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UNCERTAINTY PRINCIPLE FOR SOLUTIONS OF THE SCHRÖDINGER EQUATION ON THE HEISENBERG GROUP

PHIIPPE JAMING & SOMNATH GHOSH

ABSTRACT. The aim of this paper is to prove two versions of the Dynamical Uncertainty Principle for the Schrödinger equation $i\partial_s u = \mathcal{L}u + Vu$, $u(s = 0) = u_0$ where \mathcal{L} is the sub-Laplacian on the Heisenberg group.

We show two results of this type. For the first one, the potential $V = 0$, we establish a dynamical version of Amrein-Berthier-Benedicks's Uncertainty Principle that shows that if u_0 and $u_1 = u(s = 1)$ have both small support then $u = 0$. For the second result, we add some potential to the equation and we obtain a dynamical version of the Paley-Wiener theorem in the spirit of the result of Kenig, Ponce & Vega [20]. Both results are obtained by suitably transferring results from the Euclidean setting.

We also establish some limitations to Dynamical Uncertainty Principles.

1. INTRODUCTION

The aim of this paper is to prove several Dynamical Uncertainty Principles for the Schrödinger operator associated to the sub-Laplacian of the Heisenberg group.

Let us recall that in Euclidean harmonic analysis, an Uncertainty Principle is a statement that *a function f and its Fourier transform \widehat{f} can not be simultaneously localised*. There are numerous statements of this form, depending on how localization of functions is measured. The most famous results of this form are the Heisenberg Uncertainty Principle when localization is measured in terms of dispersion, and Hardy's Uncertainty Principle when localization is measured in terms of fast decay. Another family of Uncertainty Principles measures localization in terms of smallness of the support. For instance, a result by Amrein-Berthier [1] and Benedicks [5] states that a function and its Fourier transform can not both have support of finite measure. This leads to the introduction of the concept of *annihilating pairs* (see Definition 2.1 below and the examples following it): a pair of sets (S, Σ) such that a function f supported in S with Fourier transform \widehat{f} supported in Σ is necessarily $f = 0$. We refer to the surveys [16, 6, 14, 24] or the book [18] for more information on the subject.

As the Uncertainty Principle is a central phenomena in Euclidean harmonic analysis, there has been a lot of activity aiming at its extension to other settings,

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among which Lie groups, and in particular to nilpotent Lie groups. There has been a lot of activity around Hardy's Uncertainty Principle (*see* [28] and references therein) or on the Theorem of Amrein-Berthier-Benedicks (*see e.g.* [25, 17]). Here a new difficulty occurs: while in the Euclidean setting, the function and its Fourier transform are both defined on \mathbb{R}^d , this is no longer the case for the Fourier transform on Lie groups.

One way to overcome this, is to take the dynamical reformulation of the Uncertainty Principle. The idea is rather simple: the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u(x, t) + \Delta_x u(x, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

is given by

$$u(x, t) = \frac{e^{-i\frac{\pi}{4}d \operatorname{sign}(t)}}{(4\pi|t|)^{d/2}} e^{i|x|^2/4t} \int_{\mathbb{R}^d} e^{i|y|^2/4t} u_0(y) e^{-i\langle x, y \rangle/2t} dy.$$

Thus $u_1(x) = u(x, 1)$ is, up to constant modulus factors that do not affect concentration, essentially the Fourier transform of u_0 . One may then restate any Uncertainty Principle for the pair (f, \widehat{f}) in terms of an Uncertainty Principle for (u_0, u_1) . Such a result is now called a Dynamical Uncertainty Principle. For instance, we have the following

(1) **Hardy's Uncertainty Principle:**

Assume that $u_0(x) = O(e^{-|x|^2/\alpha^2})$ and that $u(x, T) = O(e^{-|x|^2/\beta^2})$. If $T/\alpha\beta > 1/4$, then $u_0 = 0$ and if $T/\alpha\beta = 1/4$, then u_0 is a multiple of $e^{-(1/\alpha^2 + i/4T)|x|^2}$.

(2) **Amrein-Berthier-Benedicks:**

Assume that u_0 and $u(x, T)$ both have finite support, then $u_0 = 0$.

This simple observation, is the starting point of two fruitful lines of research.

On one hand, in a seminal series of papers, Escauriaza, Kenig, Ponce and Vega [10, 11, 12] have extended Hardy's Theorem and proved Dynamical Uncertainty Principles associated to the Schrödinger Equation with potential V : $i\partial_t u = \Delta_x u + Vu$ with various conditions on V (as well as for other dispersive equations). It is noteworthy to say that in this case, the result does not follow from its non-dynamical counterpart and requires a different strategy of proof using only real variable techniques. It turns out that this strategy is so powerful that, in the absence of potential, one may even recover Hardy's original result [9].

On the other hand, this formulation is amenable to extension to other settings in which there is a natural notion of Laplacian (e.g. on graphs or Lie groups), as the variable space does not change with time. The first result in that direction is due to S. Chanillo [8]. The dynamical version of Hardy's Uncertainty Principle (without potential), on the H -type groups has been first investigated by S. Ben Saïd, S. Thangavelu and V.N. Dogga [4] and for general step 2 nilpotent Lie groups by J. Ludwig and D. Müller [23]. For the Dynamical Uncertainty Principle with potential, see A. Fernandez-Bertolin *et al* [13].

Our aim here is precisely to pursue this direction of research for the notion of annihilating pairs when dealing with the Schrödinger equation associated to the sub-Laplacian on the Heisenberg group. Before describing our results, we need to recall some notions on the Heisenberg group as can be found *e.g.* in [15].

The $(2n + 1)$ -dimensional Heisenberg group, denoted by \mathbb{H}^n , is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group law

$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' - \frac{1}{2} \sum_{j=1}^n (x_j y'_j - y_j x'_j) \right).$$

Under this multiplication \mathbb{H}^n is a nilpotent unimodular Lie group, the Haar measure being the Lebesgue measure $dx dy dt$. It will be convenient to write $z = (x, y) \in \mathbb{R}^{2n}$, $(z, t) = (x, y, t) \in \mathbb{H}^n$ and the Haar measure is then denoted by $dz dt$ on $\mathbb{R}^{2n} \times \mathbb{R}$.

