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A Prototype for Integral Elimination

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Integral Equation Modeling

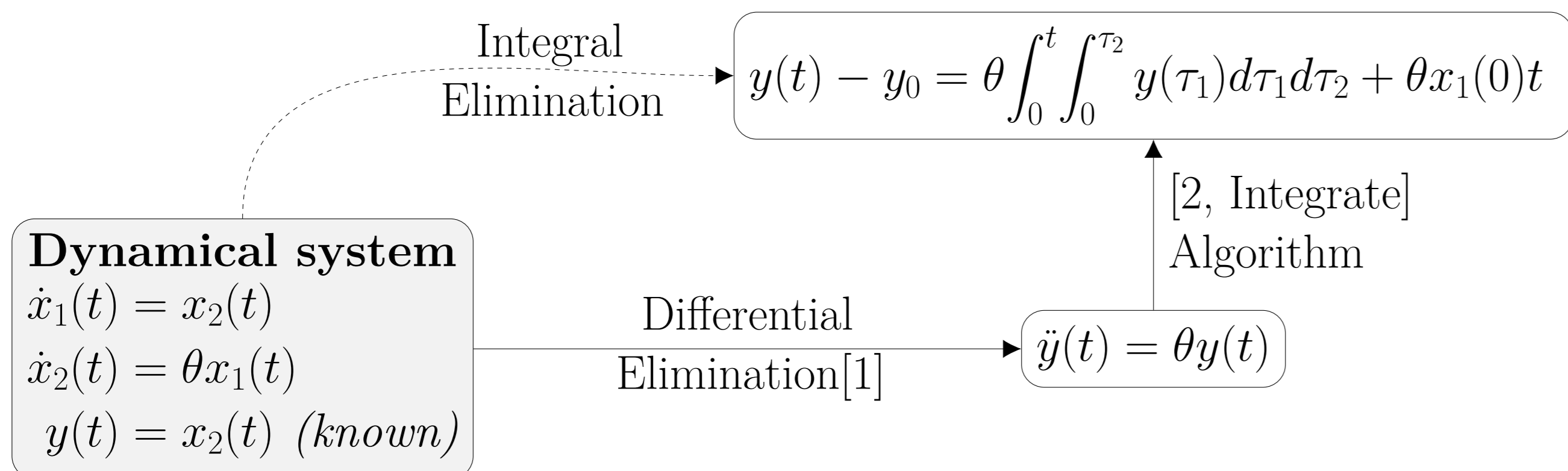


Figure: Modeling, elimination of x_1 and x_2 , and [2, Integrate] algorithm.

Many models (such as the SIR epidemiology model) consist of non-linear differential equations, involving unknown numerical parameters. Recent work investigates the treatment of integro-differential equations i.e. equations involving both derivation and integration operators. The introduction of integral equations helps to increase the expressiveness of the models, improve the estimation of parameters and reduce the size of the intermediate equations.

