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# Source Theory: A Tractable and Positive Ambiguity Theory

Aurélien Baillon,<sup>a</sup> Han Bleichrodt,<sup>b</sup> Chen Li,<sup>c</sup> Peter P. Wakker<sup>c,\*</sup>

<sup>a</sup>emlyon business school, CNRS, Université Lumière Lyon 2, Université Jean Monnet Saint-Etienne, GATE, 69007 Lyon, France; <sup>b</sup>Department of Economics (FAE), University of Alicante, 03690 Alicante, Spain; <sup>c</sup>Erasmus School of Economics, Erasmus University, 3000 DR Rotterdam, Netherlands

\*Corresponding author

Contact: baillon@em-lyon.com,  <https://orcid.org/0000-0002-0169-9760> (AB); hanbleichrodt@ua.es,  <https://orcid.org/0000-0002-2700-412X> (HB); c.li@ese.eur.nl,  <https://orcid.org/0000-0002-7830-2581> (CL); wakker@ese.eur.nl,  <https://orcid.org/0000-0002-2936-5498> (PPW)

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**Abstract.** This paper introduces source theory, a new theory for decision under ambiguity (unknown probabilities). It shows how Savage’s subjective probabilities, with source-dependent nonlinear weighting functions, can model Ellsberg’s ambiguity. It can do so in Savage’s framework of state-contingent assets, permits nonexpected utility for risk, and avoids multistage complications. It is tractable, shows ambiguity attitudes through simple graphs, is empirically realistic, and can be used prescriptively. We provide a new tool to analyze weighting functions: *p*-matchers. They give Arrow–Pratt-like transformations but operate “within” rather than “outside” functions. We further show that ambiguity perception and inverse S probability weighting, seemingly unrelated concepts, are two sides of the same “insensitivity” coin.

**History:** Accepted by Manel Baucells, behavioral economics and decision analysis.

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**Supplemental Material:** The online appendix is available at <https://doi.org/10.1287/mnsc.2023.03307>.

**Keywords:** subjective beliefs • ambiguity aversion • Ellsberg paradox • source of uncertainty

## 1. Introduction

Many decisions, from job choice to investment decisions to medical decisions, involve choices between uncertain alternatives. Since Savage (1954), decision analysis models uncertainty using expected utility (EU). Under EU, decisions can be rationalized by subjective beliefs. The Ellsberg (1961) paradox challenged EU, both normatively and descriptively (Trautmann and van de Kuilen 2015, Gilboa and Marinacci 2016). It showed that people are ambiguity averse and behave as if they trust their subjective probabilities under ambiguity less than the objective probabilities under risk.

This paper presents a new theory of ambiguity: source theory (ST). It shows how probability weighting functions can be used to analyze uncertainty and ambiguity. For example, suppose that a manager has to compare the quality of production components that can be obtained from a local supplier or a foreign supplier, where the

only concern is possible mechanical failure. She may specify her beliefs about such failures, and those beliefs may be the same for the two suppliers. She may still strictly prefer the local supplier because of familiarity. ST allows the beliefs for the two suppliers to be expressed by subjective probabilities, which may be the same. Given identical outcomes, classical EU then cannot explain the strict preference. ST can by using different probability weighting functions (i.e., more pessimistic for the foreign supplier than for the local supplier).

We show that (i) contrary to current thinking, ambiguity can be tractably analyzed in the Savage (1954) framework for uncertainty; (ii) ST gives a theoretical foundation to the empirically popular source method and makes it suitable for prescriptive applications; (iii) ST can model source preference, which generalizes ambiguity aversion; (iv) ST can also model insensitivity, which captures ambiguity perception, is empirically

necessary, and for instance, can explain underprevention; and (v) ST can be applied to all (natural) sources of uncertainty, not just (artificial) urns.

Section 2 reviews the insights, debates, and challenges in the literature and gives the background and motivation for our approach. It also provides applications of ST. Standard concepts are in Section 3. Section 4 formalizes the theoretical framework underlying ST, including sources. We then define source preference and source insensitivity. We do so in both a comparative (Section 5) and an absolute (Section 6) sense. Section 7 obtains our first main result, Theorem 1, a preference foundation of ST.

Section 8 shows that pmatchers provide proper tools to analyze uncertainty attitudes under ST. The pmatchers map subjective probabilities, formally called  $\alpha$ -neutral, of one source into gambling-equivalent (same willingness to bet)  $\alpha$ -neutral probabilities of another source. Our other main results, Theorems 2 and 3 in Section 8, show that under ST, source preference is equivalent to a disliked (roughly, lower) pmatcher, and insensitivity of sources (ambiguity perception) is equivalent to an insensitive (inverse S-shaped) pmatcher. Gutierrez and Kemel (2024) applied pmatchers empirically and showed their usefulness. We axiomatically justify their choice. Section 9 highlights the tractability of ST. Calculations are easy, and plots of pmatchers allow us to visually display ambiguity attitudes (see Figure 1 in Section 9). Section 10 discusses related frameworks, and Section 11 concludes. Proofs are in Appendix B. Technical details and further references are in the Online Appendix.

## 2. History and Motivation

Ambiguity is central in decision under uncertainty. There are many ambiguity theories (Gilboa and Marinacci 2016). Most use the Anscombe and Aumann (1963) framework, which simplifies the mathematical analysis but makes assumptions that restrict its applicability (Section 10). The Savage (1954) framework avoids these restrictions but has been considered too complex for formal analyses of ambiguity. Thus, Karni et al. (2022, p. 229) wrote: “Savage’s most brilliant measure-theoretic approach was not so easily extended beyond its original domain and this was a main reason why so little happened in the field for decades after his 1954 masterpiece.”

By contrast, the Savage framework has often been used in empirical applications of the source method (Einhorn and Hogarth 1985, Abdellaoui et al. 2011) (see Online Appendix D.1 for further references). This method assumes that subjective probabilities exist within sources of uncertainty. However, agents use different weighting functions to evaluate subjective probabilities across sources. For example, in the Ellsberg paradox, people prefer betting on an urn with a known composition to betting on an urn with an unknown composition.

Classical (nonweighted) subjective probabilities cannot explain this behavior. The source method explains it by a more pessimistic weighting function for the unknown urn than for the known urn.

The source method has been suggested since the origins of the ambiguity literature (Fellner 1961, p. 672; Kahneman and Tversky 1979, p. 289). It can analyze any type of uncertainty, not just artificial urns, making it suitable for the analysis of real-world problems, which often involve choices between different ambiguous alternatives. For example, the home bias in finance means that people systematically prefer to invest in domestic stocks over foreign stocks. Both investments are ambiguous, but investors prefer one source of uncertainty to the other. This source preference can be modeled by a different, more pessimistic weighting function for foreign stocks than for domestic stocks.

Despite its wide empirical use, the source method has not yet received a theoretical foundation. There has not yet been a formal definition (our Definition 1), there has been no preference foundation (our Theorem 1), and there has not been a way to characterize people’s source preference. Surveys of ambiguity theories have not mentioned it (Karni et al. 2014, Machina and Siniscalchi 2014, Gilboa and Marinacci 2016). Because of the lack of a theoretical foundation, the source method has not been used in prescriptive applications or subjected to critical empirical tests. It may come as a surprise that such a widely applied empirical method lacks theoretical modeling. This absence is because of the aforementioned perceived difficulty of applying Savage’s framework to formal analyses of ambiguity.

ST gives a theoretical basis for the source method. We give a preference foundation for ST and for its main attitudinal components. Preference foundations are considered necessary for prescriptive applications. Expected utility obtained a normative status only after the Savage (1954) behavioral foundation, allowing decision analysis to thrive. We give theoretical credibility to the source method and make it prescriptively applicable (for non-Bayesians): for instance, in climate change, the biggest field of prescriptive applications of ambiguity today (Berger et al. 2017, Chandy et al. 2019). After measuring preferences with the source method, we can study their properties and judge their normative appeal.

Besides theoretical soundness, empirical testability, and the normative status of a model, preference foundations also show the key concepts to be measured, such as the Savage (1954) subjective probabilities or the Gilboa and Schmeidler (1989) sets of priors. Our theorems show that pmatchers are the key concept for ambiguity. For example, they can capture the Ellsberg paradoxes, the home bias (French and Poterba 1991, footnote 4), and the competence effect (i.e., how one’s perceived level of knowledge about a source of uncertainty influences

choices (de Lara Resende and Wu 2010, definition 3 and equation 1.1). In general, the source method gave good explanatory power for effort provision (Chen and Zhong 2024, equation 3 and prediction 2), belief updating (Baillon et al. 2018b, equation 2), game theory (Ivanov 2011, p. 367, the fourth paragraph and assumption 1), and many other applications (Online Appendix D.1).

We give behavioral foundations of two components of uncertainty attitudes. The first captures aversion/preference. Ambiguity aversion in the Ellsberg paradox is a special case of source preference, for a risky source over an ambiguous source. The second component is insensitivity. Under ambiguity, people struggle to discriminate between different levels of likelihoods. In the extreme, they treat all nondegenerate likelihoods as 50/50. Insensitivity captures ambiguity perception (Baillon et al. 2018b) and adds a subjective element, namely cognitive ability.

Aversion and insensitivity have distinct implications. The home bias is an example of source preference for domestic over foreign stocks. Insensitivity can explain why people are reluctant to take measures toward climate change. If people do not sufficiently distinguish changes in likelihood, then they underestimate the benefits of climate measures. Aversion cannot explain such underestimation because it favors reductions of uncertainty. Extreme insensitivity leads people to only value measures that fully eliminate threats and to dismiss all other mitigation measures.

Insensitivity has received much attention in the recent literature (Trautmann and van de Kuilen 2015). Many studies measured it and showed its relevance for real-life decisions (Dimmock et al. 2016, Watanabe and Fujimi 2024) (Online Appendix D.2). However, no mathematical tools to analyze insensitivity have been provided in the literature as yet. This paper is the first to define insensitivity formally. The analysis is, admittedly, complex but is necessary for applications. Our Theorems 1 and 4 show that insensitivity is also indispensable to model ambiguity theoretically. The problems of insensitivity should be dealt with rather than ignored.

### 3. Basic Definitions

$S$  denotes a *state space*. Its subsets are *events*.<sup>1</sup> Later assumptions imply that  $S$  is infinite. A *weighting function*  $W$  is defined on all events and satisfies  $W(\emptyset) = 0$ ,  $W(S) = 1$ , and  $A \supset B \Rightarrow W(A) \geq W(B)$ . *Probability measures*  $P$  are weighting functions that satisfy additivity. Following Savage (1954), we do not impose countable additivity on probability measures.