It is sometimes convenient to identify \mathbb{R}^{2n} with \mathbb{C}^n writing $z = x + iy$ and then $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n} \times \mathbb{R}$. In this case, the group law on $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$ is given by

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} \text{Im}(z \cdot \bar{w}))$$

and the Haar measure is still denoted by $dz dt$.

The Lie algebra associated to \mathbb{H}^n is generated by the vector fields

$$X_j := \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t} \quad , \quad Y_j := \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t} \quad j = 1, 2, \dots, n$$

and $T = \frac{\partial}{\partial t}$. The sub-Laplacian for the Heisenberg group \mathbb{H}^n , is given by

$$\mathcal{L} = \sum_{j=1}^n X_j^2 + Y_j^2 = \Delta_{\mathbb{R}^{2n}} + \frac{|z|^2}{4} \frac{\partial^2}{\partial t^2} - \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial t}.$$

It is well-known that \mathcal{L} is hypo-elliptic, self-adjoint and non-negative. Further $e^{i\mathcal{L}}$ generates a unitary semi-group on $L^2(\mathbb{H}^n)$ and the solution of the free Schrödinger equation for \mathcal{L}

$$(1.1) \quad i\partial_s u(z, t, s) + \mathcal{L}u(z, t, s) = 0$$

with initial condition $u(z, t, 0) = u_0(z, t)$, $u_0 \in L^2(\mathbb{H}^n)$ is given by $u(\cdot, \cdot, s) = e^{is\mathcal{L}}u_0$.

The following Dynamical Uncertainty Principle of Hardy type associated to \mathcal{L} has been shown in [4]:

Theorem 1.1 (Ben Saïd & Thangavelu & Dogga). *Let $u_0 \in L^2(\mathbb{H}^n)$ and u be the solution of (1.1) with initial condition $u(z, t, 0) = u_0(z, t)$. Assume that, for some time $T > 0$,*

$$|u_0(z, t)| = O(e^{-|x|^2/4a^2 - \delta|t|}) \quad \text{and} \quad |u(z, t, T)| = O(e^{-|x|^2/4b^2 - \delta|t|})$$

for some $\delta > 0$ and $ab < T$, then $u = 0$.

The result has been extended to all step 2 nilpotent Lie groups in [23] and to Schrödinger equation with certain potentials on H-type groups in [13]. Among the tasks we tackle in this paper is to show that the extra factor $e^{-\delta|t|}$ is essential, though δ can be arbitrarily small. We will then show that this is no longer the case when considering a dynamical version of Amrein-Berthier-Benedicks' Uncertainty Principle on the Heisenberg group where the central variable plays no role:

Theorem 1.2. *Let $S, \Sigma \subset \mathbb{R}^{2n}$ be two sets of finite measure. Let $u_0 \in L^2(\mathbb{H}^n)$ and u be the solution of (1.1) with initial condition $u(z, t, 0) = u_0(z, t)$. Assume that $\text{supp } u_0 \subset S \times \mathbb{R}$ and that, for some time $s > 0$ $\text{supp } u(\cdot, \cdot, s) \subset \Sigma \times \mathbb{R}$. Then $u = 0$*

Our actual result is more general and provides a way of transferring other results about some annihilating pairs from the Fourier transform to Dynamical Uncertainty Principles associated to the sub-Laplacian of the Heisenberg group. However, we will show that a more quantitative results (so-called strong annihilating pairs) do not transfer to this more general setting.

We then investigate the influence of the potential. Even in the Euclidean setting, not much is known, excepted for a result by Kenig, Ponce and Vega [20]. We will here show that this result can also be transferred to the Heisenberg group. This result is a form of Paley-Wiener theorem:

Theorem 1.3. *Let $u \in C([0, 1] : L^2(\mathbb{H}^n))$ be a solution of*

$$(1.2) \quad i\partial_s u(z, t, s) + \mathcal{L}u(z, t, s) + V(z, s)u(z, t, s) = 0,$$

where $(z, t) \in \mathbb{H}^n$, $s \in [0, 1]$ and $V : \mathbb{H}^n \times [0, 1] \rightarrow \mathbb{C}$ is independent of the central variable t .

Assume further that the potential V is bounded, i.e., $\|V\|_{L^\infty(\mathbb{C}^n \times [0, 1])} < \infty$ and

$$(1.3) \quad \lim_{\rho \rightarrow \infty} \int_0^1 \|V(\cdot, s)\|_{L^\infty(\mathbb{C}^n \setminus B_\rho)} ds = 0.$$

Suppose

$$(1.4) \quad \sup_{0 \leq s \leq 1} \int_{\mathbb{H}^n} |u(z, t, s)|^2 dz dt < \infty,$$

$$(1.5) \quad \int_{\mathbb{H}^n} e^{2a_0 x_1} |u(z, t, 0)|^2 dz dt < \infty \quad \text{for some } a_0 > 0,$$

and

$$(1.6) \quad \text{supp}(u(\cdot, \cdot, 1)) \subseteq \{(z, t) \in \mathbb{H}^n : |z_1| \leq a_1, |t| \leq a_2\}$$

for some $0 < a_1, a_2 < \infty$.

Then $u = 0$.

The remaining of this paper is organized as follows. The next section is devoted to the proof of Theorem 1.2 with Section 2.2 devoted to establishing some limitations on the Dynamical Uncertainty Principle on the Heisenberg Group. In Section 3, we prove Theorem 1.3.

2. DYNAMICAL UNCERTAINTY PRINCIPLES ASSOCIATED TO \mathcal{L}

In this section, we will need the following notation. For a function $f \in L^1(\mathbb{H}^n)$, we write

$$f^\lambda(x, y) = \int_{\mathbb{R}} f(x, y, t) e^{it\lambda} dt$$

for the partial Fourier transform of f in the central variable t . The convolution of two functions $f, g \in L^1(\mathbb{H}^n)$ is defined by

$$f * g(z, t) = \int_{\mathbb{H}^n} f(w, s) g((w, s)^{-1}(z, t)) dw ds$$

and an easy computation shows that

$$(f * g)^\lambda(z) = f^\lambda *_\lambda g^\lambda(z) := \int_{\mathbb{C}^n} f^\lambda(z - w) g^\lambda(w) e^{i\frac{\lambda}{2} \text{Im}(z \cdot \bar{w})} dw.$$

This last expression is called the λ -twisted convolution of f^λ and g^λ .