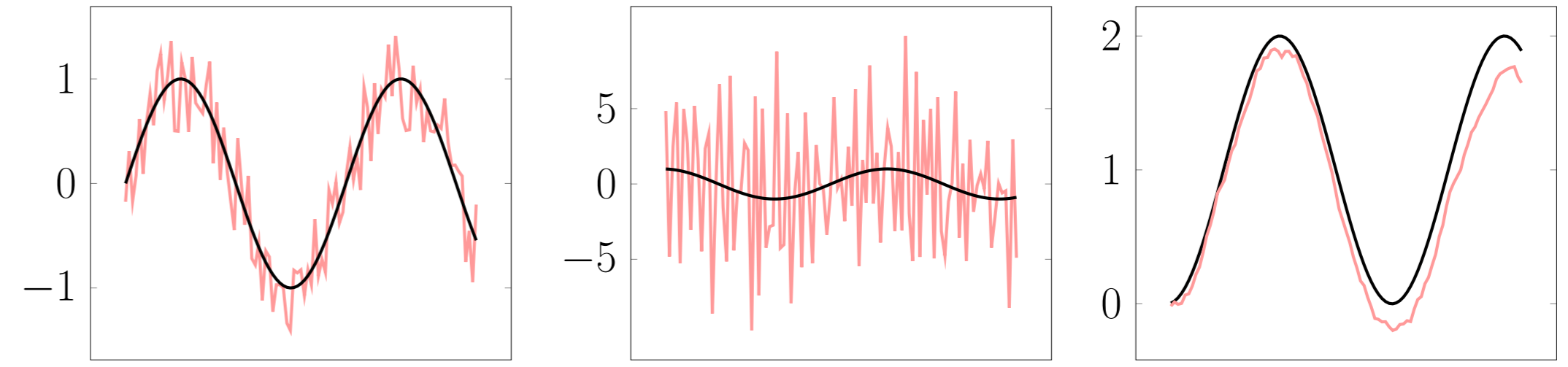


Figure: A signal, its derivative and its integral (with noise in red).

integral_elimination Algorithm

Our algorithm `integral_elimination` which performs the integral elimination

- relies on an ordering on the integral monomials,
- follows a classical Knuth-Bendix approach by treating critical pairs until the set of equations stabilises,
- detects and adds exponential terms during the process,
- uses the formal operator f which encodes the primitive cancelling at $t = 0$.

Our prototype has been implemented in Python using `SymPy` and is able to handle nontrivial examples as the one below. The implementation is available at <https://codeberg.org/louis-roussel/IntegralElimination>.

François Lemaire and Louis Roussel. "Contribution to Integral Elimination". Accepted at CASC 2024. Apr. 2024. URL: <https://hal.science/hal-04570612>

Reduction Simple Case

$$P : y^2 \int (y \int x) - y$$

How to rewrite P by $Q = \int x - y$?

Solution: Use

$$y^2 \int (yQ) = y^2 \int (y \int x) - y \int y^2$$

to reduce P .

Reduced-Product

$$P : \int (x \int y) - y$$

How to rewrite P by $Q = \int x - y$?

Solution: Using the formula below compute $Q \cdot (\int y) = \int (x \int y) + \int (y \int x) - y \int y$ and $\int (yQ) = \int (y \int x) - \int y^2$. Finally, use

$$Q \cdot (\int y) - \int (yQ) = \int (x \int y) - \int (y \int y) + \int y^2$$

to reduce P .

$$\text{IBP: } \int a \cdot \int b = \int (a \int b) + \int (b \int a)$$

Reduced-Power

$$P : \int xy - x$$

How to rewrite P by $Q = \int x - y$?

Solution: First compute

$$Q^2 = 2 \int (x \int x) - 2y \int x + y^2,$$

$$2 \int (xQ) = 2 \int (x \int x) - 2 \int xy$$

and $2yQ = 2y \int x - 2y^2$. Then, use

$$Q^{\circledast} = Q^2 - 2yQ + 2 \int (xQ) = 2 \int xy - y^2$$

to reduce P .

Critical Pairs (one of three cases)

$$\begin{cases} P : \int x^2 - y \int y = 0 \\ Q : \int xy - x = 0 \end{cases}$$

Solution: First compute $P^{\circledast} = -2 \int (x^2 y \int y) + (y \int y)^2$. Then use

$Q^{\circledast} = -2 \int x^2 y + x^2$ to compute the reduced-product

$$Q^{\circledast} \cdot (\int y) - \int (yQ^{\circledast}) = -2 \int (x^2 y \int y) + x^2 \int y - \int x^2 y.$$

Finally introduce

$$\begin{aligned} S(P, Q) &= -\frac{1}{2} P^{\circledast} - \left(-\frac{1}{2} (Q^{\circledast} \cdot (\int y) - \int (yQ^{\circledast})) \right) \\ &= \frac{1}{2} (x^2 \int y - \int x^2 y - (y \int y)^2). \end{aligned}$$

Exponentials

If $A(t) = A(0) + \int (A(t)G(t) + F(t))$
then $A(t) = A(0)u(t) + u(t) \int (v(t)F(t))$
where $u(t) = e^{\int G(t)}$ and $v(t) = e^{-\int G(t)}$

$$T : \begin{cases} R_1 : x = x_0 + \theta \int x \\ R_2 : y = y_0 + \int xy \end{cases}$$