$\Gamma$  denotes a set of consequences, or *outcomes*, and can be finite or infinite. We assume that all outcomes are gains, leaving reference dependence to future studies. An *act* maps  $S$  to  $\Gamma$  and is finite valued. We denote acts by  $x = (E_1 : x_1, \dots, E_n : x_n)$ , mapping every  $s \in E_j$  to

$x_j \in \Gamma$ . It is implicit in this notation that the  $E_j$ 's partition  $S$  and that the  $x_j$ 's are outcomes. Further,  $\alpha_E \beta$  denotes  $(E : \alpha, E^c : \beta)$ . A *preference relation*  $\succsim$  of an agent is given over acts, with  $\leq, >, <$ , and  $\sim$  as usual. We assume that  $\succsim$  is a *weak order* (transitive and complete). As usual, we identify constant acts with outcomes so that  $\succsim$  also denotes a preference relation over outcomes.

We assume *biseparable utility* with a *utility function*  $U : \Gamma \rightarrow \mathbb{R}$  and a *weighting function*  $W$  such that preferences over binary acts  $\gamma_E \beta$ , with  $\gamma \succsim \beta$ , maximize

$$W(E)U(\gamma) + (1 - W(E))U(\beta). \quad (1)$$

An event is *null* if its outcomes never affect preference. We assume that  $\succsim$  satisfies *strong monotonicity*: strictly improving an outcome of an act on a nonnull event strictly improves the act.

In Equation (1),  $W(E)$  is called the *decision weight* of event  $E$  when *ranked best* (i.e., when yielding the best outcome(s)). We interpret it as the share of the agent's attention given to event  $E$  if it is ranked best. We sometimes denote it as  $\pi^b(E)$  or  $\pi^b$  for short if  $E$  is understood. The complementary share of attention,  $1 - W(E)$ , is the *decision weight* of event  $E^c$  when *ranked worst* (yielding the worst outcome(s)), also denoted  $\pi^w(E^c)$  or  $\pi^w$  for short if  $E^c$  is understood. For acts  $(E_1 : x_1, \dots, E_n : x_n)$  with more than two outcomes  $x_1 > \dots > x_n$  there are "middle" events with neither best nor worst outcomes. The remaining share of attention is divided among them. Empirical studies usually find that the attention paid to a fixed event  $E_i$  when ranked middle is approximately constant (even though theoretically, it could depend on several aspects of the act). In informal interpretations, we, therefore, sometimes use the term "middle" attention/weight  $\pi^m(E_i)$  without further specification or  $\pi^m$ . The preference conditions defined later will specify all comparisons between  $\pi^b$ ,  $\pi^m$ , and  $\pi^w$ , and later theorems will confirm that these comparisons capture the relevant aspects for uncertainty and ambiguity attitudes.

*Rank-dependent utility* (RDU; or Choquet expected utility), a special case of biseparable utility, holds if  $(E_1 : x_1, \dots, E_n : x_n)$ , with  $x_1 \succsim \dots \succsim x_n$ , is evaluated by  $\sum_{j=1}^n \pi_j U(x_j)$  with  $\pi_j = W(E_j \cup \dots \cup E_1) - W(E_{j-1} \cup \dots \cup E_1)$ ; here,  $\pi_1 = W(E_1) = \pi^b(E_1)$ . *Expected utility* (EU) holds if, further,  $W$  is a probability measure. EU holds if and only if the decision weight of a fixed event is the same for all acts: in particular, when ranked best or worst. This weight is then always the probability of that event. Deviations from EU are characterized by the way in which the decision weight of a fixed event varies over different acts. We analyze and interpret uncertainty attitudes from this perspective.

For events  $A, B$ , we define  $A \succsim B$  ( $A$  is *preferred* to  $B$ ) if  $\gamma_A \beta \succsim \gamma_B \beta$  for some  $\gamma > \beta$ . Under biseparable utility,  $\succsim$  is represented by  $W$ , implying that it is a weak order, and

the preference is the same for all  $\gamma > \beta$  (Savage 1954, condition P4). The *event interval*  $[E, G]$  contains all events  $F$  with  $W(E) \leq W(F) \leq W(G)$  (i.e.,  $E \leq F \leq G$ ). The *Archimedean axiom* holds if there is no infinite sequence of disjoint nonnull events  $E_1, E_2, \dots$  with  $E_i \sim E_j$  for all  $i, j$ . The axiom is necessary and not restrictive in uniform sources (defined later), the main topic of this paper. For simplicity, we assume it throughout.

The function  $\pi^w(E) = 1 - W(E^c)$  is called the *dual* of  $W$ . Although duality is not needed in our formal analysis, it facilitates conceptual understanding. Insensitivity conditions defined later always involve two conditions that are, in fact, one condition but imposed both on the weighting function and its dual.

We assume the Gilboa (1987) Savage-type richness: (i) There are at least three nonequivalent outcomes. (ii) *convex rangedness* holds: for all events  $A \subset C$  and  $W(A) \leq \mu \leq W(C)$ , there exists  $A \subset B \subset C$  with  $W(B) = \mu$ . RDU (i.e., prospect theory for gains) is the primary model of interest for our analysis. Intuitive discussions that need specific models will be targeted to this model.

Our formal results are valid under all special cases of biseparable utility that besides rank-dependent utility/Choquet expected utility for uncertainty (Gilboa 1987, Schmeidler 1989, Tversky and Kahneman 1992 for gains), include various multiple prior models, such as  $\alpha$ -maxmin EU (Ghirardato et al. 2004) and its special cases of maxmin EU (Gilboa and Schmeidler 1989), Hurwicz expected utility (Gul and Pesendorfer 2015), and neoadditive utility (Chateauneuf et al. 2007). For risk, which we take as a special case of uncertainty and as part of our model, our results are valid under most of the popular models (Wakker 2010, observation 7.11.1).

## 4. Source Theory

We first formalize the concept of a source. Formally, *sources* are algebras of events. Uncertainty attitudes may differ for different sources. For instance, Tversky and Fox (1995) showed that basketball fans are ambiguity averse for Ellsberg urns but ambiguity seeking for basketball games. Such findings illustrate that ambiguity theories have to reckon with source dependence. Our formal results will always assume that weighting functions satisfy convex rangedness when restricted to sources. That is, we only consider “rich” infinite sources. We sometimes use finite sources in illustrations. An act  $x$  is *from a source* if it is measurable w.r.t. (with respect to) that source (i.e.,  $x^{-1}(\alpha)$  is in the source for each outcome  $\alpha$ ).

In principle, our results can be applied to any algebra of events, taking it as a source. In applications, people will usually specify sources that are of special interest to them, and then, sources are exogenous. In the Ellsberg paradox, they are also exogenous, determined by information about urns irrespective of preference. Therefore, sources will mostly be exogenous, similar to

commodities in consumer theory, and this is our primary interpretation. Other authors have preferred endogenous interpretations of sources. Most of our results concern uniform sources (defined later), which can be taken as endogenous. Observation 5 lists the related results, valid if sources are endogenous and if they are exogenous. Grant et al. (2022) adopted a similarly flexible interpretation of sources.

We sometimes, but not always, assume that a special source, called *risk* and denoted  $\mathcal{R}$ , is present. For  $\mathcal{R}$ , probabilities of its events  $R$  are known, denoted  $K(R)$ . *Risky acts* are acts from that source. We then assume *stochastic dominance*; that is, for  $P = K$  and all risky acts  $x, y$ :

$$[\text{for all } \alpha \in \Gamma : P(x \geq \alpha) \geq P(y \geq \alpha)] \Rightarrow x \geq y. \quad (2)$$

It implies that risky acts that induce the same probability distribution over outcomes are indifferent (apply Equation (2) both ways). In other words, preferences over risky acts depend only on the probability distribution that they induce over outcomes. We usually identify risky acts with their induced probability distributions. We call  $w$  a *probability weighting function* if  $w : [0, 1] \rightarrow [0, 1]$ ,  $w$  is strictly increasing,  $w(0) = 0$ , and  $w(1) = 1$ . Our assumptions imply Observation 1.

**Observation 1.** For  $\mathcal{R}$ , there exists a unique continuous probability weighting function  $w$  such that for all risky events  $R$ ,

$$W(R) = w(K(R)). \quad (3)$$

Because  $R_1 \geq R_2 \Leftrightarrow K(R_1) \geq K(R_2)$  for risky events  $R_1$  and  $R_2$ , we can rewrite Equation (2) for risky acts  $x, y$  as

$$[\text{for all } \alpha \in \Gamma : \{s \in S : x(s) \geq \alpha\} \supseteq \{s \in S : y(s) \geq \alpha\}] \Rightarrow x \geq y. \quad (4)$$

*Cumulative dominance* holds if Equation (4) also holds for all acts  $x, y$  that are not risky. We assume cumulative dominance throughout the paper.

A source  $S$  is *uniform* if there exists a probability measure  $P_S$  on  $S$  such that Equation (2) holds with  $P = P_S$  for all acts  $x, y$  from that source.

**Definition 1.** *Source theory* holds for  $S$  if besides biseparable utility holding for  $S$ ,  $S$  is uniform.

Uniformity has sometimes been called local probabilistic sophistication. Theorem 1 will explain our term “uniform.” In the definition,  $P_S$  may concern subjective probabilities, or it may merely be a mathematical device without any particular interpretation. Risk is a special case of uniformity. As with risk, preferences over acts from a uniform source  $S$  are entirely determined by the probability distributions over outcomes induced by  $P_S$  under ST, but they depend on the source  $S$ . Consequently, as for risk, we can define a continuous source-

dependent probability weighting function  $w_S$  under ST such that we have:

$$\text{for a uniform } \mathcal{S}: W = w_S \circ P_S. \quad (5)$$

The proof is identical to Observation 1 and is omitted. We call  $w_S$  the *source function* of  $\mathcal{S}$ .  $w = w_{\mathcal{R}}$  is the *source function for risk*. The source  $\mathcal{S}$  is *ambiguity neutral* if it is uniform, the risky source  $\mathcal{R}$  is present in the domain of events, and  $w_S = w$ . That is, the probabilities  $P_S$  are treated as if objective. An agent is *ambiguity neutral* if the algebra of all events is ambiguity neutral. In general, for any uniform source  $\mathcal{S}$ , we call  $P_S$  the *a-neutral probability measure* because it would serve as a regular, objective probability measure had the agent been ambiguity neutral (with unchanged risk attitude).