2.1. Annihilating pairs. Let us recall the following classical notion when dealing with Uncertainty Principles [18]:

Definition 2.1. Let S, Σ be two measurable subsets of \mathbb{R}^d . Then

(i) (S, Σ) is a (weak) *annihilating pair* if,

$$\text{supp } f \subset S \quad \text{and} \quad \text{supp } \widehat{f} \subset \Sigma$$

implies $f = 0$.

(ii) (S, Σ) is called a *strong annihilating pair* if there exists $C = C(S, \Sigma)$ such that

$$(2.1) \quad \|f\|_{L^2(\mathbb{R}^d)} \leq C(\|f\|_{L^2(\mathbb{R}^d \setminus S)} + \|\widehat{f}\|_{L^2(\mathbb{R}^d \setminus \Sigma)}).$$

Note that (S, Σ) is (strongly) annihilating if and only if (Σ, S) is (strongly) annihilating.

Example 2.2. Here are some classical examples:

(i) If S is compact and $\mathbb{R}^d \setminus \Sigma$ has an accumulation point, then (S, Σ) is a weak annihilating pair since $\text{supp } f \subset S$ implies that \widehat{f} is analytic.

(ii) *Logvinenko-Sereda*, [22].

If Σ is compact then (S, Σ) is a strong annihilating pair if and only if $E = \mathbb{R}^d \setminus S$ is dense in the sense that there exists $a, \gamma > 0$ such that, for every $x \in \mathbb{R}^d$, $|E \cap x + [-a, a]^d| \geq \gamma|[-a, a]^d|$.

In this case, an essentially sharp estimate of the constant is due to Kovrijkine [21] in dimension $d = 1$ and the proof can be extended to arbitrary dimension: if

$\Sigma \subset [-c, c]^d$ then $C(S, \Sigma) \leq \left(\frac{\kappa}{\gamma}\right)^{\kappa(ac+1)}$ where κ is a constant that depends on d only.

(iii) *Amrein-Berthier-Benedicks*, [1, 5].

If $S, \Sigma \subset \mathbb{R}^d$ are two sets of finite measure, then (S, Σ) is a strong annihilating pair.

Further, an essentially sharp estimate of the constant is due to Nazarov [26] in dimension $d = 1$ (see [19] for $d \geq 1$ where sharpness is unknown): $C(S, \Sigma) \leq \kappa e^{\kappa|S||\Sigma|}$ where κ is a constant that depends on d only.

(iv) *Bonami-Demange*, [6].

A slight elaboration on Benedicks' proof was given in [6]: Consider the class of sets

$$\mathcal{F} = \{E \subset \mathbb{R}^d : \text{for almost every } x \in \mathbb{R}^d, (x + \mathbb{Z}^d) \cap E \text{ is finite}\}.$$

The class \mathcal{F} contains sets of finite measures, as well as \mathcal{F}' , the class of sets that concentrate on a finite number of lines, where $E \in \mathcal{F}'$ is defined via the following property:

There is a finite set of unit vectors $e_1, \dots, e_N \subset \mathbb{R}^d$, and functions η_1, \dots, η_k with $\lim_{|x| \rightarrow +\infty} \eta_j(x) = 0$ such that if $x \in E$ then $|x - \langle x, e_k \rangle e_k| \leq \eta_k(x)$ for some $k \in \{1, \dots, N\}$.

For instance, for any $C > 0$, $\{(x, y) \in \mathbb{R}^2 : |xy| \leq C\} \in \mathcal{F}'$. Then Bonami-Demange proved that if $S, \Sigma \in \mathcal{F}$, then (S, Σ) is a weak annihilating pair.

For each of these pairs, one should notice that if T is an invertible linear transform, then $(S, T\Sigma)$ and (TS, Σ) are still annihilating pairs.

(v) *Shubin-Vakilian-Wolff*, [27].

Define $\rho(x) = \min(1, |x|^{-1})$ and say that E is ε -thin ($0 < \varepsilon < 1$) if, for every $x \in \mathbb{R}^d$,

$|E \cap B(x, \rho(x))| \leq \varepsilon |B(x, \rho(x))|$. The sets $\{(x, y) \in \mathbb{R}^2 : |xy| \leq C\}$ are examples of thin sets in this sense. Then, there is an $0 < \varepsilon_0 \leq 1$ such that, if $0 < \varepsilon < \varepsilon_0$ is small enough and S, Σ are ε -thin, (S, Σ) is a strong annihilating pair.

It is reasonable to conjecture that one can take $\varepsilon_0 = 1$ so that there is no restriction on ε . If this conjecture were true, then $(S, T\Sigma)$ and (TS, Σ) would still be annihilating pairs for any invertible linear transform T . The current result requires T to have determinant near enough to 1.

We can now state our transference result:

Theorem 2.3. *Let $S, \Sigma \subset \mathbb{R}^{2n}$ be such that, for every invertible linear transform T , $(S, T\Sigma)$ is a weak annihilating pair.*

Let $u \in C^1([0, +\infty) : L^2(\mathbb{H}^n))$ be a solution of the free Schrödinger equation for the sub-Laplacian of the Heisenberg group \mathbb{H}^n ,

$$i\partial_s u(z, t, s) + \mathcal{L}u(z, t, s) = 0$$

with initial condition $u(z, t, 0) = u_0(z, t)$, $u_0 \in L^2(\mathbb{H}^n)$. Assume that

$$\text{supp } u_0 \subset S \times \mathbb{R} \quad \text{and} \quad \text{supp } u(\cdot, \cdot, s_0) \subset \Sigma \times \mathbb{R}$$

for some $0 < s_0 < \infty$. Then $u = 0$.

The main difficulty here is that one needs to use intricate distribution theory to write the solution via a kernel and that this kernel is not nice enough to be exploited (see [3] for more details about the kernel). To overcome this difficulty we will take a Fourier transform in the central variable. This changes the Schrödinger equation into another equation that can be solved via an integral kernel and this can be linked to the Euclidean Fourier transform. The price to pay is that our condition on the support of u_0 is a bit restrictive and does not cover general sets of finite measure in \mathbb{H}^n .

