}{\pi^2}$ and $\frac{1}{\pi}$ that motivate further applications of our method. For example, applying EKHAD-normalization in (25) and setting $(\alpha, \beta, k) = (\frac{1}{2}, 1, 0)$ and applying the relation in (3), we obtain that

$$\frac{2048}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{1}{729} \right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \end{matrix} \right]_n \times \\ (372736n^6 + 815616n^5 + 682752n^4 + 275168n^3 + 55536n^2 + 5310n + 207).$$

Similarly, by applying EKHAD-normalization in (25) and setting $(\alpha, \beta, k) = (\frac{1}{4}, 1, 0)$ and applying the relation in (3), we obtain that

$$\frac{2048}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{729} \right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \end{matrix} \right]_n \times \\ (186368n^5 + 384512n^4 + 293344n^3 + 100760n^2 + 14890n + 651).$$

Similarly, by applying EKHAD-normalization to

$$\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 2n \\ n+1, n+1, 1 \end{matrix} \right]_k, \quad (26)$$

we recover Guillera's WZ pair (F, G) such that

$$F(n, k) = \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 + \frac{k}{2}, \frac{1}{2} + \frac{k}{2} \\ 1, 1, 1, k+1, k+1 \end{matrix} \right]_n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right]_k \frac{96n^3}{2n+k}$$

applied by Guillera [12] to prove the remarkable result such that

$$\frac{48}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{27}{64} \right)^n \left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_n (74n^2 + 27n + 3).$$

This leads us to generalize (26) by setting

$$\mathcal{F}_{\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4}(n, k) = \left[\begin{matrix} \alpha, 1 - \alpha, (\beta_1 + \beta_2)n + \gamma_1 \\ \beta_1 n + \gamma_2, \beta_2 + \gamma_3, \gamma_4 \end{matrix} \right]_{k+\delta n} \quad (27)$$

and by applying EKHAD-normalization. For example, for the $(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta) = (\frac{1}{2}, 1, 2, 0, 1, 1, 1, 0)$ case of (27), we obtain from the WZ relation in (3) that

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{256}{729}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3} \end{matrix} \right]_n \times \\ (7568n^6 + 17184n^5 + 15256n^4 + 6748n^3 + 1574n^2 + 186n + 9).$$

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John M. Campbell

Department of Mathematics and Statistics
 Dalhousie University
 6299 South St, Halifax, NS B3H 4R2

`jh241966@dal.ca`