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PROJECTIVE UNIFICATION IN BI-INTUITIONISTIC LOGIC

DAMIANO FORNASIERE, QUENTIN GOUGEON, AND MIGUEL MARTINS

ABSTRACT. The *bi-intuitionistic propositional calculus* **bi-IPC** is the natural symmetric extension of the intuitionistic logic **IPC**. We provide both an axiomatic and semantic characterization of the consistent axiomatic extensions of **bi-IPC** which admit projective unification. They are exactly those which contain a theorem of the form $(\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$, for some $n \in \mathbb{N}$ – or, equivalently, which are valid on Kripke frames of n -bounded zigzag depth, a simple order-theoretic condition. We also prove that **bi-IPC** does not have a unitary unification type by showing that while the formula $\neg\sim p \rightarrow (\neg\sim)p$ is a unifiable formula in **bi-IPC**, it cannot have a most general unifier.

1. INTRODUCTION

In propositional logics, a formula φ is said to be *unifiable* if there exists a valid instance $\sigma(\varphi)$ of φ . In this case, the substitution σ is called a *unifier* of φ , and it is said to be *most general* if every unifier of φ is an instance of σ . Some most general unifiers have the stronger property of being *projective* [Ghi97], which makes them easier to identify. Unification is closely related to the problem of *admissibility*: an inference rule $\frac{\varphi}{\psi}$ is admissible if every unifier of φ is also a unifier of ψ – which means, intuitively, that the rule turns theorems into theorems. If φ admits a most general unifier σ , then checking that $\frac{\varphi}{\psi}$ is admissible amounts to check that σ is a unifier of ψ . Even better, if φ admits a projective unifier, then $\frac{\varphi}{\psi}$ is admissible if and only if ψ is a global consequence of φ .

The problem of recognizing admissible rules attracted the interest of intuitionistic logicians, after Friedman put it under the spotlight in a 1975 list of mathematical problems [Fri75]. This problem was eventually solved by Citkin [Cit78] and Rybakov [Ryb84]. Later on, Ghilardi made notable steps toward understanding unification in the intuitionistic propositional calculus **IPC** [Ghi99, Ghi04] and in many of its fragments [Ghi97]. In this paper, we will begin the study of unification in the *bi-intuitionistic propositional calculus* **bi-IPC**, the natural symmetric extension of **IPC**.

The logic **bi-IPC** can be obtained by expanding **IPC** with the *co-implication* connective \leftarrow (also called *exclusion*, or *subtraction*), by ensuring that it behaves dually to the intuitionistic implication \rightarrow and by adding the *double negation* inference rule “from φ infer $\neg\sim\varphi$ ” (we use $\sim\varphi$ as a shorthand for the formula $\top \leftarrow \varphi$, and call it the *co-negation* of φ). This process yields a conservative extension of **IPC** with significantly greater expressive power, a feature that can be easily observed in [Wol98], where Gödel’s embedding of **IPC** into the modal logic **S4** is extended to an embedding of **bi-IPC** into the temporal modal logic tense-**S4**.

As mentioned above, **bi-IPC** can be thought of as the natural symmetric extension of **IPC**. This is because, thanks to the co-implication \leftarrow , the logic **bi-IPC** achieves a symmetry between its connectives (namely, each of $\wedge, \rightarrow, \neg, \perp$ has its dual $\vee, \leftarrow, \sim, \top$, respectively), a property that **IPC** notably lacks. Moreover, this symmetry is reflected by the fact that **bi-IPC** is *algebraized* (in the sense of [BP89]) by the variety of *bi-Heyting algebras* (Heyting algebras whose order duals are also Heyting algebras). In fact, the lattice of *bi-intermediate logics* (consistent axiomatic extensions of **bi-IPC**) is dually isomorphic to that of nontrivial subvarieties of bi-Heyting algebras (see, e.g., [Rau74b]). The theory of these algebras was developed by Rauszer and others in a series of paper [Bea80, Köh80, Rau74a, Rau74b, Rau77, San85], motivated by their connection with bi-intuitionistic logic. However, bi-Heyting algebras also arise naturally in other fields of research. For

example, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, *e.g.*, [Tay16]). Similarly, every quantum system can be associated with a complete bi-Heyting algebra [Dör16]. Many other examples can be found, especially in the field of topos theory [Law86, Law91, RZ96].

And although **bi-IPC** has recently gathered the attention of some logicians (see, *e.g.*, [Bad16, BGJ22, Gor00, GP10, Shr16]), to our knowledge no advances have been made in the study of its unification problems. We start filling this vacuum by proving two results: we provide an axiomatic characterization of all the projective bi-intermediate logics (Section 3), and show that **bi-IPC** does not have a unitary unification type (Section 4). In more detail, we prove that the projective bi-intermediate logics are exactly those whose Kripke models have a natural bound for the size of the “zigzags”, a property that is definable by a type of formulas in the language of **bi-IPC**, namely, those of the form $(\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$, for $n \in \mathbb{N}$. As for the proof that **bi-IPC** is not unitary, we show that the formula $p \rightarrow \neg\sim p$ is unifiable but does not admit a most general unifier.

Notably, our methods diverge significantly from those used by Ghilardi in his study of unification in **IPC** [Ghi04]. In fact, due to the different properties of the two logical systems, most of Ghilardi’s machinery does not admit an adaptation into the bi-intuitionistic setting. For example, throughout his paper he makes crucial use of the fact that **IPC** is sound and complete with respect to finite rooted Kripke frames. And while **bi-IPC** also has the finite model property, it does not admit a completeness result to such a simple class as that of finite rooted frames (or finite co-rooted frames, or even frames $\langle W, \leq \rangle$ such that $W = \uparrow w \cup \downarrow w$ for some $w \in W$). Yet another tool that we failed to adapt to our setting, which was essential for his characterization of projective formulas in **IPC**, were the substitutions θ_φ^a . Given a formula $\varphi(p_1, \dots, p_n)$ in the language of **IPC**, Ghilardi defines for each $a \subseteq \{p_1, \dots, p_n\}$ a substitution θ_φ^a by

$$\theta_\varphi^a(p) := \begin{cases} \varphi \rightarrow p & \text{if } p \in a, \\ \varphi \wedge p & \text{if } p \notin a, \end{cases}$$

which all satisfy certain desirable properties. He then proceeds to define θ_φ as a suitable finite composition of the above defined θ_φ^a , and prove that φ is projective iff θ_φ unifies φ . However, after many attempts, we could not find a general bi-intuitionistic analogue for the θ_φ^a that would satisfy, for all φ , the aforementioned desirable properties necessary to characterize projective formulas. While some definitions of θ_φ^a would work (in the sense of satisfying the desirable properties) when φ was a \leftarrow -free formula (and others for \rightarrow -free formulas), they would fail if this was not the case.

2. BACKGROUND

2.1. Bi-intuitionistic logics and their models. We fix a denumerable set **Var** of *variables* (which we will usually denote by $p, q, r \dots$). The language \mathcal{L} of the *propositional bi-intuitionistic logic* is then defined by:

$$\varphi ::= p \mid \perp \mid \top \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftarrow \varphi),$$

where $p \in \mathbf{Var}$. Given $\varphi \in \mathcal{L}$, we denote by $\text{var}(\varphi)$ the set of variables occurring in φ . When $P \subseteq \mathbf{Var}$, we denote by \mathcal{L}_P the set of all formulas φ such that $\text{var}(\varphi) \subseteq P$. We use the abbreviations $\neg\varphi := \varphi \rightarrow \perp$, $\sim\varphi := \top \leftarrow \varphi$ and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. For $n \in \mathbb{N}$, we define $(\neg\sim)^n \varphi$ recursively as follows:

$$(\neg\sim)^0 \varphi := \varphi \quad \text{and} \quad (\neg\sim)^{n+1} \varphi := \neg(\sim(\neg\sim)^n \varphi).$$

The *bi-intuitionistic propositional calculus* **bi-IPC** is the least set of formulas in the language \mathcal{L} , that contains **IPC**, the axioms below, and is closed under modus ponens, uniform substitutions, and the *double negation rule* “from φ infer $\neg\sim\varphi$ ”.

$$(1) \quad p \rightarrow (q \vee (p \leftarrow q)),$$

- (2) $(p \leftarrow q) \rightarrow \sim(p \rightarrow q)$,
- (3) $((p \leftarrow q) \leftarrow r) \rightarrow (p \leftarrow q \vee r)$,
- (4) $\neg(p \leftarrow q) \rightarrow (p \rightarrow q)$.