Proof. For $s \geq 0$, $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{R}$, let

$$u^\lambda(z, s) = \int_{\mathbb{R}} u(z, t, s) e^{i\lambda t} dt$$

be the inverse Fourier transform of $u(z, t, s)$ in the central variable and u_0^λ be the inverse Fourier transform of u_0 in the central variable. Then

$$(2.2) \quad u^\lambda(z, s_0) = u_0^\lambda(z) *_{\lambda} h_{is_0}^\lambda(z)$$

for a.e. $\lambda \in \mathbb{R} \setminus \{0\}$, where

$$(2.3) \quad h_{is_0}^\lambda(z) = (4\pi)^{-n} \left(\frac{\lambda}{i \sin(\lambda s_0)} \right)^n e^{i\frac{\lambda}{4} \cot(\lambda s_0) |z|^2}.$$

Note that this formula is valid for $\lambda \in \mathbb{R} \setminus \frac{\pi}{s_0} \mathbb{Z}$.

Next note that $\text{supp } u^\lambda(\cdot, 0) \subset S$ and $\text{supp } u^\lambda(\cdot, s_0) \subset \Sigma$. From (2.2) and (2.3), we have

$$(2.4) \quad \begin{aligned} u^\lambda(z, s_0) &= \int_{\mathbb{C}^n} u_0^\lambda(w) h_{is_0}^\lambda(z-w) e^{-\frac{i\lambda}{2} \text{Im}(z \cdot \bar{w})} dw \\ &= (4\pi)^{-n} \left(\frac{\lambda}{i \sin(\lambda s_0)} \right)^n \int_{\mathbb{C}^n} u_0^\lambda(w) e^{i\frac{\lambda}{4} \cot(\lambda s_0) |z-w|^2} e^{-\frac{i\lambda}{2} \text{Im}(z \cdot \bar{w})} dw \\ &= c_{\lambda, s_0} e^{i\frac{\lambda}{4} \cot(\lambda s_0) |z|^2} \int_{\mathbb{C}^n} u_0^\lambda(w) e^{i\frac{\lambda}{4} \cot(\lambda s_0) |w|^2} e^{-\frac{i\lambda}{2} (\cot(\lambda s_0) \text{Re}(z \cdot \bar{w}) + \text{Im}(z \cdot \bar{w}))} dw, \end{aligned}$$

where $c_{\lambda, s_0} = (4\pi)^{-n} \lambda^n / (i \sin(\lambda s_0))^n$. Now write $z = x + iy$, $w = \xi + i\eta$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and observe that

$$\begin{aligned} \cot(\lambda s_0) \operatorname{Re}(z \cdot \bar{w}) + \operatorname{Im}(z \cdot \bar{w}) &= \cot(\lambda s_0) (\langle x, \xi \rangle + \langle y, \eta \rangle) + (\langle y, \xi \rangle - \langle x, \eta \rangle) \\ &= \langle (\cot(\lambda s_0) I_{2n} + J)(x, y), (\xi, \eta) \rangle. \end{aligned}$$

Further define γ_λ on $\mathbb{R}^n \times \mathbb{R}^n$ as

$$\gamma_\lambda(u, v) = e^{i \frac{\lambda}{4} \cot(\lambda s_0) (|u|^2 + |v|^2)}$$

and note that this function takes its values in the set of complex numbers of modulus 1. We can now write (2.4) as

$$(2.5) \quad u^\lambda(x, y, s_0) = c_{\lambda, s_0} \gamma_\lambda(x, y) \mathcal{F}[\gamma_\lambda u_0^\lambda](J_\lambda(x, y))$$

where \mathcal{F} is the Euclidean Fourier transform on \mathbb{R}^{2n} and $J_\lambda = \frac{\lambda}{2}(J + \cot(\lambda s_0) I_{2n})$.

Observe that

$$\det J_\lambda = \left(\frac{\lambda}{2}\right)^{2n} (\cot^2(\lambda s_0) + 1)^n \neq 0$$

so that J_λ is invertible. Thus, as $u^\lambda(\cdot, s_0)$ is supported in Σ , $\mathcal{F}[\gamma_\lambda u_0^\lambda]$ is supported in $J_\lambda \Sigma$. On the other hand, $\gamma_\lambda u_0^\lambda$ is supported in S . As $(S, J_\lambda \Sigma)$ is an annihilating pair, it follows that $\gamma_\lambda u_0^\lambda = 0$ for almost every λ , thus $u_0 = 0$. \square

Remark 2.4. The derivation of Formula of the kernel h_{is_0} can be found *e.g.* in [23] even replacing \mathbb{H}^n by a more general step 2 nilpotent Lie group.

One would of course like to obtain a quantitative counterpart of this result. We will show that this is not fully possible below.

Let us show a partial result that will highlight the difficulties. Take $S, \Sigma \subset \mathbb{R}^{2n}$ two sets of finite measure. Writing u^λ as a Fourier transform as in (2.5) and noticing that $|\gamma_\lambda| = 1$, we may apply the higher dimensional extension of Nazarov's Uncertainty Principle [19] to obtain

$$(2.6) \quad \|u_0^\lambda\|_{L^2(\mathbb{R}^{2n})}^2 \leq \kappa e^{\kappa|S||J_\lambda \Sigma|} (\|u_0^\lambda\|_{L^2(\mathbb{R}^{2n} \setminus S)}^2 + \|\mathcal{F}[\gamma_\lambda u_0^\lambda]\|_{L^2(\mathbb{R}^{2n} \setminus J_\lambda \Sigma)}^2).$$

A simple computation shows that

$$\|\mathcal{F}[\gamma_\lambda u_0^\lambda]\|_{L^2(\mathbb{R}^{2n} \setminus J_\lambda \Sigma)}^2 = c \|u^\lambda(\cdot, s_0)\|_{L^2(\mathbb{R}^{2n} \setminus \Sigma)}^2$$

for some constant c independent of λ and s_0 . On the other hand

$$|J_\lambda \Sigma| = |\Sigma| \left(\frac{\lambda}{2}\right)^{2n} (\cot^2(\lambda s_0) + 1)^n,$$

so that (2.6) reads

$$(2.7) \quad \|u_0^\lambda\|_{L^2(\mathbb{R}^{2n})}^2 \leq \kappa e^{\kappa \lambda^{2n} (\cot^2(\lambda s_0) + 1)^n |S||\Sigma|} \left(\|u_0^\lambda\|_{L^2(\mathbb{R}^{2n} \setminus S)}^2 + \|u^\lambda(\cdot, s_0)\|_{L^2(\mathbb{R}^{2n} \setminus \Sigma)}^2 \right).$$

Unfortunately, due to the term $\lambda^{2n} (\cot^2(\lambda s_0) + 1)^n$, one can not use Parseval in the central variable to obtain an estimate of $\|u_0\|_{L^2(\mathbb{H}^n)}^2$ in terms of $\|u_0\|_{L^2((\mathbb{R}^{2n} \setminus S) \times \mathbb{R})}^2$ and $\|u(\cdot, s_0)\|_{L^2((\mathbb{R}^{2n} \setminus \Sigma) \times \mathbb{R})}^2$.