Chew and Sagi (2008) demonstrated the importance of uniformity for ambiguity and thus, “revived” the use of probabilities to analyze ambiguity. ST combines this insight with the classical idea of probability weighting for ambiguity.

$\mathcal{S}$  is an *EU source*, or *expected utility holds in  $\mathcal{S}$* , if EU represents preferences over all acts from  $\mathcal{S}$ . Then, the source is uniform, and  $w_S$  is the identity function. EU for risk thus means that  $w$  is the identity. We summarize the assumptions made so far. They are assumed explicitly in theorems and implicitly elsewhere. Section 10 discusses behavioral foundations of biseparable utility. Using those, we can obtain a complete behavioral foundation of ST and of our other results.

**Assumption 1** (Structural Assumption).  $S$  is a state space with subsets events, and  $\Gamma$  is an outcome set. Acts  $x = (E_1 : x_1, \dots, E_n : x_n)$  are finite-valued maps from  $S$  to  $\Gamma$ , endowed with a weak order  $\succsim$ , the preference relation.  $\Gamma$  contains at least three nonindifferent outcomes, and the Archimedean axiom holds. Sources are subalgebras of events. Biseparable utility holds, with a utility function  $U : \Gamma \rightarrow \mathbb{R}$ , a weighting function  $W$  on  $S$ , and a representation  $W(E)U(\gamma) + (1 - W(E))U(\beta)$  over binary acts  $\gamma_E\beta$  ( $\gamma \succ \beta$ ).  $W$  is convex ranged for every source. The relation  $\succsim$  is extended to outcomes via constant acts, maximizing  $U$ , and to events via bets on them, maximizing  $W$ . Strong monotonicity, cumulative dominance, and the Archimedean axiom hold.

## 5. Comparative Uncertainty and Ambiguity Attitudes

This section analyzes the main novelty of uncertainty relative to risk: within-person between-sources comparisons. This was a big but not always sufficiently appreciated novelty in Ellsberg (1961).

### 5.1. Introduction

We take risk as a single source of uncertainty, partly for tractability reasons. The domain of ambiguity, however, is too rich to be taken as one single source, similarly as the domain of nonmonetary commodities is too rich to

be taken as one. Therefore, we distinguish between different sources of ambiguity. This leads to within-person between-sources comparisons.

We consider comparisons between two sources  $\mathcal{A}$  and  $\mathcal{C}$ . Although our analysis is symmetric between the sources, an asymmetric presentation is more convenient. In elucidations, we take  $\mathcal{C}$  as an established source used for calibration and  $\mathcal{A}$  as a new source to be compared with  $\mathcal{C}$ . Ambiguity concerns the special case of  $\mathcal{C} = \mathcal{R}$ . Generic elements of  $\mathcal{A}$  are  $A, A_i, A_j$ , and those of  $\mathcal{C}$  are  $C, C_i, C_j$ .

The preference conditions introduced next completely cover all comparisons between best-, middle-, and worst-rank decision weights  $\pi^b, \pi^m, \pi^w$ : source dispreference concerns  $\pi^b$  losing more weight than  $\pi^w$  (Section 5.2), and insensitivity concerns  $\pi^m$  losing more weight than either  $\pi^b$  or  $\pi^w$  (Section 5.3). Theorems 1 and 4 will further suggest that the two conditions completely capture uncertainty attitudes.

### 5.2. Source Preference

We first consider changes in decision weights for twofold partitions  $(A_1, A_2)$  and  $(C_1, C_2)$  with  $\pi^b(A_1) = \pi^b(C_1)$ .<sup>2</sup> It means  $\pi^w(A_2) = \pi^w(C_2)$  so that the two partitions involve the same decision weights. We call such twofold partitions *matching*. If  $\mathcal{C} = \mathcal{R}$ , then  $K(C_1)$  is the *matching probability* of  $A_1$ . It is the gambling-equivalent objective probability of  $A_1$ .

**Definition 2.** (Source) preference for  $\mathcal{C}$  over  $\mathcal{A}$  holds, or  $\mathcal{C}$  is preferred to  $\mathcal{A}$ , if for all partitions  $(C_1, C_2)$  from  $\mathcal{C}$  and  $(A_1, A_2)$  from  $\mathcal{A}$ , we have

$$W(A_1) = W(C_1) \Rightarrow W(A_2) \leq W(C_2). \quad (6)$$

Thus, if in the two matching partitions we change the ranks of  $A_2$  and  $C_2$  from worst to best, then  $W(A_2) = \pi^b(A_2) < \pi^b(C_2) = W(C_2)$  may occur, whereas we had  $\pi^w(A_2) = \pi^w(C_2)$ . It means that  $A_2$  loses more weight than  $C_2$  (or gains less weight) when becoming best. There is less attention for favorable events, and more for unfavorable events, for source  $\mathcal{A}$  than for source  $\mathcal{C}$ , leading to more dislike of  $\mathcal{A}$ . Formally, source preference (or preference for short) for  $\mathcal{C}$  over  $\mathcal{A}$  allows for such inequalities but precludes any reversed inequalities.

The Ellsberg two-color paradox illustrates source preference for known urns over unknown urns, where we recall that  $W$  captures the preference for gambling on events. Assume that winning on red ( $A_1$ ) from an urn with an unknown composition but symmetric in its two colors, red and black, is indifferent to winning on red from a known urn ( $C_1$ ) with 45% red balls so that 0.45 is the matching probability of  $A_1$  and by symmetry, also of  $A_2$ . The complementary  $C_2$  (winning on black from the known urn: 55%) will then be preferred to winning on black from the unknown urn ( $A_2$ ) as in Equation (6).

Source indifference means source preference both ways. We then also say that the two sources are *equally preferred*. The most well-known case of source preference is ambiguity aversion. Definitions of absolute ambiguity attitudes follow by comparisons to risk.

**Definition 3.** *Ambiguity aversion* holds for source  $\mathcal{A}$  if the risky source  $\mathcal{R}$  is preferred to  $\mathcal{A}$ . *Ambiguity aversion* holds if it holds for the source of all events.

*Ambiguity seeking* is the opposite: source preference for  $\mathcal{A}$  over  $\mathcal{R}$ . *Ambiguity indifference* for source  $\mathcal{A}$  means that there is both ambiguity aversion and ambiguity seeking for  $\mathcal{A}$ . Then,  $\mathcal{A}$  is equally preferred as  $\mathcal{R}$ . Comparative results for ambiguity and uncertainty coincide: source preference for source  $\mathcal{C}$  over source  $\mathcal{A}$  is the same as *more ambiguity aversion* for  $\mathcal{A}$  than for  $\mathcal{C}$ .

Given the richness and monotonicity that we assume, source preference for  $\mathcal{C}$  over  $\mathcal{A}$  is equivalent to the following condition (Appendix A):

$$W(A_1) \geq W(C_1) \Rightarrow W(A_2) \leq W(C_2). \quad (7)$$

Similar conditions have been used in many models in the literature.

### 5.3. Insensitivity

Besides source preference (including ambiguity aversion), which is a motivational component, uncertainty attitudes also include a cognitive component, insensitivity, which reflects a lack of discriminatory power. The behavioral implication of insensitivity is extremity orientedness, focusing on extreme events rather than middle events.<sup>3</sup> Such focusing moves the perfectly linear perception of likelihood in expected utility in the direction of a flat default of “just don’t know” in the middle, where middle events become one blur.

Many ambiguity models in the literature contain a component of ambiguity perception determined by perceived vagueness of information (Marinacci 2015). For instance, the size of the set of priors in the  $\alpha$ -maxmin model can be interpreted this way. Increasing the set of priors increases insensitivity, with more weight for extreme events. Our insensitivity component generalizes ambiguity perception by allowing dependence on the agent’s cognitive ability, a subjective factor. Baillon et al. (2018a) and many other studies (Online Appendix D.3) have shown that, empirically, insensitivity does indeed depend on cognitive ability. Hence, for empirical work, this subjective generalization of ambiguity perception is desirable. For risk, where vagueness of probabilities does not play any role, insensitivity leads to inverse S-shaped probability weighting, and this is the prevailing empirical pattern (Fehr-Duda and Epper 2012). Gonzalez and Wu (1999, pp. 136–139) provided an excellent explanation of insensitivity for risk. Insensitivity is stronger under ambiguity (Trautmann and van de Kuilen 2015)<sup>4</sup> as it is reinforced by the perceived uncertainty about the

probabilities. Thus, our insensitivity component brings together ambiguity perception and inverse S probability weighting. Henkel (2024) found a strong empirical relation between these concepts. Wakker (2010, p. 292) provided a detailed discussion of insensitivity for uncertainty, and for risk (Wakker 2010, section 7.9). Many papers have shown its empirical and practical relevance (end of Section 2).

For insensitivity, we compare middle events with extreme events. To consider middle events, we need threefold partitions  $(A_1, A_2, A_3)$  and  $(C_1, C_2, C_3)$ . We consider partitions where  $\pi^b(A_1) = \pi^b(C_1)$  and  $\pi^w(A_3) = \pi^w(C_3)$ . Then, the remaining share of attention, informally denoted  $\pi^m = 1 - \pi^b - \pi^w$ , will also be the same for the two partitions, and the partitions involve the same decision weights. We, therefore, also call such threefold partitions *matching*.

The following condition entails that changing middle weights  $\pi^m$  to extreme weights  $\pi^b$  or  $\pi^w$  involves more gain of weight in the source with more insensitivity. The insensitivity region  $[B, D]$  below serves to ensure, through the inequalities involving the events in the premises, that middle events are really middle and not (too close to) best or worst. For instance, consider Equation (8) below, comparing the change from  $\pi^m(A_2)$  to  $\pi^b(A_2)$  with the change from  $\pi^m(C_2)$  to  $\pi^b(C_2)$ . Here,  $\pi^m(A_2)$  and  $\pi^m(C_2)$  were calibrated to be equal so that they cancel in the comparison, and we can directly compare  $\pi^b(A_2) = W(A_2)$  with  $\pi^b(C_2) = W(C_2)$ . Then, the inequality with  $W(D)$  ensures that the worst events  $A_3$  and  $C_3$  are big enough to separate  $A_2$  and  $C_2$  when middle from being (close to) worst. This way, we avoid comparisons between  $\pi^b$ ’s and  $\pi^w$ ’s (which concern source preference). Below, Equation (9) is Equation (8) but applied to the dual of  $W$ .