Using a straightforward Kripke semantical argument, it can be shown that **bi-IPC** is a conservative extension of **IPC**. Additionally, the *classical propositional calculus* **CPC** can be identified with the proper extension of **bi-IPC** obtained by adding the *law of excluded middle* $p \vee \neg p$. Notably, in **CPC** the co-implication \leftarrow is term-definable, since $(p \leftarrow q) \leftrightarrow (p \wedge \neg q) \in \mathbf{CPC}$. Consequently, the double negation rule becomes superfluous, as it reduces to “from φ infer φ ”.

A *super-bi-intuitionistic logic* is a set of formulas \mathbf{L} in the language \mathcal{L} , that contains **bi-IPC** and is closed under the three inference rules listed above. In this case, the elements $\varphi \in \mathbf{L}$ are called the *theorems* of \mathbf{L} . We call \mathbf{L} *consistent* if $\perp \notin \mathbf{L}$ and *inconsistent* otherwise. A super-bi-intuitionistic logic \mathbf{L}' is an *extension* of \mathbf{L} when $\mathbf{L} \subseteq \mathbf{L}'$. Consistent super-bi-intuitionistic logics are called *bi-intermediate logics*, and it can be shown that a super-bi-intuitionistic logic \mathbf{L} is a bi-intermediate logic if and only if $\mathbf{L} \subseteq \mathbf{CPC}$. Given a set of formulas Σ , we denote by $\mathbf{L} + \Sigma$ the least (with respect to inclusion) bi-intuitionistic logic containing $\mathbf{L} \cup \Sigma$ (which always exists), and call it an *axiomatic extension* of \mathbf{L} . If Σ is a singleton $\{\varphi\}$, we simply write $\mathbf{L} + \varphi$. Two formulas φ and ψ are *\mathbf{L} -equivalent* when $(\varphi \leftrightarrow \psi) \in \mathbf{L}$, in which case we write $\varphi \equiv_{\mathbf{L}} \psi$. Finally, we call \mathbf{L} *locally tabular* if whenever $P \subseteq \mathbf{Var}$ is finite, there are only finitely formulas in \mathcal{L}_P up to \mathbf{L} -equivalence.

Notable bi-intermediate logics include the *bi-intuitionistic Gödel-Dummett logic* $\mathbf{bi-GD} := \mathbf{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p)$, extensively studied in [BMM24], and, for each $n \in \mathbb{N}$, the bi-intermediate logic of *n -bounded zigzag depth* $\mathbf{BZ}_n := \mathbf{bi-IPC} + (\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$, known for being connected to the existence of so-called *discriminator terms* [Tay16].

In order to introduce the notion of logical consequence that we will use throughout this paper, we first need to define *intuitionistic Kripke frames* and *models*. A *frame* (for short) is a pair $\mathfrak{F} = \langle W, \leq \rangle$, where W is a set and \leq is a *partial order on W* (i.e., a binary relation on W that is reflexive, transitive, and antisymmetric). In other words, a frame is just a *poset*. Given a frame $\mathfrak{F} = \langle W, \leq \rangle$ and a subset $U \subseteq W$, we denote the *upset generated by U* by

$$\uparrow U := \{w \in W : \exists u \in U \text{ s.t. } u \leq w\},$$

and if $U = \uparrow U$, then U is called an *upset*. If $U = \{u\}$, we simply write $\uparrow u$. The set of upsets of \mathfrak{F} will be denoted by $Up(\mathfrak{F})$. The notion of a *downset* and the arrow operator \downarrow are defined analogously, and if U is both an upset and a downset, we call it an *updownset*. We will always use the convention that the arrow operators defined above bind stronger than the other set theoretic operations. For example, the expressions $\uparrow U \setminus V$ and $\downarrow U \cap V$ are to be read as $(\uparrow U) \setminus V$ and $(\downarrow U) \cap V$, respectively.

For $n \in \mathbb{N}$, we define $(\downarrow\uparrow)^n U$ recursively as follows:

$$(\downarrow\uparrow)^0 U := U \quad \text{and} \quad (\downarrow\uparrow)^{n+1} U := \downarrow(\uparrow(\downarrow\uparrow)^n U).$$

Intuitively, $(\downarrow\uparrow)^n U$ is the set of points in W that can be reached from a point in U after n -many zigzags, in the sense that $w \in (\downarrow\uparrow)^n U$ if and only if there are $u \in U$ and $x_1, y_1, \dots, x_n, y_n \in W$ such that

$$u \leq x_1 \geq y_1 \leq x_2 \geq y_2 \leq \dots \leq x_n \geq y_n = w.$$

An *intuitionistic Kripke model* (or *model* for short) is a tuple $\mathfrak{M} = \langle W, \leq, V \rangle$ where $\mathfrak{F} = \langle W, \leq \rangle$ is a frame and $V : P \rightarrow Up(\mathfrak{F})$, for some $P \subseteq \mathbf{Var}$. We refer to such maps as *valuations* (of P) on \mathfrak{F} , to such models as *models* (of P) on \mathfrak{F} , and use the shorthand notation $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$. The notion of *satisfaction* of a formula φ at a point $w \in W$ is defined as usual when the main connective of φ is in $\{\perp, \top, \wedge, \vee, \rightarrow\}$, and if φ is of the form $\alpha \leftarrow \beta$, we define

$$\mathfrak{M}, w \models \alpha \leftarrow \beta \iff \exists u \leq w \text{ s.t. } \mathfrak{M}, u \models \alpha \text{ and } \mathfrak{M}, u \not\models \beta.$$

Moreover, the notions of *satisfaction* and *validity* of a formula (resp. a set of formulas) in an intuitionistic model or frame are defined as usual, hence we omit them.

One easily verified equivalence that we will use throughout is the following: for every model $\mathfrak{M} = \langle W, \leq, V \rangle$, every $w \in W$, and every formula φ , we have

$$(1) \quad \mathfrak{M}, w \vDash (\neg \sim)^n \varphi \iff \forall u \in (\downarrow \uparrow)^n w \ (\mathfrak{M}, u \vDash \varphi).$$

Much like the case of modal logic, bi-intuitionistic logic admits both a local and global semantical consequence relation. We will only be concerned with the latter, which we proceed to define.

Definition 2.1. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ and \mathbf{L} be a bi-intermediate logic. We write $\Gamma \vdash_{\mathbf{L}} \varphi$ when, for every model \mathfrak{M} on a frame \mathfrak{F} that validates \mathbf{L} , if $\mathfrak{M} \vDash \Gamma$ then $\mathfrak{M} \vDash \varphi$. When $\Gamma = \emptyset$, we simply write $\vdash_{\mathbf{L}} \varphi$.

A proof of the following completeness result can be found in [Rau77].

Theorem 2.2. *Let $\varphi \in \mathcal{L}$ and \mathbf{L} be a bi-intermediate logic. Then $\varphi \in \mathbf{L}$ if and only if $\vdash_{\mathbf{L}} \varphi$.*

Next we state the version of the Deduction Theorem we will use in this paper. Many erroneous versions of the Deduction Theorem for bi-intuitionistic logic have appeared in the literature. In [GS20], the authors identify one of the reasons that lead to the existence of these conflicting results, namely, the fact that in a standard Hilbert calculus system, the double negation rule of bi-intuitionistic logics admits multiple interpretations. To eliminate these possible ambiguities, they view bi-intuitionistic logics as consequence relations via the notion of *generalized Hilbert calculi systems* in \mathcal{L} (we note that this is the same language we defined at the start of this section), and introduce the *strong bi-intuitionistic logic* \vdash_s . It is then proved in [GS20, Thm. 7.7] that

$$(2) \quad \Gamma, \varphi \vdash_s \psi \iff \exists n \in \mathbb{N} \text{ s.t. } \Gamma \vdash_s (\neg \sim)^n \varphi \rightarrow \psi,$$

for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}$. Since it follows immediately from [GS20, Thm. 8.8] and from Definition 2.1 that $\Gamma \vdash_{\mathbf{bi-IPC}} \psi$ if and only if $\Gamma \vdash_s \psi$, we get the Deduction Theorem for **bi-IPC**:

$$\Gamma, \varphi \vdash_{\mathbf{bi-IPC}} \psi \iff \exists n \in \mathbb{N} \text{ s.t. } \Gamma \vdash_{\mathbf{bi-IPC}} (\neg \sim)^n \varphi \rightarrow \psi.$$