There is however one exception: assume there is an $a > 0$ such that $u_0^\lambda = 0$ for $|\lambda| > a$ and that $s_0 a < \pi$. First note that, for $|\lambda| < a$,

$$\begin{aligned} \kappa \lambda^{2n} (\cot^2(\lambda s_0) + 1)^n &= \kappa \left(\frac{1}{s_0^2} \left(\frac{\sin(\lambda s_0)}{\lambda s_0} \right)^{-2} \cos^2(\lambda s_0) + \lambda^2 \right)^n \\ &\leq \kappa' := \kappa \frac{2^n a^{2n}}{\sin^{2n}(s_0 a)}. \end{aligned}$$

But then

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{H}^n)}^2 &= \int_{\mathbb{R}} \|u_0^\lambda\|_{L^2(\mathbb{R}^{2n})}^2 d\lambda = \int_{-a}^a \|u_0^\lambda\|_{L^2(\mathbb{R}^{2n})}^2 d\lambda \\ &\leq \kappa e^{\kappa'|S|\Sigma|} \int_{-a}^a \left(\|u_0^\lambda\|_{L^2(\mathbb{R}^{2n} \setminus S)}^2 + \|u^\lambda(\cdot, s_0)\|_{L^2(\mathbb{R}^{2n} \setminus \Sigma)}^2 \right) d\lambda \\ &\leq \kappa e^{\kappa'|S|\Sigma|} \int_{\mathbb{R}} \left(\|u_0^\lambda\|_{L^2(\mathbb{R}^{2n} \setminus S)}^2 + \|u^\lambda(\cdot, s_0)\|_{L^2(\mathbb{R}^{2n} \setminus \Sigma)}^2 \right) d\lambda \\ &= \kappa e^{\kappa'|S|\Sigma|} \left(\|u_0\|_{L^2((\mathbb{R}^{2n} \setminus S) \times \mathbb{R})}^2 + \|u(\cdot, s_0)\|_{L^2((\mathbb{R}^{2n} \setminus \Sigma) \times \mathbb{R})}^2 \right). \end{aligned}$$

We have thus proved the following:

Proposition 2.5. *Let $S, \Sigma \subset \mathbb{R}^{2n}$ be two sets of finite measure. Let $a, s_0 > 0$ be such that $s_0 a < \pi$. Let $u_0 \in L^2(\mathbb{H}^n)$ be such that $u_0^\lambda = 0$ for $|\lambda| > a$. Let u be the solution of $i\partial_s u + \mathcal{L}u = 0$ with initial condition u_0 . Then*

$$\|u_0\|_{L^2(\mathbb{H}^n)}^2 \leq \kappa e^{\kappa'|S|\Sigma|} \left(\|u_0\|_{L^2((\mathbb{R}^{2n} \setminus S) \times \mathbb{R})}^2 + \|u(\cdot, s_0)\|_{L^2((\mathbb{R}^{2n} \setminus \Sigma) \times \mathbb{R})}^2 \right)$$

where κ' depend on n, a, s_0 only and κ on n only.

Remark 2.6 (Observability inequality). Fix $\lambda \in \mathbb{R} \setminus \{0\}$, and consider the λ -twisted Laplacian or the special Hermite operator on \mathbb{R}^{2n} ,

$$L_\lambda = \Delta_{\mathbb{R}^{2n}} - \frac{|\lambda|^2}{4}(|x|^2 + |y|^2) + i\lambda \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right)$$

Then the solution of the Schrödinger equation

$$(2.8) \quad \begin{aligned} i\partial_s u^\lambda(x, y, s) + L_\lambda u^\lambda(x, y, s) &= 0, \\ u^\lambda(x, y, 0) &= u_0^\lambda(x, y) \in L^2(\mathbb{R}^{2n}), \end{aligned}$$

is obtained in (2.5) as

$$u^\lambda(x, y, s) = c_{\lambda, s} \gamma_\lambda(x, y) \mathcal{F}[\gamma_\lambda u_0^\lambda](J_\lambda(x, y))$$

for $s \notin \frac{\pi}{\lambda} \mathbb{N}$. Hence, if $S, \Sigma \subset \mathbb{R}^{2n}$ be such that, for every invertible linear transform T , $(S, T\Sigma)$ is a strong annihilating pair, we can obtain the following observability inequality at two time points for L_λ (see [29] for this kind of inequalities for the standard Laplacian and its applications).

Let $s_2 > s_1 \geq 0$ with $s_2 - s_1 \notin \frac{\pi}{\lambda} \mathbb{N}$. Then there exists a constant $C = C(A, \Sigma, \lambda, s_1, s_2) > 0$ such that for all solution u^λ of (2.8),

$$(2.9) \quad \|u_0^\lambda\|_{L^2(\mathbb{R}^{2n})}^2 \leq C \left(\|u^\lambda(\cdot, s_1)\|_{L^2(\mathbb{R}^{2n} \setminus S)}^2 + \|u^\lambda(\cdot, s_2)\|_{L^2(\mathbb{R}^{2n} \setminus \Sigma)}^2 \right).$$

Note that such sets (S, Σ) are described in Example 2.2, and the constant C can be calculated explicitly for some instances.

2.2. Limitations to Dynamical Uncertainty Principles. In this section, we will show that there are limitations to Dynamical Uncertainty Principles for the Schrödinger equation (1.1). This is based on the following example built on an observation from [2].