**Definition 4.** Source  $\mathcal{A}$  is *more insensitive* than source  $\mathcal{C}$  with insensitivity region  $[B, D]$  if for all partitions  $(C_1, C_2, C_3)$  from  $\mathcal{C}$  and  $(A_1, A_2, A_3)$  from  $\mathcal{A}$ :

$$\begin{aligned} W(A_1) &= W(C_1) \ \& \ W(A_3^c) \\ &= W(C_3^c) \leq W(D) \Rightarrow W(A_2) \geq W(C_2) \end{aligned} \quad (8)$$

and

$$\begin{aligned} W(B) &\leq W(A_1) = W(C_1) \ \& \ W(A_3^c) \\ &= W(C_3^c) \Rightarrow W(A_2^c) \leq W(C_2^c). \end{aligned} \quad (9)$$

Thus, for  $\mathcal{A}$ , there is more focus on extreme events (i.e., more insensitivity), with  $A_2$  gaining more weight than  $C_2$  when turning from middle to extreme (best in Equation (8) and worst in Equation (9)). A verbal statement of Definition 4 is as follows: assume two threefold matching partitions. If a middle event changes rank with an extreme event, where it is safely bounded away from the other extreme, then more weight is gained as there is more insensitivity.

To illustrate Definition 4, consider source  $\mathcal{A}$ , the variation of a stock index in a given hour, and source  $\mathcal{C}$ , the result of rolls of two 10-sided dies giving a number between 0 and 99. Baillon et al. (2018a) found, on average, indifference between  $A_1 =$  “the index decreases by strictly more than 0.2%” and  $C_1 =$  “the die falls on 38 or less” and also, between  $A_3^c =$  “the index either increases by less than 0.2% or decreases” and  $C_3^c =$  “the die falls on 64 or less.” Subtracting  $C_1$  from  $C_3^c$  has a clear impact, giving event  $C_2$  (die: 39–64) with only 26 chances of winning. However, subtracting the ambiguous  $A_1$  from the larger but still ambiguous  $A_3^c$ , giving the ambiguous event  $A_2$  ( $-0.2\% \leq$  index change  $\leq 0.2\%$ ), will be felt less clearly, yielding a preference to bet on the vaguely small  $A_2$  rather than on the clearly small  $C_2$ .

The inequality  $W(C_3^c) \leq W(D)$  in Equation (8) precludes cases such as  $C_3 = A_3 = \emptyset$ , in which case  $C_2$  and  $A_2$  would actually be ranked worst in the matching partitions and not be genuinely middle. In fact, we would then have twofold partitions and would observe source preference. Within the insensitivity region  $[B, D]$ , there will be less discriminatory power, with  $W$  shallower for  $\mathcal{A}$  than for  $\mathcal{C}$  events. The larger  $[B, D]$  is, the more restrictive and informative the above definition is. Empirically, we can usually take events  $B$  and  $D$  with matching probabilities 0.05 and 0.95, which are strong enough for most applications. We can often even take  $B$  empty.

**Definition 5.** *Ambiguity-generated insensitivity (a-insensitivity) holds for source  $\mathcal{A}$  if it is more insensitive than  $\mathcal{R}$  (w.r.t. insensitivity region  $[B, D]$ ); a-insensitivity holds if there is a-insensitivity for the source of all events (w.r.t. insensitivity region  $[B, D]$ ).*

Comparative results for ambiguity and uncertainty again coincide: *more a-insensitivity for  $\mathcal{A}$  than for  $\mathcal{C}$  is the same as more insensitivity.*

## 6. Absolute Uncertainty Attitudes

Absolute conditions follow from comparative conditions by choosing a neutrality point. Before, we chose the risk source as the neutrality point for ambiguity. The following proposition, which readily follows from substitution, suggests that EU sources are natural neutrality points for general uncertainty. The proposition guarantees that it does not matter which EU source is selected as neutrality source.

**Proposition 1.** *Any two EU sources are equally preferred and equally insensitive with the maximal insensitivity region  $[\emptyset, S]$ .*

We now apply the comparative Definition 2, taking EU sources as the neutrality point: *source preference holds for  $\mathcal{A}$  (in an absolute sense) if there exists an EU source  $\mathcal{C}$  such that  $\mathcal{A}$  is preferred to  $\mathcal{C}$ . By Proposition 1, this then holds for all EU sources  $\mathcal{C}$  (i.e., it is independent of which*

*$\mathcal{C}$  is chosen).*<sup>5</sup> The existence clause regarding  $\mathcal{C}$  may still seem to be problematic for empirical purposes: how to find any such  $\mathcal{C}$ ? Fortunately, substitution in Equation (6) readily shows (using  $W(C_2) = 1 - W(C_1) = 1 - W(A_1)$ ) that the condition is equivalent to

$$W(A) \geq 1 - W(A^c) \text{ for all } A \in \mathcal{A}. \quad (10)$$

Thus, no specification of any EU source  $\mathcal{C}$  is needed after all. The definition is extended this way if no EU source  $\mathcal{C}$  is available in the preference domain.

Unfortunately, there is no convenient verb related to source preference. Hence, we formally say that  $\mathcal{A}$  is *liked* and that the restriction of  $W$  to  $\mathcal{A}$  is *liked* if Equation (10) holds.  $W$  is *liked* if Equation (10) holds for all events  $A$ . *Disliked* results from reversed inequalities.

We use the same approach for insensitivity, applying the comparative Definition 4 with EU sources as neutrality points. Source  $\mathcal{A}$  and also  $W$ 's restriction to  $\mathcal{A}$  are *insensitive with insensitivity region  $[B, D]$*  (in an absolute sense) if there exists an EU source  $\mathcal{C}$  such that  $\mathcal{A}$  is more insensitive than  $\mathcal{C}$  with insensitivity region  $[B, D]$ . Again, by Proposition 1, this then holds for all EU sources  $\mathcal{C}$  (i.e., it is independent of which  $\mathcal{C}$  is chosen). Substitution in Equation (8) readily shows (using  $W(C_2) = W(C_3^c) - W(C_1) = W(A_3^c) - W(A_1)$  in Equation (8), with dual equalities in Equation (9)) that the condition is equivalent to:

$$W(A_2) \geq W(A_1 \cup A_2) - W(A_1) \quad \text{whenever } W(A_1 \cup A_2) \leq W(D) \quad (11)$$

and

$$1 - W(A_2^c) \geq W(A_1 \cup A_2) - W(A_1) \quad \text{whenever } W(A_1) \geq W(B) \quad (12)$$

for all partitions  $(A_1, A_2, A_3)$  from  $\mathcal{A}$ .

Again, no specification of an EU source  $\mathcal{C}$  is needed, and the definition is extended this way.

The inequalities compare the decision weight of  $A_2$  when ranked middle and when ranked extreme, safely bounded away from the other extreme. The conditions ensure that  $W$  is shallow and “insensitive” for events in  $\mathcal{A}$  between  $B$  and  $D$  (i.e., on the insensitivity region  $[B, D]$ ). Equation (11) (and similarly, Equation (13) below) without the boundary restriction ( $W(A_1 \cup A_2) \leq W(D)$ ) is sometimes called subadditivity. Insensitivity amounts to imposing subadditivity and its dual but imposing boundary conditions to avoid that the two conditions “bite” each other.

For risk,  $w$  is *liked* if  $w(p) \geq 1 - w(1 - p)$  for all  $p$  and *disliked* if the reversed inequality holds. Further,  $w$  is *insensitive with insensitivity region  $[b, d]$*  ( $0 \leq b < d \leq 1$ ) if for all probabilities  $p_1, p_2, p_3$  summing to 1:

$$w(p_2) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 + p_2 \leq d \quad (13)$$

and

$$1 - w(p_1 + p_3) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 \geq b. \quad (14)$$

For  $w$ , the insensitivity region  $[b, d]$  is a subinterval of the reals. If risk is available as a source with objective probability measure  $K$ , then  $E[b, d]$  denotes the corresponding event interval  $[B, D]$  (i.e.,  $K(B) = b$  and  $K(D) = d$ ). In other words, it contains the events with matching probabilities between  $b$  and  $d$ .

## 7. A Behavioral Foundation of Source Theory and Its Main Attitudinal Comparisons

Preference conditions to capture the aforementioned comparative properties readily follow because all conditions were in terms of inequalities and equalities for  $W$  that immediately translate into preferences and indifferences between events. We, therefore, use the same terms. *Source preference holds for  $C$  over  $A$*  if for all partitions  $(A_1, A_2)$  from  $A$  and  $(C_1, C_2)$  from  $C$ :

$$A_1 \sim C_1 \Rightarrow A_2 \leq C_2. \quad (15)$$

There is more insensitivity for source  $A$  than for source  $C$  (or  $A$  is more insensitive than  $C$ ) with insensitivity region  $[B, D]$  if for all partitions  $\{C_1, C_2, C_3\}$  from  $C$  and  $\{A_1, A_2, A_3\}$  from  $A$ ,

$$A_1 \sim C_1 \ \& \ A_3^c \sim C_3^c \leq D \Rightarrow A_2 \geq C_2 \quad (16)$$

and

$$B \leq A_1 \sim C_1 \ \& \ A_3^c \sim C_3^c \Rightarrow A_2^c \leq C_2^c. \quad (17)$$

Again, the left-hand side of Equation (16) specifies that  $A_2$  and  $C_2$  have the same weight when ranked middle and that they are safely bounded away from the extreme/worst rank because the worse  $A_3$  and  $C_3$  are big enough. The right-hand side then specifies that  $A_2$  gains more weight than  $C_2$  when moved from the middle to the other extreme (the best) rank. Equation (17) is the same condition imposed dually. Again, when comparing with one extreme, one bounds away from the other extreme. The following result trivially follows from substitution.

**Observation 2.**  $W$  shows more source preference (or insensitivity) for one source over another if and only if preferences do.

We further have Observation 3.

**Observation 3.** The source preference and source insensitivity relations are transitive. For insensitivity, the new insensitivity region is the intersection of the other two.

Two sources are *equally preferred* or *equally insensitive* if the comparative relations hold in both directions. In

several results presented later, insensitivity regions should be large enough to avoid triviality. An insensitivity region  $[B, D]$  is *regular* for source  $S$  if for every fourfold partition  $(E_1, E_2, E_3, E_4)$  of  $S$  from  $S$ , we have  $E_j \geq B$  and  $E_j^c \leq D$  for at least one  $j$ . Intuitively, the region should capture at least the middle half of the event domain. If we take  $S$  uniform and  $B$  and  $D$  from  $S$ , then  $P_S(B) \leq \frac{1}{4} \leq \frac{3}{4} \leq P_S(D)$  follows (take  $P_S(E_j) = \frac{1}{4}$  for all  $j$ ). Empirically, insensitivity regions are commonly found to be larger.

