# Distributed Transition System with Tags and Value-wise Metric, for Privacy Analysis

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**Abstract**

We introduce a logical framework named *Distributed Labeled Tagged Transition System* (DLTTS), using concepts from Probabilistic Automata, Probabilistic Concurrent Systems, and Probabilistic labelled transition systems. We show that DLTTS can be used to formally model how a given piece of *private* information $P$ (e.g., a set of tuples) stored in a given database $D$ can get captured progressively by an adversary $A$ repeatedly querying $D$, enhancing the knowledge acquired from the answers to these queries with relational deductions using certain additional non-private data. The database $D$ is assumed protected with generalization mechanisms. We also show that, on a large class of databases, metrics can be defined 'value-wise', and more general notions of adjacency between data bases can be defined, based on these metrics. These notions can also play a role in differentially private protection mechanisms.

*Keywords*: Database, Privacy, Transition System, Probability, Distribution.

## 1 Introduction

Data anonymization has been investigated for decades, and many privacy models have been proposed ($k$-anonymity, *differential privacy*, ...) whose goals are to protect sensitive information. In this paper, our goal is not to define a new privacy model, but rather to propose a logical framework (agnostic to the privacy model) to formally model how the information stored in a database can get captured progressively by any agent repeatedly querying the database. This model can also be used to quantify reidentification attacks on a database. We start with the observation that databases are distributed over several 'worlds' in general, and querying such databases leads to answers which would also be distributed; and conceivably, to such distributed answers one could assign probability distributions of pertinence to the query. We present a logical framework that formally models how a given *private* information $P$, stored on a given (distributed) database $D$, can get captured progressively by an adversary querying the database repeatedly. Named DLTTS (*Distributed Labeled Tagged Transition System*), the framework borrows ideas from several domains, such as: Probabilistic Automata of Segala, Probabilistic Concurrent Systems, and Probabilistic labelled transition systems. To every node on a DLTTS will be attached a *tag* (which we will formally define later) representing the 'current' knowledge of

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the adversary, acquired from the responses to his/her queries by the answering mechanism, at the nodes traversed on a run; this knowledge can be 'saturated' by the adversary at any node on the run, using (a given class of) relational deductions in combination with public information from some external databases, given in advance.

In our developments below, the data in the databases will all be with finite domain, of any of the following 'basic' types: numerical, non-numerical, or literal. Some data could be structured in a complex taxonomical relation (e.g. an ontology), but we shall only consider simple tree-structured taxonomies in this work. Part of the data could also be 'anonymized' via a generalization mechanism (e.g. finite intervals or finite sets, instead of precise values), over the basic types. We therefore consider the types of the data in such an extended/overloaded sense. (cf. Example 1 below) The paper is structured as follows:

• Preliminaries and notations are presented in Section 2.

• DLTTS are defined formally in Section 3. They are meant as models for how the information stored in a (distributed) database $D$ can get captured progressively, by an adversary querying repeatedly that database. We introduce a 'blackbox' mechanism as an 'auxiliary' to a DLTTS; denoted as $\mathcal{O}$, it will be conveniently referred to as an oracle. A motivating and illustrative 'running' example is given in subsection 2.2,

• A 'value-based' distance function $\rho$ is defined in Section 4, first between data of 'compatible types' in a natural manner, and subsequently extended as a (partial) distance function between 'compatible sets' of data tuples. $\rho$ is assumed known only to the oracle $\mathcal{O}$, to which can be assigned a role of control on the runs of adversary $A$: At any node along a run of $A$ on the DLTTS, where the saturated knowledge of $A$ gets too $\rho$-close to private/secret data in $D$ (in a sense made precise in Section 4), force the run to terminate. (At any other node, $\mathcal{O}$ would indicate possible transitions with their distributions; for $A$ to continue the run at his/her own choice.)

• The notions of $\epsilon$-distinguishability, $\epsilon$-LDP, and $\epsilon$-DP for databases are recalled in Sections 5 and 6; and they are refined in combination with a vision based on metrics, in Section 7. The notion of $\epsilon_\rho$-distinguishability is defined, and shown to be finer than $\epsilon_h$-distinguishability. Here $h$ stands for the (partial) Hamming metric assumed definable between the type-compatible tuples concerned.

• In Section 8, which is the core additional contribution to the previous version of this article [1], we show that a DLTTS-based vision can be applied to some practical problems that a database administrator may be confronted with, when analyzing privacy protection using generalization in databases, such as the following:

Assumption: The 'quasi-identifier' (qid) columns of non-sensitive data in the base $D$ are anonymized via generalization. Adversary $A$ is assumed to have access only to the qids, moreover with specified probability thresholds on certain entries.

Objective: To estimate the maximal probability threshold for $A$ to get access to the sensitive values of some of the entries in $D$ (and possibly taking subsequent actions such as restricting access by not answering any more queries).

• Related Work and Comments constitute Section 9.

• Section 10 concludes the paper.

**Positioning w.r.t. our previous work.** The work presented in the current paper is an extensively revisited version of our earlier work [1] presented in a conference. In

that work, the formalism of DLTTS was fully developed, as well as a 'value-wise' (partial) metric between type compatible sets of data, for a large class of databases. The same role continues to be played by the DLTTS in the current paper, with some additional precisions. But the role played by the 'value-wise' metric has been *significantly modified* in the current version: this metric is now assumed known *only* to (the system administrator and) the oracle mechanism $\mathcal{O}$ controlling the runs on a DLTTS modeling a query-sequence of an adversary on the base. And, more importantly, contrary to [1], the oracle $\mathcal{O}$ no longer gives any information of any kind to the adversary (such as on how close or how far (s)he is, from information intended to remain secret), at any node on the DLTTS modeling the query runs of the adversary. Thus our new proposition can now be used as an 'assistant' for a database administrator to control and limit query execution on a sensitive database. In particular, Section 8 is a new contribution; Section 4 has been extended, and other sections have been restructured and augmented with novel DLTTS examples and illustrations to enhance readability. Sections 5, 6 and 7 are presented essentially as in our previous work [1].

## 2 Preliminaries

In this section, we present concepts useful to understand the work presented in this article, and introduce a simple running example.

### 2.1 Useful concepts definitions

We assume given a database $D$, with its attributes set $\mathcal{A}$, usually divided in three disjoint groups: the subgroup $\mathcal{A}^{(i)}$ of *identifiers*, $\mathcal{A}^{(qi)}$ of *quasi-identifiers*, and $\mathcal{A}^{(s)}$ of *sensitive attributes*. The tuples of database $D$ will be generally denoted as $t$, and their attributes denoted respectively as $t^i, t^{qi}$, and $t^s$ in the three subgroups of $\mathcal{A}$. The attributes $t^i$ on any tuple $t$ of $D$ are conveniently viewed as defining a 'user' or a 'client' stored in database $D$. Quasi-identifiers[1] are informally defined as a set of public attributes, which in combination with other attributes and/or external information, can allow to re-identify all or some of the users to whom the information refers.

By a *privacy policy* $P = P_A(D)$ on $D$ with respect to a given agent/adversary $A$ is meant the stipulation that for a certain *given set* of tuples $\{t \in P \subset D\}$, the sensitive attributes $t^s$ on any such $t$ shall remain inaccessible to $A$ ('even after further deduction' – see below).

The logical framework we propose below, to model the evolution of the 'knowledge' that an adversary $A$ can gain by repeatedly querying the given base $D$ will be called *Distributed Labeled-Tagged Transition System* (DLTTS); repeated querying is intended, in general, to capture sensitive data meant to remain hidden, under the privacy policy $P$. The base signature $\Sigma$ for the framework is assumed to be first-order, with countably many variables, finitely many constants (including dummy symbols such as '$\star$'), and no non-constant function symbols. By 'knowledge' of $A$ we shall mean the data that $A$ retrieves as answers to his/her successive queries, as well as other data that can be

[1]The notion of quasi-identifier attributes was introduced in informal terms, by T. Dalenius in [9]. Suffices, for now, to see them as attributes that are not identifiers nor sensitive.

derived under relational operations on these answers, and some others derivable from these using relational combinations with data (possibly involving certain users of $D$) from finitely many *external databases given in advance*, denoted as $B_1, \ldots, B_m$, to which the adversary $A$ has free access. These relational and querying operations are all assumed done with a well-delimited fragment of the relational language SQL; this *SQL fragment is assumed part of the signature* $\Sigma$. In addition, if $n \geq 1$ is the length of the data tuples in $D$, finitely many predicate symbols $\mathcal{K}_i, 1 \leq i \leq n$, each $K_i$ of arity $i$, will also be part of the signature $\Sigma$; in the work presented here they will be the only predicate symbols in $\Sigma$; the role of these symbols is to allow us to see any data tuple of length $r, 1 \leq r \leq n$, as a variable-free first-order formula with top symbol $\mathcal{K}_r$, with all arguments assumed typed implicitly (with the headers of $D$). But in practice, we shall drop these top symbols $\mathcal{K}_i$ and see any data tuple (not part of the given privacy policy $P_A(D)$) directly as a first-order variable-free formula over $\Sigma$; tuples $t$ that are elements of $P_A(D)$ are just written as $\neg t$, in general. We also assume that the given external bases $B_1, \ldots, B_m$ – to which $A$ could resort, for deducing additional information with relational operations – are of the same signature $\Sigma$ as $D$. Thus all the knowledge $A$ can derive from repeated queries on $D$ can be expressed as first-order variable-free formulas over $\Sigma$.

