



Stochastic Modeling in Geophysical Fluid Dynamics

Long Li

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Stochastic Modeling in Geophysical Fluid Dynamics

Seminar at the Research Center for Mathematics and
Interdisciplinary Sciences

Long Li

INRIA, IRMAR, ODYSSEY Team, Rennes, France

September 1, 2024, Qingdao, China



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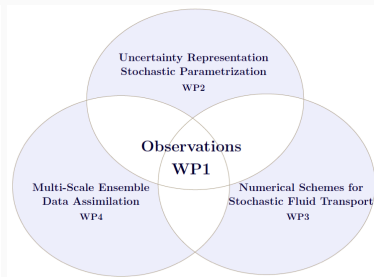
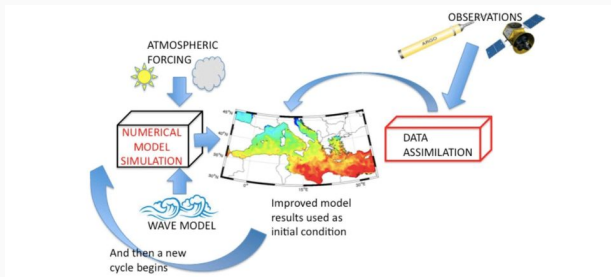
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- 2012 – 2015: **Bachelor's in Mathematics**, Université Jean Monnet Saint Etienne, France
- 2015 – 2017: **Master's in Mathematics and Applications**, Université Grenoble Alpes, France
- 2017 – 2020: **PhD in Mathematics and Their Interactions**, Université de Rennes I, France
- Since 2021: **Starting Research Positions**, Centre Inria de l'Université de Rennes, France

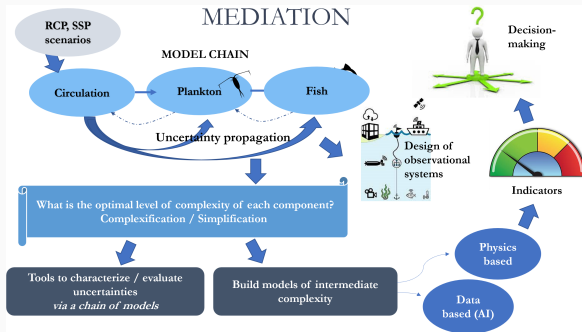
Research Projects

Stochastic transport in upper ocean dynamics (STUOD): International collaborative project, funded by EXCELLENT SCIENCE – European Research Council (ERC), PIs: Bertrand Chapron, Dan Crisan, Darryl Holm, Etienne Mémin



Methodological developments for a robust and efficient digital twin of the ocean (MEDIATION):

National collaborative project, funded by the “Ocean & Climate” French Priority Research Program, PIs: Laurent Debreu, Franck Dumas, Patrick Marchesiello



- **Interests:** Stochastic Modeling, Ensemble Forecasting, Uncertainty Quantification, Sensitivity Analysis, Data Assimilation, Numerical Methods, Scientific Computing
- **Achievements:** 7 high-impact SCI papers (2 as first/corresponding author, 2 as co-first author, 2 as second author), 4 submitted preprints (2 as first/corresponding author, 1 as second author), 8 book chapters (1 as first author, 2 as co-first author, 2 as second author)
- **Other Activities:** Peer reviewer for the *Journal of Advances in Modeling Earth Systems*; Co-supervised Francesco Tucciarone's PhD thesis, titled "Stochastic parameterization of ocean models using high-resolution observations" in collaboration with Etienne Mémin

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Background

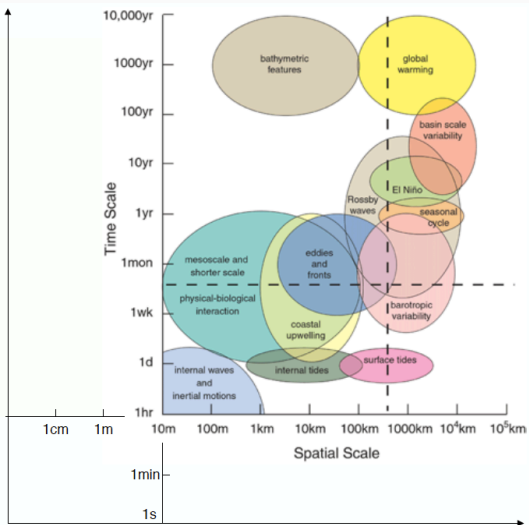
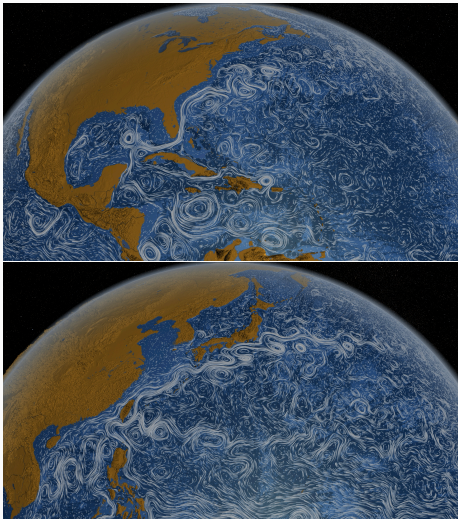
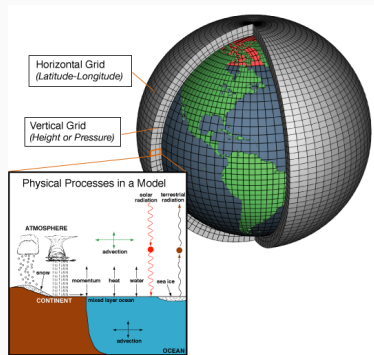


Figure: [Video1] [Video2]

Motivations

- Rigorously model the impact of unresolved small-scale processes on large-scale circulation
- Enhance uncertainty representation to construct physically meaningful ensembles
- Strong interest in applying ensemble methods for forecasting and data assimilation

⇒ Shift from a deterministic, causal physics approach to a stochastic representation of fluid dynamics



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Stochastic flow under location uncertainty [Mémin, 2014]

- Stochastic Lagrangian flow:

$$d\mathbf{X}_t = \underbrace{\mathbf{v}(\mathbf{X}_t, t) dt}_{\text{resolved}} + \underbrace{\sigma(\mathbf{X}_t, t) dB_t}_{\text{unresolved}}$$

\mathbf{X} : Lagrangian displacement within bounded domain $\mathcal{S} \subset \mathbb{R}^d (d = 2, 3)$

B : Cylindrical Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ taking values in the Hilbert space $\mathcal{H} := L^2(\mathcal{S})$

- Correlation operator:

$$\sigma(\mathbf{x}, t)f = \int_{\mathcal{S}} \check{\sigma}(\mathbf{x}, \mathbf{y}, t)f(\mathbf{y}) d\mathbf{y}, \quad f \in \mathcal{H}, \quad \mathbf{x} \in \mathcal{S}$$

$\check{\sigma}$: bounded symmetric integral kernel, σ : Hilbert-Schmidt operator

Stochastic flow under location uncertainty

- Spectral decomposition:

$$\sigma(\mathbf{x}, t) \, dB_t = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) d\beta_t^n$$