Now, if $\mathbf{L} = \mathbf{bi-IPC} + \Sigma$ is a bi-intermediate logic, let us denote by \vdash_s^Σ the generalized Hilbert calculus system in \mathcal{L} obtained by adding Σ to the set of axioms of \vdash_s , and call it a *consistent axiomatic extension* of \vdash_s . A straightforward adaptation of the proof of [GS20, Thm. 8.8] (which uses both the standard canonical model construction and an appropriate version of a Truth Lemma) shows that $\Gamma \vdash_{\mathbf{L}} \psi$ if and only if $\Gamma \vdash_s^\Sigma \psi$, for every $\Gamma \cup \{\psi\} \subseteq \mathcal{L}$. Because of this equivalence, and since the syntactic argument used to prove (2) can be easily applied to consistent axiomatic extensions of \vdash_s , we obtain the following generalization of [GS20, Thm. 7.7]:

Theorem 2.3 (Deduction Theorem). *If $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}$ and \mathbf{L} is a bi-intermediate logic, then*

$$\Gamma, \varphi \vdash_{\mathbf{L}} \psi \iff \exists n \in \mathbb{N} \text{ s.t. } \Gamma \vdash_{\mathbf{L}} (\neg \sim)^n \varphi \rightarrow \psi.$$

Next we introduce the algebraic models of bi-intermediate logics, as well as their duals spaces.

Definition 2.4. A *bi-Heyting algebra* is a tuple $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \leftarrow, 0, 1 \rangle$ whose $(\wedge, \vee, 0, 1)$ -reduct is a bounded distributive lattice, and such that \mathbf{A} validates the following equations:

- | | |
|---|--|
| (1) $p \rightarrow p \approx 1$, | (5) $p \leftarrow p \approx 0$, |
| (2) $p \wedge (p \rightarrow q) \approx p \wedge q$, | (6) $p \vee (q \leftarrow p) \approx p \vee q$, |
| (3) $q \wedge (p \rightarrow q) \approx q$, | (7) $q \vee (q \leftarrow p) \approx q$, |
| (4) $p \rightarrow (q \wedge r) \approx (p \rightarrow q) \wedge (p \rightarrow r)$, | (8) $(q \vee r) \leftarrow p \approx (q \leftarrow p) \vee (r \leftarrow p)$. |

A standard example of these structures is $\mathbf{Up}(\mathfrak{F}) := \langle Up(\mathfrak{F}), \cap, \cup, \rightarrow, \leftarrow, \emptyset, W \rangle$, the *bi-Heyting algebra of upsets* of a frame $\mathfrak{F} = \langle W, \leq \rangle$, obtained by defining the implications as:

- $U \rightarrow V := W \setminus \downarrow(U \setminus V) = \{w \in W : \forall u \in W, \text{ if } w \leq u \text{ and } w \in U, \text{ then } u \in V\}$;
- $U \leftarrow V := \uparrow(U \setminus V) = \{w \in W : \exists u \leq w \text{ s.t. } u \in U \text{ and } u \notin V\}$.

Definition 2.5. An ordered topological space $\mathcal{X} = \langle X, \tau, \leq \rangle$ is a *bi-Esakia space* if it is compact and satisfies the following conditions (where $ClopUp(\mathcal{X})$ denotes the set of its clopen upsets):

- if U is clopen, then both $\downarrow U$ and $\uparrow U$ are clopen;
- *Priestley separation axiom, i.e.*,

$$\forall x, y \in X \ (x \not\leq y \implies \exists V \in ClopUp(\mathcal{X}) \ (x \in V \text{ and } y \notin V)).$$

Definition 2.6. Let $\langle X, \leq \rangle$ and $\langle Y, \leq \rangle$ be posets. A map $f: X \rightarrow Y$ is called a *bi-p-morphism*, denoted by $f: \langle X, \leq \rangle \rightarrow \langle Y, \leq \rangle$, if it satisfies the following conditions:

- *Order preserving:* $\forall x, z \in X \ (x \leq z \implies f(x) \leq f(z))$;
- *Up:* $\forall x \in X, \forall y \in Y \ (f(x) \leq y \implies \exists z \in \uparrow x \ (f(z) = y))$;
- *Down:* $\forall x \in X, \forall y \in Y \ (y \leq f(x) \implies \exists z \in \downarrow x \ (f(z) = y))$.

A continuous bi-p-morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ between bi-Esakia spaces is called a *bi-Esakia morphism*.

It is known that the celebrated Esakia duality restricts to a duality between the category of bi-Heyting algebras and bi-Heyting homomorphisms, and that of bi-Esakia spaces and bi-Esakia morphisms [Esa75, Esa19]. Here, we will only recall the contravariant functors which establish this duality. Given a bi-Heyting algebra \mathbf{A} , we denote its *bi-Esakia dual* by $\mathbf{A}_* := \langle A_*, \tau, \subseteq \rangle$, where A_* is the set of prime filters of \mathbf{A} and τ is the topology generated by the subbasis

$$\{\langle a \rangle : a \in A\} \cup \{A_* \setminus \langle a \rangle : a \in A\},$$

where $\langle a \rangle := \{F \in A_* : a \in F\}$. Notably, it can be shown that $ClopUp(\mathbf{A}_*) = \{\langle a \rangle : a \in A\}$. Furthermore, if $f: \mathbf{A} \rightarrow \mathbf{B}$ is a bi-Heyting homomorphism, then its dual is the restricted inverse image map $f_* := f^{-1}: \mathbf{B}_* \rightarrow \mathbf{A}_*$. Conversely, if \mathcal{X} is a bi-Esakia space, we denote its *bi-Heyting dual* by $\mathcal{X}^* := \langle ClopUp(\mathcal{X}), \cap, \cup, \rightarrow, \leftarrow, \emptyset, X \rangle$, where the implications are defined as before, that is,

- $U \rightarrow V := X \setminus \downarrow(U \setminus V)$,
- $U \leftarrow V := \uparrow(U \setminus V)$,

for every $U, V \in ClopUp(\mathcal{X})$. Moreover, if $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a bi-Esakia morphism, then its dual is the restricted inverse image map $f^* := f^{-1}: \mathcal{Y}^* \rightarrow \mathcal{X}^*$.

As previously mentioned, bi-Heyting algebras and bi-Esakia spaces are the algebraic and geometric models of bi-intermediate logics, respectively. This is because, for a bi-intermediate logic \mathbf{L} , we have that $\vdash_{\mathbf{L}} \varphi$ if and only if every bi-Heyting algebra that *validates* \mathbf{L} must also *validate* φ . Moreover, a formula is valid in a bi-Heyting algebra \mathbf{A} if and only if it is *valid* in its bi-Esakia dual \mathbf{A}_* . For the proofs of these facts, as well as proper definitions of the terminology, see, *e.g.*, [Rau74b, Esa75].

2.2. Unification. From now on, all formulas will be assumed to be in the language of bi-intuitionistic logic. Let \mathbf{L} be a bi-intermediate logic. A *substitution* is any map $\sigma: \mathcal{L}_P \rightarrow \mathcal{L}_Q$ where P and Q are finite and σ commutes with all logical connectives. Let $\sigma: \mathcal{L}_P \rightarrow \mathcal{L}_Q$ and $\tau: \mathcal{L}_P \rightarrow \mathcal{L}_{Q'}$ be two substitutions. We say that σ is *at least as general* as τ , and write $\sigma \preceq \tau$, if there exists a substitution $\mu: \mathcal{L}_Q \rightarrow \mathcal{L}_{Q''}$ such that $\vdash_{\mathbf{L}} \tau(p) \leftrightarrow \mu\sigma(p)$ for all $p \in P$. Now let $\varphi \in \mathcal{L}$. A *unifier* of φ is a substitution $\sigma: \mathcal{L}_{\text{var}(\varphi)} \rightarrow \mathcal{L}$ such that $\vdash_{\mathbf{L}} \sigma(\varphi)$. A set Σ of unifiers of φ is said to be a *basis* if it satisfies the following conditions:

- for all unifiers τ of φ , there exists $\sigma \in \Sigma$ such that $\sigma \preceq \tau$;
- for all $\sigma, \sigma' \in \Sigma$, if $\sigma \preceq \sigma'$ then $\sigma = \sigma'$.

