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# Quadratic obstructions to Small-Time Local Controllability for the multi-input bilinear Schrödinger equation

Théo Gherdaoui\*

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## Abstract

We investigate the small-time local controllability (STLC) near the ground state of a bilinear Schrödinger equation when the linearized system is not controllable. It is well known that, for single-input systems, quadratic terms in the state expansion can then lead to obstructions to the STLC of the nonlinear system. In this work, we extend this phenomenon to the multi-input setting, presenting the first example of multi-input quadratic obstructions for PDEs. Our results build upon our previous study of such obstructions for ODEs and provide a functional framework for analyzing them in the bilinear Schrödinger equation.

**Keywords:** Bilinear Schrödinger equation, quadratic obstructions, small-time local exact controllability, power series expansion.

## 1 Introduction

### 1.1 Model and problem

Let  $r \in \mathbb{N}^*$ . In this article, we study the following Schrödinger equation

$$\begin{cases} i\partial_t \psi(t, x) = -\partial_x^2 \psi(t, x) - (u(t), \mu(x)) \psi(t, x), & t \in (0, T), x \in (0, 1), \\ \psi(t, 0) = \psi(t, 1) = 0, & t \in (0, T), \\ \psi(0, x) = \psi_0(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where  $T > 0$ ,  $\mu : (0, 1) \rightarrow \mathbb{R}^r$ ,  $u : (0, T) \rightarrow \mathbb{R}^r$ ,  $\psi : (0, T) \times (0, 1) \rightarrow \mathbb{C}$  and  $\|\psi_0\|_{L^2} = 1$ . We will use the notations  $u = (u^1, \dots, u^r)$  and  $\mu = (\mu_1, \dots, \mu_r)$  in all the document. When well defined, the solution is denoted  $\psi$ . When required, we will write  $\psi(\cdot; u, \psi_0)$  to refer to this solution to emphasize its dependence on the different parameters.

This equation describes the evolution of the wave function  $\psi$  of a quantum particle, in a 1D infinite square potential well  $(0, 1)$ , subjected to  $r$  electric fields with amplitudes  $u^\ell(t)$ . The functions  $\mu_\ell$ , called "dipolar moments", model the interaction between the particle's wave function  $\psi$  and the electric fields  $u^\ell(t)$ .

This is a multi-input nonlinear control system:

- the state is the wave function  $\psi : (0, T) \rightarrow \mathcal{S}$ , where  $\mathcal{S}$  denotes the  $L^2(0, 1)$  sphere,
- the controls are  $u^\ell : (0, T) \rightarrow \mathbb{R}$ , they act bilinearly on the state.

The ground state is the particular free – with  $u \equiv 0$  – trajectory  $\psi_1(t, x) := \varphi_1(x)e^{-i\lambda_1 t}$  where  $\varphi_1(x) := \sqrt{2}\sin(\pi x)$  and  $\lambda_1 := \pi^2$ . We are interested in the small-time local controllability around the ground state.

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\*Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

## 1.2 Notations and definitions

We consider, for  $k \in \mathbb{N}$ , the Sobolev space  $H^k((0, T), \mathbb{R})$ , equipped with its usual norm and  $H_0^k((0, T), \mathbb{R})$  the adherence of  $C_c^\infty((0, T), \mathbb{R})$  for this norm.

Except explicit precision, we will work with complex valued space-dependent functions. We consider the space  $L^2(0, 1)$ , endowed with the hermitian scalar product. We define the operator

$$A := -\partial_x^2 \text{ with domain } D(A) := H^2(0, 1) \cap H_0^1(0, 1). \quad (1.2)$$

The eigenvalues and eigenvectors of  $A$  are

$$\lambda_j := (j\pi)^2, \quad \varphi_j := \sqrt{2} \sin(j\pi \cdot), \quad j \geq 1. \quad (1.3)$$

The family  $(\varphi_j)_{j \geq 1}$  is an orthonormal basis of  $L^2(0, 1)$ . We denote by

$$\psi_j(t, x) := \varphi_j(x) e^{-i\lambda_j t}, \quad (t, x) \in (0, T) \times (0, 1), \quad j \geq 1,$$

the solutions to the free – *i.e.* with controls  $u \equiv 0$  – Schrödinger equation (1.1) with initial data  $\varphi_j$  at time  $t = 0$ . The solution  $\psi_1$  is called the fundamental state, or ground state. We finally define the spaces  $H_{(0)}^s(0, 1) := D(A^{s/2})$  for  $s \geq 0$ , equipped with the norm

$$\|\varphi\|_{H_{(0)}^s} := \left( \sum_{j=1}^{+\infty} |j^s \langle \varphi, \varphi_j \rangle|^2 \right)^{1/2}.$$

With these notations, the equation (1.1) is well-posed, in the sense of the following proposition, proved for the single-input case in [6, Proposition 2] and which can be adapted to the case of (1.1).

**Proposition 1.1** (Well-posedness). *Let  $T > 0$ ,  $\mu \in H^3((0, 1), \mathbb{R})^r$ ,  $u \in L^2((0, T), \mathbb{R})^r$  and  $\psi_0 \in H_{(0)}^3(0, 1)$ . There exists a unique weak solution  $\psi \in C^0([0, T], H_{(0)}^3(0, 1))$  to the equation (1.1).*

We can now define the notion of small-time local controllability we are working on.

**Definition 1.2.** *Let  $E_T$  be a family of normed vector spaces of functions defined on  $[0, T]$  for  $T > 0$ . The bilinear Schrödinger equation (1.1) is  $E$ -STLC in  $H_{(0)}^3(0, 1)$  around the ground state if for every  $T, \varepsilon > 0$ , there exists  $\delta > 0$  such that for every  $\psi_f \in H_{(0)}^3(0, 1) \cap \mathcal{S}$  with  $\|\psi_f - \psi_1(T)\|_{H^3} < \delta$ , there exist  $u \in (E_T)^r$  such that  $\|u\|_{E_T} \leq \varepsilon$  and  $\psi(T; u, \varphi_1) = \psi_f$ .*

Under appropriate assumptions on the dipolar moments  $\mu_\ell$ , the STLC around the ground state of (1.1) can be proved by applying the linear test. This is the purpose of the following statement, that can be proved by adapting with  $r$  controls the strategy developed in [6, 17] with one control.