$(\phi_n)_n$: a family of eigenfunctions (scaled by eigenvalues) of σ

$(\beta_t^n)_n$: a sequence of independent standard Brownian motions

- Diffusion tensor (derived from noise quadratic variation):

$$\frac{1}{2} \left\langle \int_0^\cdot \sigma^i(\mathbf{x}, s) \, dB_s, \int_0^\cdot \sigma^j(\mathbf{x}, s) \, dB_s \right\rangle_t = \int_0^t a_{ij}(\mathbf{x}, s) \, ds$$

$$\mathbf{a}(\mathbf{x}, t) = \frac{1}{2} \int_{\mathcal{S}} \check{\sigma}(\mathbf{x}, \mathbf{y}, t) \check{\sigma}^T(\mathbf{y}, \mathbf{x}, t) \, d\mathbf{y} = \frac{1}{2} \sum_{n \in \mathbb{N}} (\phi_n \phi_n^T)(\mathbf{x}, t)$$

\mathbf{a} is symmetric non-negative definite and has a unit of $\text{m}^2 \text{s}^{-1}$

In the following, we only consider incompressible unresolved flows: $\nabla \cdot \boldsymbol{\sigma} = 0$

Stochastic Reynolds Transport Theorem [Bauer, Li et al., 2020]

The rate of change of a random scalar q within a volume $\mathcal{V}(t)$ transported by the stochastic flow \mathbf{X}_t is described by

$$d\left(\int_{\mathcal{V}(t)} q(\mathbf{x}, t) d\mathbf{x}\right) = \int_{\mathcal{V}(t)} \left(Dq + q \nabla \cdot \mathbf{v}^* dt\right) d\mathbf{x}$$

with the **stochastic transport operator** D given by

$$Dq = dq + (\mathbf{v}^* dt + \boldsymbol{\sigma} dB_t) \cdot \nabla q - \nabla \cdot (\mathbf{a} \nabla q) dt$$

and the *effective advection drift* $\mathbf{v}^* := \mathbf{v} - \nabla \cdot \mathbf{a}$.

- Derived from Itô-Wenzell formula (Generalized Itô formula) [Kunita, 1997]
- Passage from Lagrangian to Eulerian coordinates \Rightarrow Derive equations of motions following conservation laws (mass, momentum, thermodynamic)
- (D) plays the role of material derivative in the deterministic setting
- $(\nabla \cdot \mathbf{a})$ can be viewed as a statistical-induced velocity due to noise inhomogeneity

Stochastic transport of passive tracers

$$Dq = 0$$

Under incompressible flow ($\nabla \cdot \mathbf{v}^* = 0$) and adequate boundary conditions (e.g., torus) for both components:

- **Conservation of energy:**

$$\begin{aligned} \frac{1}{2} d(\|q\|_{\mathcal{H}}^2) &= \int_S (q dq + \frac{1}{2} d\langle q \rangle_t) d\mathbf{x} \\ &= \left(\underbrace{\int_S q \nabla \cdot (\mathbf{a} \nabla q) d\mathbf{x}}_{\text{energy loss by diffusion}} + \underbrace{\int_S (\nabla q)^T \mathbf{a} \nabla q d\mathbf{x}}_{\text{energy intake by noise}} \right) dt = 0 \end{aligned}$$

\Rightarrow Fluctuation-dissipation theorem encoded directly by stochastic transport

Stochastic transport of passive tracers

- **Energy transfer in an ensemble:**

$$\begin{aligned}d(\|q\|_{\mathcal{H}}^2) = 0 &\Rightarrow d(\mathbb{E}\|q\|_{\mathcal{H}}^2) = 0 \\&\Rightarrow d\left(\int_{\mathcal{S}} \text{Var}(q) d\mathbf{x}\right) = -d(\|\mathbb{E}[q]\|_{\mathcal{H}}^2)\end{aligned}$$

The tracer is continuously randomized as it interacts with unresolved scales

- **Iterative conservations** [Li, 2021]:

$$\begin{aligned}Dq = 0 &\Rightarrow D\left(\frac{1}{p}q^p\right) = q^{p-1}Dq = 0, \quad p > 1 \\&\Rightarrow d\left(\int_{\mathcal{S}} \frac{1}{p}q^p d\mathbf{x}\right) = 0\end{aligned}$$

It shares the same conservation properties as the deterministic one

Other formulations

- **Stratonovich representation** [Bauer, Li et al., 2020]

$$Dq = dq + (\mathbf{v}^* dt + \boldsymbol{\sigma} \circ dB_t) \cdot \nabla q$$

- **Girsanov transformation**

$$W_t = B_t + \int_0^t \Gamma_s ds, \quad t \in [0, T]$$

Γ : \mathcal{H} -valued predictable process with $\mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_0^T \|\Gamma_t\|_{\mathcal{H}}^2 dt \right) \right] < \infty$

Change of probability measures $\left[\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(\int_0^T \langle \Gamma_t, dB_t \rangle_{\mathcal{H}} - \frac{1}{2} \int_0^T \|\Gamma_t\|_{\mathcal{H}}^2 dt \right) \right]$:

$\Rightarrow W_t$ is an \mathcal{H} -valued cylindrical Brownian motion under \mathbb{Q}

\Rightarrow **Stochastic transport including Girsanov drift** (under \mathbb{Q}):

$$Dq = dq + ((\mathbf{v}^* - \boldsymbol{\sigma} \Gamma_t) dt + \boldsymbol{\sigma} dW_t) \cdot \nabla q - \nabla \cdot (\mathbf{a} \nabla q) dt$$

Incorporates slow components of subgrid scales into the initial stochastic formulation [Li et al., 2023a, b], and is useful for nudging techniques in data assimilation [Dufée et al. 2023]

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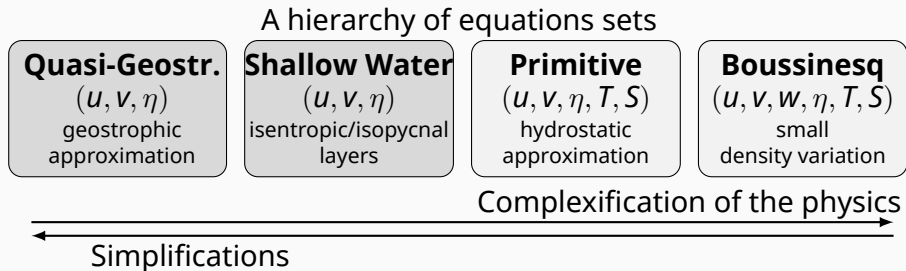
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Geophysical approximated models



- Obtained from power series expansion in small number and scaling
- Same strategy and proper scaling of the noise term [Li et al. 2023]
- Global energy conservation of random systems [Brecht, Li et al. 2021]
- Mathematically well-posed [Debussche et al., 2023, 2024, Lang et al. 2023]

Stochastic rotating shallow water (RSW) model

- Conservation of momentum

$$D\mathbf{u} + f\mathbf{u}^\perp dt = -g\nabla\eta dt$$

- Conservation of mass

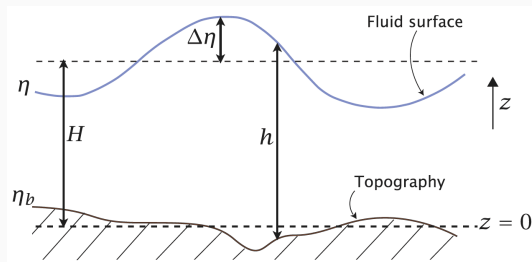
$$Dh + h\nabla \cdot \mathbf{u} dt = 0$$

- Non-divergent constraints

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (\nabla \cdot \mathbf{a}) = 0$$