The DLTTS framework will be shown to be well suited for capturing the ideas of acquiring knowledge and of policy violation, in an elegant and abstract setup. The definition of this framework (Section 3) considers only the case where the data, as well as the answers to the queries, do not involve any notion of 'noise' – by 'noise' we mean the perturbation of data by some *external* random mechanism. Sections 5 and 6 extend these results to the case where noise can be used as part of the privacy mechanism. The DLTTS we consider will be modeling the lookout for the sensitive attributes of certain given users on a base, by a single adversary, with finitely many queries. It is straightforward to extend the vision to model query-sequences by multiple 'non-communicating' users, seeking to capture possibly different privacy policies. The formal definition of the DLTTS is best motivated by first presenting our 'running example' in informal style:

## 2.2 A Running Example

**Example 1**. Table 1 below is the record kept by the central Hospital of a Faculty, on recent consultations by the faculty staff of three Departments, in a University. 'Name' is an identifier attribute, 'Ailment' is sensitive, the others are QIDs; 'Ailment' is categorical with 3 branches: Heart-Disease, Cancer, and Viral-Infection; this latter in turn is categorical too, with 2 branches: Flu and CoVid. By convention, such taxonomical relations are assumed known to public. (For simplicity of the example, we assume that *all* Faculty staff are on the consultation list of the Hospital.)

| Name | Age | Gender | Dept. | Ailment |
|---|---|---|---|---|
| Joan | 24 | F | Chemistry | Heart-Disease |
| Michel | 46 | M | Chemistry | Cancer |
| Aline | 23 | F | Physics | Flu |
| Harry | 53 | M | Maths | Flu |
| John | 46 | M | Physics | CoVid |

Table 1: Hospital's 'secret' record

The Hospital intends to keep 'secret' information concerning CoVid infected faculty mem-

bers; and the tuple $\neg(John, \star, \star, \star, CoVid)$ is decided as its privacy policy. Other privacy policies are of course possible (e.g. $\neg(John, 46, M, \star, CoVid)$) and would lead to other analysis formulations. Table 2 is published by the Hospital for the public, where the 'Age' attributes are anonymized as (integer) intervals [2]; and 'Ailment' is anonymized by an upward push in the taxonomy.

A certain person $A$, who met John at a faculty banquet, suspected John to have been infected with CoVid; (s)he thus decides to consult the published record of the hospital for information.

| Line | Age | Gender | Dept. | Ailment |
|---|---|---|---|---|
| $\ell_1$ | $[20-30]$ | F | Chemistry | Heart-Disease |
| $\ell_2$ | $[40-50]$ | M | Chemistry | Cancer |
| $\ell_3$ | $[20-30]$ | F | Physics | Viral-Infection |
| $\ell_4$ | $[50-60]$ | M | Maths | Viral-Infection |
| $\ell_5$ | $[40-50]$ | M | Physics | Viral-Infection |

Table 2: Hospital's published record

Knowing that the 'John' (s)he met is 'male' and that Table 2 must contain some information on John's health, $A$ has as choice lines 2, 4 and 5 ($\ell_2, \ell_4, \ell_5$) of Table 2. $A$ being in the lookout for a 'CoVid-infected male', this choice is reduced to the last two tuples of the table – which are a priori indistinguishable because of anonymization (as 'Viral-Infection'). Now, $A$ had the impression that the John (s)he met 'was not too old', so feels the last tuple $\ell_5$ is twice more likely, and assumes 'John must be from the Physics Dept.', so goes to consult the following CoVid-cases record 'publicly visible' at the faculty.

| Dept. | CoVid cases |
|---|---|
| Physics | M : 1  F : 0 |

Table 3: Faculty CoVid-cases

And that confirms $A$'s suspicion concerning John. □

One of the objectives of this article is to define a formal model to capture this kind of reasoning. In the next section, we shall present our Distributed Labeled-Tagged Transition System (DLTTS) model, which we believe is well suited to assist in the modelling and detection of possible privacy policy breaches; we will show in Section 8 how the model can be used by e.g. a database administrator to monitor potential privacy breaching queries.

## 3 Distributed Labeled-Tagged Transition Systems

The DLTTS framework presented in this section synthesizes ideas coming from several domains, such as the Probabilistic Automata of Segala [15], Probabilistic Concurrent Systems, and Probabilistic labelled transition systems [4, 5]. Although the underlying signature for the DLTTS can be rich, in the current paper we shall work with a limited first-order signature (as mentioned in the Introduction) denoted $\Sigma$, with countably many variables, finitely many constants, including certain additional 'dummies', no non-constant function symbols, and a finite set of propositional (predicate) symbols. Let $\mathcal{E}$ be

[2]Note that here the generalized interval chosen for Age is non deterministic for extremum values.

the set of all variable-free formulas over $\Sigma$, and `Ext` a given subset of $\mathcal{E}$. We assume given a decidable procedure $\mathcal{C}$ whose role is to 'saturate' any finite set $G$ of variable-free formulas into a finite set $\overline{G}$, by adding a finite (possibly empty) set of variable-free formulas, using *relational operations* on $G$ and `Ext`. This procedure $\mathcal{C}$ will be *internal* at every node on a DLTTS, it is assumed executed using a given *finite set of internal actions*. There will be an oracle $\mathcal{O}$ as mentioned earlier, to 'check' if the given privacy policy on the database is violated at the current node.

In the definition below, $L$ will stand for a given (finite) set of ground (variable-free) first-order statements over $\Sigma$, its elements will be called *labels*. For any set $S$, $Distr(S)$ will stand for the set of all probability distributions with finite support, over the subsets of $S$.

**Definition 1** *A Distributed Labeled-Tagged Transition System (DLTTS), over a given signature $\Sigma$, is formed of:*

- *a finite (or denumerable) set $S$ of states, an 'initial' state $s_0 \in S$, and a special state $\otimes \in S$ named 'Stop'.*
- *a finite set $Act$ of action symbols (disjoint from $\Sigma$), with a special action $\delta \in Act$ named 'violation'.*
- *a (probabilistic) transition relation $\mathcal{T} \subset S \times Act \times Distr(S)$.*
- *A transition $\mathfrak{t} = (s, \alpha, \mathfrak{t}(s)) \in \mathcal{T}$ is said to be 'from' the state $s \in S$, and every $s' \in \mathfrak{t}(s)$ is a $\mathfrak{t}$-successor of $s$. The 'branch' of $\mathfrak{t}$ from $s$ to $s'$ is 'labeled' with a label $l(s, s') \in L$.*
- *a tag $\tau(s)$ attached to every state $s \in S$ other than $\otimes$, formed of finitely many ground first-order formulas over $\Sigma$. The tag at $s_0$ is $\{\top\}$, the tag at $\otimes$ is the empty set $\emptyset$.*
- *at every state $s \in S$ other than $\otimes$ a special action symbol $\iota = \iota_s \in Act$,* internal *at $s$, 'saturates' $\tau(s)$ into a set $\overline{\tau}(s)$ using the procedure $\mathcal{C}$.*

The formulas in the tag $\overline{\tau}(s)$ attached to any state $s$ will all have the same probability as assigned (by the distribution) to $s$. If the set $\overline{\tau}(s)$ of formulas turns out to be inconsistent, the oracle $\mathcal{O}$ will impose $(s, \delta, \otimes)$ as the only transition from $s$, which stands for 'violation' and 'Stop'. No outgoing transition or internal action at the halting state $\otimes$.

Remark 1: (a) We shall assume our DLTTS to be fully probabilistic, in the following sense: Given a state $s$ and a given distribution $E' \in Distr(S)$, there is at most one probabilistic transition from $s$ with $E'$ as its set of successors.

(b) It will also be assumed that the *tags at the states of a DLTTS are tight, in the following sense*, wrt the labels (and the procedure $\mathcal{C}$): For any state $s$, any transition $\mathfrak{t}$ from $s$ and any $\mathfrak{t}$-successor $s'$ of $s$, we have $\tau(s') = \overline{\tau}(s) \cup l(s, s')$, except when $s'$ is the state Stop $\otimes$.