**Proposition 1.3.** *Let  $\mu \in H^3((0, 1), \mathbb{R})^r$  be such that*

$$\exists C > 0, \quad \forall j \in \mathbb{N}^*, \quad \sum_{\ell=1}^r |\langle \mu_\ell \varphi_1, \varphi_j \rangle| \geq \frac{C}{j^3}. \quad (1.4)$$

*The Schrödinger equation (1.1) is  $L^2$ -STLC in  $H_{(0)}^3(0, 1)$  around the ground state.*

*Idea of proof for Proposition 1.3.* Assumption (1.4) guarantees that the linearized system – see (2.2) – is controllable with states in  $H_{(0)}^3(0, 1) \cap \mathcal{S}$  and controls in  $L^2$ . Then, one can prove that the end-point map is of class  $C^1$  between the following spaces

$$\Theta_T : u \in L^2((0, T), \mathbb{R})^r \mapsto \psi(T; u, \varphi_1) \in H_{(0)}^3(0, 1) \cap \mathcal{S}.$$

Consequently, by applying the inverse mapping theorem, we obtain the local controllability of the nonlinear system (1.1).  $\square$

### 1.3 Main result

For all the document,  $k, K, r \in \mathbb{N}^*$  are fixed integers. To ensure that equation (1.1) is well-posed in  $H_{(0)}^3(0, 1)$  – see Proposition 1.1 – we assume that

$$(\mathbf{H})_{\text{reg}} : \quad \mu \in H^3((0, 1), \mathbb{R})^r.$$

In this article, we study the case where the linearized system around the ground state – see (2.2) – is not controllable, so we consider an integer  $K \geq 1$  and we assume

$$(\mathbf{H})_{\text{lin}} : \quad \langle \mu_\ell \varphi_1, \varphi_K \rangle = 0, \quad 1 \leq \ell \leq r.$$

In this case, the linearized system misses one direction  $\langle \psi(t; u, \varphi_1), \varphi_K \rangle \in \mathbb{C}$ , *i.e.* the first order term in the Taylor expansion at 0 of the map  $u \mapsto \langle \psi(t; u, \varphi_1), \varphi_K \rangle$  vanishes. To prove an obstruction to STLTC, we will use a "power series expansion" of order 2, *i.e.* the second order term in the Taylor expansion at 0 of  $u \mapsto \langle \psi(t; u, \varphi_1), \varphi_K \rangle$ . We make assumptions about the quadratic expansion of the solution. To state them, we need to introduce the following definitions.

**Definition 1.4.** For all  $1 \leq \ell, L \leq r$ , we define the following sequences

$$c^{\ell, L} := \left( c_j^{\ell, L} \right)_{j \geq 1} := \left( \langle \mu_\ell \varphi_K, \varphi_j \rangle \langle \mu_L \varphi_1, \varphi_j \rangle \right)_{j \geq 1}.$$

We will denote  $c^{\ell, \ell} = c^\ell$  and  $c_j^{\ell, \ell} = c_j^\ell$  for  $j \geq 1$  and  $1 \leq \ell \leq r$ .

Quadratic hypotheses are formulated as properties on series. To ensure convergence, we assume the following

$$(\mathbf{H})_{\text{conv}} : \quad \sum_{j=1}^{+\infty} \left| c_j^{\ell, L} \right| j^{4k} < +\infty, \quad 1 \leq \ell, L \leq r.$$

**Remark 1.5.** By Cauchy–Schwarz’s inequality,  $(\mathbf{H})_{\text{conv}}$  holds when  $\mu_\ell \varphi_K$  and  $\mu_L \varphi_1$  belong to  $H_{(0)}^{2k}(0, 1)$ , for every  $1 \leq \ell, L \leq r$ . In particular, it is enough that  $\mu_\ell \in H^{2k}(0, 1)$  with vanishing odd derivatives, or  $\mu_\ell \in \mathcal{C}_c^\infty(0, 1)$ , for every  $1 \leq \ell \leq r$ .

**Definition 1.6** (Quadratic brackets). Assuming  $(\mathbf{H})_{\text{conv}}$ , we define the following series which converge,

$$\gamma_p^{\ell, L} := \sum_{j=1}^{+\infty} \left( (\lambda_K - \lambda_j)^{\lfloor \frac{p+1}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{p}{2} \rfloor} c_j^{\ell, L} - (\lambda_K - \lambda_j)^{\lfloor \frac{p}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{p+1}{2} \rfloor} c_j^{L, \ell} \right),$$

for  $0 \leq p \leq 2k - 1$  and  $1 \leq \ell \leq L \leq r$ . When  $\ell = L$ , we will note  $\gamma_p^\ell$  instead of  $\gamma_p^{\ell, \ell}$ .

**Remark 1.7.** When  $p$  is even,  $\gamma_p^\ell = 0$  for every  $1 \leq \ell \leq r$ .

We are now able to state the hypotheses concerning the quadratic terms:

$$(\mathbf{H})_{\text{null}} : \quad \forall 1 \leq \ell \leq L \leq r, \quad \forall 0 \leq p \leq 2k - 2, \quad \gamma_p^{\ell, L} = 0,$$

$$(\mathbf{H})_{\text{pos}} : \quad q : a \in \mathbb{R}^r \mapsto \sum_{\ell=1}^r \gamma_{2k-1}^\ell \frac{a_\ell^2}{2} + \sum_{1 \leq \ell < L \leq r} \gamma_{2k-1}^{\ell, L} a_\ell a_L \text{ is a definite quadratic form on } \mathbb{R}^r.$$

**Remark 1.8.** If  $(\mathbf{H})_{\text{pos}}$  holds,  $\text{sgn}(\gamma_{2k-1}^1)q$  is a positive-definite quadratic form on  $\mathbb{R}^r$ .

**Remark 1.9.** When  $r = 2$ ,  $(\mathbf{H})_{\text{pos}}$  holds iff  $\left( \gamma_{2k-1}^{1, 2} \right)^2 < \gamma_{2k-1}^1 \gamma_{2k-1}^2$ .

At first glance, these assumptions may seem mysterious and technical, but they can be interpreted in terms of Lie brackets as detailed in Section 1.5.

The main result of this article is the following theorem. It can be seen as a generalization to the case  $r > 1$  of quadratic obstructions known in the scalar-input  $r = 1$  case – see Section 1.6.

**Theorem 1.10.** *Let  $k, K, r \in \mathbb{N}^*$ ,  $\mu_1, \dots, \mu_r$  satisfying  $(\mathbf{H})_{\text{reg}}$ ,  $(\mathbf{H})_{\text{lin}}$ ,  $(\mathbf{H})_{\text{conv}}$ ,  $(\mathbf{H})_{\text{null}}$  and  $(\mathbf{H})_{\text{pos}}$ . Then, the multi-input bilinear Schrödinger equation (1.1) is not  $H^{2k}$ -STLC in  $H_{(0)}^3(0, 1)$  around the ground state.*

Ideas for proving the existence of such functions  $\mu_1, \dots, \mu_r$  are given in Appendix A.1.