⇒ Conservation of total energy (KE + PE) [Brecht, Li et al. 2021]:

$$d\left(\int_S \frac{1}{2}(h|\mathbf{u}|^2 + gh^2) d\mathbf{x}\right) = 0$$



$\mathbf{u} = (u, v)$, $\mathbf{u}^\perp = (-v, u)$, $\nabla = (\partial_x, \partial_y)$, f : Coriolis parameter, g : gravity value

Linear wave solutions of stochastic RSW [Mémin, Li et al., 2024]

- Linearised system with $\nabla \cdot \mathbf{a} = 0$:

$$d\mathbf{u} + (f\mathbf{u}^\perp + g\nabla\eta - \nabla \cdot (\mathbf{a}\nabla\mathbf{u})) dt + \nabla\mathbf{u} \cdot \boldsymbol{\sigma} dB_t = 0$$

$$d\eta + (H\nabla \cdot \mathbf{u} - \nabla \cdot (\mathbf{a}\nabla\eta)) dt + \nabla\eta \cdot \boldsymbol{\sigma} dB_t = 0$$

$$d(H\|\mathbf{u}\|_{\mathcal{H}}^2 + g\|\eta\|_{\mathcal{H}}^2) = 0$$

- Mean system:

$$\partial_t \mathbb{E}[\mathbf{u}] + f\mathbb{E}[\mathbf{u}]^\perp + g\nabla \mathbb{E}[\eta] - \nabla \cdot (\mathbf{a}\nabla \mathbb{E}[\mathbf{u}]) = 0$$

$$\partial_t \mathbb{E}[\eta] + H\nabla \cdot \mathbb{E}[\mathbf{u}] - \nabla \cdot (\mathbf{a}\nabla \mathbb{E}[\eta]) = 0$$

$$d(H\|\mathbb{E}[\mathbf{u}]\|_{\mathcal{H}}^2 + g\|\mathbb{E}[\eta]\|_{\mathcal{H}}^2) \leq 0 \Rightarrow d \int_{\mathcal{S}} \left(H(\text{Var}(u) + \text{Var}(v)) + g\text{Var}(\eta) \right) d\mathbf{x} \geq 0$$

- Monochromatic homogeneous noise

$$\sigma(\mathbf{x})dB_t = i\mathbf{k}_\sigma^\perp \alpha \exp(i\mathbf{k}_\sigma \cdot \mathbf{x}) d\beta_t + \text{c.c.}, \quad \mathbf{a} = \frac{1}{2}|\alpha|^2 \mathbf{k}_\sigma^\perp (\mathbf{k}_\sigma^\perp)^\tau$$

- Mean wave ansatz

$$\mathbb{E}[\mathbf{q}](\mathbf{x}, t) = \hat{\mathbf{q}}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)), \quad \mathbf{q} := (\mathbf{u}, \eta)$$

Injecting them into the mean system, we get $L\hat{\mathbf{q}}_0 = 0$, solving the eigenvalue problem for L , we obtain

Mean Poincaré waves

$$\omega = \pm \sqrt{gH|\mathbf{k}|^2 + f^2} - i|\alpha|^2 (\mathbf{k}_\sigma \times \mathbf{k})^2$$

$$\mathbb{E}[\eta](\mathbf{x}, t) = \hat{\eta}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \exp(-|\alpha|^2 (\mathbf{k}_\sigma \times \mathbf{k})^2 t)$$

$$\mathbb{E}[\mathbf{u}](\mathbf{x}, t) = \frac{\hat{\eta}_0}{H|\mathbf{k}|^2} (\omega \mathbf{k} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) + f\mathbf{k}^\perp \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)) \exp(-|\alpha|^2 (\mathbf{k}_\sigma \times \mathbf{k})^2 t)$$

\Rightarrow Mean waves that are not in phase with the noise are damped, whereas the others correspond to classical wave solutions. The decay rate of mean waves depends on noise's variance and scales.

- Constant noise

$$\sigma dB_t = i\alpha \mathbf{k}_\sigma^\perp d\beta_t + \text{c.c.}, \quad \mathbf{a} = \frac{1}{2}\alpha^2 \mathbf{k}_\sigma^\perp (\mathbf{k}_\sigma^\perp)^T$$

- Stochastic wave ansatz

$$\mathbf{q}(\mathbf{x}, t) = \hat{\mathbf{q}}_0 \exp\left(i(\mathbf{k} \cdot \mathbf{x} - \omega t - \text{Re}\{i\gamma\beta_t\})\right)$$

- Itô formula

$$d\mathbf{q} = -\left((i\omega + \frac{1}{2}|\gamma|^2) dt + i\text{Re}\{i\gamma d\beta_t\}\right) \mathbf{q}$$

Pathwise Poincaré waves under constant noise

$$\omega = \pm \sqrt{gH|\mathbf{k}|^2 + f^2}, \quad \gamma = \alpha \mathbf{k}_\sigma \times \mathbf{k},$$

$$\eta(\mathbf{x}, t) = \hat{\eta}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t + \gamma \text{Im}\{\beta_t\})$$

$$\mathbf{u}(\mathbf{x}, t) = \frac{\hat{\eta}_0}{H|\mathbf{k}|^2} (\omega \mathbf{k} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t + \gamma \text{Im}\{\beta_t\}) + f \mathbf{k}^\perp \sin(\mathbf{k} \cdot \mathbf{x} - \omega t + \gamma \text{Im}\{\beta_t\}))$$

⇒ Constant noise only randomise wave's temporal phase.

- “Smooth” homogeneous noise ($\epsilon \ll 1$):

$$\sigma(\mathbf{x}) dB_t = i\alpha \mathbf{k}_\sigma^\perp \exp(i\epsilon \mathbf{k}_\sigma \cdot \mathbf{x}) d\beta_t + \text{c.c.}, \quad \mathbf{a} = \frac{1}{2}|\alpha|^2 \mathbf{k}_\sigma^\perp (\mathbf{k}_\sigma^\perp)^T$$

- Approximation of stochastic ansatz

$$\mathbf{q}(\mathbf{x}, t) \approx \hat{\mathbf{q}}_0 \exp\left(i(\mathbf{k} \cdot \mathbf{x} - \omega t - \gamma \operatorname{Re}\{i(1 + i\epsilon \mathbf{k}_\sigma \cdot \mathbf{x})\beta_t\})\right)$$

Pathwise Poincaré waves under homogeneous noise

$$\gamma = \alpha \mathbf{k}_\sigma \times \mathbf{k}, \quad \omega = \pm \sqrt{gH|\tilde{\mathbf{k}}_t|^2 + f^2}, \quad \tilde{\mathbf{k}}_t := \mathbf{k} + \epsilon \mathbf{k}_\sigma \gamma \operatorname{Re}\{\beta_t\}$$

$$\eta(\mathbf{x}, t) = \hat{\eta}_0 \cos(\tilde{\mathbf{k}}_t \cdot \mathbf{x} - \omega t + \gamma \operatorname{Im}\{\beta_t\})$$

$$\mathbf{u}(\mathbf{x}, t) = \frac{\hat{\eta}_0}{H|\tilde{\mathbf{k}}_t|^2} (\omega \tilde{\mathbf{k}}_t \cos(\tilde{\mathbf{k}}_t \cdot \mathbf{x} - \omega t + \gamma \operatorname{Im}\{\beta_t\}) + f \tilde{\mathbf{k}}_t^\perp \sin(\tilde{\mathbf{k}}_t \cdot \mathbf{x} - \omega t + \gamma \operatorname{Im}\{\beta_t\}))$$

\Rightarrow Pathwise waves that are not in phase with the noise are scattered, whereas the others correspond to classical wave solutions.