In particular, if $\Sigma = \{\sigma\}$ is a one-element basis, then σ is called a *most general unifier* of φ . A special kind of most general unifier is that of so-called projective unifiers. A substitution $\sigma: \mathcal{L}_P \rightarrow \mathcal{L}_P$ is called *φ -projective* if we have $\varphi \vdash_{\mathbf{L}} p \leftrightarrow \sigma(p)$ for all $p \in \text{var}(\varphi)$. If in addition $\vdash_{\mathbf{L}} \sigma(\varphi)$, then

σ is called a *projective unifier* of φ . In this case, σ is known to be most general for φ . We call φ *unifiable* if it admits a unifier, *unitary* if it admits a most general unifier, and *projective* if it admits a projective unifier. Accordingly, the logic \mathbf{L} is said to be *unitary* (resp. *projective*) if every unifiable formula is unitary (resp. projective).

Proposition 2.7 ([Ghi00]). *A substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_P$ is φ -projective if and only if we have $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \sigma(\psi)$ for all $\psi \in \mathcal{L}_P$.*

A recurrent question is that of recognizing unifiable formulas. Formally, the decision problem of *unifiability* is defined as follows:

- **Input:** a formula φ ,
- **Output:** yes if φ admits a unifier, no otherwise.

In intuitionistic logic, unifiability can be reduced to consistency in **CPC**, and the same trick also works for bi-intuitionistic logic. This shows that unifiability is NP-complete in all super-bi-intuitionistic logics.

Theorem 2.8. *A formula φ is unifiable in \mathbf{L} if and only if φ is consistent with **CPC**.*

Proof. Suppose that φ admits a unifier σ in \mathbf{L} . Then $\vdash_{\mathbf{L}} \sigma(\varphi)$, and since $\mathbf{L} \subseteq \mathbf{CPC}$, it follows that $\vdash_{\mathbf{CPC}} \sigma(\varphi)$. As a consequence, φ is consistent with **CPC** – for otherwise we would have $\vdash_{\mathbf{CPC}} \neg\varphi$ and thus $\vdash_{\mathbf{CPC}} \neg\sigma(\varphi)$.

Conversely, suppose that φ is consistent with **CPC**. Then there exists a propositional valuation $V : \text{var}(\varphi) \rightarrow \{\perp, \top\}$ which satisfies φ . The valuation V can be extended to a substitution $\sigma : \mathcal{L}_{\text{var}(\varphi)} \rightarrow \mathcal{L}_{\emptyset}$, so that $\vdash_{\mathbf{CPC}} \sigma(\varphi)$. Since $\sigma(\varphi)$ is variable-free, it follows that $\vdash_{\mathbf{IPC}} \sigma(\varphi)$ by [CZ97, Cor. 2.27], and thus $\vdash_{\mathbf{L}} \sigma(\varphi)$. \square

Closely related to the problem of unification is that of admissibility. An inference rule $\frac{\varphi}{\psi}$ is said to be *admissible* if every unifier of φ is also a unifier of ψ , and *derivable* if we have $\varphi \vdash_{\mathbf{L}} \psi$. Formally, the decision problem of *admissibility* is defined as follows:

- **Input:** an inference rule $\frac{\varphi}{\psi}$,
- **Output:** yes if $\frac{\varphi}{\psi}$ is admissible, no otherwise.

The condition of being derivable is easier to decide than that of being admissible, since it simply amounts to check an implication. Every derivable rule is admissible, but the converse does not need to hold in general. Further, a rule $\frac{\varphi}{\psi}$ is called *passive* if its premise φ admits no unifier – which means that the rule can never be instantiated at all. Passive rules are vacuously admissible and can thus be safely omitted. All these observations motivate the following definition, introduced by Dzik in [Dzi06].

Definition 2.9. The logic \mathbf{L} is called *almost structurally complete* if every admissible non-passive rule is derivable.

This notion echoes that of *structural completeness*, which originates from [Pog71] and was extensively studied in various logical systems – see [Ryb97] for the modal and intuitionistic case.

Proposition 2.10. *If \mathbf{L} is almost structurally complete and decidable, then admissibility is decidable in \mathbf{L} .*

Proof. The following procedure decides whether $\frac{\varphi}{\psi}$ is admissible: first, check whether φ is unifiable, using Theorem 2.8. If not, then $\frac{\varphi}{\psi}$ is admissible. If yes, then by assumption, $\frac{\varphi}{\psi}$ is admissible if and only if $\varphi \vdash_{\mathbf{L}} \psi$. Since \mathbf{L} is decidable, this can be checked effectively. \square

The following result connects admissibility to projective unification.

Proposition 2.11 ([Dzi06, Lemma 13]). *If \mathbf{L} is projective, then \mathbf{L} is almost structurally complete.*

Proof. Let $\frac{\varphi}{\psi}$ be an admissible non-passive inference rule. By assumption, there exists a projective unifier σ of φ . Then by Proposition 2.7, we have $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \sigma(\varphi)$. Since $\frac{\varphi}{\psi}$ is admissible and σ is a unifier of ψ , it is also a unifier of ψ . Therefore $\varphi \vdash_{\mathbf{L}} \psi$, as desired. \square

3. PROJECTIVITY IN BI-INTUITIONISTIC LOGIC

We are finally ready to characterize the projective bi-intermediate logics: they are exactly those which contain a formula of the form $(\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$, for some $n \in \mathbb{N}$. In other words, extensions of the bi-intermediate logics of *n-bounded zigzag depth* $\mathbf{BZ}_n := \mathbf{bi-IPC} + (\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$ (it is routine to check that a frame $\mathfrak{F} = \langle W, \leq \rangle$ validates \mathbf{BZ}_n iff $(\downarrow\uparrow)^n w = (\downarrow\uparrow)^{n+1} w$, for all $w \in W$). In [Tay16], it is shown that for a bi-intermediate logic, extending \mathbf{BZ}_n is equivalent to having a *discriminator term*. For this reason, a result in [Dzi11] already entails that all these logics will be projective. Nevertheless, we include the following proof to keep the paper self-contained and to provide a clear intuition on why the presence of a formula $(\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$ in a bi-intermediate logic ensures its projectivity. We present an adaptation of a similar proof for modal logic introduced in [BG22], which is based on duality. For the sake of concision, we will only sketch the main steps of the reasoning, and redirect the reader to the corresponding proofs in [BG22].

Throughout this section, we work with a fixed but arbitrary bi-intermediate logic \mathbf{L} . Given $P \subseteq \mathbf{Var}$, we denote by \mathbf{A}_P the Lindenbaum algebra of \mathbf{L} with variables in P , obtained by quotienting the set \mathcal{L}_P by the equivalence relation $\equiv_{\mathbf{L}}$ and equipping it with the appropriate algebraic operations. Given $\varphi \in \mathcal{L}_P$, we denote by $[\varphi]$ the equivalence class of φ modulo $\equiv_{\mathbf{L}}$. We then denote by \equiv_{φ} the smallest equivalence congruence on \mathcal{L}_P such that $[\varphi] \equiv_{\varphi} [\top]$. Accordingly, every substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_Q$ can be identified to a homomorphism $\sigma : \mathbf{A}_P \rightarrow \mathbf{A}_Q$, defined by $\sigma([\varphi]) := [\sigma(\varphi)]$. Given $a \in \mathbf{A}_P$, we denote by $\pi_{\varphi}(a)$ the equivalence class of a modulo \equiv_{φ} , and this defines a homomorphism $\pi_{\varphi} : \mathbf{A}_P \rightarrow \mathbf{A}_P / \equiv_{\varphi}$.

Lemma 3.1. *For all $\varphi, \psi, \theta \in \mathcal{L}_P$, the following are equivalent:*

- (1) $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$,
- (2) $[\psi] \equiv_{\varphi} [\theta]$.

Proof sketch. From 1 to 2, this follows from Theorem 2.3. From 2 to 1, define a relation \equiv on \mathbf{A}_P by $[\varphi] \equiv [\psi]$ iff $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. We can check that \equiv is a congruence, and so by definition it contains \equiv_{φ} . \square

Proposition 3.2. *Let $\varphi \in \mathcal{L}$ and $P := \text{var}(\varphi)$. Then:*

- (1) a substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_Q$ is a unifier of φ iff $\text{Ker } \pi_{\varphi} \subseteq \text{Ker } \sigma$;
- (2) a substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_P$ is φ -projective iff $\pi_{\varphi}\sigma = \pi_{\varphi}$.

Proof sketch.