**Remark 1.11.** *When  $r = 1$ , hypothesis  $(\mathbf{H})_{\text{pos}}$  is equivalent to  $\gamma_{2k-1}^1 \neq 0$ . This is the same assumption as in [18, Theorem 1.3]. In this statement, Bournissou concludes the non- $H^{2k-3}$ -STLC (resp. non- $W^{-1, \infty}$ -STLC) for  $k \geq 2$  (resp.  $k = 1$ ). Her result hints that  $H^{2k}$  might not be the weakest control regularity for which one can deny STLC under our assumptions. However, her proof uses arguments specific to the  $r = 1$  case (namely an auxiliary system) to estimate the cubic remainder as  $(\int u)^3$  – whereas we obtain  $u^3$  – see (2.7). The optimal norm with which one can estimate the cubic remainder for multi-input systems is already a difficult question for ODEs.*

## 1.4 Heuristic

The essence of Theorem 1.10 can be found in the following example of a finite-dimensional control-affine system

$$\begin{cases} x_1' &= u^1 \\ x_2' &= x_1 \\ x_3' &= u^2 \\ x_4' &= (x_1^2 + \frac{1}{2}x_1x_3 + 2x_3^2) - 2u^2x_3 - x_2^2 - u^2x_1^2 \end{cases}. \quad (1.5)$$

Let  $T > 0$ ,  $u_1^1(t) := \int_0^t u^1$  and  $u_1^2(t) := \int_0^t u^2$  for  $t \in [0, T]$ . Explicit integration from 0 yields

$$\begin{aligned} x_4(T) &= \int_0^T \left( (u_1^1)^2 + \frac{1}{2}u_1^1u_1^2 + 2(u_1^2)^2 \right) - u_1^2(T)^2 - \int_0^T \left( \int_0^t u_1^1 \right)^2 dt - \int_0^T u^2 (u_1^1)^2 \\ &\geq \left( \frac{3}{4} - T^2 - \|(u^1, u^2)\|_{L^\infty} \right) \|(u_1^1, u_1^2)\|_{L^2(0, T)}^2 - u_1^2(T)^2 \\ &\geq C \|(u_1^1, u_1^2)\|_{L^2(0, T)}^2 - u_1^2(T)^2, \end{aligned}$$

for any  $C \in (0, \frac{3}{4})$ , for small enough times and controls in  $L^\infty$ . As  $x_3(T) = u_1^2(T)$ , all final states in  $\{(x_1, \dots, x_4) \in \mathbb{R}^4; x_4 + x_3^2 < 0\}$  are not reachable from 0. The system (1.5) is not  $L^\infty$ -STLC around 0 - see Definition 1.13.

To prove Theorem 1.10, we use a "power series expansion" of order 2, *i.e.* we are interested in the Taylor expansion of order 2 at 0 of the map  $u \mapsto \psi(T; u, \varphi_1)$ . To give more details, we need the following definition.

**Definition 1.12.** *For  $T > 0$  and  $f \in L^1((0, T), \mathbb{R})$ , one defines the iterated primitives  $f_n$  vanishing at  $t = 0$  by induction as*

$$f_0 := f \quad \text{and} \quad \forall n \in \mathbb{N}, t \in [0, T], \quad f_{n+1}(t) := \int_0^t f_n(\tau) d\tau. \quad (1.6)$$

When  $f = (f^1, \dots, f^r) \in L^1((0, T), \mathbb{R})^r$ ,  $f_n$  will denote  $(f_n^1, \dots, f_n^r)$ .

Then, it is as if the following terms are the dominant part of the quadratic expansion of the solution in the direction  $\varphi_K e^{-i\lambda_1 T}$

$$\sum_{\ell=1}^r \sum_{p=1}^k i^{2p-1} \gamma_{2p-1}^\ell \int_0^T \frac{u_p^\ell(t)^2}{2} dt + \sum_{1 \leq \ell < L \leq r} \sum_{p=0}^{2k-1} i^p \gamma_p^{\ell, L} \int_0^T u_{\lfloor \frac{p}{2} \rfloor + 1}^\ell(t) u_{\lfloor \frac{p+1}{2} \rfloor}^L(t) dt.$$

The cancellation assumption  $(\mathbf{H})_{\text{null}}$  reduces this sum to the simpler expression

$$i(-1)^{k+1} \sum_{\ell=1}^r \gamma_{2k-1}^\ell \int_0^T \frac{u_k^\ell(t)^2}{2} dt + i(-1)^{k+1} \sum_{1 \leq \ell < L \leq r} \gamma_{2k-1}^{\ell, L} \int_0^T u_k^\ell(t) u_k^L(t) dt, \quad (1.7)$$

and  $(\mathbf{H})_{\text{pos}}$  allows use to take advantage of a signed term because (1.7) and  $(\mathbf{H})_{\text{lin}}$  lead to

$$(-1)^{k+1} \text{sgn}(\gamma_{2k-1}^1) \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle \simeq \text{sgn}(\gamma_{2k-1}^1) \int_0^T q(u_k^1, \dots, u_k^r)(t) dt \geq 0.$$

Using the interpretation of the series in terms of Lie brackets given by (1.9), we can recognize the leading terms of the Magnus-type representation formula, that is used to prove Theorem 1.19. See [7] for more details.

## 1.5 Comparison with the finite-dimensional case

Theorem 1.10 is an adaptation to the multi-input bilinear Schrödinger equation of a theorem that proves quadratic obstructions to small-time local controllability for multi-input control-affine systems in [34] for  $r = 2$ . Here is the framework and key definitions. One considers the system

$$x'(t) = f_0(x(t)) + \sum_{\ell=1}^r u^\ell(t) f_\ell(x(t)), \quad (1.8)$$

where the state is  $x(t) \in \mathbb{R}^d$ , the controls are scalar functions  $u^\ell(t) \in \mathbb{R}$  and  $f_\ell$  are real-analytic vector fields on a neighborhood of 0 in  $\mathbb{R}^d$  such that  $f_0(0) = 0$ . The last hypothesis ensures that 0 is an equilibrium of the free system, *i.e.* with  $u \equiv 0$ .

For each  $t > 0$ ,  $u \in L^1((0, t), \mathbb{R})^r$ , there exists a unique maximal mild solution to (1.8) with initial data 0, which we will denote by  $x(\cdot; u)$ . As we are interested in small-time and small controls, the solution is well-defined up to time  $t$ .