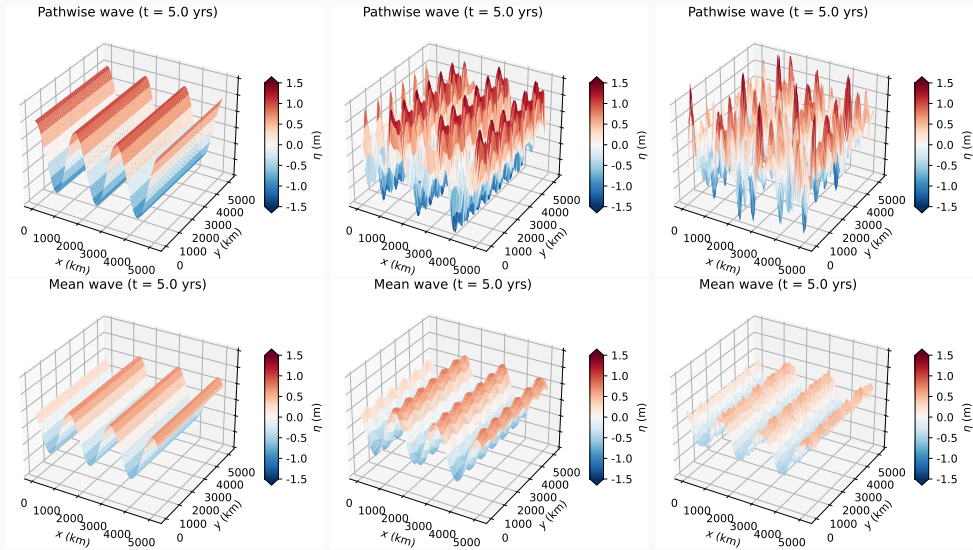


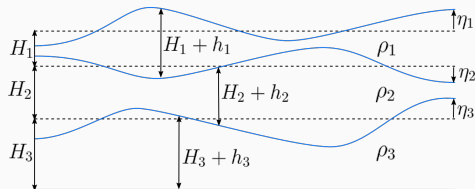
Figure: Illustration for pathwise surface elevation (top) and its ensemble-mean (bottom) with constant noise (left), homogeneous monochromatic noise (middle) and homogeneous multi-scale noise (right).

Stochastic multilayer SW model

$$D\mathbf{u} - f\mathbf{v} dt = -\partial_x \mathbf{p} dt$$

$$D\mathbf{v} + f\mathbf{u} dt = -\partial_y \mathbf{p} dt$$

$$D\mathbf{h} + (\mathbf{h} + \mathbf{H})(\partial_x \mathbf{u} + \partial_y \mathbf{v}) dt = 0$$



$$\mathbf{u} = (u_1, \dots, u_n)^T, \mathbf{v} = (v_1, \dots, v_n)^T, \mathbf{h} = (h_1, \dots, h_n)^T$$

$$\mathbf{p} = M\mathbf{h}, \quad \mathbf{h} = \mathbf{H}\mathbf{A}\mathbf{p}, \quad A = \frac{1}{\rho_0} \begin{bmatrix} \frac{1}{H_1 g'_1} + \frac{1}{H_1 g'_2} & \frac{-1}{H_1 g'_2} & \cdot & \cdot \\ \frac{-1}{H_2 g'_2} & \frac{1}{H_2} \left(\frac{1}{g'_2} + \frac{1}{g'_3} \right) & \frac{-1}{H_2 g'_3} & \cdot \\ \cdot & \dots & \dots & \dots \\ \cdot & \cdot & \frac{-1}{H_n g'_n} & \frac{1}{H_n g'_n} \end{bmatrix}$$

Compact form:

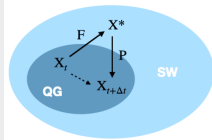
$$d\mathbf{X} = F(\mathbf{X}) dt + \Sigma(\mathbf{X}) dB_t, \quad \mathbf{X} = (\mathbf{u}, \mathbf{v}, \mathbf{h})^T$$

Stochastic multilayer quasi-geostrophic (QG) model

Formulated as a projection of the stochastic SW [Thiry, Li et al., 2024]

$$d\mathbf{X} = P \circ F(\mathbf{X}) dt + P \circ \Sigma(\mathbf{X}) dB_t, \quad \mathbf{X} = (\mathbf{u}, \mathbf{v}, h)^T$$

$$P = G \circ (Q \circ G)^{-1} \circ Q \quad \Rightarrow \quad P \circ P = P$$



- Potential Vorticity operator: $Q(\mathbf{X}) = f_0(\partial_x \mathbf{v} - \partial_y \mathbf{u}) - f_0^2 \frac{h}{H}$
- Geostrophic operator: $G(\mathbf{p}) = \left(-\frac{1}{f_0} \partial_y \mathbf{p}, \frac{1}{f_0} \partial_x \mathbf{p}, H \mathbf{A} \mathbf{p} \right)^T$
- Helmholtz operator: $Q \circ G(\mathbf{p}) = (\Delta - f_0^2 A) \mathbf{p}$

⇒ QG projector filters gravity waves by solving Helmholtz eq., like Leray projector filters sound waves by solving Poisson eq.

⇒ Resulting random system preserves total energy pathwise

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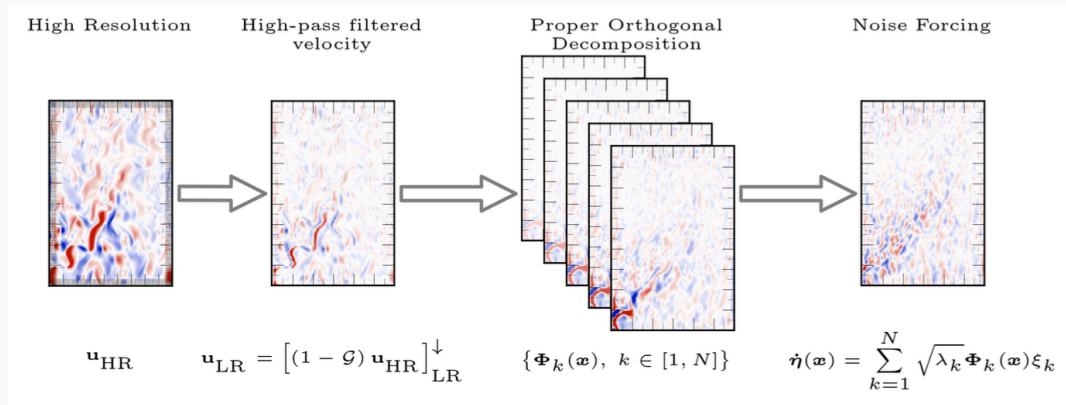
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Empirical Orthogonal Functions (EOF)