- (1) If σ is a unifier of φ , then $\text{Ker } \sigma$ contains the pair $([\varphi], [\top])$, and so it contains \equiv_{φ} by definition. Conversely, if $\text{Ker } \pi_{\varphi} \subseteq \text{Ker } \sigma$, then in particular $([\varphi], [\top]) \in \text{Ker } \sigma$ and thus $\vdash_{\mathbf{L}} \sigma(\varphi)$.
- (2) Follows from Proposition 2.7 and Lemma 3.1. \square

The *canonical bi-Esakia space* of \mathbf{L} on P is the dual $\mathcal{X}_P := (\mathbf{A}_P)_* = \langle W_P, \tau_P, \leq_P \rangle$ of \mathbf{A}_P . Given $\varphi \in \mathcal{L}_P$ we denote by $\langle \varphi \rangle := \{w \in W_P : [\varphi] \in w\}$ the *extension* of φ . We then introduce $\langle \varphi \rangle^{\infty} := \bigcap_{n \in \mathbb{N}} \langle (\neg\sim)^n \varphi \rangle$.

Lemma 3.3. *The set $\langle \varphi \rangle^{\infty}$ is an updownset.*

Proof. Let $w \in \langle \varphi \rangle^{\infty}$ and suppose that $w \leq_P u$ or $u \leq_P w$. Given $n \in \mathbb{N}$, we know by assumption that $w \in \langle \varphi \rangle^{\infty} \subseteq \langle (\neg\sim)^{n+1} \varphi \rangle$. Hence, the behavior of the double negation (see (1)) entails

$\downarrow\uparrow w \subseteq \langle (\neg\sim)^n \varphi \rangle$. Thus, for both possibilities on u , it follows $u \in \downarrow\uparrow w \subseteq \langle (\neg\sim)^n \varphi \rangle$. Since n was arbitrary, we now have $u \in \langle \varphi \rangle^\infty$, and we conclude that this is indeed an updownset. \square

Lemma 3.4. *Let $w \in W_P$. The following are equivalent:*

- (1) w is closed under \equiv_φ ,
- (2) $w \in \text{Im}(\pi_\varphi)_*$.

Proof sketch. The direction from Item 2 to Item 1 is easy. From Item 1 to Item 2, we set $u := \pi_\varphi[w]$, and we prove that u is a prime filter on $\mathbf{A}_P/\equiv_\varphi$ and that $w = (\pi_\varphi)_*(u)$. \square

Proposition 3.5. *We have $\text{Im}(\pi_\varphi)_* = \langle \varphi \rangle^\infty$.*

Proof sketch. Easily follows from Theorem 2.3, Lemma 3.1 and Lemma 3.4. \square

Theorem 3.6. *Let $\varphi \in \mathcal{L}$ and let $P := \text{var}(\varphi)$. Then:*

- (1) a substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_Q$ is a unifier of φ if and only if $\text{Im} \sigma_* \subseteq \langle \varphi \rangle^\infty$;
- (2) a substitution $\sigma : \mathcal{L}_P \rightarrow \mathcal{L}_P$ is φ -projective if and only if $\langle \varphi \rangle^\infty \subseteq \text{fp} \sigma_*$.

Proof sketch. Item 1 follows from Proposition 3.2 and an appropriate use of the Prime Filter Theorem [CZ97, Th. 7.41]. Item 2 is a straightforward consequence of Proposition 3.2 and Proposition 3.5. \square

Accordingly, we will call a *dual unifier* of φ any morphism $f : \mathcal{X}_Q \rightarrow \mathcal{X}_P$ such that $\text{Im} f \subseteq \langle \varphi \rangle^\infty$, and a *projective dual unifier* of φ any morphism $f : \mathcal{X}_P \rightarrow \mathcal{X}_P$ such that $\text{Im} f \subseteq \langle \varphi \rangle^\infty \subseteq \text{fp} f$.

Proposition 3.7. *If $\mathbf{BZ}_n \subseteq \mathbf{L}$, then \mathbf{L} is projective.*

Proof. Let φ be a formula and suppose it has a unifier $\tau' : \mathcal{L}_P \rightarrow \mathcal{L}_Q$, where $P := \text{var}(\varphi)$. We select an arbitrary substitution $\mu : \mathcal{L}_Q \rightarrow \mathcal{L}_P$, and then set $\tau := \mu\tau' : \mathcal{L}_P \rightarrow \mathcal{L}_P$, noting it is also a unifier of φ . Hence, we have a dual unifier $g := \tau^* : \mathcal{X}_P \rightarrow \mathcal{X}_P$ of φ . We define a map $f : \mathcal{X}_P \rightarrow \mathcal{X}_P$ by $f(w) := w$ if $w \in \langle \varphi \rangle^\infty$ and $f(w) := g(w)$ otherwise.

We prove that f is a bi-p-morphism. Let $w, u \in \mathcal{X}_P$. We first assume that $w \leq_P u$. By Lemma 3.3, we have $w \in \langle \varphi \rangle^\infty$ iff $u \in \langle \varphi \rangle^\infty$. If $w, u \in \langle \varphi \rangle^\infty$, then $f(w) \leq_P f(u)$ amounts to $w \leq_P u$. If $w, u \notin \langle \varphi \rangle^\infty$, then $f(w) \leq_P f(u)$ is equivalent to $g(w) \leq_P g(u)$ by the definition of f , and the latter condition holds because g is a bi-p-morphism. So f is order preserving.

Now suppose that $f(w) \leq_P u'$. If $w \in \langle \varphi \rangle^\infty$, then $f(w) = w$ and $w \leq_P u'$, whence $u' \in \langle \varphi \rangle^\infty$ by Lemma 3.3 and thus $f(u') = u'$. If instead $w \notin \langle \varphi \rangle^\infty$, then $f(w) = g(w)$, and since g is a bi-p-morphism we obtain the existence of $u \in \mathcal{X}_P$ with $w \leq_P u$ and $g(u) = u'$. The aforementioned lemma now forces $u \notin \langle \varphi \rangle^\infty$, whence $f(u) = g(u)$ and thus $f(u) = u'$. The down condition is proved similarly, hence we omit it.

Now let $\psi \in \mathcal{L}_P$. We have

$$f^{-1}[\langle \psi \rangle] = (\langle \psi \rangle \cap \langle \varphi \rangle^\infty) \cup (f^{-1}[\langle \psi \rangle] \setminus \langle \varphi \rangle^\infty).$$

Since $\mathbf{BZ}_n \subseteq \mathbf{L}$, we have $\langle \varphi \rangle^\infty = \langle (\neg\sim)^n \varphi \rangle \in \tau_P$ and therefore $f^{-1}[\langle \psi \rangle] \in \tau_P$. Finally, it is immediate that $\text{Im} f \subseteq \langle \varphi \rangle^\infty \subseteq \text{fp} f$. This proves that φ is projective. \square

To prove the converse of Proposition 3.7, we introduce, for each $n \in \mathbb{N}$, the formulas

$$\theta_n := (\neg\sim)^{n+2}p \rightarrow (\neg\sim)^{2n+4}\neg(\neg\sim)^n\neg p.$$

Lemma 3.8. *If $\theta_n \in \mathbf{L}$ then $\mathbf{BZ}_{2n+3} \subseteq \mathbf{L}$.*

Proof. By contraposition, suppose that $\mathbf{BZ}_{2n+3} \not\subseteq \mathbf{L}$. Then there is a model $\mathfrak{M} = \langle W, \leq, V \rangle$ whose frame validates \mathbf{L} and such that $\mathfrak{M}, w' \not\models (\neg\sim)^{2n+3}q \rightarrow (\neg\sim)^{2n+4}q$, for some $w' \in W$. Hence, we have $\mathfrak{M}, w \models (\neg\sim)^{2n+3}q$ and $\mathfrak{M}, w \not\models (\neg\sim)^{2n+4}q$ for some $w \geq w'$, and condition (1) now forces the existence of a point $u \in (\downarrow\uparrow)^{2n+4}w$ such that $\mathfrak{M}, u \not\models q$.