(c) One final assumption: 'No infinite set can get generated from a finite set' by the procedure $\mathcal{C}$ for gaining further knowledge, at any state $s \in S$. (This corresponds to the assumption of *bounded inputs outputs*, as in e.g., [2, 3].) □

**DLTTS and Repeated queries on a database**: The states of the DLTTS will stand for the various 'moments' of the querying sequence, while the tags attached to the

states will stand for the knowledge $A$ has acquired on the data of $D$ 'thus far'. This knowledge consists partly in the answers to the queries (s)he made so far, then saturated with additional knowledge using the (finitely many) internal relational operations of the procedure $\mathcal{C}$, between the answers retrieved (as tuples/subtuples in $D$) by $A$ for his/her queries, and suitable tuples from the given external databases $B_1, \ldots, B_m$. If the saturated knowledge of $A$, namely $\overline{\tau}(s)$, at a current state $s$ on the DLTTS is not inconsistent, then the transition from $s$ to its successor states represents the probability distribution of the likely answers $A$ would expect to get for the next query.

Note that we make no assumption on whether the repeated queries by $A$ on $D$ are treated *interactively, or non-interactively*, by the DBMS. It appears that the logical framework would function exactly alike, in both cases.

**Example 1**(bis): Figure 1 below shows a DLTTS functioning on the Check-for-CoVid problem of Example 1 above The edges in the figure are marked with the 'query expressions' of $A$, the labels on the branches are part of the answers to $A$'s current queries:

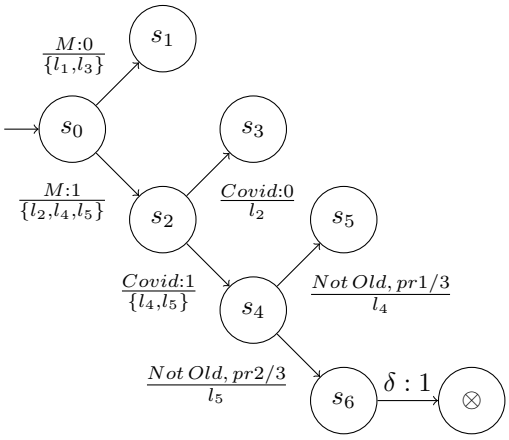


Figure 1: A DLTTS for Check-for-CoVid

**Proposition 1** *Suppose given a database $D$, a finite sequence of repeated queries on $D$ by an adversary $A$, and a first-order relational formula $P = P_A(D)$ over the signature $\Sigma$ of $D$, expressing the privacy policy of $D$ with respect to $A$. Let $\mathcal{W}$ be the DLTTS modeling the various queries of $A$ on $D$, and the evolution of the knowledge of $A$ on the data of $D$, resulting from these queries and the internal actions at the states of $\mathcal{W}$, as described above.*

*(i) The given privacy policy $P_A(D)$ on $D$ is violated under some run on $\mathcal{W}$, if and only if the failure state $\otimes$ on $\mathcal{W}$ is reachable on $\mathcal{W}$ under that run.*

*(ii) The satisfiability of the set of formulas $\overline{\tau}(s) \cup \{\neg P\}$ is decidable, at any state $s$ on the DLTTS, under the assumptions of Remark 1(b).*

*Proof*: Assertion (i) is just restatement. Observe now, that at any state $s$ on $\mathcal{W}$, the tags $\tau(s)$, $\overline{\tau}(s)$ are both finite sets of first-order *variable-free formulas* over $\Sigma$, without non-constant function symbols. Indeed, to start with, the knowledge of $A$ consists of the responses received for his/her queries, in the form of a finite set of data tuples/subtuples

from the given bases; and by our assumption of Remark 1(c), no infinite set can be generated by saturating this initial knowledge with the procedure $\mathcal{C}$. Assertion (ii) follows then from the known result that the inconsistency of any given finite set of *variable-free* first-order Datalog formulas is decidable, e.g., by the analytic tableaux procedure. (Only the absence of variables is essential.) □

# 4 Value-wise Metrics on Databases

We assume given a (distributed) database $D$, with the datatypes as mentioned in the Introduction (with a privacy policy $P$ specified on $D$). Our objective in this section is to look for a 'quantitative measure' for comparing type comparable tuples in $D$, based on which it might be possible to define a (partial) notion of distance/metric $\rho$ between sets of tuples/subtuples in $D$. Our objective is motivated by the following considerations. Suppose an adversary $A$ launches a sequence of queries on $D$, with a view to capture some of the sensitive data in the policy $P$; and suppose a DLTTS models the sequence of queries-and-answers (as described in Section 3). Assumed given $\epsilon \geq 0$ 'sufficiently small', as a 'threshold' for approximation, the manner in which the DLTTS functions can then be refined in two different ways as described below:

(1) The role of the oracle mechanism $\mathcal{O}$ can be made 'sharper', than just registering the violation of the policy $P$: if the current (saturated) knowledge of $A$ at some node $s$ on the DLLTS is at $\rho$-distance $\leq \epsilon$, then the oracle $\mathcal{O}$ could force the transition from $s$ to the state $\otimes$ (Stop); such a transition would then be named $\epsilon$-*violation* of privacy. In operational terms, a DLTTS could inform the database administrator, or some access control system, which in turn could take actions such as refusing to answer subsequent queries

(2) Two distinct 'query instances' by $A$, at some given node $s$ on the DLTTS, could receive the same output(answer) from the answering mechanism, as concerns the secret data from $P$. The two instances will then be said to be $\epsilon$-indistiguishable, in a sense and under certain conditions that we define formally in the following section 5. In such a case, the labels on the outgoing branches at $s$, corresponding to the knowledge gained by $A$ from these queries, will be considered $\epsilon$-*equivalent* in $\overline{\tau}(s)$, for privacy.

It will be actually shown in section 5, that the notion of $\epsilon$-equivalence can be rendered finer, into an $\epsilon_\rho$-*equivalence*, by combining it with the value-wise metric $\rho$ constructed in the current section.

Remember that the knowledge of $A$, at any node on the DLTTS is represented as a set of tuples, and that the data forming any tuple are assumed 'implicitly typed with the headers' of the database $D$. For 'quantitatively' comparing two tuples of the same length, we shall assume there is a natural, injective, *type-preserving* map from one of them onto the other; this map will remain implicit in general, and two such tuples will be said to be *type-compatible*. If the two tuples are not of the same length, one of them will be projected onto (or restricted to) a suitable subtuple, so as to be type-compatible and comparable with the other; if this turns out to be impossible, the two tuples will be said to be uncomparable.

We shall propose a comparison method based on an appropriately defined notion of 'distance' between two *sets of type-compatible tuples*. For that, we shall first define a 'distance' between any two type-compatible tuples – more precisely, define a notion of

distance between any two data values under every given header of $D$. As a first step, we shall therefore begin by defining, for every given header of $D$, a binary 'distance' function on the *set of all values that get assigned to the attributes under that header*, all along the sequence of $A$'s queries. This distance function to be defined will be a *metric*: non-negative, symmetric, and satisfying the so-called Triangle Inequality (cf. below). The '*direct-sum*' of these metrics, taken over all the headers of $D$, will then define a metric $d$ on the set of all type-compatible tuples of data assigned to the various attributes, under all the headers of $D$, all along the sequence of $A$'s queries. The 'distance' $d(t, t')$, from any given tuple $t$ in this set to another type-compatible tuple $t'$, will be defined as the value of this direct-sum metric on the pair of tuples $(t, t')$; it will be calculated 'column-wise' by definition, on $D$ and also on the intermediary databases along $A$'s query sequence. Note that it will a priori give us an $m$-tuple of numbers, where $m$ is the number of headers (number of columns) in the database $D$.

A single number can then be derived as the sum of the entries in this $m$-tuple $d(t, t')$. This sum will be denoted as $\overline{d}(t, t')$, and defined as the distance from the tuple $t$ to the tuple $t'$ in the database $D$. Finally, if $S, S'$ are any two given finite sets of type-compatible tuples, of data that get assigned to the various attributes (along the sequence of $A$'s queries), we define the distance from the set $S$ to the set $S'$ as the number $\rho(S, S') = min\{\, \overline{d}(t, t') \mid t \in S,\, t' \in S' \,\}$

Now, for clarity of presentation, in order to define the 'distance' between the data values under every given header of $D$, we divide the headers of $D$ into four classes, as below:

. 'Nominal': identities, names, attributes receiving *literal* data *not in any taxonomy* (e.g., gender, city, ...), and finite sets of such data;

. 'Numerval' : attributes receiving *numerical* values, or bounded intervals of (finitely many) numerical values;

. 'Numerical': attributes receiving *single numerical values* (numbers).

. 'Taxoral': attributes receiving *literal data in a taxonomy relation.*

• For defining the 'distance' between any two values $v, v'$ assigned to an attribute under a given 'Nominal' header of $D$, for the sake of uniformity we agree to consider every value as a *finite set* of singleton values; in particular, a singleton value '$x$' will be seen as the set $\{x\}$. Given two such values $v, v'$, note first that the so-called *Jaccard Index* between them is the number $jacc(v, v') = |(v \cap v')/(v \cup v')|$, often called a 'measure of their similarity'; but this index is not a metric, because the *triangle inequality* is not satisfied; however, the Jaccard metric $d_{Nom}(v, v') = 1 - jacc(v, v') = |(v \Delta v')/(v \cup v')|$ does satisfy that property, and will suite our purposes. Thus defined, $d_{Nom}(v, v')$ is a 'measure of the dissimilarity' between the sets $v$ and $v'$.