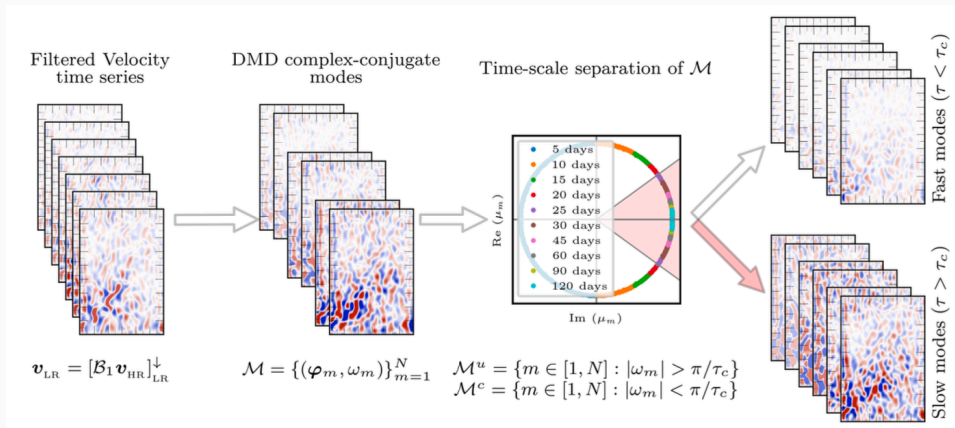
Linear procedure that provides a basis of the “most probable” realizations of the input data \Rightarrow Data-driven stationary noise model



Girsanov drift can be build by $\sigma(\mathbf{x})\Gamma = \sum_{k=1}^N \langle \overline{\mathbf{u}}_{\text{LR}}^t, \Phi_k \rangle \Phi_k(\mathbf{x})$

Dynamic Mode Decomposition (DMD)

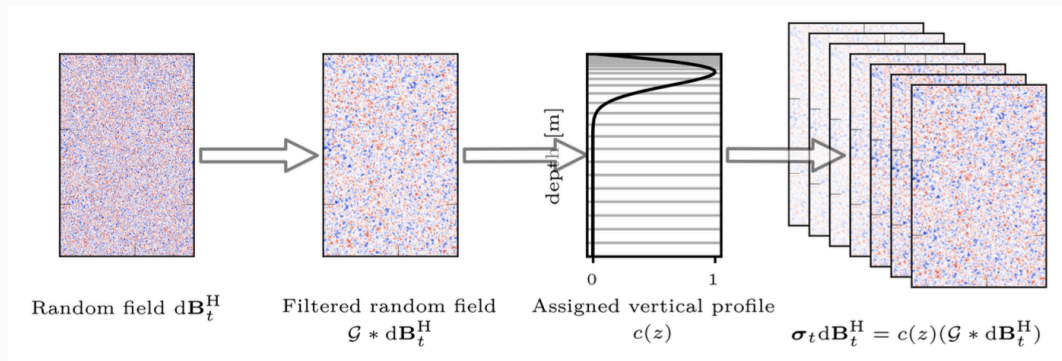
Builds a linear dynamical system from data and provides expansion in terms of linear recurrent solution.



$$\sigma \Gamma_t = \sum_{m \in \mathcal{M}^c} \exp(i\omega_m t) \boldsymbol{\varphi}_m(\mathbf{x}), \quad \sigma dB_t = \sum_{m \in \mathcal{M}^u} \exp(i\omega_m t) \boldsymbol{\varphi}_m(\mathbf{x}) d\beta_t^m$$

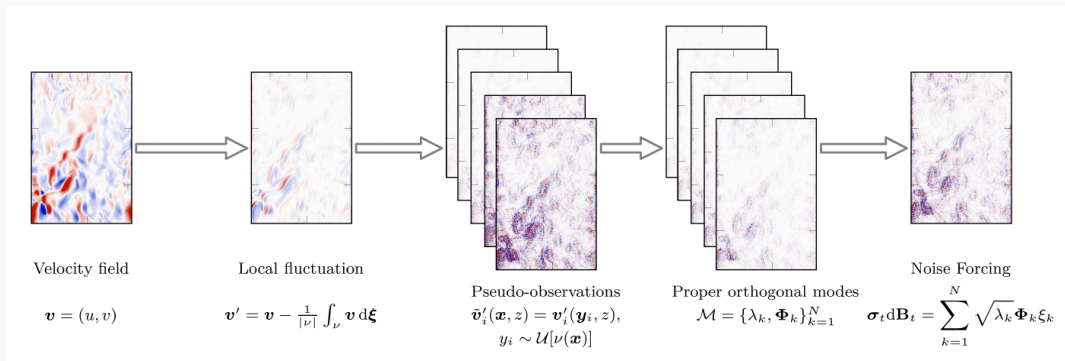
Vertical Profile Parameterization (VPP)

Create a horizontal Gaussian noise with a vertical profile imposed connect the noise to the current state of the flow.



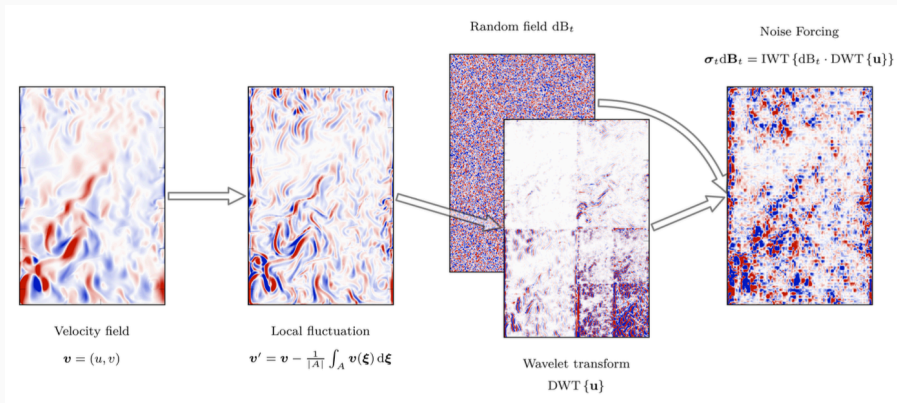
Pseudo-Observation (PSO)

Build a set of random observations (pseudo-observations) and apply EOF online (at each timestep).



Wavelet Transform (WLT)

Wavelet basis function characterized by a fixed shape and by a scale, allows detect and randomise edges and filaments.



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[Brecht, Li et al., 2021] Barotropically unstable jet at mid-latitude on the sphere, flow destabilized by a small perturbation of the initial field, RSW model including a biharmonic eddy viscosity in the momentum equation

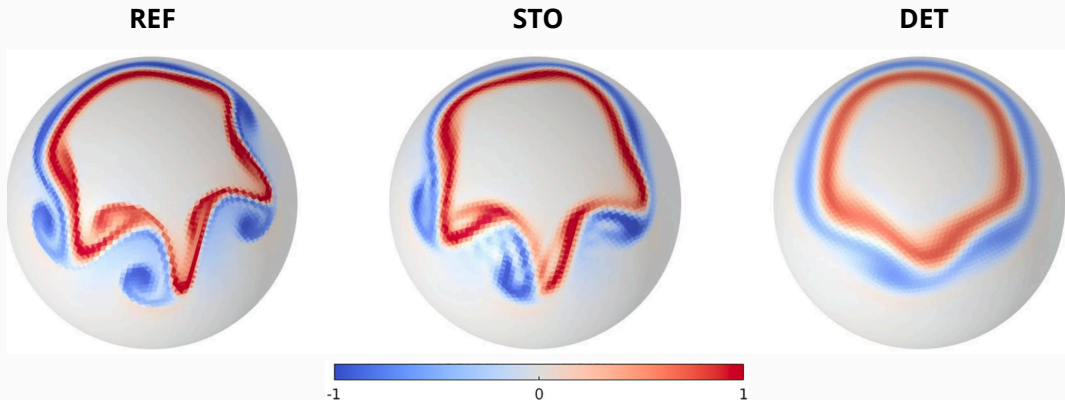


Figure: Evolution of vorticity during one week for different models. REF: high-resolution (60 km) reference; STO: low-resolution (240 km) stochastic model; DET: low-resolution (240 km) deterministic model.