Let σ be the substitution defined by $\sigma(p) := (\neg\sim)^{n+1}q$, and notice that the above fact $\mathfrak{M}, w \models (\neg\sim)^{2n+3}q$ is equivalent to $\mathfrak{M}, w \models (\neg\sim)^{n+2}\sigma(p)$. We will establish $\mathfrak{M}, w \not\models \sigma(\theta_n)$ by proving that

$$\mathfrak{M}, w \not\models (\neg\sim)^{2n+4}\neg(\neg\sim)^n\neg\sigma(p).$$

To this end, we show $\mathfrak{M}, u \not\models \neg(\neg\sim)^n\neg\sigma(p)$, which suffices because $u \in (\downarrow\uparrow)^{2n+4}w$. In fact, we will prove $\mathfrak{M}, u \models (\neg\sim)^n\neg\sigma(p)$. For suppose that $v \in (\downarrow\uparrow)^n u$ and let $t \geq v$. Since $t \in (\downarrow\uparrow)^{n+1}u$ and $\mathfrak{M}, u \not\models q$, it follows that $\mathfrak{M}, t \not\models (\neg\sim)^{n+1}q$, which in turn implies $\mathfrak{M}, v \models \neg(\neg\sim)^{n+1}q$, i.e., that $\mathfrak{M}, v \models \neg\sigma(p)$. Thus, $v \in (\downarrow\uparrow)^n u$ now entails $\mathfrak{M}, u \models (\neg\sim)^n\neg\sigma(p)$, as desired.

As we assumed $\langle W, \leq \rangle \models \mathbf{L}$, the above argument ensures $\not\models_{\mathbf{L}} \sigma(\theta_n)$, hence also $\not\models_{\mathbf{L}} \theta_n$. \square

Proposition 3.9. *If \mathbf{L} is projective, then there exists $k \in \mathbb{N}$ such that $\mathbf{BZ}_k \subseteq \mathbf{L}$.*

Proof. Suppose that \mathbf{L} is projective. Since the formula $\varphi := p \rightarrow \neg\sim p$ is unifiable in \mathbf{L} (just substitute \top for p), by our assumption it also admits a projective unifier σ . Then $\varphi \vdash_{\mathbf{L}} p \leftrightarrow \sigma(p)$, and by Theorem 2.3, there exists $n \in \mathbb{N}$ such that $\vdash_{\mathbf{L}} (\neg\sim)^n \varphi \rightarrow (p \leftrightarrow \sigma(p))$.

Let us assume towards a contradiction that $\mathbf{BZ}_{2n+3} \not\subseteq \mathbf{L}$, so Lemma 3.8 entails $\not\models_{\mathbf{L}} \theta_n$, i.e.,

$$\not\models_{\mathbf{L}} (\neg\sim)^{n+2}p \rightarrow (\neg\sim)^{2n+4}\neg(\neg\sim)^n\neg p.$$

Because of this, there must be a model $\mathfrak{M} = \langle W, \leq, V \rangle$ whose frame validates \mathbf{L} and such that $\mathfrak{M}, w' \not\models \theta_n$, for some $w' \in W$. So, there exists $w \geq w'$ satisfying $\mathfrak{M}, w \models (\neg\sim)^{n+2}p$ and $\mathfrak{M}, w \not\models (\neg\sim)^{2n+4}\neg(\neg\sim)^n\neg p$. Using condition (1), we see that this forces the existence of a point $u \in (\uparrow\downarrow)^{2n+4}w$ such that $\mathfrak{M}, u \not\models \neg(\neg\sim)^n\neg p$, which in turn yields $\mathfrak{M}, v \models (\neg\sim)^n\neg p$, for some $v \geq u$.

Now, recall that $\mathfrak{M}, w \models (\neg\sim)^{n+2}p$, i.e., $(\downarrow\uparrow)^{n+2}w \subseteq V(p)$, by (1). We claim that this ensures $\mathfrak{M}, w \models (\neg\sim)^n \varphi$. To see this, notice the following equivalences:

$$\begin{aligned} \mathfrak{M}, w \models (\neg\sim)^n \varphi &\iff \mathfrak{M}, w \models (\neg\sim)^n (p \rightarrow \neg\sim p) \\ &\iff \forall x \in (\downarrow\uparrow)^n w (x \in V(p) \Rightarrow \downarrow\uparrow x \subseteq V(p)) \\ &\iff \forall x \in (\downarrow\uparrow)^n w (\downarrow\uparrow x \subseteq V(p)) \\ &\iff (\downarrow\uparrow)^{n+1} w \subseteq V(p), \end{aligned}$$

where the first and last equivalences follow immediately from the definitions, the second by (1), and the third from $(\downarrow\uparrow)^n w \subseteq (\downarrow\uparrow)^{n+2} w \subseteq V(p)$. Similarly, we claim that the fact $\mathfrak{M}, v \models (\neg\sim)^n\neg p$ yields $\mathfrak{M}, v \models (\neg\sim)^n \varphi$. This is because (1) implies

$$\mathfrak{M}, v \models (\neg\sim)^n \varphi \iff \forall x \in (\downarrow\uparrow)^n v (x \in V(p) \Rightarrow \downarrow\uparrow x \subseteq V(p)),$$

but $\mathfrak{M}, v \models (\neg\sim)^n\neg p$ is equivalent to $(\downarrow\uparrow)^n v \subseteq V(\neg p)$. Hence, the right side of the previous display is vacuously true, and we indeed have $\mathfrak{M}, v \models (\neg\sim)^n \varphi$.

Finally, using our assumption that σ is a projective unifier of φ , in particular, that $\vdash_{\mathbf{L}} (\neg\sim)^n \varphi \rightarrow (p \leftrightarrow \sigma(p))$, it follows from $\mathfrak{M}, w \models (\neg\sim)^{n+2}p$ (hence also $\mathfrak{M}, w \models p$) and $\mathfrak{M}, w \models (\neg\sim)^n \varphi$ that $\mathfrak{M}, w \models \sigma(p)$. As this assumption also ensures that $\vdash_{\mathbf{L}} \sigma(p) \rightarrow \neg\sim \sigma(p)$, a straightforward induction yields $\vdash_{\mathbf{L}} \sigma(p) \rightarrow (\neg\sim)^{2n+5} \sigma(p)$. Hence $\mathfrak{M}, w \models (\neg\sim)^{2n+5} \sigma(p)$, and since $v \in (\uparrow\downarrow)^{2n+5} w$, it follows that $\mathfrak{M}, v \models \sigma(p)$. But we proved above that $\mathfrak{M}, v \models (\neg\sim)^n \varphi$, so the projectivity of σ forces $\mathfrak{M}, v \models \sigma(p) \rightarrow p$, and from $\mathfrak{M}, v \models \sigma(p)$ we can now infer $\mathfrak{M}, v \models p$, contradicting $\mathfrak{M}, v \models (\neg\sim)^n\neg p$. We conclude $\mathbf{BZ}_{2n+3} \subseteq \mathbf{L}$, as desired. \square

From Proposition 3.7 and Proposition 3.9, we finally derive the following result.

Theorem 3.10. *The following are equivalent:*

- (1) \mathbf{L} is an extension of $\mathbf{BZ}_n = \mathbf{bi-IPC} + (\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$ for some $n \in \mathbb{N}$,
- (2) \mathbf{L} has projective unification.

Below we highlight two remarkable consequences of this theorem.

Corollary 3.11.

- (1) All extensions of $\mathbf{bi-GD} = \mathbf{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p)$ have projective unification.
- (2) All locally tabular super-bi-intuitionistic logics have projective unification.

Proof.

- (1) From [BMM24, Cor. 3.8], we know that $\mathbf{BZ}_1 \subseteq \mathbf{bi-GD}$. Then the result follows from Theorem 3.10.
- (2) Assume that \mathbf{L} is locally tabular. The set $\mathcal{L}_{\{p\}}$ is then finite up to \mathbf{L} -equivalence, and so there exists $n \in \mathbb{N}$ and $k > 0$ such that $(\neg\sim)^n p \equiv_{\mathbf{L}} (\neg\sim)^{n+k} p$. In particular, this entails $\vdash_{\mathbf{L}} (\neg\sim)^n p \rightarrow (\neg\sim)^{n+1} p$, and thus $\mathbf{BZ}_n \subseteq \mathbf{L}$. By Theorem 3.10, it follows that \mathbf{L} is projective. \square

The above characterization of the projective bi-intermediate logics is in contrast with the intuitionistic case. In [Wro95], it is shown that the projective *intermediate logics* (i.e., consistent axiomatic extensions of \mathbf{IPC}) are exactly those which extend the *intuitionistic Gödel-Dummett logic* $\mathbf{GD} := \mathbf{IPC} + (p \rightarrow q) \vee (q \rightarrow p)$. And while it follows from the previous corollary that being an extension of the bi-intuitionistic Gödel-Dummett logic $\mathbf{bi-GD}$ is a sufficient condition for a bi-intermediate logic to be projective, it is not necessary. For example, since $(\neg\sim)^2 p \rightarrow (\neg\sim)^3 p$ is a theorem of $\mathbf{bi-IPC} + \neg((q \leftarrow p) \wedge (p \leftarrow q))$, Theorem 3.10 ensures that this bi-intermediate logic is projective, but it is not an extension of $\mathbf{bi-GD}$ (see, e.g., [BMM24]). The result on locally tabular bi-intermediate logics displays one more discrepancy between the two settings, as many locally tabular intermediate logics are known to be non-projective and even nullary [DKW24].