• Let $\mathcal{T}_{Nom}$ be the set of all data assigned to the attributes under the 'Nominal' headers of $D$, along $A$'s queries sequence. Then the above defined binary function $d_{Nom}$ extends to a metric on the set of all type-compatible data-tuples from $\mathcal{T}_{Nom}$, defined as the 'direct-sum' taken over the 'Nominal' headers of $D$.

• If $\mathcal{T}_{Num}$ is the set of all data assigned to the attributes under the 'Numerval' headers along the sequence of queries by $A$, we define in a similar manner (as above) a 'distance' metric $d_{Num}$ on the set of all type-compatible data-tuples from $\mathcal{T}_{Num}$: we first define

$d_{Num}$ on any couple of values $u, v$ assigned to the attributes under a given 'Numerval' header of $D$, then extend it to the set of all type-compatible data-tuples from $\mathcal{T}_{Num}$ (as the direct-sum taken over the 'Numerval' headers of $D$). This will be done exactly as above under the 'Nominal' headers: suffices to visualize any finite interval value as a particular way of presenting a set of numerical values (integers, usually). In particular, a single value '$a$' under a 'Numerval' header will be seen as the interval value $[a]$.) Thus defined the (Jaccard) metric $d_{Nom}([a,b],[c,d])$ will be a measure of 'dissimilarity' between $[a,b]$ and $[c,d]$.

• Between numerical data $x, x'$ under the 'Numerical' headers, the distance we shall work with is the euclidean metric $|x-x'|$, *normalized as*: $d_{eucl}(x,x') = |x-x'|/D$, where $D > 0$ is a fixed finite number, bigger than the maximal euclidean distance between the numerical data on the databases and on the answers to $A$'s queries.

• For the data under the 'Taxoral' headers, we choose as distance function the metric $d_{wp}$ that we define in the Appendix (Section 11), based on the well-known notion of Wu-Palmer symmetry between the nodes of a Taxonomy tree,

REMARK 2: (a) Note that the 'datawise distance functions' defined above are all *with values in the real interval* $[0,1]$. This is one reason for our choice of the distance metric on Taxonomy trees, it is of importance, cf. Section 7.

(b) The Hamming metric between datatuples (and databases) is generally well-defined only for databases with all data of a single (numerical or string) type, and all tuples of the same length. However, in this paper we shall be using a generalized notion of that metric, by extending that usual notion, in a natural manner 'data-wise' and 'column-wise' as a partial metric, just as we did for the distance function $\rho$ above. We shall denote it as $d_h$. For instance, we shall have $d_h([1,2],a),([2,3],a)) = 1, d_h([1,2],a),([2,3],b)) = 2$, whereas $d_h((bd,a),([2,3],b))$ is undefined, etc.

## 5 $\epsilon$-local-differential privacy, $\epsilon$-indistinguishability

In this section we extend the result of Proposition 1 to cases where the violation of a policy can be up to a 'threshold of approximation' $\epsilon \geq 0$, in the sense defined in the previous section. We stick to the same notation as above. The set $\mathcal{E}$ of all variable-free formulas over $\Sigma$ is thus a disjoint union of subsets of the form $\mathcal{E} = \cup\{\mathcal{E}_i^{\mathcal{K}} \mid 0 < i \leq n, \mathcal{K} \in \Sigma\}$, the index $i$ in $\mathcal{E}_i^{\mathcal{K}}$ standing for the common length of the formulas in the subset, and $\mathcal{K}$ for the common root symbol of its formulas; each set $\mathcal{E}_i^{\mathcal{K}}$ will be seen as a database of $i$-tuples.

As above, we consider the situation where the queries of an adversary intend to capture certain given (sensitive) values in the database $D$. The following definitions of $\epsilon$-indistinguishability (and of $\epsilon$-distinguishability) of two different query instances for the answering mechanism $\mathcal{M}$, as well as that of $\epsilon$-DP that will be defined in the next subsection, are essentially reformulations of the same (or similar) notions defined in [10, 11].

**Definition 2** *(i) Suppose the probabilistic mechanism $\mathcal{M}$, answering A's queries on the base D outputs(answers with) the same tuple $\alpha \in \mathcal{E}$ for two different input instances $v, v'$. Given $\epsilon \geq 0$, the two instances will be said to be $\epsilon$-indistinguishable wrt $\alpha$, if and only if:*

$$Prob[\mathcal{M}(v) = \alpha] \leq e^{\epsilon} Prob[\mathcal{M}(v') = \alpha] \text{ and}$$
$$Prob[\mathcal{M}(v') = \alpha] \leq e^{\epsilon} Prob[\mathcal{M}(v) = \alpha].$$

*Otherwise, the two instances $v, v'$ are said to be $\epsilon$-distinguishable for output $\alpha$.*

*(ii) The probabilistic answering mechanism $\mathcal{M}$ is said to satisfy* $\epsilon$-local differential privacy ($\epsilon$-LDP) *for $\epsilon \geq 0$, if and only if: for any two instances $v, v'$ of $\mathcal{M}$* that lead to the same output*, and any set $\mathcal{S} \subset Range(\mathcal{M})$, we have*

$$Prob[\mathcal{M}(v) \in \mathcal{S}] \leq e^{\epsilon} Prob[\mathcal{M}(v') \in \mathcal{S}].$$

The two small examples below illustrate $\epsilon$-lndistinguishability:

(i) The two queries based on sub-tuples ([50–60], M, Maths) and ([40–50], M, Physics), from the Hospital's published record in Example 1 (Table 2) have both Viral–Infection as output, with respective probabilities $1/3, 2/3$; thus they are $\epsilon$-indistinguishable for any $\epsilon \geq ln(2)$; and $\epsilon$-distinguishable for any $0 \leq \epsilon < ln(2)$.

(ii) The 'Randomized Response' mechanism $RR$ ([16]) can be modelled as follows. Input is $(X, F_1, F_2)$ where $X$ is a Boolean, and $F_1, F_2$ are flips of a coin ($H$ or $T$). $RR$ outputs $X$ if $F_1 = H$, $True$ if $F_1 = T$ and $F_2 = H$, and $False$ if $F_1 = T$ and $F_2 = T$. This mechanism is $ln(3)$-LDP : the instances $(True, H, H)$, $(True, H, T)$, $(True, T, H)$ and $(True, T, T)$ are $ln(3)$-indistinguishable for output $True$. $(False, H, H)$, $(False, H, T)$, $(False, T, H)$and $(False, T, T)$ are $ln(3)$-indistinguishable for output $False$.

## 6 $\epsilon$-Differential Privacy

The notion of *$\epsilon$-indistinguishability of two given databases $D, D'$* for an adversary, is more general than that of $\epsilon$-indistinguishability of pairs of query instances giving the same output (defined above). $\epsilon$-indistinguishability for pairs of databases $D, D'$ is usually defined only for bases that are *adjacent* in a certain sense (cf. below).

There seems to be no uniquely defined notion of adjacence on pairs of databases; in fact, several are known and in use in the literature. Actually, a notion of adjacence can be defined in a generic parametrizable manner (as in e.g., [6]), as follows. Assume given a map $\mathbf{f}$ from the set $\mathcal{D}$ of all databases of $m$-tuples (for some given $m > 0$), into some given metric space $(X, d_X)$. The (symmetric) binary relation on pairs of databases in $\mathcal{D}$, defined by $\mathbf{f}_{adj}(D, D') = d_X(\mathbf{f}(D), \mathbf{f}(D'))$ is then said to give a *measure of adjacence* between these bases. The relation $\mathbf{f}_{adj}$ is said to define an 'adjacency relation'.