⇒ STO better captures the large scale structure of REF than DET

⇒ STO produces more efficient ensemble spread containing observations than PIC (perturbed initial condition).

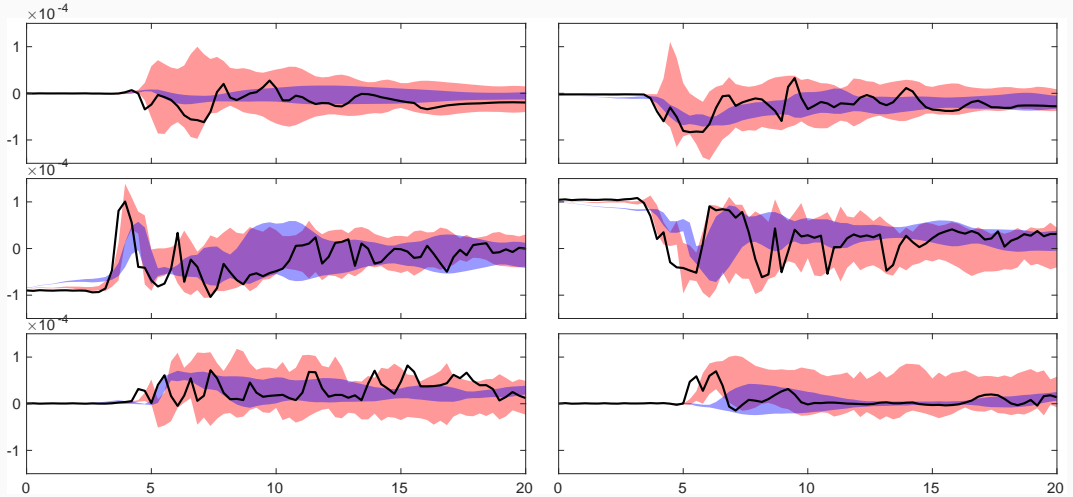


Figure: Comparison of ensemble spread evolution of the vorticity field for the **STO-EOF** (red area) runs and the **PIC-EOF** (blue area) runs, at six different locations. The observation trajectories are shown by the black lines.

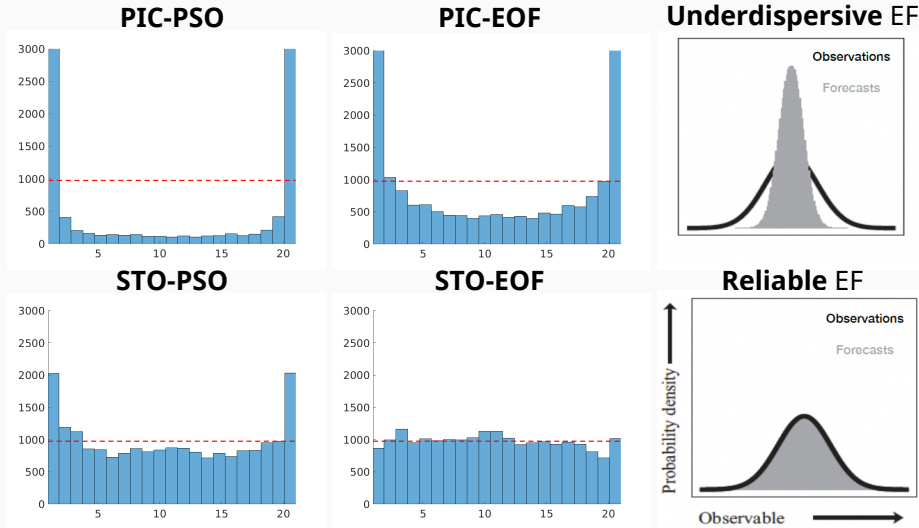


Figure: Comparison of the rank histograms for different models after 2 weeks.

⇒ STO-EOF better than STO-PSO better than PIC-EOF better than PIC-PSO

For a reliable ensemble forecast, the mean squared error (MSE) of the ensemble matches the mean intra-ensemble variance (MEV) up to a factor:

$$\text{MSE} = \frac{1}{N} \sum_{n=1}^N (\widehat{\mathbb{E}}[q_n] - q_n^o)^2$$

$$\text{MEV} = \frac{1}{N} \sum_{n=1}^N \widehat{\text{Var}}(q_n)$$

$$\text{MSE} \approx \frac{M+1}{M} \text{MEV}$$

q_n^o : n -th observation, $q_n^{(m)}$: n -th forecast of the m -th ensemble member

⇒ STO provides a good trade-off between model error representation and ensemble spread

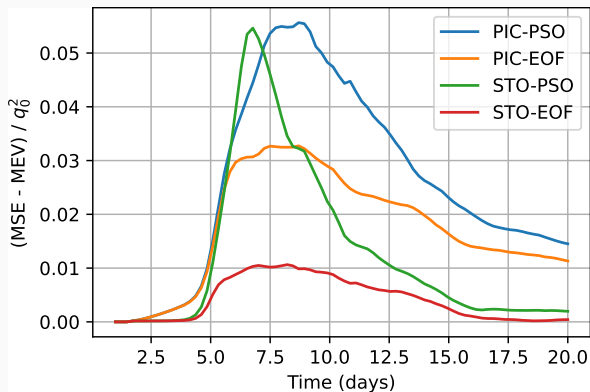


Figure: Comparison of the differences between the mean square error (MSE) and the mean ensemble variance (MEV) of the ensemble vorticity fields for different models.

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- Multilayer QG model in double-gyre configuration incorporating steady symmetric wind forcing and biharmonic eddy viscosity
- Western boundary current problem: challenges in reproducing the jet and enhancing the intrinsic variability in coarse-resolution models
- The structure of noise and Girsanov drift is calibrated from data and a projection method is proposed to parameterize their dynamics [Li et al., 2023]

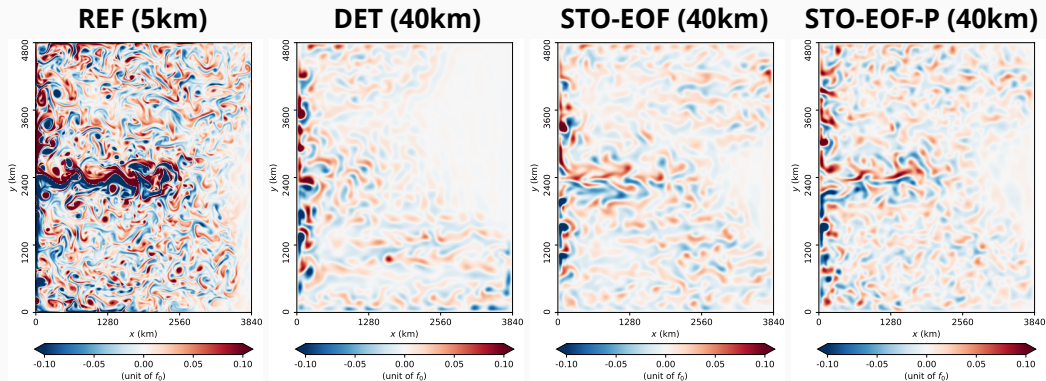


Figure: Snapshots of upper layer relative vorticity provided by different simulations after 60-years run.