**Definition 1.13** ( $W^{m,p}$ -STLC). *Let  $m \in \llbracket -1, +\infty \llbracket$  and  $p \in [1, +\infty]$ . The system (1.8) is  $W^{m,p}$ -STLC around 0 if for every  $T, \varepsilon > 0$ , there exists  $\delta > 0$ , such that, for every  $x^* \in B(0, \delta)$ , there exist  $u \in W^{m,p}((0, T), \mathbb{R})^r$  with  $\|u\|_{W^{m,p}} \leq \varepsilon$  and  $x(T; u) = x^*$ .*

The historical definition of STLC is the case where  $m = 0$  and  $p = \infty$  – see [26]. Let  $X := \{X_0, \dots, X_r\}$  be a set of  $r + 1$  non-commutative indeterminates.

**Definition 1.14** (Free Lie algebra). *We consider  $\mathcal{A}(X)$  the unital associative algebra of polynomials of the indeterminates  $X_\ell$ . For  $a, b \in \mathcal{A}(X)$ , one defines the Lie bracket of  $a$  and  $b$  as  $[a, b] := ab - ba$ . Let  $\mathcal{L}(X)$  be the free Lie algebra generated by  $X$  over the field  $\mathbb{R}$ .*

**Definition 1.15** (Lie bracket of vector fields). *Let  $f, g : \Omega \rightarrow \mathbb{R}^d$  be  $C^\infty$  vector fields on an open subset  $\Omega$  of  $\mathbb{R}^d$ . One defines*

$$[f, g] : x \in \Omega \mapsto Dg(x) \cdot f(x) - Df(x) \cdot g(x).$$

**Definition 1.16** (Evaluated Lie bracket). *Let  $f_0, \dots, f_r$  be  $C^\infty(\Omega, \mathbb{R}^d)$  vector fields on an open subset  $\Omega$  of  $\mathbb{R}^d$ . For  $b \in \mathcal{L}(X)$ , we define  $f_b := \Lambda(b)$ , where  $\Lambda : \mathcal{L}(X) \rightarrow C^\infty(\Omega, \mathbb{R}^d)$  is the unique homomorphism of Lie algebras such that  $\Lambda(X_\ell) = f_\ell$ , for  $0 \leq \ell \leq r$ .*

For example, if  $b = [[[X_1, X_r], X_0], [X_{r-1}, X_0]]$ , one has  $f_b = [[[f_1, f_r], f_0], [f_{r-1}, f_0]]$ .

**Definition 1.17** (Bracket integration  $b0^\nu$ ). *For  $b \in \mathcal{L}(X)$  and  $\nu \in \mathbb{N}$ , we use the unconventional short-hand  $b0^\nu$  to denote the right-iterated bracket  $[\dots [b, X_0], \dots, X_0]$ , where  $X_0$  appears  $\nu$  times.*

For example, if  $b = [[X_r, X_0], [X_0, X_r]]$ , then  $b0^2 = [[[[X_r, X_0], [X_0, X_r]], X_0], X_0]$ .

**Definition 1.18.** *We define the following brackets*

- a.  $M_\ell^\ell := X_\ell 0^\ell$ ;  $(\ell, l) \in \llbracket 1, r \rrbracket \times \mathbb{N}$ ,
- b.  $W_{p,l}^\ell := [M_{p-1}^\ell, M_p^\ell] 0^l$ ;  $(\ell, p, l) \in \llbracket 1, r \rrbracket \times \mathbb{N}^* \times \mathbb{N}$ ,
- c.  $C_{p,l}^{\ell,L} := (-1)^p \left[ M_{\lfloor \frac{p+1}{2} \rfloor}^\ell, M_{\lfloor \frac{L}{2} \rfloor}^L \right] 0^l$ ;  $(\ell, L) \in \llbracket 1, r \rrbracket^2$  such that  $\ell < L$ ,  $(p, l) \in \mathbb{N}^2$ .

The following theorem generalizes [34, Theorem 1.15], originally stated for  $r = 2$ . By applying the same methods, we obtain the extended result presented here, which relies on a Magnus-type representation formula of the state.<sup>1</sup> The proof is designed to prepare an easier transfer to PDEs.

**Theorem 1.19.** *Let  $k, r \in \mathbb{N}^*$ . Let  $f_0, \dots, f_r$  be real-analytic vector fields over  $\mathbb{R}^d$  such that  $f_0(0) = 0$ . We define the set*

$$\mathcal{N}_k := \text{Span} \left\{ f_{M_\ell^\ell}(0); (\ell, l) \in \llbracket 1, r \rrbracket \times \mathbb{N} \right\} \cup \text{Span} \left\{ f_{W_{p,l}^\ell}(0); (\ell, p, l) \in \llbracket 1, r \rrbracket \times \llbracket 1, k-1 \rrbracket \times \mathbb{N} \right\} \\ \cup \text{Span} \left\{ f_{C_{p,l}^{\ell,L}}(0); (\ell, L) \in \llbracket 1, r \rrbracket^2 \text{ such that } \ell < L, (p, l) \in \llbracket 0, 2k-2 \rrbracket \times \mathbb{N} \right\}.$$

If there exists  $\mathbb{P} : \mathbb{R}^d \rightarrow \mathbb{R}$  a linear form such that  $\mathbb{P}|_{\mathcal{N}_k} \equiv 0$  and

$$(a_1, \dots, a_r) \in \mathbb{R}^r \mapsto \sum_{\ell=1}^r \mathbb{P} \left( f_{W_{k,0}^\ell}(0) \right) \frac{a_\ell^2}{2} + \sum_{1 \leq \ell < L \leq r} \mathbb{P} \left( f_{C_{2k-1,0}^{\ell,L}}(0) \right) a_\ell a_L \text{ is a definite quadratic form on } \mathbb{R}^r,$$

then, the affine system (1.8) is not  $H^{2k}$ -STLC around 0.

**Remark 1.20.** *The example of system (1.5) can be studied again using this statement. In this case,  $r = 2$ ,  $k = 1$  and*

$$\mathcal{N}_k = \text{Span}(e_1, e_2, e_3).$$

Moreover,  $[f_1, f_2](0) = 0$ . Then, for every  $l \in \mathbb{N}$ ,  $f_{C_{0,l}^{1,2}}(0) = 0$ . Finally, one has  $f_{W_{1,0}^1}(0) = 2e_4$ ,  $f_{W_{1,0}^2}(0) = 4e_4$  and  $f_{C_{1,0}^{1,2}}(0) = \frac{1}{2}e_4$ . Then, the linear form  $\mathbb{P} := \langle \cdot, e_4 \rangle$  is suitable because, for every  $(x, y) \in \mathbb{R}^2$ ,

$$x^2 + \frac{1}{2}xy + 2y^2 \geq \frac{3}{4}x^2 + \frac{7}{4}y^2,$$

so the associated quadratic form is a positive-definitive quadratic form on  $\mathbb{R}^2$ .