Under the assumption of EU for risk (or another source), commonly made in the literature on ambiguity, the absolute conditions can readily be obtained by applying Observation 2 to comparisons with those EU preferences. How to obtain a tractable general behavioral foundation of these conditions in general without assuming EU for risk, is an open question to us. We are now ready for our first main result, a preference foundation of ST.

**Theorem 1.** Under structural Assumption 1, the following two statements are equivalent for a source  $S$ .

- i. Source theory (Definition 1) holds for  $S$ .
- ii.  $S$  is equally preferred and insensitive to itself w.r.t. a regular insensitivity region.

The proof shows that the probability measure in a uniform source is uniquely determined. There have been several behavioral foundations of general probabilistic sophistication (Machina and Schmeidler 1992; Chew and Sagi 2008; Grant et al. 2022, theorem 5). Buchak (2013) provided a behavioral foundation under rank-dependent utility, an important special case of our model. The main feature of our behavioral foundation is that it captures the meaning of probabilistic sophistication (within one source) directly in terms of ambiguity attitudes: the source must be equal to itself regarding ambiguity aversion and a-insensitivity. There should not be more sensitivity or preference for some of its events than for others: hence, the term uniform. Theorem 1 suggests that our two components, source preference and insensitivity, completely capture the essence of ambiguity/uncertainty attitudes.

## 8. pmatchers to Capture Uncertainty Attitudes Under Source Theory

This section provides comparative uncertainty/ambiguity results for ST, resulting in a new tool to analyze uncertainty: pmatchers. It has often been observed that in doing so, one should control for beliefs. We indeed do so in all results in this paper.

**Assumption 2** (For This Section).  $ST$  holds for sources  $C$  and  $A$  with generic events  $C, C_1, C_2, \dots$  and  $A, A_1, A_2, \dots$ ,  $a$ -neutral probability measures  $P_C$  and  $P_A$ , and source functions  $w_C$  and  $w_A$ .

We first present the second and third main results of this paper, illustrated in Figure 1 in Section 9, and then

discuss them, especially the resulting new concept of pmatchers (denoted  $\varphi$ ).

**Theorem 2.** *Under structural Assumption 1 and Assumption 2, the following two statements are equivalent for sources  $\mathcal{A}$  and  $\mathcal{C}$ .*

- i.  $\mathcal{C}$  is preferred to  $\mathcal{A}$ .
- ii. There exists a disliked transformation  $\varphi$  such that  $w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$ .

Ambiguity aversion as in the Ellsberg paradox, the competence effect, and the home bias can all be accommodated by source preference: one prefers to deal with uncertainties from a source  $\mathcal{C}$  that is unambiguous, that one feels more competent about, or that is about domestic stocks than from an opposite source  $\mathcal{A}$ . Thus, preferences as in Equation (15) and inequalities as in Equation (6) will result. Theorem 2 shows that pmatchers capture these phenomena.

**Theorem 3.** *Under structural Assumption 1 and Assumption 2, the following two statements are equivalent for sources  $\mathcal{A}$  and  $\mathcal{C}$ .*

- i.  $\mathcal{A}$  is more insensitive than  $\mathcal{C}$ .
- ii. There exists an insensitive transformation  $\varphi$  such that  $w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$ .

Furthermore, if the insensitivity region for the preference condition is  $[B, D]$  where the boundary events (i.e.,  $B, D$ ) are from source  $\mathcal{A}$ , then the insensitivity region for  $\varphi$  is  $[P_{\mathcal{A}}(B), P_{\mathcal{A}}(D)]$ .

The literature on ambiguity and its applications has as yet mostly focused on the source preference component. However, Trautmann and van de Kuilen (2015) and many others emphasized the importance of insensitivity (end of our Section 2). We expect more insensitivity under lower competence and for unfamiliar (foreign) stocks and, thus, predict preference for lower competence gains and unfamiliar gains if the perceived likelihoods (inputs of  $\varphi$  in statement (ii) in Theorem 3 above) are low, contrary to current thinking in the literature. We leave further study of such implications to future work.

In Theorem 3, for any general insensitivity region  $[B', D']$ , we can always take  $B$  and  $D$  from  $\mathcal{A}$  (take  $B \sim B'$  and  $D \sim D'$ ), and then, the insensitivity region for  $\varphi$  is  $[P_{\mathcal{A}}(B), P_{\mathcal{A}}(D)]$ . In this sense, the theorem can handle any general insensitivity region.

The theorems identify as a central tool for analyzing uncertainty the transformation  $\varphi$ , which satisfies

$$\varphi = w_{\mathcal{C}}^{-1} \circ w_{\mathcal{A}} \text{ and } w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi. \quad (18)$$

By convex rangedness and strong monotonicity,  $\varphi$  is well defined, continuous, and strictly increasing. It is also uniquely determined in all results in this section. It calibrates for every a-neutral probability  $p$  in source  $\mathcal{A}$ , the gambling-equivalent a-neutral probability  $\varphi(p)$  in source  $\mathcal{C}$ . That is, we take  $\mathcal{C} \sim \mathcal{A}$ , and then, for  $P_{\mathcal{A}}(A) = p$ , we have  $P_{\mathcal{C}}(C) = \varphi(p)$ . We, therefore, call  $\varphi$  the pmatcher from  $\mathcal{A}$  to  $\mathcal{C}$ . We henceforth use this notation  $\varphi$ . This  $\varphi$

can readily be obtained empirically if the a-neutral probabilities of the sources are available. Dimmock et al. (2016, theorem 3.1) showed that matching probabilities conveniently capture ambiguity attitudes. This concerns the special case where source  $\mathcal{C}$  is risk. Thus, our Theorems 2 and 3 have generalized their result to general uncertainty, showing that pmatchers generalize matching probabilities and are suited to analyze uncertainty attitudes in general. For easy reference, we display the following result, which readily follows from the preceding discussion.

**Observation 4.** *Under structural Assumption 1 and Assumption 2,  $\varphi(P_{\mathcal{A}}(A)) = P_{\mathcal{C}}(C)$  if and only if  $A \sim C$ . Consequently,  $\varphi(P_{\mathcal{A}}(A)) > P_{\mathcal{C}}(C)$  if and only if  $A > C$ , and  $\varphi(P_{\mathcal{A}}(A)) < P_{\mathcal{C}}(C)$  if and only if  $A < C$ .*

The following theorem shows that we completely capture ambiguity and uncertainty attitudes through the two components of preference and insensitivity. We do not impose Assumption 2 in the following theorem because it is implied by the other conditions.

**Theorem 4.** *Under structural Assumption 1, sources  $\mathcal{A}$  and  $\mathcal{C}$  are equally preferred and insensitive with a regular insensitivity region if and only if both are uniform and  $w_{\mathcal{A}} = w_{\mathcal{C}}$ .  $\mathcal{A}$  and  $\mathcal{C}$  are then equally insensitive with the maximal insensitivity region  $[\emptyset, S]$ .*

It is desirable that we can compare sources and possibly find that they are at the same level regarding relevant properties. The theorem shows that such a comparison is only possible for uniform sources, underscoring their importance. Example 1 will show that insensitivity is indispensable for this purpose.

Ambiguity concerns comparisons of uncertainty with risk (i.e.,  $\mathcal{C} = \mathcal{R}$ ). Hence, our results for general uncertainty immediately imply the following results for ambiguity.

**Corollary 1.** *Assume  $\mathcal{C} = \mathcal{R}$ . Under structural Assumption 1 and Assumption 2, we have*

$$w_{\mathcal{A}} = w_{\mathcal{R}} \circ \varphi. \quad (19)$$

Here,  $w_{\mathcal{A}}$ , capturing the uncertainty attitude, combines the risk attitude  $w_{\mathcal{R}}$  and the ambiguity attitude  $\varphi$ , where the pmatcher  $\varphi$  is the matching probability function. Ambiguity aversion holds if and only if matching probabilities are disliked; a-insensitivity holds if and only if matching probabilities are insensitive. Ambiguity neutrality holds for source  $\mathcal{A}$  if and only if it is equally preferred<sup>6</sup> and insensitive as risk (with a regular insensitivity region). They are then equally insensitive with the maximal insensitivity region  $[\emptyset, S]$ . Ambiguity neutrality holds for all events if and only if the source of all events is equally preferred and insensitive as risk.

The following example shows that aversion/source preference alone is not enough to capture ambiguity and uncertainty attitudes or their neutrality.

**Example 1.** Suppose an agent behaves according to EU for a known urn (risk) but has an insensitive, symmetric,<sup>7</sup> but nonlinear source function  $w_A$  for an unknown urn. Then,  $\gamma_A \beta \sim \gamma_p \beta \Rightarrow \gamma_{A \cdot \beta} \sim \gamma_{1-p} \beta$  so that ambiguity indifference holds. This has often been defined as ambiguity neutrality in the literature. However, the agent is less sensitive to the unknown urn, and ambiguity greatly impacts the agent.

The example underscores a new insight that we obtain from analyzing insensitivity: we must distinguish between ambiguity indifference and the strictly more restrictive ambiguity neutrality. This distinction is usually not made in the literature.

As mentioned before, sources can be taken to be endogenous in our analysis. An *endogenous uniform source* is any algebra of events that is uniform, with convex rangedness of the restriction of  $W$ . All results of this section can then be applied.

**Observation 5** (Endogenous Results). The results of this section (Theorem 2, Theorem 3, Theorem 4, Observation 4, and Corollary 1) remain valid if  $\mathcal{A}$  and  $\mathcal{C}$  are endogenous uniform sources.

### 9. Tractability of Source Theory

Besides being empirically tractable, ST also provides tractable calculations. The formula

$$\int_0^{\infty} w_S(G_{f,U}(\alpha)) d\alpha \tag{20}$$

captures the rank-dependent utility value of an act  $f$  from a uniform source  $S$ , assuming nonnegative outcomes and utilities. Here, with  $P_S$  the a-neutral probability measure on  $S$ ,  $G_{f,U}$  denotes the dual ( $1 - \dots$ ) of the distribution function that  $f$  induces over outcome utilities by  $P_S$ , and  $w_S$  denotes the source function. The only addition to classical Bayesian expected utility calculations concerns the added transformation  $w_S$ . This addition is straightforward and easier than, for instance, adding higher-order distributions and calculating double integrals as in the smooth model or solving extra maximization and minimization problems for every evaluation of every single act as in multiple prior models. A restriction of Equation (20) is that the (dual) distribution should be available. Alternative formulas for rank-dependent utility that do not use distribution functions require a ranking of outcomes, and authors have complained about this (Spiliopoulos and Hertwig 2023, footnote 12). However, again, ranking outcomes is easier than carrying out extra integrations or solving extra optimization problems.