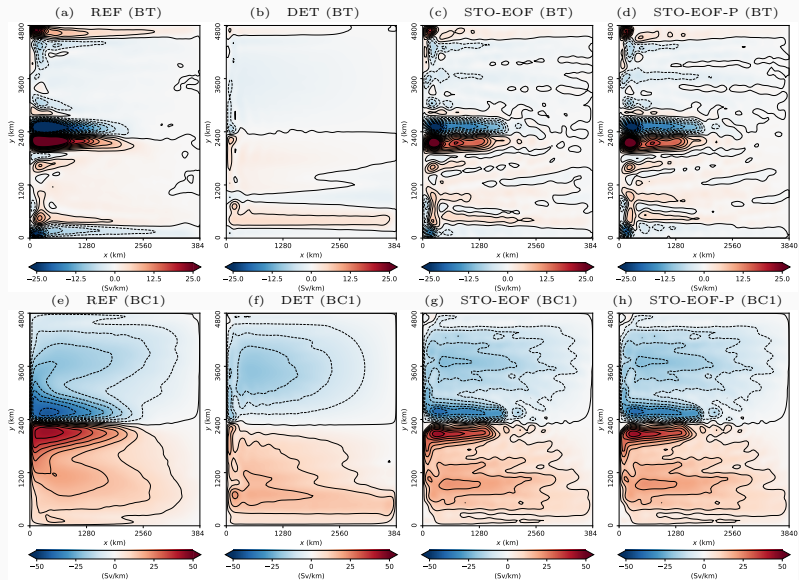


Figure: Comparison of time-mean contour of barotropic (top) and first baroclinic (bottom) streamfunctions for different models.

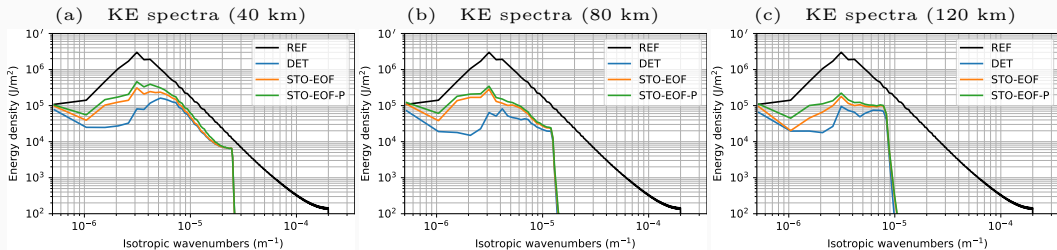


Figure: Temporal mean of vertically integrated kinetic energy (KE) spectrum for different models at different resolutions.

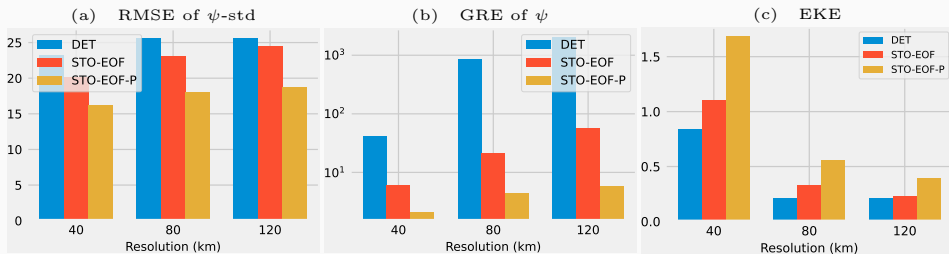


Figure: Comparison of variability measures across different coarse models. Lower RMSE and GRE values indicate better model performance. The EKE of the REF model is $8.26 J/m^3$.

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Stochastic Ekman-Stokes model [Li et al., 2024]

- A novel formulation encodes the interactions between wind, waves, and currents through the introduction of the uncertainty of unresolved motions (*p*-SPDE):

$$d\mathbf{u}_e = \left(-if(\mathbf{u}_e + \mathbf{u}_s) + \partial_z(a_{zz}\partial_z(\mathbf{u}_e + \mathbf{u}_s)) \right) dt - (if\sigma_{\mathbf{x}} + \sigma_z\partial_z(\mathbf{u}_e + \mathbf{u}_s)) dB_t$$

$$\rho_0 a_{zz} \partial_z \mathbf{u}_e|_{z=0} = \boldsymbol{\tau}_w, \quad \rho_0 a_{zz} \partial_z \mathbf{u}_e|_{z=-H} = 0$$

\mathbf{u}_e : Ekman current, \mathbf{u}_s : Stokes drift (wave), $\boldsymbol{\tau}_w$: wind stress

- Comparison with a benchmark model (*p*-PDE):

$$\partial_t \mathbf{u}_e + if(\mathbf{u}_e + \mathbf{u}_s) = \partial_z(a_{zz}\partial_z \mathbf{u}_e)$$

with the same boundary conditions.

- Both models incorporate established physical parameterizations: Markovian wind process, random wave direction, and K-Profile vertical mixing

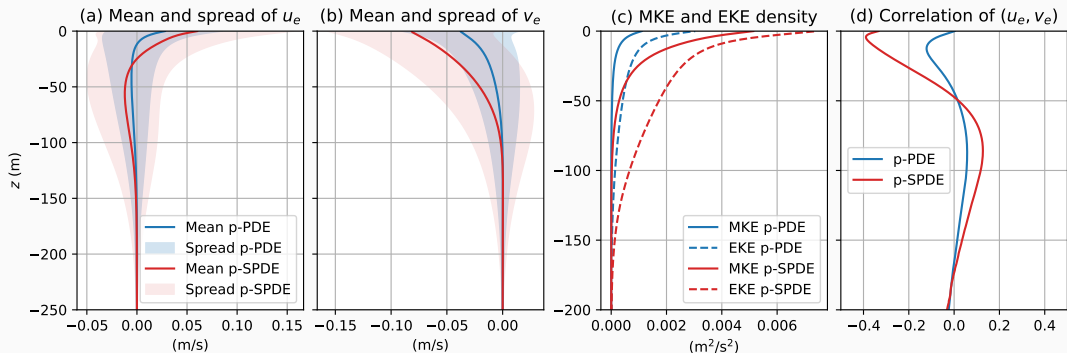


Figure: Comparison of the ensemble mean and spread for the Ekman velocity components, the MKE and EKE density, as well as the Pearson correlation coefficient for the Ekman velocity components, using different random models.

⇒ The p-SPDE model yields smoother mean Ekman velocity profiles, higher uncertainty, increased MKE and EKE, and stronger correlations.

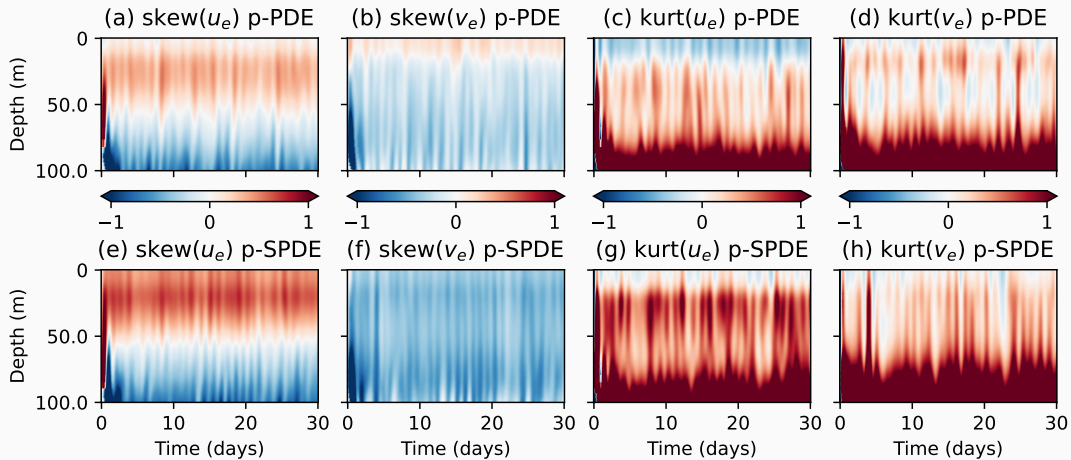


Figure: Comparison of the ensemble skewness and kurtosis for the Ekman velocity components over time and depth using different random models.