**Definition 3** *Let $\mathbf{f}_{adj}$ be a given adjacency relation on a set $\mathcal{D}$ of databases, and $\mathcal{M}$ a probabilistic answering mechanism for queries on the bases in $\mathcal{D}$. Two bases $D, D' \in \mathcal{D}$ are said to be $\mathbf{f}_{adj}$-indistinguishable under $\mathcal{M}$, if and only if, for any possible output $\mathcal{S} \subset Range(\mathcal{M})$, we have*

$$Prob[\mathcal{M}(D) \in \mathcal{S}] \leq e^{\mathbf{f}_{adj}(D,D')} Prob[\mathcal{M}(D') \in \mathcal{S}]$$
$$Prob[\mathcal{M}(D') \in \mathcal{S}] \leq e^{\mathbf{f}_{adj}(D,D')} Prob[\mathcal{M}(D \in \mathcal{S}].$$

*The mechanism $\mathcal{M}$ is said to satisfy* $\mathbf{f}_{adj}$-differential privacy *($\mathbf{f}_{adj}$-DP), if and only if the above conditions are satisfied for* every pair of databases *$D, D'$ in $\mathcal{D}$, and any possible output $\mathcal{S} \subset Range(\mathcal{M})$.*

*Comments*: (i) Given $\epsilon \geq 0$, the 'usual' notions of *$\epsilon$-indistinguishability and $\epsilon$-DP* correspond, in general, to the choice of adjacency $\mathbf{f}_{adj} = \epsilon d_h$, where $d_h$ is the *Hamming metric* on databases (namely, the number of 'records' where $D$ and $D'$ differ) and the assumption

$d_h(D, D') \leq 1$, (cf. [6]). But in this paper we shall be using a generalized version of that Hamming metric, well-defined as a (partial) metric between all bases concerned, as e.g. between the tuples and subtuples of the database of our running Example 1.)

(ii) In Section 7, we propose a more general notion of adjacency based on the value-wise (data-wise) metric $\rho$ we defined in Section 4.

(iii) On disjoint databases, one can work with different adjacency relations, using different maps to the same (or different) metric space(s),

(iv) The mechanism $RR$ described above is actually $ln(3)$-DP, not only $ln(3)$-LDP. To check $DP$, we have to check all possible pairs of numbers of the form $(Prob[\mathcal{M}(x) = y], Prob[\mathcal{M}(x') = y])$, $(Prob[\mathcal{M}(x) = y'], Prob[\mathcal{M}(x') = y])$, $(Prob[\mathcal{M}(x) = y], Prob[\mathcal{M}(x') = y'])$, etc., where the $x, x'....$ are the input instances for $RR$, and $y, y', ...$ the outputs. The mechanism $RR$ has $2^3$ possible input instances for $(X, F_1, F_2)$ and two outputs (*True, False*); we thus have 16 pairs of numbers, among which the distinct ones are: $(1/4, 1/4), (1/4, 3/4), (3/4, 1/4)$, and $(3/4, 3/4)$; if $(a, b)$ is any such pair, obviously $a \leq e^{ln(3)}b$. Thus $RR$ is indeed $ln(3)$-DP. □

# 7 Metrics for Indistinguishability and DP

Given a probabilistic mechanism $\mathcal{M}$ answering the queries on databases, and an $\epsilon \geq 0$, recall that the $\epsilon$-indistinguishability of any two given databases under $\mathcal{M}$, and the notion of $\epsilon$-DP for $\mathcal{M}$, were both defined in Definition 2 (Section 6); based first on a hypothetical map $\mathbf{f}$ from the set of all the databases concerned, into some given metric space $(X, d_X)$, and an 'adjacency relation' on databases defined as $\mathbf{f}_{adj}(D, D') = d_X(\mathbf{f}D, \mathbf{f}D')$, which was subsequently instantiated to $\mathbf{f}_{adj} = \epsilon d_h$, where $d_h$ is the (generalized) Hamming metric between the bases concerned, cf. Remark 2(b)

In this section, our objective is to propose a more general notion of adjacency, based on the metric $\rho$ we defined in Section 4, between type-compatible tuples on databases with data of multiple types. In other words, our $\mathcal{D}$ here will be the set of all databases, *not necessarily all with the same number of columns, and also with data of several possible types* as mentioned in the Introduction. We define then a (partial) binary relation $\mathbf{f}^{\rho}_{adj}(D, D')$ between databases $D, D'$ in the set $\mathcal{D}$ by setting $\mathbf{f}^{\rho}_{adj}(D, D') = \rho(D, D')$, visualizing $D, D'$ as sets of type-compatible data tuples.

Given $\epsilon$, we can then define the notion of $\epsilon_\rho$-indistinguishability of two databases $D, D'$ under a (probabilistic) answering mechanism $\mathcal{M}$, as well as the notion of $\epsilon_\rho$-DP for $\mathcal{M}$, exactly as in Definition 2, by replacing $\mathbf{f}_{adj}$ first with the relation $\mathbf{f}^{\rho}_{adj}$, and subsequently with $\epsilon\rho$. The notions thus defined are *more general* than those presented earlier in Section 6 with the choice $\mathbf{f}_{adj} = \epsilon d_h$. An example will illustrate this point.

**Example 5**. We go back to the 'Hospital's public record' of our previous Example 1, and the two sub-tuples ([50–60], M, Maths) and ([40–50], M, Physics), from the Hospital's published record in Example 1 (Table 2). The mechanism $\mathcal{M}$ answering two queries for 'Virus-Ailment information involving men', returns the tuples $l_4, l_5$ with the probability distribution $1/3, 2/3$, respectively. Let us look for the minimum value of $\epsilon \geq 0$, for which these tuples will be $\epsilon_\rho$-indistinguishable under the mechanism $\mathcal{M}$.

We first compute the $\rho$-distance between the two tuples:

$$\rho(l_4, l_5) = \overline{d}(l_4, l_5) = (1 - \tfrac{1}{20}) + 0 + 1 + 0 = 39/20.$$

The condition for $l_4$ and $l_5$ to be $\epsilon_\rho$-indistinguishable under $\mathcal{M}$ is thus:

$$(1/3) \leq e^{(39/20)\epsilon} * (2/3), \quad (2/3) \leq e^{(39/20)\epsilon} * (1/3).$$

Which gives: $\epsilon \geq (20/39) * ln(2)$. That is, for $\epsilon \geq (20/39) * ln(2)$, the two tuples $l_4$ and $l_5$ will be $\epsilon_\rho$-indistinguishable; and for values of $\epsilon$ with $0 \leq \epsilon < (20/39) * ln(2)$, these tuples will be $\epsilon_\rho$-distinguishable.

Now, the Hamming metric is definable between these two tuples: they differ only at two places, we have $d_h(l_4, l_5) = 2$. So, they are $\epsilon_h$-indistinguishabilty (wrt $d_h$), for $\epsilon \geq 0$, if and only if: $(2/3) \leq e^{2\epsilon} * (1/3)$, i.e., $\epsilon \geq (1/2) * ln(2)$ .

In other words, if these two tuples are $\epsilon_\rho$-indistinguishable wrt $\rho$ under $\mathcal{M}$ for some $\epsilon$, then they will be $\epsilon_h$-indistinguishable wrt $d_h$ for the same $\epsilon$. But the converse is not true, since $(1/2) * ln(2) < (20/39) * ln(2)$. Said otherwise: *$\mathcal{M}$ $\epsilon$-distinguishes more finely when combined with $\rho$, than with $d_h$.* □

REMARK 3: The statement "$\mathcal{M}$ $\epsilon$-distinguishes more finely with $\rho$, than with $d_h$", is *always true* – not just in Example 4 – For the following reasons: the records that differ 'at some given position' on two bases $D, D'$ are always at distance 1 for the Hamming metric $d_h$, by definition, whatever be the type of data stored at that position. Now, if the data stored at that position 'happened to be' numerical, the usual euclidean distance between the two data could have been (much) bigger than their Hamming distance 1; precisely to avoid such a situation, our definition of the metric $d_{eucl}$ on numerical data 'normalized' the euclidean distance, to ensure that their $d_{eucl}$-distance will not exceed their Hamming distance. Thus, all the 'record-wise' metrics we have defined above have their values in $[0, 1]$, as we mentioned earlier; so, whatever the type of data at corresponding positions on any two bases $D, D'$, the $\rho$-distance between the records will never exceed their Hamming distance. That suffices to prove our statement above. The Proposition below formulates all this, more precisely:

**Proposition 2** *Let $\mathcal{D}_m$ be the set of all databases with the same number $m$ of columns, over a finite set of given data, and $\mathcal{M}$ a probabilistic mechanism answering queries on the bases in $\mathcal{D}$. Let $\rho$ be the metric (defined above) and $d_h$ the Hamming metric, between the databases in $\mathcal{D}$, and suppose given an $\epsilon \geq 0$.*

*- If two databases $D, D' \in \mathcal{D}_m$ are $\epsilon_\rho$-indistinguishable under $\mathcal{M}$ wrt $\rho$, then they are also $\epsilon$-indistinguishable under $\mathcal{M}$ wrt $d_h$.*

*- If the mechanism $\mathcal{M}$ is $\epsilon_\rho$-DP on the bases in $\mathcal{D}_m$ (wrt $\rho$), then it is also $\epsilon$-DP (wrt $d_h$) on these bases.*

The idea of 'normalizing' the Hamming metric between numerical databases (with the same number of columns) has already been suggested in several works (cf. e.g., [6]) for the same reasons. When only numerical databases are considered, the metric $\rho$ that we have defined above is the same as the 'normalized Hamming metric' of [6]. Our metric $\rho$ must actually be seen as a generalization of that notion, to directly handle bases with more general types of data, such as anonymized, taxonomies, etc.