We conclude this section with a decidability result that exploits both the results we have just proved and the material of Section 2.2. We call a bi-intermediate logic \mathbf{L} *tabular* if there exists a finite intuitionistic frame \mathfrak{F} such that \mathbf{L} is exactly the set of formulas φ with $\mathfrak{F} \models \varphi$. In this case, we say that \mathbf{L} is the logic of the frame \mathfrak{F} .

Corollary 3.12. *Admissibility is decidable in all tabular bi-intermediate logics in polynomial space.*

Proof. Let $\mathfrak{F} = \langle W, \leq \rangle$ be a finite intuitionistic frame and let \mathbf{L} be the logic of \mathfrak{F} . Then \mathbf{L} is locally tabular, hence projective by Corollary 3.11, and thus almost structurally complete by Proposition 2.11.

Further, we know that unifiability in \mathbf{L} is decidable in NP time (Theorem 2.8), and thus also in polynomial space. Since \mathbf{L} is the logic of \mathfrak{F} , checking that $\varphi \vdash_{\mathbf{L}} \psi$ amounts to check that for all valuations V of $P := \text{var}(\varphi) \cup \text{var}(\psi)$ on \mathfrak{F} , if $\mathfrak{F}, V \models \varphi$ then $\mathfrak{F}, V \models \psi$. Note that the size of V is $|W| \times |P|$, and thus linear in $|P|$ since $|W|$ is a constant. Hence, this procedure also runs in polynomial space. As a result, the algorithm described in Proposition 2.10 recognizes admissible rules in polynomial space, as desired. \square

4. $\mathbf{bi-IPC}$ IS NOT UNITARY

Since $\mathbf{bi-IPC}$ does not have bounded zigzag depth, we already know from Proposition 3.9 that it is not projective. In this section we prove the stronger claim that it is in fact not unitary. First, we introduce the notion of implicational depth of formulas as well as the process of *bi-unraveling* an intuitionistic frame, a well-known technique from modal logic that was adapted to intuitionistic frames in [OB22].

Definition 4.1. We define the *implicational depth* $\text{depth}(\varphi)$ of a formula $\varphi \in \mathcal{L}$ by induction:

- $\text{depth}(p) := 0$ for all variables p ,
- $\text{depth}(\perp) := 0$,
- $\text{depth}(\varphi \wedge \psi) := \text{depth}(\varphi \vee \psi) := \max \{ \text{depth}(\varphi), \text{depth}(\psi) \}$,
- $\text{depth}(\varphi \rightarrow \psi) := \text{depth}(\varphi \leftarrow \psi) := 1 + \max \{ \text{depth}(\varphi), \text{depth}(\psi) \}$.

The essential information provided by the implicational depth is how far a formula can ‘see’ in a model. Basically, if φ has implicational depth n , then its truth value in a pointed model $\langle \mathfrak{M}, w \rangle$ only depends on $(\downarrow\uparrow)^n w$, *i.e.*, the points located up to n zigzags away from w . To make this property precise, we introduce below the notation $\mathfrak{M}, w \triangleright_n \mathfrak{M}', w'$ to express the fact that the restriction of \mathfrak{M} to $(\downarrow\uparrow)^n w$ and of \mathfrak{M}' to $(\downarrow\uparrow)^n w'$ are isomorphic as models.

Definition 4.2. Let $\mathfrak{M} = \langle W, \leq, V \rangle$ and $\mathfrak{M}' = \langle W', \leq', V' \rangle$ be two intuitionistic models on $P \subseteq \mathbf{Var}$. Let $w \in W$, $w' \in W'$ and $n \in \mathbb{N}$. We write $\mathfrak{M}, w \triangleright_n \mathfrak{M}', w'$ if there exists a surjection $f : (\downarrow\uparrow)^n w \rightarrow (\downarrow\uparrow)^n w'$ such that:

- (1) $f(w) = w'$;
- (2) for all $u, v \in (\downarrow\uparrow)^n w$, we have $u \leq v$ iff $f(u) \leq f(v)$;
- (3) for all $u \in (\downarrow\uparrow)^n w$ and $p \in P$, we have $u \in V(p)$ iff $f(u) \in V'(p)$.

Proposition 4.3. Let $P \subseteq \mathbf{Var}$. Let $\langle \mathfrak{M}, w \rangle$ and $\langle \mathfrak{M}', w' \rangle$ be two intuitionistic pointed models on P , and assume that $\mathfrak{M}, w \triangleright_n \mathfrak{M}', w'$. Then for all $\varphi \in \mathcal{L}_P$ such that $\text{depth}(\varphi) \leq n$, we have $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}', w' \models \varphi$.

Proof. By induction on φ . If φ is a variable, the result follows from Item 3 of Definition 4.2. If φ is of the form \perp , $\psi \wedge \theta$, or $\psi \vee \theta$, then it follows immediately from the induction hypothesis. So assume that φ is of the form $\psi \rightarrow \theta$. Let $f : (\downarrow\uparrow)^n w \rightarrow (\downarrow\uparrow)^n w'$ be a map with the properties described in Definition 4.2. Given $u \geq w$, we have $(\downarrow\uparrow)^{n-1} u \subseteq (\downarrow\uparrow)^n w$. Since f is surjective, f preserves the order and $f(w) = w'$, it follows that $(\downarrow\uparrow)^{n-1} f(u) \subseteq (\downarrow\uparrow)^n w'$. Hence, f specializes to a map from $(\downarrow\uparrow)^{n-1} u$ to $(\downarrow\uparrow)^{n-1} f(u)$ satisfying the conditions of Definition 4.2, and therefore $\mathfrak{M}, u \triangleright_{n-1} \mathfrak{M}', f(u)$. Since $\text{depth}(\varphi) \leq n$, we have $\text{depth}(\psi), \text{depth}(\theta) \leq n - 1$, and so the induction hypothesis yields

$$\mathfrak{M}, u \models \psi \iff \mathfrak{M}', f(u) \models \psi \text{ and } \mathfrak{M}, u \models \theta \iff \mathfrak{M}', f(u) \models \theta.$$

This holds for all $u \geq w$, and since f is surjective, it follows from the semantics of \rightarrow that $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}', w' \models \varphi$. The reasoning for \leftarrow is analogous, hence we omit it. \square

The bi-unraveling of a pointed model $\langle \mathfrak{M}, w \rangle$ consists of all the paths that one can construct by starting from w and moving up and down throughout the model. This construction is of interest because it ensures the existence of points that are arbitrarily far from w (in terms of how many zigzags separate them from w), while preserving the truth of bi-intuitionistic formulas, when evaluated at w . An example of the first steps of a bi-unraveling is depicted in Figure 1.

Definition 4.4 ([OB22, Sect. 3]). Let $\mathfrak{M} = \langle W, \leq, V \rangle$ be an intuitionistic model on $P \subseteq \mathbf{Var}$, and let $w \in W$. We denote by W' the set of finite sequences $w_0 \dots w_n$ where $w_0, \dots, w_n \in W$, $w_0 = w$ and for all $i \in [1, n - 1]$, we have that $w_i \neq w_{i+1}$ and that $w_i \leq w_{i+1}$ or $w_i \geq w_{i+1}$. We then denote by \leq' the smallest reflexive and transitive relation on W' satisfying, for all $w_0 \dots w_n \in W'$ with $n > 0$:

$$\begin{aligned} w_0 \dots w_{n-1} \leq' w_0 \dots w_n & \text{ in case } w_{n-1} \leq w_n, \\ w_0 \dots w_{n-1} \geq' w_0 \dots w_n & \text{ in case } w_{n-1} \geq w_n. \end{aligned}$$

Finally, we define the valuation $V' : P \rightarrow \mathcal{P}(W')$ by $V'(p) := \{w_0 \dots w_n \in W' : w_n \in V(p)\}$. The tuple $\overline{\mathfrak{M}}^w := \langle W', \leq', V' \rangle$ is then an intuitionistic model, called the *bi-unraveling* of \mathfrak{M} around w .