Let us make the link between Theorem 1.19 and the main result of this article. When  $\ell < L$ , the Lie brackets  $\gamma_p^{\ell,L}$  are specific to the multi-input system (1.1) ( $r > 1$ ), they can be used to recover a complex lost direction at linear order – see [35] with  $r = 2$ . They can be considered as good in a STLC point of view. *A contrario*, the Lie brackets  $\gamma_{2k-1}^{\ell,\ell}$  are known to be potential obstructions to small-time local controllability for the bilinear Schrödinger equation in the single-input case, in an appropriate functional setting – see [18].

We now give an interpretation of the series  $\gamma_p^{\ell,L}$  in terms of Lie brackets. For  $A$  and  $B$  two operators, we define  $\underline{\text{ad}}_A^l(B)$  as:  $\underline{\text{ad}}_A^0(B) = B$  and  $\underline{\text{ad}}_A^{l+1}(B) = \underline{\text{ad}}_A^l(B)A - A\underline{\text{ad}}_A^l(B)$ . Under appropriate assumptions on the function  $\mu_\ell$  for domain compatibility (a finite number of derivatives of odd order have to vanish on the boundary) the assumption  $(\mathbf{H})_{\text{null}}$  can be interpreted as Lie brackets because

$$\forall 0 \leq p \leq 2k-1, 1 \leq \ell \leq L \leq r, \quad \gamma_p^{\ell,L} = (-1)^p \left\langle [\underline{\text{ad}}_A^{\lfloor \frac{p+1}{2} \rfloor}(B_\ell), \underline{\text{ad}}_A^{\lfloor \frac{p}{2} \rfloor}(B_L)] \varphi_1, \varphi_K \right\rangle, \quad (1.9)$$

where  $A$  is defined in (1.2) and  $B_\ell$  is the multiplication operator by  $\mu_\ell$  in  $L^2(0, 1)$ . We refer to [35, Propositions A.7, A.8, A.10 and A.11] for a precise proof.

Assume that  $\mu_\ell \in C_c^\infty(0, 1)$  are smooth functions. Heuristically, we can think of the bilinear Schrödinger equation (1.1) as a control-affine system on the form (1.8) with  $f_0 = A$ , where the operator  $A$  is defined in (1.2) and  $f_\ell = B_\ell$  with  $1 \leq \ell \leq r$ , where  $B_\ell$  is the multiplication operator by  $\mu_\ell$  in  $L^2(0, 1)$ . The equilibrium is no longer 0 but the free trajectory  $\psi(T; 0, \varphi_1) = \varphi_1 e^{-i\lambda_1 T}$ . In this situation, the linear form is  $\mathbb{P} := \langle \cdot, \varphi_K \rangle$ . The hypotheses are the same in the finite and infinite-dimensional cases.

<sup>1</sup>A proof of the theorem in the case where  $r$  is an arbitrary nonzero natural integer is available in Appendix F of my thesis manuscript.

1. The equality (1.9), [35, Lemma A.9] and the assumptions  $(\mathbf{H})_{\text{lin}}$  and  $(\mathbf{H})_{\text{null}}$  give

$$\forall(\ell, l) \in \llbracket 1, r \rrbracket \times \mathbb{N},$$

$$\mathbb{P} \left( \underline{\text{ad}}_A^l(B_\ell) \varphi_1 \right) = \left\langle \underline{\text{ad}}_A^l(B_\ell) \varphi_1, \varphi_K \right\rangle = (\lambda_1 - \lambda_K)^l \langle \mu_\ell \varphi_1, \varphi_K \rangle = 0,$$

$$\forall(\ell, p, l) \in \llbracket 1, r \rrbracket \times \llbracket 1, k-1 \rrbracket \times \mathbb{N},$$

$$\mathbb{P} \left( \underline{\text{ad}}_A^l \left( \left[ \underline{\text{ad}}_A^{p-1}(B_\ell), \underline{\text{ad}}_A^p(B_\ell) \right] \right) \varphi_1 \right) = (\lambda_1 - \lambda_K)^l \gamma_{2p-1}^\ell = 0,$$

$$\forall(\ell, L) \in \llbracket 1, r \rrbracket^2 \text{ such that } \ell < L, \forall(p, l) \in \llbracket 0, 2k-2 \rrbracket \times \mathbb{N},$$

$$\mathbb{P} \left( \underline{\text{ad}}_A^l \left( \left[ \underline{\text{ad}}_A^{\lfloor \frac{p+1}{2} \rfloor}(B_\ell), \underline{\text{ad}}_A^{\lfloor \frac{p}{2} \rfloor}(B_L) \right] \right) \varphi_1 \right) = (\lambda_1 - \lambda_K)^l (-1)^p \gamma_p^{\ell, L} = 0.$$

This is the equivalent of the condition  $\mathbb{P}|_{\mathcal{N}_k} \equiv 0$ .

2. The equality (1.9) and the hypothesis  $(\mathbf{H})_{\text{pos}}$  ensure that the hypotheses about the quadratic forms are the same.

## 1.6 State of the art

### 1.6.1 Topological obstructions to exact controllability

In [1], Ball, Marsden and Slemrod proved obstructions to local exact controllability of linear PDEs with bilinear controls. For instance, if the multiplicative operators  $\mu_\ell$  are bounded on  $H_{(0)}^s(0, 1)$ , then system (1.1) is not exactly controllable in  $\mathcal{S} \cap H_{(0)}^s(0, 1)$ , with controls  $u \in L_{\text{loc}}^p(\mathbb{R}, \mathbb{R})^r$  and  $p > 1$ . The fundamental reason behind is that, under these assumptions, the reachable set has empty interior in  $H_{(0)}^s(0, 1)$ . The case of  $L_{\text{loc}}^1$ -controls ( $p = 1$ ) was incorporated in [19] and extensions to nonlinear equations were proved in [23, 24]. Turicini adapted this statement to Schrödinger equations in [44]. After the important work [1], different notions of controllability were studied for the single-input bilinear Schrödinger equation such as

- *exact controllability* in more regular spaces, on which the  $\mu_\ell$  do not define bounded operators – see [6, 16, 35],
- *approximate controllability*.