# 8 Using DLTTS for privacy analysis

Our aim now is to show that a DLTTS-based vision can be applied to certain practical problems, such as the following:

- Given: a database $D$ with its qid columns anonymized. And an adversary $A$ with access only to (some among) the qids, and specified probability thresholds on the entries in $D$.

- Objective: To estimate the maximal probability threshold for $A$'s getting access to the sensitive values of (some of) the entries in $D$.

The ideas needed for achieving this objective are all best brought out by the simple and concrete example presented next:

## 8.1 A Motivating example

EXAMPLE: An enterprise **E** stores a database $D$, containing a sensitive value as an integer between 1 and 10, standing for the (anonymized) responses of its employees to a questionnaire on their working conditions. The qid–attributes of $D$ are *Id, Sex, Age*, the anonymized sensitive value is 'Response'. *Age* and *Id* may be anonymized when $D$ is rendered public.

| **id** | **Sex** | **Age** | **Response** |
|---|---|---|---|
| $\ell_1$ | F | [30, 40] | 1 |
| $\ell_2$ | F | [30, 40] | 8 |
| $\ell_3$ | M | [30, 40] | 3 |
| $\ell_4$ | M | [40, 50] | 7 |

**Objective:** *An estimation for the maximal probability, for different attackers to infer the response of one or more randomly chosen employees of* **E***, with runs on suitably constructed DLTTS.*

With such a purpose, the administrator of $D$ conceives three different 'test attacks', where the initial knowledges/beliefs of the attackers on the qid's are specified, so as to be 'sufficiently complementary'. Based on these tests, an empirical strategy will be formulated in subsection 8.3, with some parametric conditions for accepting/refusing a given query for accessing any *given* response. It is assumed that all the employees in **E** have responded to the questionnaire; none of the attackers $A, B, C$ is assumed to have any a priori knowledge of the 'Response' of any particular employee.

## 8.2 Privacy attack analysis

We detail next how DLTTS can be used to analyze privacy attack: we present several realistic attackers, and compare them.

**Evaluating the success of an attack:** There are two types of attackers : Attackers that have some knowledge of the individual in the database for whom they wish to retrieve information (Attackers $A$ and $B$ below), and Attackers with no knowledge other than the database distribution (Attacker $C$). The efficiency of an attack is evaluated by first computing (using the DLTTS) the probability that an attacker interested in re-identifying a given individual will have, using no external knowledge. This gives a baseline probability of success. We then compute (using a different DLTTS) the maximal probability with which an 'informed' attacker retrieves this knowledge. If such an attack has a 'better'

probability of retrieving the correct knowledge, then that attack is considered as a success. (This notion will be formalized in Section 8.3.)

**Attacker** $A$ works with the following initial knowledge(belief):

| | |
|---|---|
| Sex = F : 80%, | Age = [30-40] : 70% |
| Sex = M : 20% | Age = [40-50] : 30% |

• $A$ aims to capture 'preferably' the response of a female employee, age 30 to 40 years.

**Attacker** $B$ works with the following initial knowledge(belief):

| | |
|---|---|
| Sex = F : 20% | Age = [30-40] : 75% |
| Sex = M : 80% | Age = [40-50] : 25% |

• $B$ aims to capture 'preferably' the response of a male employee, age 30 to 40 years.

**Attacker** $C$ works with Initial qid knowledge inferred from the base :

| | |
|---|---|
| Sex = F : 50% | Age = [30-40] : 75% |
| Sex = M : 50% | Age = [40-50] : 25% |

• $C$ to capture the response of 'any' employee of 'any' sex, and of 'any' age. $C$ has no knowledge beyond the distribution of the qid values in the database. In particular, $C$ has no knowledge about possible correlations between the QIDs.

(We shall agree to rename attacker $C$ the "*Basic Analyser*").

**Preliminary Remarks for attack analysis** For a DLTTS-analysis on privacy policies with probability thresholds as intended above, it is natural (even necessary) to 'compare' the probability distributions available prior to choosing a particular transition. Given an outgoing transition $\tau$ at any given node $s$ on a DLTTS, its distribution is a *multiset* of the probability measures on the branches of $\tau$ that we denote as $\mathcal{M}_\tau$. Since our DLTTS are assumed to be 'fully probabilistic', the transition $\tau$ is uniquely determined by $\mathcal{M}_\tau$ and the set $\tau(s)$ of successor states to $s$ (and the labels on the branches from $s$ to its successors).

Now, the multisets of probability distributions (for the transitions available at any given node on a DLTTS) are totally ordered by the multiset extension $\succ$ of the natural order $>$ on numbers (reals or integers). If $\tau, \tau'$ are two outgoing transitions from a state $s$ on a DLTTS, we shall *define $\tau$ to have priority over $\tau'$* if and only if $\mathcal{M}_\tau \succ \mathcal{M}_{\tau'}$. Two different outgoing transitions from a state $s$ can have the same multiset distribution, but with different successor states; if so, neither has priority over the other, they will not be '$\succeq$-distinguishable'.

For computing the maximal probability threshold for any attack to capture sensitive data, it is thus necessary to choose, at any given node on a DLTTS, an outgoing transition $\tau$ such that $\mathcal{M}_\tau$ is maximal for the ordering $\succ$. This will be the case in the attacks presented below.

On the other hand, 'tags' do not play any role in the example considered in this subsection, so the DLTTS constructed for this example make no references to tags and saturation with 'outside' knowledge).

On any DLTTS $\mathcal{A}$ constructed below, and any state $s$ on $\mathcal{A}$ where the *incoming transition has a singleton label set* of the form $\{\ell\}$, a special outgoing transition, referred to as *response*$(\ell)$, is assumed available; its objective is to give access to the 'response'

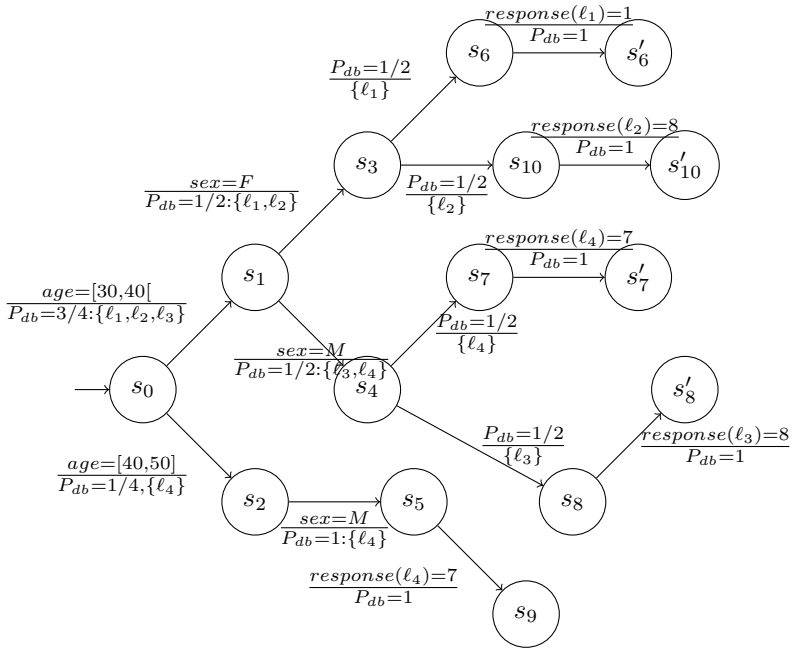


Figure 2: DLTTS-C: Basic Analyser $C$ captures all responses

for the entry $\ell$ in base $D$. This special outgoing transition $response()$ is in fact a switch, 'turned ON, by default'. (But it can be turned OFF, at certain states by the oracle mechanism $\mathcal{O}$ of the DLTTS, on 'considerations of strategy for secrecy' – that we shall present/discuss in section 8.3 below. Pending that discussion, the switch response() is assumed ON by default; it will be an outgoing transition with probability 1.)

**Basic Analyser $C$:** We first consider the case of Basic Analyser $C$, with initial knowledge inferred from the base $D$, and its QIDs. The DLTTS-C (Figure 2) shows how $C$ gets access to the responses of all employees, and respective probability thresholds. The DLTTS construction is based only on the database $D$; the probabilities on its various branches, denoted as $P_{db}$, are all inferred from $D$.

Starting from the initial state $s_0$ on DLTTS-C, the probability for access to the response of employee $\ell_1$ is the product of the probabilities along the branches traversed by the run, namely: $(3/4) * (1/2) * (1/2) = (3/16)$,which is also same for access to the response for entry $\ell_2$; access to the responses for $\ell_3$ and $\ell_4$ are also the same. The transitions of DLTTS-C are of $\succ$-maximal priority at all the nodes. The maximal probability threshold for access to *any of the four responses*, as reported by Basic Analyser $C$, turns out to be 3/16.