First, we write R_1 and R_2 as $R_1 : x = x_0 + \int (x \cdot \theta + 0)$ and

$R_2 : y = y_0 + \int (y \cdot x + 0)$. Then, T is enlarged with $\begin{cases} R_3 : x = (x_0 u_1) \\ R_4 : y = (y_0 u_2) \end{cases}$ where

$u_1 = e^{\int \theta}$ and $u_2 = e^{\int x}$. Using R_3 in R_2 , and using R_3 in R_4 and u_2 give the following equations in y and θ

$$y = y_0 + x_0 \int (e^{\int \theta} y) \text{ and } y = y_0 e^{\int (x_0 e^{\int \theta})}$$

Example Treated with our Integral Elimination Prototype

$$\begin{cases} \dot{S} = \mu - \beta_I SI - \beta_W SW - \mu S + \alpha R \\ \dot{I} = \beta_W SW + \beta_I SI - \gamma I - \mu I \\ \dot{W} = \zeta(I - W) \\ \dot{R} = \gamma I - \mu R - \alpha R \\ y = \kappa I \end{cases}$$

Differential I/O equation:

- I/O differential equation and identifiability studied in [3] using `StructuralIdentifiability` package in Julia,
- 9 megabytes: Sum of 209 350 terms containing y^{10} , y_1^{10} , y_1^8 , y_1^5 , y_1^4 .

The integral I/O equation is much smaller:

$$\begin{aligned} & -\alpha \beta_W \gamma \zeta \int (u_1 u_3 \int v_1 y \int u_2 v_3 \int v_2 y) - \alpha \beta_I \kappa r_0 \int (u_3 y \int u_2 v_3) - \alpha \beta_W \gamma \kappa w_0 \int (u_1 u_3 \int u_2 v_3 \int v_2 y) \\ & - \beta_I \kappa \mu \int (u_3 y \int v_3) - \alpha \beta_W \gamma \zeta \int (u_1 u_3 \int u_2 v_3 \int v_2 y \int v_1 y) - \beta_W \kappa \zeta s_0 \int (u_1 u_3 \int v_1 y) - \alpha \beta_W \kappa^2 r_0 w_0 \int (u_1 u_3 \int u_2 v_3) \\ & - \alpha \beta_W \kappa \zeta r_0 \int (u_1 u_3 \int u_2 v_3 \int v_1 y) - \alpha \beta_W \kappa \zeta r_0 \int (u_1 u_3 \int v_1 y \int u_2 v_3) - \beta_W \kappa^2 s_0 w_0 \int (u_1 u_3) - \beta_W \kappa^2 \mu w_0 \int (u_1 u_3 \int v_3) \\ & - \beta_W \kappa \mu \zeta \int (u_1 u_3 \int v_1 y \int v_3) - \alpha \beta_W \gamma \zeta \int (u_1 u_3 \int u_2 v_3 \int v_1 y \int v_2 y) - \beta_I \kappa s_0 \int (u_3 y) - \beta_W \kappa \mu \zeta \int (u_1 u_3 \int v_3 \int v_1 y) \\ & - \alpha \beta_I \gamma \int (u_3 y \int u_2 v_3 \int v_2 y) + \gamma \kappa \int (y) - \kappa^2 I_0 + \kappa \mu \int (y) + \kappa y = 0 \end{aligned}$$

$$\begin{cases} v_1 = e^{\zeta \int 1} \\ v_2 = e^{(\alpha + \mu) \int 1} \\ v_3 = e^{(\mu \int 1 + \frac{\beta_I}{\kappa} \int y + \frac{\beta_W}{\kappa} \int y - \frac{\beta_W}{\kappa} e^{-\zeta \int 1} (\int y e^{\zeta \int 1}))} \\ \quad \times e^{(\frac{\beta_W \mu(0)}{\zeta} - \frac{\beta_W \mu(0)}{\zeta} e^{-\zeta \int 1})} \\ u_1 v_1 = u_2 v_2 = u_3 v_3 = 1 \end{cases}$$