### 1.6.2 Exact controllability in more regular spaces, by linear test

For the single-input bilinear Schrödinger equation, local exact controllability was first proved in [2, 3] with Nash-Moser techniques, to deal with an apparent derivative loss problem and then in [6] with a classical inverse mapping theorem, thanks to a regularizing effect. By grafting other ingredients onto this core strategy, global (resp. local) exact controllability in regular spaces was proved for different models in [38, 40] (resp. [17]). This strategy has also been used to obtain local controllability results for nonlinear Schrödinger equations, as in [30], or for coupled Schrödinger equations, as in [37].

### 1.6.3 Power series expansion of order 2 or 3

First, let us look at the case of **single-input systems**. A power series expansion of order 2 allows to recover a lost direction *in large time*: this strategy is used in [12] for the single-input bilinear Schrödinger equation. This method is also used for other equations, such as KdV, in [20]. If the order 2 vanishes, a power series of order 3 can be used to recover the *small-time* local controllability, for example in [27], for KdV. If the order 2 doesn't cancel out, but the term of order 3 is strong enough, this expansion can also give the *small-time* local controllability – see [16], for the single-input bilinear Schrödinger equation.

In the context of **multi-input systems**, we use a power series expansion of order 2 in [35] to recover in *small-time* a direction that is lost at the linear order. The purpose of this article is to give other algebraic assumptions to prove obstructions for STLC.



### 1.6.4 Quadratic obstructions to STLC

In [25], Coron denied the  $L^\infty$ -small-time local controllability result for the single-input bilinear Schrödinger equation, with a particular dipolar moment  $\mu_\ell$ , thanks to a drift. In [12], Beauchard and Morancey gave general assumptions on  $\mu_1$  to deny  $L^\infty$ -STLC for the same system. In [18], Bournissou proved that this drift also occurs with small control in  $W^{-1,\infty}$ . In [11], Beauchard, Marbach and Perrin proved an obstruction for the single-input bilinear Schrödinger equation with Neumann boundary conditions, using a power series of order 2. Quadratic terms have also been used to create obstructions to the controllability of other single-input systems, for example in [8] for ODEs, in [36] for the Burgers' equation, [9] for the heat equation, [28] for KdV and [41] for KdV-Neumann. With the exception of a non-physical PDE designed for, [9, Section 5 and 6] for single-input systems, the quadratic terms generally do not recover small-time controllability.

In [34], we prove quadratic obstructions for multi-input control-affine systems in the finite-dimensional case (ODEs). Our strategy is to adapt [34, Theorem 1.15] for the multi-input bilinear Schrödinger equation. The strong analogy has already been developed in Section 1.5.

### 1.6.5 Approximate controllability

The first results of global approximate controllability of bilinear Schrödinger equations were obtained in large time – see [14, 21, 32, 39, 43]. For particular systems, a large time is indeed necessary for the approximate controllability – see [4, 5]. Small-time approximate controllability between eigenstates for Schrödinger equations on the torus is proved by Duca and Nersesyan in [29], by means of an infinite-dimensional geometric control approach (saturation argument). Related results have been subsequently established in [15, 22, 31]. Recently, Beauchard and Pozzoli provided the first examples of small-time globally approximately controllable bilinear Schrödinger equations in [13].

## 2 Proof of the main theorem

One recalls that  $\lambda_j$  and  $\varphi_j$  are defined in (1.3). In all the document, for  $j \in \mathbb{N}^*$ , we will note

$$\omega_j := \lambda_j - \lambda_1 \quad \text{and} \quad \nu_j := \lambda_K - \lambda_j. \quad (2.1)$$

### 2.1 Expansion of the solution

We are going to make an asymptotic expansion of the solution to (1.1). Let  $u \in L^2(0, T)^r$  be fixed controls. The first-order term  $\Psi \in C^0\left([0, T], H_{(0)}^3(0, 1)\right)$  is the solution to the linearized system of (1.1) around the free trajectory  $(\psi_1, u \equiv 0)$ , *i.e.*

$$\begin{cases} i\partial_t \Psi(t, x) = -\partial_x^2 \Psi(t, x) - (u(t), \mu(x))\psi_1(t, x), & t \in (0, T), x \in (0, 1), \\ \Psi(t, 0) = \Psi(t, 1) = 0, & t \in (0, T), \\ \Psi(0, x) = 0, & x \in (0, 1). \end{cases} \quad (2.2)$$

The solution is given by:  $\forall t \in [0, T]$ ,

$$\Psi(t) = i \sum_{\ell=1}^r \sum_{j=1}^{+\infty} \left( \langle \mu_\ell \varphi_1, \varphi_j \rangle \int_0^t u^\ell(\tau) e^{i\omega_j \tau} d\tau \right) \psi_j(t). \quad (2.3)$$

We expand the development of the solution to the quadratic term. The second-order term  $\xi \in C^0\left([0, T], H_{(0)}^3(0, 1)\right)$  is the solution to the following system

$$\begin{cases} i\partial_t \xi(t, x) = -\partial_x^2 \xi(t, x) - (u(t), \mu(x))\Psi(t, x), & t \in (0, T), x \in (0, 1), \\ \xi(t, 0) = \xi(t, 1) = 0, & t \in (0, T), \\ \xi(0, x) = 0, & x \in (0, 1). \end{cases}$$

The idea is that  $\psi(T; u, \varphi_1) \simeq \psi_1(T) + \Psi(T) + \xi(T)$ . Thus,

$$\Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle \simeq 0 + 0 + \Im \langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle, \quad (2.4)$$

the first term being 0 since it is real and the second one by hypothesis  $(\mathbf{H})_{\text{lin}}$ . For  $1 \leq \ell, L \leq r$ , one defines  $H_{\ell, L} : (t, s) \in [0, T]^2 \mapsto -e^{-i\omega_K T} \sum_{j=1}^{+\infty} c_j^{\ell, L} e^{i\nu_j t + i\omega_j s}$ .

**Remark 2.1.** *The kernels introduced in this article differ by a phase  $e^{-i\omega_K T}$ , compared with those introduced in [35] in the case where  $r = 2$ .*

We finally use the notation, for  $f, g \in L^2((0, T), \mathbb{R})$ ,  $1 \leq \ell, L \leq r$ ,

$$\mathcal{F}_T^{\ell, L}(f, g) := \int_0^T f(t) \left( \int_0^t H_{\ell, L}(t, \tau) g(\tau) d\tau \right) dt.$$

With these notations,

$$\langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle = \sum_{\ell=1}^r \mathcal{F}_T^{\ell, \ell}(u^\ell, u^\ell) + \sum_{1 \leq \ell < L \leq r} \left( \mathcal{F}_T^{\ell, L}(u^\ell, u^L) + \mathcal{F}_T^{L, \ell}(u^L, u^\ell) \right). \quad (2.5)$$

To quantify the error term in the equation (2.4), we use the following classical estimates. The reader will find proof of this, for example, in [35, Lemma 3.7].