**Attacker $B$:** With DLTTS-B (Figure 3), Attacker $B$ gets access to the responses of employees, preference to males of age 30 to 40. ($P_b$ = probabilities from $B$'s assigned objective.)

Maximal probability thresholds computed by $B$: Males 6/10, Females 1/10. Note: The threshold 6/10 for males is higher than the 3/16 reported by $C$.

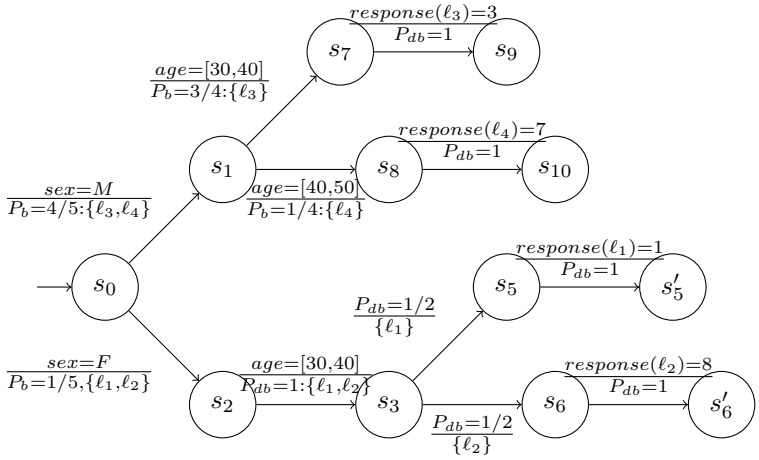


Figure 3: DLTTS-B for $B$'s capture of responses

The actual probabilities for all possible responses are: $Pr(response = 3|M) = 6/10$, $Pr(response = 7|M) = 1/5, Pr(response = 1|F) = 1/10$ and $Pr(response = 8|F) = 1/10$.

**Attacker** $A$**:** With DLTTS-A (Figure 4) Attacker $A$ gets access to the responses of employees, preference to females of age 30 to 40. ($P_a$ = probabilities from $A$'s assigned objective.)

Maximal probability thresholds computed by $A$: Males 7/50, Females 2/5.
Note: The threshold 2/5 for females is higher than the 3/16 reported by $C$.

The actual probabilities for possible responses are: $Pr(response = 3|M) = 7/50$, $Pr(response = 7|M) = 3/50, Pr(response = 1|F) = 2/5$ and $Pr(response = 8|F) = 2/5$.

In view of our earlier remark, we may deduce from these details, that:

- $B$'s attack succeeds at node $s_7$ ($response(\ell_3)$) and node $s_8$ ($response(\ell_4)$), on DLTTS-B.

- $A$'s attack succeeds at node $s_5$ ($response(\ell_1)$) and node $s_6$ ($response(\ell_2)$), on DLTTS-A.

Such 'defeats' for the Basic Analyser can be avoided if the administrator of $D$ (and the oracle mechanism $\mathcal{O}$ in the DLTTS) implement a strategy for better protecting the access to the special values in the base $D$ ('responses', in this example). We present such a strategy in the following subsection.

## 8.3 A Strategy for better 'Secrecy' Protection

For any attacker $N$, and a DLTTS-N constructed by $N$ modeling his/her query-runs on the given base $D$, if $s$ is a node where the incoming transition has a singleton label set $\{\ell\}$, we shall denote by $Pr(s, l; N)$ the probability computed at $s$ with $\succ$-priority transitions along the runs to $s$ from the initial state. We shall denote by $Max_{pr}(\ell; N)$

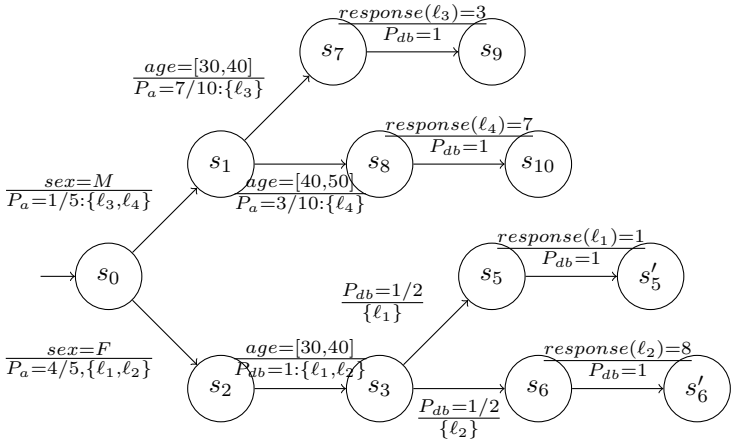


Figure 4: DLTTS-A for $A$'s capture of responses

the maximum of all these probabilities, taken over all such nodes $s$ on DLTTS-N.

(1) Let $s$ be a node on DLTTS-B, with a singleton $\{\ell\}$ labeling the incoming transition.
IF $Max_{pr}(\ell; C) < Pr(s, \ell; B)$,
THEN *Switch OFF* the outgoing transition *response(l)* at $s$.

(2) Do the same on DLTTS-A, with respect to DLTTS-C.

**Details:**

(i) We have: $3/16 = Max_{pr}(\ell_3; C) < Pr(s_7, \ell_3; B) = 3/5$,
and $3/16 = Max_{pr}(\ell_4; C) < Pr(s_8, \ell_4; B) = 1/5$.

(i) We have: $3/16 = Max_{pr}(\ell_1; C) < Pr(s_5, \ell_1; A) = 2/5$,
and $3/16 = Max_{pr}(\ell_2; C) < Pr(s_6, \ell_2; A) = 2/5$.

It follows that the outgoing transition 'response()' can be switched OFF, if we apply the the strategy above at nodes $s_7, s_8$ on DLTTS-B, and at nodes $s_5, s_6$ on DLTTS-A. □

We are in a position now to formulate an empirical strategy for better protecting access to the 'special' values, for any database. The formulation below is for *any* general database $D$ with a 'column of protected values' (anonymized or not) that we shall still refer to as 'responses', and *any* attacker $N$. It is assumed in this formulation that the administrator of the base $D$ has made an appropriate choice for the 'Basic Analyser' $C$.

**The Strategy:**

• Let $s$ be a node on a DLTTS-N under construction by an Attacker $N$ for access to the responses, where the incoming transition at $s$ on DLTTS-N has a singleton label $\ell$.

• Suppose the probability $Pr(s, \ell; N)$ at $s$, computed along the runs to $s$ with the supposed initial knowledge of $N$, and $\succ$-priority transitions all along, satisfies the condition:

$$Pr(s, \ell; N)) > Max_{pr}(\ell; C).$$

Then *switch OFF* the outgoing transition $response(\ell)$ at the node $s$ on DLTTS-N.

## 9 Related Work and Comments

Our work started with the observation that databases could be distributed over several 'worlds', so querying such bases leads in general to answers which would also be distributed; and to the distributed answers one could conceivably assign probability distributions of relevance to the query. It seemed thus natural to view the probabilistic automata of Segala ([14, 15]), with outputs, as an appropriate logical structure for analyzing formally, the evolution of distributed information under the transitions of these automata. Distributed Transition Systems (DTS) appeared a little later, but most of them had as objective the behavioral analysis of the distributed transitions, based on traces or on simulation/bisimulation. Quasi- or pseudo- or hemi- metrics, suitably defined, turned out to be essential for the reasonings employed, for instance, as in [4, 5, 8]. On the other hand, approaches based on metrics have been studied, in particular in [6], for refining notions of differential privacy and of adjacency, for databases in 'standard formats' (numerical or strings., all of the same dimension).

Our lookout for a metric based vision for privacy analysis has been influenced by many of these works, although not with the same objective. As the developments in this work show, our *syntax*-based metric can almost directly handle data of 'mixed types': they can be numbers or literals , but can also be 'anonymized' as intervals or sets; they can also be taxonomically related to each other on a tree structure.

As has been shown in Section 7, the value-wise (partial) metric, constructed in Section 4 on type compatible sets of data, has led us to a finer notion of $\epsilon$-distinguishabilty on mechanisms answering queries. The practical application for the DLTTS vision that we have presented in Section 8 of the current paper, is an addition to our earlier work [1]. Although rather simple, it must be sufficiently illustrative of how the DLTTS vision can be used. On the other hand, the syntactic developments presented in Section 8 have certain similarities, in our opinion, with the semantic considerations presented in [7].

We have not considered any notions of noisy channels perturbing numerical data in the databases; but it is not difficult to extend the DLTTS setup – and the mechanism $\mathcal{M}$ answering the queries – of our work, to handle noise additions. It will then be assumed that the internal (saturation) procedure $\mathcal{C}$, at every state in the DLTTS, incorporates the three well-known noise adding mechanisms: the Laplace, Gauss, and exponential mechanisms, with the assumption that noise additions to numerical values is done in a *bounded* fashion – as in e.g., [12], so as to be from a finite prescribed domain around the values. It will then be assumed that the tuples with noisy data are also in the base signature $\mathcal{E}$. The notion of $\epsilon$-local-indistinguishabilty between tuples with noisy data can also be defined in such an extended setup.