Lemma 4.5. Let \mathfrak{M} be an intuitionistic model and suppose that there exists $u \geq w$ such that $w \neq u$. Then for all $n \in \mathbb{N}$, there exists a point v in $\overline{\mathfrak{M}}^w$ such that $v \in (\downarrow\uparrow)^{n+1} w \setminus (\downarrow\uparrow)^n w$.

Proof. Simply take $v := \underbrace{wuw \dots uw}_{n+1 \text{ times}}$. \square

Proposition 4.6 ([OB22, Lemma 7]). For all formulas φ , we have $\mathfrak{M}, w \models \varphi$ iff $\overline{\mathfrak{M}}^w, w \models \varphi$.

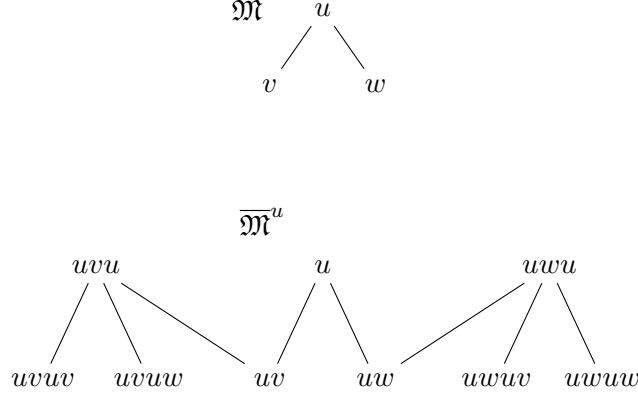


FIGURE 1. Bi-unraveling of \mathfrak{M} around u

Theorem 4.7. *The logic bi-IPC is not unitary.*

Proof. Consider the formula $\varphi := p \rightarrow \neg \sim p$. Define the substitutions σ_{\perp} and σ_{\top} by $\sigma_{\perp}(p) := \perp$ and $\sigma_{\top}(p) := \top$. It is easy to see that σ_{\perp} and σ_{\top} are unifiers of φ . In fact, we claim that they form a basis $\{\sigma_{\perp}, \sigma_{\top}\}$ for φ . To see this, let σ be an arbitrary unifier of φ and set $\psi := \sigma(p)$. We start by proving that either $\vdash_{\text{bi-IPC}} \psi$ or $\vdash_{\text{bi-IPC}} \neg \psi$, since this leads to either $\sigma_{\top} \preceq \tau$ or $\sigma_{\perp} \preceq \tau$, respectively.

Let $n := \text{depth}(\psi)$. Towards a contradiction, suppose that $\not\vdash_{\text{bi-IPC}} \psi$ and $\not\vdash_{\text{bi-IPC}} \neg \psi$. By Theorem 2.2, there exist two models $\mathfrak{M}_0 = \langle W_0, \leq_0, V_0 \rangle$ and $\mathfrak{M}_1 = \langle W_1, \leq_1, V_1 \rangle$ as well as $w_0 \in W_0$ and $w'_1 \in W_1$ such that $\mathfrak{M}_0, w_0 \not\models \psi$ and $\mathfrak{M}_1, w'_1 \not\models \neg \psi$. Hence, there is $w_1 \geq w'_1$ such that $\mathfrak{M}_1, w_1 \models \psi$. We also assume without loss of generality that W_0 and W_1 are disjoint.

Our strategy is to add a ‘bridge’ connecting the two models \mathfrak{M}_0 and \mathfrak{M}_1 . Since $\psi = \sigma(p)$, where σ is a unifier of $\varphi = p \rightarrow \neg \sim p$, we know that $\vdash_{\text{bi-IPC}} \psi \rightarrow \neg \sim \psi$. Because of this, and using the fact $\mathfrak{M}_1, w_1 \models \psi$ established above, we will be able to ‘propagate’ the truth of ψ from w_1 to w_0 and obtain a contradiction, since $\mathfrak{M}_0, w_0 \not\models \psi$. However, this only works if we can ensure that the truth value of ψ at w_0 and w_1 remains unaffected by this procedure, and this is where bi-unraveling comes into play.

First, note that we can always assume the existence of $u_0 \geq w_0$ such that $w_0 \neq u_0$ (if not, it suffices to introduce a copy u_0 of w_0 with this property). The same reasoning applies to w_1 . Hence, we can apply Lemma 4.5 to obtain the existence of a point v_0 in $\overline{\mathfrak{M}}_0^{w_0} = \langle W'_0, \leq'_0, V'_0 \rangle$ such that $v_0 \in (\downarrow \uparrow)^{n+1} w_0 \setminus (\downarrow \uparrow)^n w_0$, and a point v_1 in $\overline{\mathfrak{M}}_1^{w_1} = \langle W'_1, \leq'_1, V'_1 \rangle$ such that $v_1 \in (\downarrow \uparrow)^{n+1} w_1 \setminus (\downarrow \uparrow)^n w_1$. We now construct a model $\mathfrak{M} = \langle W, \leq, V \rangle$ as follows:

- $W := W'_0 \cup W'_1 \cup t$, where t is a new point;
- \leq is the smallest reflexive and transitive relation on W that contains \leq'_0 and \leq'_1 , and such that $v_0 \leq t$ and $v_1 \leq t$;
- $V(p) := V'_0(p) \cup V'_1(p) \cup \{t\}$.

The procedure is depicted in Figure 2. It is easy to check that \mathfrak{M} is an intuitionistic model. We also have $\overline{\mathfrak{M}}_0^{w_0}, w_0 \triangleright_n \mathfrak{M}, w_0$ and $\overline{\mathfrak{M}}_1^{w_1}, w_1 \triangleright_n \mathfrak{M}, w_1$, as a consequence of $v_0 \notin (\downarrow \uparrow)^n w_0$ and $v_1 \notin (\downarrow \uparrow)^n w_1$. Since $\mathfrak{M}_0, w_0 \not\models \psi$ and $n = \text{depth}(\psi)$, it follows from Proposition 4.6 and Proposition 4.3 that $\mathfrak{M}, w_0 \not\models \psi$. By the same reasoning, we obtain $\mathfrak{M}, w_1 \models \psi$. Again using the fact that $\vdash_{\text{bi-IPC}} \psi \rightarrow \neg \sim \psi$, Theorem 2.2 now yields $\mathfrak{M} \models \psi \rightarrow \neg \sim \psi$. But we now have $\mathfrak{M}, w_1 \models \psi$ and $w_0 \in (\downarrow \uparrow)^{2n+2} w_1$, hence an easy argument shows that $\mathfrak{M}, w_0 \models \psi$, a contradiction.

We have established that $\{\sigma_{\perp}, \sigma_{\top}\}$ is a complete set of unifiers. It remains to prove that σ_{\perp} and σ_{\top} are incomparable. For suppose that $\sigma_{\perp} \preceq \sigma_{\top}$. Then we have $\sigma_{\top} \simeq \mu \sigma_{\perp}$ for some substitution

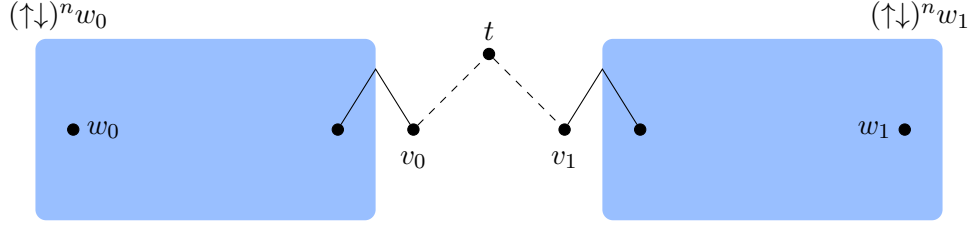


FIGURE 2. Construction of the bridge

μ . Since $\mu(\perp) = \perp$, it follows that $\top \equiv_{\mathbf{bi-IPC}} \perp$, a contradiction. If $\sigma_{\top} \preceq \sigma_{\perp}$, the reasoning is the analogous, hence we omit it. \square

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DAMIANO FORNASIERE: MILA - QUEBEC AI INSTITUTE, MONTREAL, CANADA
Email address: damiano.fornasiere@mila.quebec

QUENTIN GOUGEON: CRNS-INPT-UT3, TOULOUSE UNIVERSITY, FRANCE
Email address: quentin.gougeon@irit.fr

MIGUEL MARTINS: DEPARTAMENT DE FILOSOFIA, FACULTAT DE FILOSOFIA, UNIVERSITAT DE BARCELONA (UB), CARRER MONTALEGRE, 6, 08001 BARCELONA, SPAIN
Email address: miguelplmartins561@gmail.com