Because of their mathematical tractability, the functionals that we use, nonlinear transformations of subjective additive probabilities, currently provide the most popular risk measures (Artzner et al. 1999) called law-

invariant or distorted risk measures. There is extensive literature on these measures, including the works of Wang et al. (2020) and Liu et al. (2021). Hence, this paper also serves as a preference foundation of those risk measures and provides new tools for analyzing them.

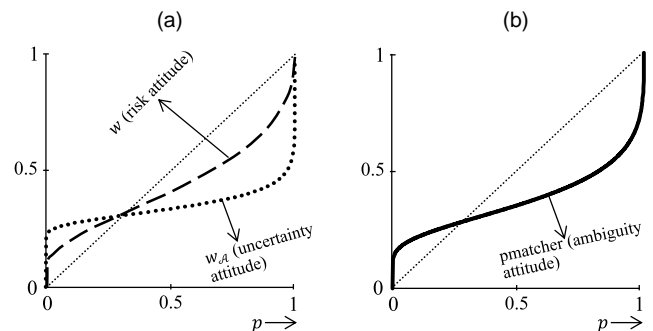
The analysis of the preceding section shows that we can easily manipulate uncertainty attitudes analytically and numerically by inserting the bold transformation  $\varphi_A$  in Equation (21) to capture how  $\mathcal{A}$  deviates from the calibration source  $\mathcal{C}$ . In particular, it has to be inserted to the right of  $w_C$  and not to the left as has been commonly done with Pratt–Arrow-type utility transformations:

$$\int_0^{\infty} w_C \circ \varphi_A(G_{f,U}(\alpha)) d\alpha. \tag{21}$$

Wakker (2004) axiomatized a simple version of such a “right-side” transformation. Adding more source dislike or insensitivity can readily be done by adding  $\varphi_A$  accordingly. Ambiguity is handled by taking  $\mathcal{C} = \mathcal{R}$ . Then, ambiguity aversion is added by adding a disliked  $\varphi_{A'}$ , as we proved axiomatically, and so on. Unfamiliar sources, as with the home bias and competence effect, will involve disliked and insensitive pmatchers  $\varphi_{A'}$  in Equation (21).

An attractive feature of ST is that we can apply it graphically. Figure 1 shows the ease with which uncertainty attitudes can be completely captured and compared visually under ST. It shows the data of subject 2 of Abdellaoui et al. (2011, figure 10).<sup>8</sup> Source  $\mathcal{A}$  is the CAC40 stock index, which passed a test of uniformity. Source  $\mathcal{C}$  is risk. Figure 1(a) shows the source functions  $w_A$  and  $w$ . Risk is disliked ( $w(p) + w(1-p) \leq 1$ ) and also exhibits insensitivity. Dislike and insensitivity are reinforced by the extra uncertainty about probabilities because of the ambiguity of CAC40. The distance between the graph and the diagonal can be taken as a measure of insensitivity. Figure 1(b) shows that the pmatcher ( $w^{-1} \circ w_A$ ; i.e., the matching probability function) indeed has the corresponding properties so that the comparative conditions of Theorems 2 and 3 hold. Figure 1(b) shows what uncertainty adds to risk (i.e., the ambiguity attitude).

**Figure 1.** Attitudes of Subject 2 of Abdellaoui et al. (2011)



Notes. (a) Source functions for CAC40 and for risk. (b) Matching probabilities for CAC40.

A non-Bayesian decision analyst who wants to use ambiguity theory in prescriptive consultancy does not have to show formulas to a client but can directly work with graphs, such as in Figure 1. Extra aversion can be captured by vertically moving the curve downward. Extra perceived uncertainty about probabilities (insensitivity) is captured by horizontally moving weight (“probability mass”) from the expected middle to the unexpected extremes, resulting in fat-tail distributions that have often been observed for returns of financial assets. Insensitivity can provide an intuitive accommodation of the pricing kernel puzzle in finance (Hens and Reichlin 2013, Polkovnichenko and Zhao 2013). Because pmatchers can be represented visually, they are suitable for policy communications as figures are more accessible to nonspecialists than formulas and tables.

## 10. Discussion

### 10.1. Complete Behavioral Foundation and Positive/Normative Status

Our main results assumed biseparable utility. For a complete behavioral foundation of source theory and our other results, a behavioral foundation of that assumption should be provided. Ghirardato and Marinacci (2001) did so for the special case of a continuum of outcomes (e.g., money) but not necessarily a continuum of events as we need. The latter can be guaranteed by adding the Gilboa (1987) event solvability: if  $\gamma > \beta$ ,  $\gamma_A \beta > x > \gamma_C \beta$ ,  $A \supset C$ , then  $\gamma_B \beta \sim x$  for some  $A \supset B \supset C$ . The Gilboa (1987) behavioral foundation of biseparable utility involved a continuum of events and the special case of RDU. Thus, for these two special cases capturing the most important applications, we can obtain complete behavioral foundations by adding the cited axiomatizations. For brevity, we did not repeat them in this paper.

There is much interest in prescriptive applications of ambiguity theory today (Gilboa and Marinacci 2016, Berger et al. 2017, Chandy et al. 2019). Source theory is tractable with visual aids (Figure 1) and can serve this purpose well. It is also descriptively useful: for instance, because insensitivity allows for adding cognitive limitations and subjective factors to ambiguity perception.

### 10.2. Risk as One Source

Following Tversky and Fox (1995, p. 271), we assumed one fixed  $w$  for all objective probabilities. We let parsimony prevail over fit here for tractability reasons. The assumption holds approximately for emotion-neutral risky events and outcomes, and we focus on those.<sup>9</sup> Objective probabilities served as the neutrality benchmark for ambiguity attitudes.

### 10.3. Models of High Generality But Low Specificity

ST provides a specification of Choquet expected utility (Gilboa 1987, Schmeidler 1989) and prospect theory

(Tversky and Kahneman 1992). Those theories use non-additive measures. However, it has often been argued that nonadditive measures are too general to be tractable beyond the simplest state spaces (Tversky and Kahneman 1992, p. 311; Ivanov 2011, p. 367; Basu and Echenique 2020). Basu and Echenique (2020) showed that this tractability problem holds even more for multiple prior models. Second-order distributions as used in the smooth model (Klibanoff et al. 2005) are yet more general and less tractable, with subjective parameters of yet higher cardinality. The problem grows for many generalizations of the aforementioned models proposed in the literature. Spiliopoulos and Hertwig (2023, p. 1198) discussed this problem in their extensive empirical study and hence, used the source method because of its tractability.<sup>10</sup>

The aforementioned ambiguity models, using high-dimensional parameters, have been used in empirical studies, but then, strong parametric assumptions had to be added, especially if the underlying models were very general. Those extra assumptions then drove the results more than the underlying model (Polisson et al. 2020, p. 1783). ST uses nonadditive weighting functions, but it adds uniformity restrictions and thus, achieves better parsimony. Abdellaoui et al. (2011) and Dimmock et al. (2016) showed that the source method is tractable enough to even allow for nonparametric measurements (i.e., without any parametric assumption added). The source method outperformed other ambiguity theories in data fitting (Sonsino et al. 2022, section 5.5) and prediction tests (Kothiyal et al. 2014, Georgalos 2019), underscoring its good trade-off between generality and parsimony.

### 10.4. Further Related Literature

Many studies on ambiguity used the Anscombe and Aumann (1963) (AA) framework. Here, acts do not assign outcomes to states but probability distributions over prizes. Acts are, thus, two stage. EU is assumed for risk,<sup>11</sup> and a backward induction evaluation is applied to the two-stage acts. Denti and Pomatto (2022) showed that this evaluation is equivalent to separable partitions of the state space on which one can condition ambiguity of information. The AA framework makes it possible to use linear algebra to analyze ambiguity, which greatly facilitates the mathematical analysis, and thus, it has propelled the ambiguity field. Multistage optimization should be studied for applications anyhow and is non-trivial for ambiguity. Yet, there is also interest in studying ambiguity in a single-stage framework, such as that of Savage (1954). Multistage stimuli are complex for tests and applications. In the words of Kreps (1988, p. 101): “imaginary objects ... [make] perfectly good sense in normative applications ... But this is a very dicey ... procedure in descriptive applications.” Epstein and Halevy (2019, p. 684) thus favored the

Savage framework: “This disconnect in the literature between Anscombe–Aumann acts and descriptive modeling in the field suggests (to us) that tests of preference models that refer only to Savage-style acts are more relevant to the potential usefulness of these models outside the laboratory.”

Regarding backward induction and in general, multi-stage optimization, as unproblematic and self-evident as they are under classical EU, so problematic and controversial they are under ambiguity and non-EU. Many studies have, thus, criticized backward induction in the AA framework (normatively: Machina 1989; descriptively: Schneider and Schonger 2019) (further references are in Online Appendix D.5). Further, especially for empirical work, it is desirable to allow for violations of EU. ST introduces its ambiguity concepts while avoiding dynamic complications and allowing violations of EU for risk. As a price to pay, our results had to be derived without resorting to linear algebra as a tool to simplify the mathematics. However, as we have shown, our tools remain tractable, providing immunity to violations of backward induction or of EU for risk.

The AA framework requires the availability of events with known probabilities, and most studies of uncertainty have as yet focused on comparing such events with events with unknown (or vague) probabilities. The unknown-probability events mostly concerned Ellsberg urns with unknown compositions because such “artificial” events have a symmetry that facilitates comparisons with known-probability events. The natural events relevant in applications, such as in climate change, usually lack such symmetry, and events with known probabilities are often not even available. Thus, for the home bias, neither domestic stocks nor foreign stocks usually have known probabilities, and known-probability events to compare with are usually not available. The AA framework then cannot be applied. ST can be applied. The general techniques of Sections 5–9 can then readily be applied and now that there are preference foundations, also can be applied for prescriptions (for non-Bayesians) as in climate change. Many authors have argued for the importance to handle natural events beyond Ellsberg urns.<sup>12</sup>

Gul and Pesendorfer (2015) did not need the Anscombe–Aumann framework. However, their requirement that all ideal events (interpreted as unambiguous) should be elicited (needed to determine their inner and outer measures) is intractable. Further, their assumption that diffuse events exist, which involves extreme unrealistic decision attitudes violating monotonicity (Grant et al. 2022), is unrealistic, both normatively and descriptively.