⇒ The p-SPDE model shows more skewed distributions and extreme value occurrences, particularly near the surface.

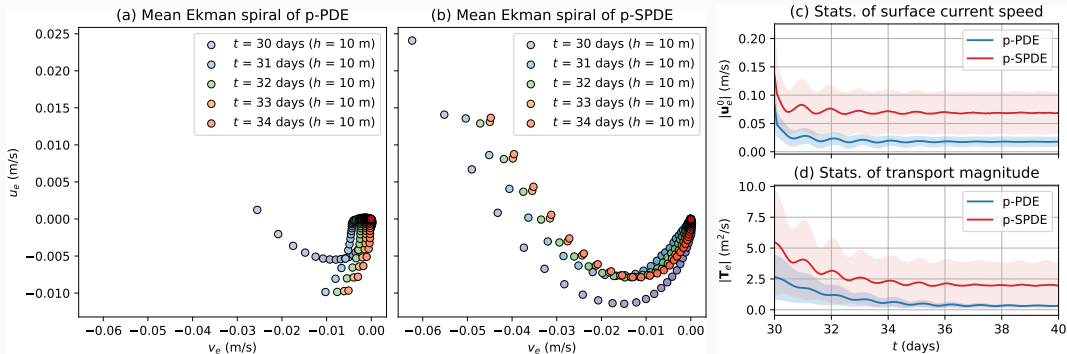


Figure: Evolution comparison after wind stopped for (a, b) mean Ekman spiral, (c) mean and spread of surface current speed, and (d) mean and spread of Ekman transport magnitude, using different random models.

⇒ After wind stops, the p-SPDE model preserves its spiral structure and exhibits higher residual values and uncertainties compared to the p-PDE, indicating a shift to wave-driven dynamics.

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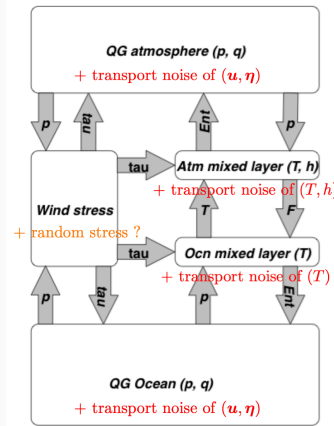
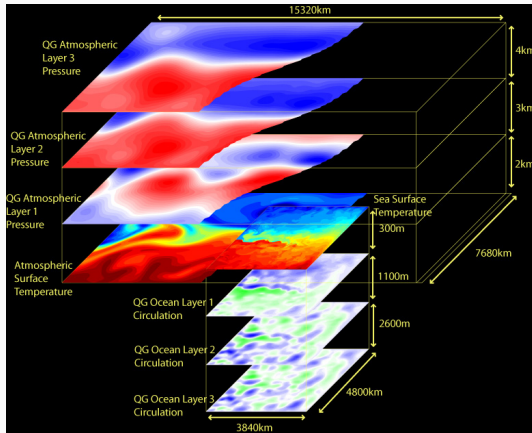
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Take home messages

- The stochastic transport formulation preserves key physical properties, such as energy conservation.
- The random models allow us to accurately reproduce the statistical distribution of large-scale flow on a coarse mesh.
- This stochastic scheme provides a good trade-off between model error representation and ensemble spread.
- Temporal and spatial characteristics of the unresolved flow component have strong influences on the numerical results.
- Some nonlinear effects, such as wave scattering and wave selection, can be represented by a linear stochastic formulation.
- By incorporating uncertainty from smaller scales, the approach helps explain interactions between intermittent winds, waves, and currents.

Perspective: Mesoscale air-sea interaction



Preliminary conclusion: Preliminary Conclusion: Introducing small-scale turbulence via stochastic transport attenuates air-sea coupling effects.

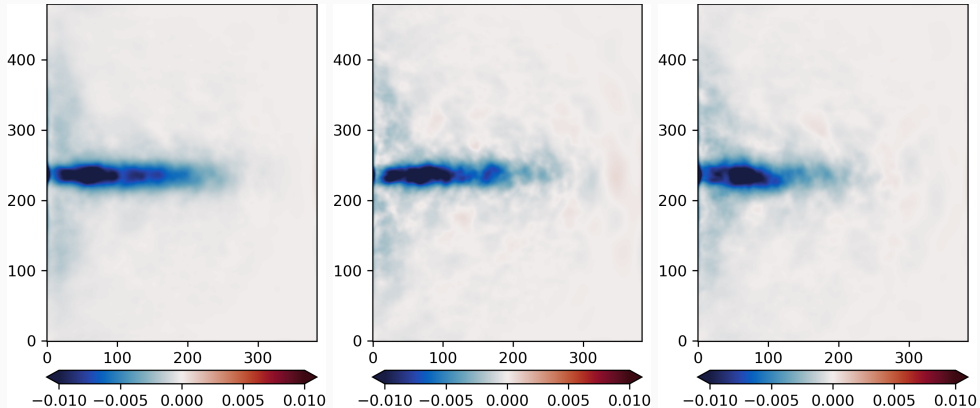
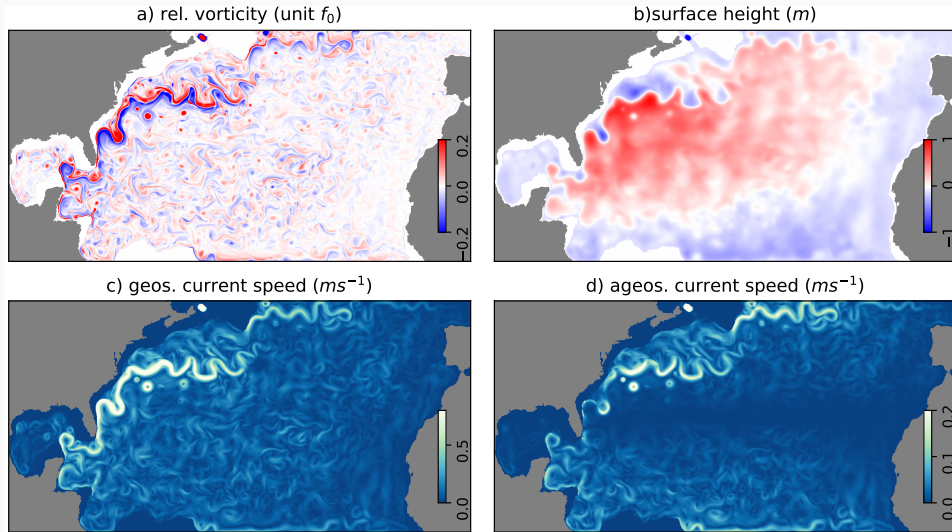


Figure: Comparison of eddy wind works $\text{Cov}(\tau_o, \mathbf{u}_o)$: (left) random initial conditions; (middle) random forcings; (right) random model.

Perspective: Data assimilation in realistic configurations



Questions?