**Proposition 2.2.** *Assume that  $\mu_1, \dots, \mu_r$  satisfy  $(\mathbf{H})_{\text{reg}}$ . Then, as  $\|u\|_{L^2} \rightarrow 0$ ,*

$$\|\psi(\cdot; u, \varphi_1) - \psi_1 - \Psi\|_{L^\infty((0, T), H_{(0)}^3)} = \mathcal{O}\left(\|u\|_{L^2(0, T)}^2\right), \quad (2.6)$$

$$\|\psi(\cdot; u, \varphi_1) - \psi_1 - \Psi - \xi\|_{L^\infty((0, T), H_{(0)}^3)} = \mathcal{O}\left(\|u\|_{L^2(0, T)}^3\right). \quad (2.7)$$

## 2.2 A new expression for the quadratic expansion

We recall that the sequences  $c^{\ell, L}$  are introduced in Definition 1.4 and the iterated primitives of a function are defined in (1.6). Finally, the quantities  $\omega_j$  and  $\nu_j$  are defined in (2.1). The purpose of this section is to show the following proposition, already proved in [18, Proposition 5.1] for  $\ell = L$ .

**Proposition 2.3.** *Let  $T > 0$ ,  $1 \leq \ell \leq L \leq r$  and  $f, g \in L^2((0, T), \mathbb{R})$ . If  $(\mathbf{H})_{\text{conv}}$  and  $(\mathbf{H})_{\text{null}}$  hold, then,*

$$\begin{aligned} \Im \left( \mathcal{F}_T^{\ell, L}(f, g) + \mathcal{F}_T^{L, \ell}(g, f) \right) &= (-1)^{k+1} \gamma_{2k-1}^{\ell, L} \int_0^T f_k(t) g_k(t) \cos(\omega_K(t-T)) dt \\ &+ \mathcal{O} \left( \sum_{p=1}^k \left( |f_p(T)|^2 + |g_p(T)|^2 \right) + T \|(f_k, g_k)\|_{L^2}^2 \right). \end{aligned} \quad (2.8)$$

We first prove the following lemma.

**Lemma 2.4.** *Let  $n \in \mathbb{N}$  and  $H \in \mathcal{C}^{2n}(\mathbb{R}^2, \mathbb{C})$ . There exists a quadratic form  $S_n$  on  $\mathbb{C}^{3n}$  such that for all  $T > 0$  and  $f, g \in L^1(0, T)$ ,*

$$\begin{aligned} \int_0^T f(t) \left( \int_0^t g(\tau) H(t, \tau) d\tau \right) dt &= - \sum_{p=1}^n \int_0^T f_p(t) g_{p-1}(t) \partial_1^{p-1} \partial_2^{p-1} H(t, t) dt \\ &- \sum_{p=1}^n \int_0^T f_p(t) g_p(t) \partial_1^p \partial_2^{p-1} H(t, t) dt + \int_0^T f_n(t) \left( \int_0^t g_n(\tau) \partial_1^n \partial_2^n H(t, \tau) d\tau \right) dt \\ &+ S_n(f_1(T), \dots, f_n(T), g_1(T), \dots, g_n(T), C_0^n(g), \dots, C_{n-1}^n(g)), \end{aligned}$$

where  $C_p^n(g) := \int_0^T g_n(\tau) \partial_1^p \partial_2^n H(T, \tau) d\tau$ .

*Proof.* We prove this lemma by induction on  $n$ . It holds for  $n = 0$  with  $S_0 = 0$ . Assume that the result is true for  $n \in \mathbb{N}$ , fixed. Let  $H \in \mathcal{C}^{2(n+1)}(\mathbb{R}^2, \mathbb{C})$ ,  $f, g \in L^1(0, T)$ . With an integration by parts,

$$\begin{aligned} \int_0^T f_n(t) \left( \int_0^t g_n(\tau) \partial_1^n \partial_2^n H(t, \tau) d\tau \right) dt &= f_{n+1}(T) \int_0^T g_n(\tau) \partial_1^n \partial_2^n H(T, \tau) d\tau \\ &\quad - \int_0^T f_{n+1}(t) \left( g_n(t) \partial_1^n \partial_2^n H(t, t) + \int_0^t g_n(\tau) \partial_1^{n+1} \partial_2^n H(t, \tau) d\tau \right) dt. \end{aligned}$$

Another integration by parts finally leads to

$$\begin{aligned} \int_0^T f_n(t) \left( \int_0^t g_n(\tau) \partial_1^n \partial_2^n H(t, \tau) d\tau \right) dt &= f_{n+1}(T) C_n^n(g) - \int_0^T f_{n+1}(t) g_n(t) \partial_1^n \partial_2^n H(t, t) dt \\ &\quad - \int_0^T f_{n+1}(t) g_{n+1}(t) \partial_1^{n+1} \partial_2^n H(t, t) dt + \int_0^T f_{n+1}(t) \left( \int_0^t g_{n+1}(\tau) \partial_1^{n+1} \partial_2^{n+1} H(t, \tau) d\tau \right) dt. \end{aligned}$$

For all  $i \in \llbracket 0, n \rrbracket$ ,  $C_i^n(g) = g_{n+1}(T) \partial_1^i \partial_2^n H(T, T) - C_i^{n+1}(g)$ , thanks to an integration by parts. Using the induction hypothesis, we obtain the result.  $\square$

Thanks to this lemma, we are now able to prove Proposition 2.3.

*Proof of Proposition 2.3.* Let  $T > 0$ ,  $1 \leq \ell \leq L \leq r$ . Using  $(\mathbf{H})_{\text{conv}}$ , one has  $H_{\ell, L}, H_{L, \ell} \in \mathcal{C}^{2k}(\mathbb{R}^2, \mathbb{C})$ . First, note that for all  $p \in \llbracket 1, k \rrbracket$ , for all  $t > 0$ ,

$$\begin{aligned} \partial_1^{p-1} \partial_2^{p-1} (H_{L, \ell} - H_{\ell, L})(t, t) &= (-1)^{p-1} e^{i\omega_K(t-T)} \gamma_{2p-2}^{\ell, L}, \\ \partial_1^{p-1} \partial_2^p H_{L, \ell}(t, t) - \partial_1^p \partial_2^{p-1} H_{\ell, L}(t, t) &= i(-1)^{p-1} e^{i\omega_K(t-T)} \gamma_{2p-1}^{\ell, L}. \end{aligned} \quad (2.9)$$