In many ambiguity theories, ambiguity attitudes depend mainly on the set of outcomes and not on the events. Examples include Chew et al. (2008, pp. 186–187), Kontek and Lewandowski (2018), and Grant et al. (2022). Kontek and Lewandowski (2018, p. 2818)

proposed to use subjective (a-neutral) probabilities as in ST. The most well-known theory of this kind is the Klibanoff et al. (2005) smooth model, where ambiguity attitudes are determined by a function  $\varphi$  operating on outcomes. An outcome interval with  $\varphi$  convex (on the utility image of this interval) gives ambiguity-seeking behavior, and an outcome interval with  $\varphi$  concave gives ambiguity aversion. Such theories cannot accommodate the fourfold pattern of ambiguity, where ambiguity aversion depends on the events concerned rather than on the outcomes or insensitivity; see König-Kersting et al. (2023).<sup>13</sup> In agreement with that reference and with Machina (2009, p. 390) and Dillenberger et al. (2017), we think that ambiguity attitudes are mainly event driven rather than outcome driven. Chew et al. (2017) found that event-driven models fit data better than the smooth model.

The Chateauneuf et al. (2007) neoadditive model is a popular and efficient special case of ST that focuses only on overweighting infimum and supremum values, as does  $\alpha$ -maxmin EU. However, this model ignores attitudes toward intermediate values that are relevant, for instance, in values at risk and their generalizations in finance.

Online Appendix B discusses the drawbacks of a widely used convexity definition for insensitivity, and Online Appendix C discusses the novelty of our conditions relative to Tversky and Wakker (1995).

## 11. Conclusion

For modeling uncertainty and ambiguity attitudes through different probability weighting functions, an idea that has been alluded to for decades because of its plausibility and that has been informally used in many empirical studies, we have provided the first formal framework and behavioral foundations. In particular, we have shown what the right formulas are (e.g., Equation (21)). No formal theory or foundation had been provided before because the proper framework for it, that of Savage (1954), was considered too difficult to handle. We showed that it can be made tractable. A pro of the Savage framework relative to the popular one of Anscombe–Aumann is that we need no multistage stimuli, and we can allow for violations of expected utility under risk. Source theory is specific enough to allow for measurements and predictions, even without parametric assumptions. Now, ambiguity and uncertainty can be analyzed tractably, both analytically and graphically and both prescriptively and empirically realistically.

Our behavioral foundations brought many new insights.

- Probabilistic sophistication and uniform sources can be characterized directly in terms of ambiguity attitudes (Theorem 1). Probabilistic sophistication captures uniform ambiguity.

- Matching partitions provide a useful tool to compare general uncertainty attitudes. They generalize the Dimmock et al. (2016) matching probabilities (Section 5).

- Pmatchers provide a useful tool to capture uncertainty attitudes quantitatively (Theorems 2 and 3 and Equation (21)). They, again, generalize the Dimmock et al. (2016) matching probabilities.

- Equation (21) shows the proper formula for using pmatchers.

- Equation (19) shows that ambiguity is the difference between uncertainty and risk.

- Insensitivity is admittedly complex to analyze, but empirical reality imposes it upon us (Trautmann and van de Kuilen 2015). Insensitivity is needed to completely capture uncertainty and ambiguity as explained at the end of Section 5.1 and confirmed by Theorems 1 and 4.

- Theorem 3 directly connects ambiguity perception and inverse S probability weighting (of the pmatcher). These are two sides of the same insensitivity coin.

- Distinguishing between ambiguity indifference and ambiguity neutrality is important (Example 1). One may neither like nor dislike ambiguity and still be impacted by it.

All of our concepts coherently fit together in source theory. Uncertainty and ambiguity can tractably and realistically be analyzed in the Savage–Gilboa framework.

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## Appendix A. Reformulations Using Weak Preferences

We give reformulations of some conditions using weak preferences instead of indifferences in the premises. They would give less powerful behavioral foundations but are better suited for empirical tests when indifferences are not easy to obtain.

**Lemma A.1.** Equation (6) is equivalent to Equation (7).

**Proof.** Equation (7) immediately implies Equation (6). Next, assume Equation (6). Assume  $W(A_1) \geq W(C_1)$ . By convex rangedness, we can move part of  $A_1$  to  $A_2$  so that the premise of Equation (6) follows. The resulting conclusion in that equation and set monotonicity of  $W$  imply Equation (7).  $\square$

We next give the corresponding reformulations of insensitivity.

**Lemma A.2.** Equation (8) is equivalent to:

$$\begin{aligned} W(C_1) \geq W(A_1) \ \& \ W(C_3^c) \leq W(A_3^c) \\ & \leq W(D) \Rightarrow W(A_2) \geq W(C_2). \end{aligned} \quad (\text{A.1})$$

Equation (9) is equivalent to:

$$\begin{aligned} W(C_3^c) \leq W(A_3^c) \ \& \ W(C_1) \geq W(A_1) \\ & \geq W(B) \Rightarrow W(A_2^c) \leq W(C_2^c). \end{aligned} \quad (\text{A.2})$$

Equation (15) is equivalent to:

For all partitions  $(A_1, A_2)$  from  $\mathcal{A}$  and  $(C_1, C_2)$  from  $\mathcal{C}$ :

$$A_1 \succcurlyeq C_1 \Rightarrow A_2 \preccurlyeq C_2. \quad (\text{A.3})$$

Equation (16) is equivalent to:

For all partitions  $(C_1, C_2, C_3)$  from  $\mathcal{C}$  and  $(A_1, A_2, A_3)$  from  $\mathcal{A}$ :

$$C_1 \succcurlyeq A_1 \ \& \ C_3^c \preccurlyeq A_3^c \leq D \Rightarrow A_2 \succcurlyeq C_2. \quad (\text{A.4})$$

Equation (17) is equivalent to:

For all partitions  $(C_1, C_2, C_3)$  from  $\mathcal{C}$  and  $(A_1, A_2, A_3)$  from  $\mathcal{A}$ ,

$$B \preccurlyeq A_1 \leq C_1 \ \& \ A_3^c \succcurlyeq C_3^c \Rightarrow A_2^c \preccurlyeq C_2^c. \quad (\text{A.5})$$

**Proof.** Equation (A.1) immediately implies Equation (8). Next, assume Equation (8). For Equation (A.1), assume  $W(C_1) \geq W(A_1)$  and  $W(C_3^c) \leq W(A_3^c)$ . By convex rangedness, we can move part of  $C_1$  to  $C_2$  to get  $W(C_1) = W(A_1)$ . Similarly,  $W(C_3^c) = W(A_3^c)$  by moving part of  $C_3$  to  $C_2$ . (The move of part of  $C_1$  to  $C_2$  did not affect  $C_3^c$ .) Equation (8) and set monotonicity of  $W$  imply Equation (A.1). The equivalence of Equations (9) and (A.2) follows similarly. It, in fact, is dual to Equations (8) and (A.1). The remaining results in terms of preferences follow from the corresponding results on  $W$ .  $\square$

## Appendix B. Proofs

**Proof of Observation 1.** We define  $w(r) = W(R)$  for event  $R$  with  $K(R) = r$ . It is well defined because  $K(R) = K(R') = r$  implies, by stochastic dominance,  $W(R) = W(R') = w(r)$ . Convex rangedness implies that  $w$ 's domain is the entire  $[0, 1]$  and that  $w$  is strictly increasing. This is elementary for  $K$  countably additive, the most important case. A general proof is in Online Appendix A. So, Equation (3) holds for all risky events  $R$ . We have  $w(0) = 0$ ,  $w(1) = 1$ . Surjectivity, implied by convex rangedness, implies that  $w$  is continuous.  $\square$

**Proof of Observation 3.** If  $\pi^b$  ( $\pi^m$ ) loses more weight relative to  $\pi^w$  ( $\pi^b$  and  $\pi^w$ ) under  $\mathcal{A}$  than under  $\mathcal{B}$  and more under  $\mathcal{B}$  than under  $\mathcal{C}$ , then also more under  $\mathcal{A}$  than under  $\mathcal{C}$ .  $\square$

**Proof of Theorem 1.** This theorem does not need biseparable utility. The only implication of convex rangedness that we use is the Gilboa (1987) event solvability. We also use cumulative dominance. The proof will be stated in terms of preference conditions. Throughout, all events are assumed to be from the source  $\mathcal{S}$ . By convex rangedness, boundary events can be assumed to be from  $\mathcal{S}$ . We write  $P$  for  $P_{\mathcal{S}}$  throughout. We first show that the conditions in statement (ii) are necessary. For source preference, assume  $A_1 \sim C_1$ . Then,  $P(A_1) = P(C_1)$ ; so, with subscript 2 indicating complements,  $P(A_2) = P(C_2)$ ,  $W(A_2) = W(C_2)$ , and  $A_2 \sim C_2$ , and so,  $A_2 \preccurlyeq C_2$  as required by source preference. For insensitivity, consider matching threefold partitions as before.  $A_1 \sim C_1$  and  $A_3^c \sim C_3^c$  imply  $P(A_1) = P(C_1)$  and  $P(A_3) = P(C_3)$ ; so,  $P(A_2) = P(C_2)$ , and hence, both  $A_2 \sim C_2$  and  $A_2^c \sim C_2^c$ . We have  $A_2 \succcurlyeq C_2$  and  $A_2^c \preccurlyeq C_2^c$  as required by more insensitivity. The boundary conditions were not needed here. This shows that the insensitivity region can be taken maximal:  $[\emptyset, S]$ . The conditions in (ii) are, indeed, necessary.

We next consider the reversed implication. We assume that  $\mathcal{S}$  is equally preferred and insensitive to itself w.r.t. a

regular insensitivity region  $[B, D]$ . We throughout use the reformulated conditions of Appendix A. By source preference,  $A \succcurlyeq C \Rightarrow A^c \preccurlyeq C^c$ . The reversed implication,  $A^c \preccurlyeq C^c \Rightarrow A \succcurlyeq C$ , follows by taking complementary events. Hence:

$$A \succcurlyeq C \Leftrightarrow A^c \preccurlyeq C^c. \quad (\text{B.1})$$

We often use Equation (B.1) implicitly below.