As part of future work, we hope to generalize the value-wise (partial) metric constructed in Section 4 of the current paper, by assigning different 'weights' to the columns of the given base. That could be one of the techniques to 'disfavor' the columns in the base that tend to be 'noisier', or of lesser interest. That would also offer the possibility of taking into account possible dependencies between some of the columns in the base. We hope to deduce still finer notions of adjacency on databases, and of $\epsilon$-distinguishabilty on query answering mechanisms, with such a refinement. As concerns the strategy for better secrecy protection, proposed in Section 8.3, the crucial assumption is on the appropriate choice of the 'Basic Analyser' by the system administrator; it seems rather specific to

the context/example considered, and notions like 'completeness' or 'soundness' of such a strategy may not be easily formalizable.

# 10 Conclusion

We have presented in this article the DLTTS model, whose goal is to capture the knowledge that a querier (considered in this work as an *honest-but-curious* attacker) accumulates when querying a database containing private information, protected by simple privacy policies aiming to protect the values of some specific tuples. We show how DLTTS can be used as a core model by a database administrator to detect privacy breaches and show how it can be used to implement a simple strategy that consists in not answering any further queries.

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# 11 Appendix

Taxonomies are frequent in machine learning. Data mining and clustering techniques employ reasonings based on measures of symmetry, or on metrics, depending on the objective. The Wu-Palmer symmetry measure on tree-structured taxonomies is one among those in use; it is defined as follows ([17]): Let $\mathcal{T}$ be a given taxonomy tree. For any node $x$ on $\mathcal{T}$, define its depth $c_x$ as the number of nodes from the root to $x$ (both included), along the path from the root to $x$. For any pair $x, y$ of nodes on $\mathcal{T}$, let $c_{xy}$ be the depth of the common ancestor of $x, y$ that is *farthest* from the root. The Wu-Palmer symmetry measure between the nodes $x, y$ on $\mathcal{T}$ is then defined as $\mathrm{WP}(x, y) = \frac{2\, c_{xy}}{c_x + c_y}$. This measure, although considered satisfactory for many purposes, is known to have some disadvantages such as not being conform to semantics in several situations.

What we are interested in, for the purposes of our current paper, is a *metric* between the nodes of a taxonomy tree, which in addition will suit our semantic considerations. This is the objective of our Lemma below. (A result that seems to be unknown, to our knowledge.)

**Lemma 1** *On any taxonomy tree $\mathcal{T}$, the binary function between its nodes defined by $d_{wp}(x, y) = 1 - \frac{2\, c_{xy}}{c_x + c_y}$* (notation as above) *is a metric.*

*Proof*: We drop the suffix *wp* for this proof, and just write $d$. Clearly $d(x,y) = d(y,x)$; and $d(x,y) = 0$ if and only if $x = y$. We only have to prove the Triangle Inequality; i.e. show that $d(x,z) \leq d(x,y) + d(y,z)$ holds for any three nodes $x, y, z$ on $\mathcal{T}$. A 'configuration' can be typically represented in its 'most general form' by the diagram below. The boldface characters $X, Y, Z, a, h$ in the diagram all stand for the *number of arcs* on the corresponding paths. So that, for the depths of $x, y, z$, and of their farthest common ancestors on the tree, we get:

$$c_x = X + h + 1, \;\; c_y = Y + h + a + 1, \;\; c_z = Z + h + a + 1,$$
$$c_{xy} = h + 1, \;\; c_{yz} = h + a + 1, \;\; c_{xz} = h + 1$$

The '+1' in these equalities is because the $X, Y, Z, a, h$ are the *number of arcs* on the paths, while the depths are the number of nodes. The $X, Y, Z, a, h$ must all be integers $\geq 0$. For the Triangle Inequality on the three nodes $x, y, z$ on $\mathcal{T}$, it suffices to prove the following two relations:

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{and} \quad d(y,z) \leq d(y,x) + d(x,z).$$

by showing that the following two algebraic inequalities hold:

$$(1)\; 1 - \tfrac{2*(h+1)}{(X+Y+2*h+a+2)} + 1 - \tfrac{2*(h+a+1)}{(Y+Z+2*h+2*a+2)} \geq 1 - \tfrac{2*(h+1)}{(X+Z+2*h+a+2)}$$
$$(2)\; 1 - \tfrac{2*(h+1)}{(X+Y+2*h+a+2)} + 1 - \tfrac{2*(h+1)}{(X+Z+2*h+2*a+2)} \geq 1 - \tfrac{2*(h+a+1)}{(Y+Z+2*h+2*a+2)}$$

The third relation $d(x,y) \leq d(x,z) + d(z,y)$ is proved by just exchanging the roles of $Y$ and $Z$ in the proof of inequality (1).

Inequality (1): We eliminate the denominators (all strictly positive), and write it out as an inequality between two polynomials $eq1, eq2$ on $X, Y, Z,\; h, a$, which must be satisfied for all their non-negative integer values:

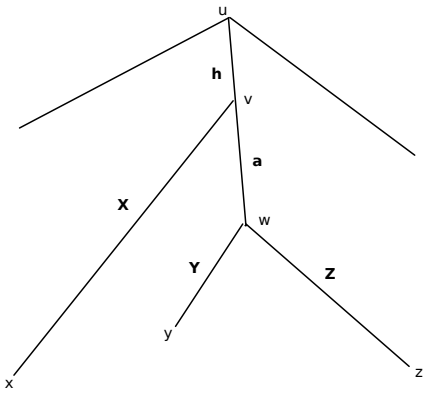
$$eq1 : (X + Y + 2*h + a + 2) * (Y + Z + 2*h + 2*a + 2) * (X + Z + 2*h + a + 2)$$
$$\begin{aligned} eq2 : {} & (h+1) * (Y + Z + 2*h + 2*a + 2) * (X + Z + 2*h + a + 2) \\ & + (h + a + 1) * (X + Y + 2*h + a + 2) * (X + Z + 2*h + a + 2) \\ & - (h+1) * (X + Y + 2*h + a + 2) * (Y + Z + 2*h + 2*a + 2) \end{aligned}$$

$eq : eq1 - 2 * eq2$. We need to check: $eq \geq 0$ ?

The equation $eq$ once expanded (e.g., under *Maxima*) appears as:

$$\begin{aligned} eq : {} & YZ^2 + XZ^2 + aZ^2 + Y^2Z + 2XYZ + 4hYZ + 2aYZ + 4YZ + X^2Z + 4hXZ + \\ & 2aXZ + 4XZ + a^2Z + XY^2 + 4hY^2 + aY^2 + 4Y^2 + X^2Y + 4hXY + 2aXY + 4XY + \\ & 8h^2Y + 8ahY + 16hY + a^2Y + 8aY + 8Y \end{aligned}$$

The coefficients are all positive, and inequality (1) is proved.




Inequality (2): We first define the following polynomial expressions:

$eq3 : (X + Y + 2 * h + a + 2) * (X + Z + 2 * h + a + 2) * (Y + Z + 2 * h + 2 * a + 2);$
$eq4 : (h + 1) * (Y + Z + 2 * h + 2 * a + 2) * (2 * X + Y + Z + 4 * h + 2 * a + 4);$
$eq5 : (h + a + 1) * (X + Y + 2 * h + a + 2) * (X + Z + 2 * h + a + 2);$

If we set $eqn : eq3 + 2 * eq5 - 2 * eq4$, we get

$$\begin{aligned} eqn : & -2(h+1) * (Z + Y + 2h + 2a + 2) * (Z + Y + 2X + 4h + 2a + 4) + \\ & (Y + X + 2h + a + 2) * (Z + X + 2h + a + 2)(Z + Y + 2h + 2a + 2) + \\ & 2(h + a + 1) * (Y + X + 2h + a + 2) * (Z + X + 2h + a + 2) \end{aligned}$$

Inequality (2) is proved by showing that $eqn$ remains non-negative for all non-negative values of $X, Y, Z, h, a$; we expand $eqn$ (with *Maxima*), to get:

$$\begin{aligned} eqn\text{: } & YZ^2 + XZ^2 + aZ^2 + Y^2Z + 2XYZ + 4hYZ + 6aYZ + 4YZ + X^2Z + 4hXZ + 6aXZ + \\ & 4XZ + 8ahZ + 5a^2Z + 8aZ + XY^2 + aY^2 + X^2Y + 4hXY + 6aXY + 4XY + 8ahY + \\ & 5a^2Y + 8aY + 4hX^2 + 4aX^2 + 4X^2 + 8h^2X + 16ahX + 16hX + 8a^2X + 16aX + 8X + \\ & 8ah^2 + 12a^2h + 16ah + 4a^3 + 12a^2 + 8a \end{aligned}$$

The coefficients are all positive, so we are done. □.