Let $\varphi \in \mathcal{S}(\mathbb{R})$ and set

$$(2.10) \quad u_0(z, t) = \int_{\mathbb{R}} e^{|\lambda|(it-|z|^2/4)} \varphi(\lambda) d\lambda.$$

A simple computation shows that

$$\mathcal{L}u_0 = -n \int_{\mathbb{R}} |\lambda| e^{|\lambda|(it-|z|^2/4)} \varphi(\lambda) d\lambda = in\partial_t u_0.$$

It follows that

$$u(z, t, s) = u_0(z, t - ns)$$

is a solution of $\mathcal{L}u(z, t, s) + i\partial_s u(z, t, s) = 0$ with $u(z, t, 0) = u_0(z, t)$.

Further, as $\varphi \in \mathcal{S}(\mathbb{R})$ we can write

$$u_0(z, t) = \mathcal{F}[e^{\lambda|z|^2/4} \varphi(\lambda) \mathbf{1}_{\mathbb{R}^-}(\lambda)](t) + \mathcal{F}[e^{-\lambda|z|^2/4} \varphi(\lambda) \mathbf{1}_{\mathbb{R}^+}(\lambda)](-t)$$

where \mathcal{F} is the Fourier transform on $L^1(\mathbb{R})$. From Parseval and Fubini, we get

$$(2.11) \quad \int_{\mathbb{H}^n} |u_0(z, t)|^2 dz dt = (2\pi)^{n+1} \int_0^{+\infty} \frac{|\varphi(\lambda) + \varphi(-\lambda)|^2}{\lambda^n} d\lambda.$$

From now on, we will take $\alpha > 0$ and further assume that φ is supported in $[\alpha, \alpha + 1]$. From the definition of u_0 , we immediately obtain that, for every $s > 0$

$$|u(z, t, s)| \leq \|\varphi\|_{L^1(\mathbb{R})} e^{-\alpha|z|^2/4}.$$

Further, integrating by parts, we have for every $k \in \mathbb{N}$,

$$u(z, t, s) = \int_{\mathbb{R}} \frac{(-\text{sign}(\lambda))^k \varphi^{(k)}(\lambda)}{(it - |z|^2/4 - ins)^k} e^{|\lambda|(-|z|^2/4 + it - ins)} d\lambda$$

and, as $|e^{|\lambda|(-|z|^2/4 + it - ins)}| \leq e^{-\alpha|z|^2/4}$ on $\text{supp } \varphi$, we obtain

$$|u(z, t, s)| \leq \left\| \varphi^{(k)} \right\|_{L^1(\mathbb{R})} \frac{e^{-\alpha|z|^2/4}}{((t - ns)^2 + |z|^4/16)^{k/2}}.$$

This shows that, in Theorem 1.1, one can not take $\delta = 0$, that is, in the dynamical version of Hardy's Uncertainty Principle on the Heisenberg group, one needs to impose a faster than polynomial decay in the central variable t .

Next, notice that the function defined in (2.10) extends into an holomorphic function in the z variable. More precisely, here $z = (x, y) \in \mathbb{R}^{2n}$ and u extends holomorphically to the cone $\Omega = \{(x + ix', y + iy'), x, x', y, y' \in \mathbb{R}^n : |x'|^2 + |y'|^2 < |x|^2 + |y|^2\}$. In particular, the function u has full support in the z -variable. This explains why the situation is a bit better for Theorem 1.2 where no restriction in the central variable is required.

However, this example also shows that there are limitations to quantitative counterparts to Theorem 1.2. Indeed, from $\text{supp } \varphi = [\alpha, \alpha + 1]$, we obtain

$$\int_{\mathbb{H}^n} |u(z, t, s)|^2 dz dt \approx \alpha^{-n} \|\varphi\|_{L^2(\mathbb{R})}^2.$$

On the other hand,

$$\begin{aligned}
& \int_{(\mathbb{R}^{2n} \setminus [-r, r]^{2n}) \times \mathbb{R}} |u(z, t, s)|^2 dz dt \\
& \leq \int_{\mathbb{R}^{2n} \setminus [-r, r]^{2n}} \int_{\mathbb{R}} \frac{\|\varphi'\|_{L^1(\mathbb{R})}^2}{(t - ns)^2 + r^4/16} dt e^{-\alpha|z|^2/2} dz \\
& \leq \frac{4\pi}{r^2} \|\varphi'\|_{L^1(\mathbb{R})}^2 \left(2 \int_r^{+\infty} e^{-\alpha t^2/2} dt \right)^{2n} \\
& \leq \frac{2^{2(n+1)}\pi}{\alpha^{2n} r^{2(n+1)}} e^{-n\alpha r^2} \|\varphi'\|_{L^1(\mathbb{R})}^2.
\end{aligned}$$

This shows that for every $s_1 < s_2$, the inequality

$$\begin{aligned}
\int_{\mathbb{H}^n} |u_0(z, t)|^2 dz dt & \leq C \left(\int_{(\mathbb{R}^{2n} \setminus Q(0, r)) \times \mathbb{R}} |u(z, t, s_1)|^2 dz dt \right. \\
& \quad \left. + \int_{(\mathbb{R}^{2n} \setminus Q(0, r)) \times \mathbb{R}} |u(z, t, s_2)|^2 dz dt \right)
\end{aligned}$$

can not hold for every u solution of $i\partial_s u(z, t, s) + \mathcal{L}u(z, t, s) = 0$ with initial condition $u(z, t, 0) = u_0(z, t)$, $u_0 \in L^2(\mathbb{H}^n)$.

3. PALEY-WIENER TYPE THEOREM

In this section, we will prove Theorem 1.3 from the introduction. We proceed through a few steps.

Step 1. *Reduction to the magnetic Laplacian (twisted Laplacian) on \mathbb{R}^{2n} .*

For $\lambda \in \mathbb{R} \setminus \{0\}$, let $u^\lambda(z, s) = \int_{\mathbb{R}} u(z, t, s) e^{i\lambda t} dt$ be the inverse Fourier transform of u in the central variable. Identify z with $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}$.