**Lemma B.1.** Assume  $E \cap H = F \cap H = \emptyset$ . Then,  $E \succcurlyeq F \Leftrightarrow E \cup H \succcurlyeq F \cup H$ .

**Proof of Lemma B.1.** First, assume  $E \sim F$ . It implies  $E \cup H \succcurlyeq F \cup H$  in the following three cases.

1.  $H \succcurlyeq B$ . By Equation (A.5), with  $A_1 = H$ ,  $A_2 = (F \cup H)^c$ ,  $A_3 = F$ ,  $C_1 = H$ ,  $C_2 = (E \cup H)^c$ ,  $C_3 = E$ .
2.  $H \succcurlyeq D^c$ . By Equation (A.4), with  $A_1 = F$ ,  $A_2 = (F \cup H)^c$ ,  $A_3 = H$ ,  $C_1 = E$ ,  $C_2 = (E \cup H)^c$ ,  $C_3 = H$ .
3.  $E \succcurlyeq B$  or  $E \succcurlyeq D^c$ . As above but with  $A_1$  and  $A_3$  interchanged and  $C_1$  and  $C_3$  too.

By symmetry of  $E$  and  $F$ , we also get  $F \cup H \succcurlyeq E \cup H$ ; that is, we get  $E \cup H \sim F \cup H$  in the above three cases. If  $E > F$ , then we take  $E \supset E' \sim F$  and apply the above cases with  $E'$  instead of  $E$ , and by monotonicity ( $E - E'$  is non-null),  $E \cup H > E' \cup H \sim F \cup H$ . Similarly,  $F > E$  implies  $F \cup H > E \cup H$ . Thus, in cases (1)–(3), we have  $E \succcurlyeq F \Leftrightarrow E \cup H \succcurlyeq F \cup H$ .

Finally, assume none of cases (1)–(3). Define  $G = (E \cup F \cup H)^c$ . By regularity, the fourfold partition  $(E, F - E, G, H)$  implies  $G \succcurlyeq B$  (also,  $G \succcurlyeq D^c$ ). By case (1), with  $G \cup H$  for  $H$ , we have  $E \succcurlyeq F \Leftrightarrow E \cup (G \cup H) \succcurlyeq F \cup (G \cup H)$ . By case (1), with  $G$  for  $H$ ,  $E \cup G$  for  $E$ , and  $F \cup G$  for  $F$ , we have  $E \cup (G \cup H) \succcurlyeq F \cup (G \cup H) \Leftrightarrow E \cup H \succcurlyeq F \cup H$ . We get  $E \succcurlyeq F \Leftrightarrow E \cup H \succcurlyeq F \cup H$  for all cases. Q.E.D.

By Lemma B.1, weak ordering of  $\succcurlyeq$ , event solvability, and Krantz et al. (1971, theorem 5.2.2; their axiom 5 follows from event solvability), there exists a unique probability measure  $P$  on source  $S$  that represents the preference relation  $\succcurlyeq$  on events. (As an aside, so does  $W$ , and hence,  $W = w_S \circ P$  for a strictly increasing  $w_S$ , continuous as in Observation 1.) Cumulative dominance implies Equation (2) w.r.t.  $P = P_S$  for all  $x, y$  from  $S$  (i.e., uniformity of  $S$ ).  $\square$

**Proof of Theorem 2.** Assume matching partitions  $(C_1, C_2)$ ,  $(A_1, A_2)$  with a-neutral probabilities  $p_1, p_2$  and  $q_1, q_2$ , respectively. We have  $\varphi(q_1) = p_1$ . The implication  $A_2 \preccurlyeq C_2$  is equivalent to  $\varphi(q_2) \leq p_2 = 1 - p_1 = 1 - \varphi(q_1) = 1 - \varphi(1 - q_2)$ . Preference for  $C$  over  $A$  is equivalent to a disliked pmatcher from  $A$  to  $C$ . For ambiguity, take  $C = \mathcal{R}$ .  $\square$

**Proof of Theorem 3.** Assume matching partitions  $(C_1, C_2, C_3)$ ,  $(A_1, A_2, A_3)$  with a-neutral probabilities  $p_1, p_2, p_3$  and  $q_1, q_2, q_3$ , respectively. Consider Equations (8) and (13).  $(C_1, C_2, C_3)$  is matching with  $(A_1, A_2, A_3)$  iff  $\phi(q_1) = p_1$  and  $\varphi(q_1 + q_2) = p_1 + p_2$ . Then,  $A_2 \succcurlyeq C_2$  if and only if  $\varphi(q_2) \geq p_2 = (p_1 + p_2) - p_1 = \varphi(q_1 + q_2) - \varphi(q_1)$ . The boundary condition  $A_3^c \preccurlyeq D$  means  $q_1 + q_2 \leq P_A(D)$ . The  $q$  probabilities are the arguments of  $\varphi$ . Hence, the worst-rank bound (d) for  $\varphi$  is  $P_A(D)$ . In general, if  $B, D$  are from a source  $S$ , then the worst-rank bound for  $\varphi$  is  $m(D)$ , where  $m$  is the pmatcher from  $S$  to  $A$ .

Equations (9) and (14) are similar (above case for dual of  $W$ ). For ambiguity, take  $C = \mathcal{R}$ .  $\square$

**Proof of Observation 4.** The definition of  $\varphi$  gives the first iff. For the second iff,  $A > C$  implies the existence of  $A' \subset A$  with  $A' \sim C$  and  $\varphi(P_A(A')) = P_C(C)$ . The second iff now mainly follows from monotonicity and transitivity. The third is similar.  $\square$

**Proof of Theorem 4.** We derive uniformity and  $w_A = w_C$  from the other conditions. (The reversed implication is direct.)  $R$  abbreviates source preference or source insensitivity.  $CRA$ ,  $ARC$ , and transitivity (Observation 3) imply  $CRC$ , similarly with  $A$ . Hence, by Theorem 1, both sources are uniform. With  $\varphi$ , the pmatcher from  $A$  to  $C$ , by two-fold source preference,  $C \sim A \Rightarrow C^c \sim A^c$  (i.e.,  $\varphi(P_A(A)) = P_C(C) \Rightarrow \varphi(1 - P_A(A)) = 1 - P_C(C)$ ); that is,  $\varphi(p) + \varphi(1 - p) = 1$ . Hence,  $\varphi(\frac{1}{2}) = \frac{1}{2}$ , and we need to prove equality of the source functions only on  $[0, \frac{1}{2}]$ .

Because  $A$  is more insensitive than  $C$ ,  $\varphi(\varepsilon) \geq \varphi(p + \varepsilon) - \varphi(p)$  for every  $p > 0$  and (“small”)  $\varepsilon > 0$  if  $p + \varepsilon$  is below the upper bound of the insensitivity region, which by regularity, exceeds  $\frac{1}{2}$ . Hence, it holds on the entire  $[0, \frac{1}{2}]$ . Because  $C$  is more insensitive than  $A$ ,  $\varphi^{-1}(\varepsilon) \geq \varphi^{-1}(p + \varepsilon) - \varphi^{-1}(p)$  for every  $p > 0$ , and (“small”)  $\varepsilon > 0$  as long as  $p + \varepsilon$  is below the upper bound of the insensitivity region, which by regularity, exceeds  $\frac{1}{2}$ . Hence, it holds on the entire  $[0, \frac{1}{2}]$ . The two inequalities can only hold if  $\varphi$  is linear on  $[0, \frac{1}{2}]$  (Online Appendix A). Because  $\varphi(\frac{1}{2}) = \frac{1}{2}$ ,  $\varphi$  must be the identity on  $[0, \frac{1}{2}]$  and then, on  $[0, 1]$ . By substitution, the insensitivity region can be taken maximal.  $\square$

## Endnotes

<sup>1</sup> Our results do not change if we endow  $S$  with an algebra or a  $\sigma$ -algebra of events and consider only measurable acts.

<sup>2</sup> We write partitions as ordered  $n$ -tuples using round brackets because the ordering of events may sometimes be relevant. In what follows, events are ranked from best to worst.

<sup>3</sup> Following Savage (1954), events in themselves carry no value. Their favorability and extremity are determined by the outcomes they yield, which depends on the act considered. This is implicitly understood throughout this paper.

<sup>4</sup> In general, phenomena for risk also occur for ambiguity but to a more pronounced degree. See Fellner (1961, p. 684) and many other references in Wakker (2010, p. 292).

<sup>5</sup> Formally, we also use here transitivity established in Observation 3 below.

<sup>6</sup> Being equally preferred as risk is also called ambiguity indifference.

<sup>7</sup> That is,  $w_A(p) + w_A(1 - p) = 1$  for all  $p$ : for instance, if  $w_A(p) = \frac{p^2}{p^2 + (1 - p)^2}$ .

<sup>8</sup> Abdellaoui et al. (2011) have  $w_A = \exp(-1.14(-\ln(p))^{0.15})$  and  $w = \exp(-1.06(-\ln(p))^{0.47})$ . By Equation (18), the pmatcher is  $\exp(-1.17(-\ln(p))^{0.32})$ . The maximal insensitivity regions are  $[0, 0.9993]$ ,  $[0, 0.965]$ , and  $[0, 0.975]$ , respectively.

<sup>9</sup> We assume a fixed outcome set. Violations have been found for events and outcomes inducing particular emotions (e.g., if referring to complex arithmetic (Armantier and Treich 2016) or particular familiarities (Chew et al. 2008)).

<sup>10</sup> They used the term two-stage model of Fox and Tversky (1998), but this model uses a decomposition  $w(P)$ , where  $w$  is the risk-probability weighting function and ambiguity is captured solely through a nonadditive,  $P$ , usually based on introspective

measurements. Spiliopoulos and Hertwig (2023) instead used  $w$  to capture ambiguity attitudes. That is, they used the source method.

<sup>11</sup> Dean and Ortoleva (2017), Hill (2019), and Wang (2022) allowed violations of EU for risk, sacrificing tractability.

<sup>12</sup> See Camerer and Weber (1992, p. 361): “There are diminishing returns to studying urns!” Ellsberg (2011, p. 223) stated that “these urn experiments ... it is long overdue to perform experiments that test for other forms of ambiguity.” See also Gilboa (2009, section 3.3.3): “Real life is not about balls and urns.” Many other examples are in Online Appendix D.4.

<sup>13</sup> The model of Kontek and Lewandowski (2018) can accommodate extremity orientedness.

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