Proposition 3.1. *Let u be a solution of (1.2) and satisfies the hypothesis of Theorem 1.3. Then there is a set $E \subset \mathbb{R} \setminus \{0\}$ of measure 0, such that, if $\lambda \in \mathbb{R} \setminus (E \cup \{0\})$, then u^λ solves*

$$(3.1) \quad i\partial_s u^\lambda(x, y, s) + \Delta_{C_\lambda} u^\lambda(x, y, s) + V(x, y, s) u^\lambda(x, y, s) = 0,$$

where $\Delta_{C_\lambda} = (\nabla - iC_\lambda)^2$ with $C_\lambda = \frac{\lambda}{2} J \begin{pmatrix} x \\ y \end{pmatrix}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Moreover,

$$(3.2) \quad \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |u^\lambda(x, y, s)|^2 dx dy < \infty,$$

$$(3.3) \quad \int_{\mathbb{R}^{2n}} e^{2a_0 x_1} |u^\lambda(x, y, 0)|^2 dx dy < \infty \quad \text{for some } a_0 > 0, \text{ and}$$

$$(3.4) \quad \text{supp}(u^\lambda(\cdot, \cdot, 1)) \subseteq \{(x, y) \in \mathbb{R}^{2n} : |\cos(\lambda)x_1 + \sin(\lambda)y_1| \leq a_3\}$$

for some $a_3 < \infty$. In addition, for each (x, y) , $u^\lambda(x, y, 1)$ can be extended holomorphically in λ to a neighbourhood of the real line.

Proof. A direct calculation shows the first part, see, for instance, [13].

For the remaining part, first note the Plancherel formula

$$\int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R}^{2n}} |u^\lambda(x, y, s)|^2 dx dy d\lambda = \int_{\mathbb{H}^n} |u(z, t, s)|^2 dz dt,$$

where $0 \leq s \leq 1$. Hence, it follows from (1.4) and (1.5) that

$$\begin{aligned} \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |u^\lambda(x, y, s)|^2 dx dy &< \infty, \quad \text{and} \\ \int_{\mathbb{R}^{2n}} e^{2a_0 x_1} |u^\lambda(x, y, 0)|^2 dx dy &< \infty \quad \text{for some } a_0 > 0, \end{aligned}$$

for a.e. λ . In addition, condition (1.6) gives

$$\begin{aligned} \text{supp}(u^\lambda(\cdot, \cdot, 1)) &\subseteq \{x \in \mathbb{R}^{2n} : |x_1| + |y_1| \leq a_3\} \\ &\subseteq \{x \in \mathbb{R}^{2n} : |\cos(\lambda)x_1 + \sin(\lambda)y_1| \leq a_3\} \end{aligned}$$

for some $a_3 < \infty$. □

Now onwards $0 < |\lambda| < \frac{\pi}{2}$ is fixed.

Step 2. *Reduction to the Hermite operator on \mathbb{R}^{2n} .*

Proposition 3.2. *Suppose u and V satisfy the hypothesis of Theorem 1.3. Let u^λ be as in Proposition 3.1. Write*

$$(3.5) \quad v(x, y, s) = u^\lambda(e^{-s\lambda J}(x, y), s).$$

Then $v \in C([0, 1] : L^2(\mathbb{R}^{2n}))$ and solves

$$(3.6) \quad \begin{aligned} i\partial_s v(x, y, s) + \Delta v(x, y, s) - \frac{|\lambda|^2}{4}(|x|^2 + |y|^2)v(x, y, s) \\ + \tilde{V}(x, y, s)v(x, y, s) = 0, \end{aligned}$$

where

$$(3.7) \quad \tilde{V}(x, y, s) = V(e^{-s\lambda J}(x, y), s).$$

Additionally, the potential \tilde{V} is bounded and satisfies

$$(3.8) \quad \lim_{\rho \rightarrow \infty} \int_0^1 \|\tilde{V}(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^{2n} \setminus B_\rho)} ds = 0.$$

Furthermore,

$$(3.9) \quad \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |v(x, y, s)|^2 dx dy < \infty,$$

$$(3.10) \quad \int_{\mathbb{R}^{2n}} e^{2a_0 x_1} |v(x, y, 0)|^2 dx dy < \infty \quad \text{for some } a_0 > 0, \quad \text{and}$$

$$(3.11) \quad \text{supp}(v(\cdot, \cdot, 1)) \subseteq \{(x, y) \in \mathbb{R}^{2n} : |x_1| \leq a_3\}$$

for some $a_3 < \infty$.

Proof. The fact that v solves (3.6) follows from [7, Proposition 5.1].

For the remaining of the proof, we will use that

$$e^{-tJ} = \begin{pmatrix} \cos(t) I_n & -\sin(t) I_n \\ \sin(t) I_n & \cos(t) I_n \end{pmatrix}.$$

Next, the boundedness of \tilde{V} is immediate from the boundedness of V . Further, for $s \in [0, 1]$,

$$\sup_{|(x, y)| \geq \rho} |\tilde{V}(x, y, s)| = \sup_{|(x, y)| \geq \rho} |V(e^{-s\lambda J}(x, y), s)| = \sup_{|(x, y)| \geq \rho} |V(x, y, s)|.$$

Here the last equality holds as $(e^{-s\lambda J})^t = e^{-s\lambda J^t} = e^{s\lambda J}$ we have $|e^{-s\lambda J}(x, y)| = |(x, y)|$. Therefore (3.8) follows from (1.3).

Since u satisfies (1.4)-(1.6), Proposition 3.1 ensures that u^λ satisfies (3.2)-(3.4). Now, for $s \in [0, 1]$, we know that

$$\det(e^{s\lambda J}) = e^{s\lambda \text{Tr}(J)} = 1.$$

Hence, from (3.2) we get

$$\begin{aligned} \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |v(x, y, s)|^2 dx dy &= \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |u^\lambda(e^{-N\lambda s}(x, y), s)|^2 dx dy \\ &= \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |u^\lambda(x, y, s)|^2 dx dy < \infty. \end{aligned}$$

Further (3.10) is identical with (3.3). To see condition (3.11), write $(\tilde{x}, \tilde{y}) = e^{-\lambda J}(x, y)$, then

$$\begin{aligned} &|\cos(\lambda)\tilde{x}_1 + \sin(\lambda)\tilde{y}_1| \\ &= |\cos(\lambda)(\cos(\lambda)x_1 - \sin(\lambda)y_1) + \sin(\lambda)(\sin(\lambda)x_1 + \cos(\lambda)y_1)| = |x_1|. \end{aligned}$$

Thus from (3.4) and (3.5) we get

$$\text{supp}(v(\cdot, \cdot, 1)) \subseteq \{(x, y) \in \mathbb{R}^{2n} : |x_1| \leq a_3\}$$

as claimed. \square

Step 3. *Reduction to the Laplacian on \mathbb{R}^{2n} .*

Proposition 3.3. *Suppose u and V satisfy the hypothesis of Theorem 1.3 and let \tilde{V} be defined in (3.7) and, for $z = (x, y) \in \mathbb{R}^{2n}$, define*

$$W(z, s) = (1 + |\lambda|^2 s^2)^{-1} \tilde{V} \left(\frac{z}{\sqrt{1 + |\lambda|^2 s^2}}, \frac{\arctan(|\lambda|s)}{|\lambda|} \right)$$

so that W is bounded with

$$(3.12) \quad \lim_{\rho \rightarrow \infty} \int_0^{\frac{1}{|\lambda|} \tan(|\lambda|)} \|W(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^{2n} \setminus B_\rho)} ds = 0.$$

Let v as in (3.5). For $0 < |\lambda| < \frac{\pi}{2}$, set

$$(3.13) \quad w(z, s) = \frac{\exp\left(i \frac{|z|^2}{4(1+|\lambda|^2 s^2)} |\lambda|^2\right)}{(1 + |\lambda|^2 s^2)^{n/2}} v \left(\frac{z}{\sqrt{1 + |\lambda|^2 s^2}}, \frac{\arctan(|\lambda|s)}{|\lambda|} \right).$$

Then $w \in C\left([0, \frac{1}{|\lambda|} \tan(|\lambda|)] : L^2(\mathbb{R}^{2n})\right)$ and solves

$$(3.14) \quad i\partial_s w(x, y, s) + \Delta w(x, y, s) + W(x, y, s)w(x, y, s) = 0,$$

Moreover,

$$(3.15) \quad \sup_{0 \leq s \leq \frac{1}{|\lambda|} \tan(|\lambda|)} \int_{\mathbb{R}^{2n}} |w(x, y, s)|^2 dx dy < \infty,$$

$$(3.16) \quad \int_{\mathbb{R}^{2n}} e^{2a_0 x_1} |w(x, y, 0)|^2 dx dy < \infty \quad \text{for some } a_0 > 0, \text{ and}$$

$$(3.17) \quad \text{supp} \left(w \left(\cdot, \cdot, \frac{1}{|\lambda|} \tan(|\lambda|) \right) \right) \subseteq \{(x, y) \in \mathbb{R}^{2n} : |x_1| \leq a_4\}$$

for some $a_4 < \infty$.

Proof. First, since u and V satisfy the hypothesis of Theorem 1.3, from Proposition 3.2 we get that v and \tilde{V} satisfy (3.8)-(3.11). Now,

$$\begin{aligned} |W(x, y, s)| &= (1 + |\lambda|^2 s^2)^{-1} \left| \tilde{V} \left(\frac{1}{\sqrt{1 + |\lambda|^2 s^2}}(x, y), \frac{1}{|\lambda|} \arctan(|\lambda|s) \right) \right| \\ &\leq \|\tilde{V}\|_{L^\infty(\mathbb{R}^{2n} \times [0, 1])} \end{aligned}$$

for $0 \leq s \leq \frac{1}{|\lambda|} \tan(|\lambda|)$. Therefore $\|W\|_{L^\infty(\mathbb{R}^{2n} \times [0, \frac{1}{|\lambda|} \tan(|\lambda|)])} < \infty$.

Put $(\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{1 + |\lambda|^2 s^2}}(x, y)$. Then $|(x, y)| \geq \rho$ implies $|(\tilde{x}, \tilde{y})| \geq \rho \cos(|\lambda|s)$. Hence (3.8) yields

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \int_0^{\frac{1}{|\lambda|} \tan(|\lambda|)} \|W(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^{2n} \setminus B_\rho)} ds \\ \leq \lim_{\rho \rightarrow \infty} \int_0^1 \|\tilde{V}(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^{2n} \setminus B_\rho)} ds = 0. \end{aligned}$$

Next, an argument as in [7, Proposition 4.3] shows that w solves (3.14).

Further, for $0 \leq s \leq \frac{1}{|\lambda|} \tan(|\lambda|)$,

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} |w(x, y, s)|^2 dx dy \\ &= \int_{\mathbb{R}^{2n}} \left| v \left(\frac{1}{\sqrt{1 + |\lambda|^2 s^2}}(x, y), \frac{1}{|\lambda|} \arctan(|\lambda|s) \right) \right|^2 \frac{dx dy}{(1 + |\lambda|^2 s^2)^n} \\ &= \int_{\mathbb{R}^{2n}} \left| v \left(x, y, \frac{1}{|\lambda|} \arctan(|\lambda|s) \right) \right|^2 dx dy \\ &\leq \sup_{0 \leq s \leq 1} \int_{\mathbb{R}^{2n}} |v(x, y, s)|^2 dx dy. \end{aligned}$$

Thus (3.15) follows from (3.9). The condition (3.16) is same as (3.10). Finally (3.17) is easy from (3.11) with $a_4 = \sec(|\lambda|)a_3$. \square

Step 4. Conclusion.

Suppose that u and V satisfy the hypothesis of Theorem 1.3. Then Proposition 3.3 yields w , defined as in (3.13), is a solution of (3.14) and has the properties listed therein. Therefore, the Paley-Wiener theorem for Schrödinger evolutions on the Euclidean space gives $w = 0$, see [20]. This infers $v = 0$, i.e., $u^\lambda = 0$ for $0 < |\lambda| < \frac{\pi}{2}$.

But, for each $(x, y) \in \mathbb{R}^{2n}$, $u^\lambda(x, y, 1)$ has an homomorphic extension in λ around the real line. We thus conclude that $u^\lambda(x, y, 1) = 0$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. This finally implies that $u = 0$.

4. DATA AVAILABILITY

No data has been generated or analysed during this study.

5. FUNDING AND/OR CONFLICTS OF INTERESTS/COMPETING INTERESTS

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