Let  $f, g \in L^2(0, T)$  and  $p \in \llbracket 1, k \rrbracket$ . With an integration by parts,

$$\begin{aligned} \int_0^T f_{p-1}(t) g_p(t) \partial_1^{p-1} \partial_2^{p-1} H_{L, \ell}(t, t) dt &= f_p(T) g_p(T) \partial_1^{p-1} \partial_2^{p-1} H_{L, \ell}(T, T) \\ &\quad - \int_0^T f_p(t) g_{p-1}(t) \partial_1^{p-1} \partial_2^{p-1} H_{L, \ell}(t, t) dt - \int_0^T f_p(t) g_p(t) \left( \partial_1^p \partial_2^{p-1} + \partial_1^{p-1} \partial_2^p \right) H_{L, \ell}(t, t) dt. \end{aligned}$$

Then, applying Lemma 2.4 with  $f \rightarrow g$ ,  $g \rightarrow f$ ,  $H_{L, \ell} \rightarrow H$ ,  $k \rightarrow n$  and using the last equality, one gets

$$\begin{aligned} \int_0^T g(t) \left( \int_0^t f(\tau) H_{L, \ell}(t, \tau) d\tau \right) dt &= \sum_{p=1}^k \int_0^T f_p(t) g_{p-1}(t) \partial_1^{p-1} \partial_2^{p-1} H_{L, \ell}(t, t) dt \\ &\quad + \sum_{p=1}^k \int_0^T f_p(t) g_p(t) \partial_1^{p-1} \partial_2^p H_{L, \ell}(t, t) dt + \int_0^T g_k(t) \left( \int_0^t f_k(\tau) \partial_1^k \partial_2^k H_{L, \ell}(t, \tau) d\tau \right) dt \\ &\quad + \mathcal{O} \left( \sum_{p=1}^k \left( |f_p(T)|^2 + |g_p(T)|^2 \right) + T \|f_k\|_{L^2}^2 \right). \end{aligned} \quad (2.10)$$

To estimate  $C_p^k(f)$ , we used the boundness of  $H_{L, \ell}$  and the Cauchy–Schwarz’s inequality. Applying Lemma 2.4 again with  $H_{L, \ell} \rightarrow H$ ,  $k \rightarrow n$ , using (2.9), (2.10) and the assumption  $(\mathbf{H})_{\text{null}}$ , one gets

$$\begin{aligned} (-1)^{k+1} e^{i\omega_K T} \left( \mathcal{F}_T^{\ell, L}(f, g) + \mathcal{F}_T^{L, \ell}(g, f) \right) &= i \gamma_{2k-1}^{\ell, L} \int_0^T f_k(t) g_k(t) e^{i\omega_K t} dt \\ &\quad + \sum_{j=1}^{+\infty} \omega_j^k \nu_j^k \int_0^T e^{i\nu_j t} \left( c_j^{\ell, L} f_k(t) \left( \int_0^t g_k(\tau) e^{i\omega_j \tau} d\tau \right) + c_j^{L, \ell} g_k(t) \left( \int_0^t f_k(\tau) e^{i\omega_j \tau} d\tau \right) \right) dt \\ &\quad + \mathcal{O} \left( \sum_{p=1}^k \left( |f_p(T)|^2 + |g_p(T)|^2 \right) + T \|(f_k, g_k)\|_{L^2}^2 \right). \end{aligned}$$

Taking the imaginary part, we obtain the result.  $\square$

## 2.3 Vectorial relations

The equations (2.5) and (2.8) give a quadratic expansion of the solution to (1.1). We then estimate the boundary terms  $|u_p^\ell(T)|^2$  that appear. To do that, we need the following lemma, which is an equivalent in the infinite-dimensional case of [34, Lemma 3.12].

**Lemma 2.5.** *Assume that the hypotheses  $(\mathbf{H})_{\text{conv}}$ ,  $(\mathbf{H})_{\text{null}}$  and  $(\mathbf{H})_{\text{pos}}$  hold. Then, the following family is  $\mathbb{R}$ -linearly independent in  $\mathbb{C}^{\mathbb{N}^k}$*

$$\left( (\langle \mu_\ell \varphi_1, \varphi_j \rangle (-i\omega_j)^p)_{j \geq 1} \right)_{(p,\ell) \in \llbracket 0, k-1 \rrbracket \times \llbracket 1, r \rrbracket}$$

*Proof.* By contradiction, assume that there exist  $(\alpha_{p,\ell})_{(p,\ell) \in \llbracket 0, k-1 \rrbracket \times \llbracket 1, r \rrbracket} \in \mathbb{R}^{k \times r} \setminus \{0\}$  s.t.

$$\forall j \geq 1, \quad \sum_{p \in \llbracket 0, k-1 \rrbracket} \sum_{\ell \in \llbracket 1, r \rrbracket} \alpha_{p,\ell} \langle \mu_\ell \varphi_1, \varphi_j \rangle (-i\omega_j)^p = 0.$$

Let  $l_0 := \max\{p \in \llbracket 0, k-1 \rrbracket; (\alpha_{p,1}, \dots, \alpha_{p,r}) \neq 0\}$ . By hypothesis, the following quantities are zero

$$C := \sum_{p,q \in \llbracket 0, l_0 \rrbracket} \sum_{\ell, L \in \llbracket 1, r \rrbracket} \alpha_{p,\ell} \alpha_{q,L} (-i)^{p+q} \sum_{j=1}^{+\infty} \left( c_j^{L,\ell} \omega_j^{p+k-l_0} \nu_j^{q+k-l_0-1} \right) = \sum_{\ell, L \in \llbracket 1, r \rrbracket} C_{\ell,L},$$

$$D := \sum_{p,q \in \llbracket 0, l_0 \rrbracket} \sum_{\ell, L \in \llbracket 1, r \rrbracket} \alpha_{p,\ell} \alpha_{q,L} (-i)^{p+q} \sum_{j=1}^{+\infty} \left( c_j^{\ell,L} \omega_j^{q+k-l_0-1} \nu_j^{p+k-l_0} \right) = \sum_{\ell, L \in \llbracket 1, r \rrbracket} D_{\ell,L}.$$

Let  $v := (v_j)_{j \geq 1}$  be a sequence of real numbers and  $m, n \in \mathbb{N}$  be two integers. Subject to convergence, we define

$$R^{m,n}(v) := \sum_{j=1}^{+\infty} v_j \omega_j^{m+k-l_0-1} \nu_j^{n+k-l_0-1}.$$

By definition of  $R$ , one has, for all  $1 \leq \ell \leq L \leq r$ ,

$$C_{\ell,L} - D_{\ell,L} = \sum_{p=0}^{l_0} \alpha_{p,\ell} \alpha_{p,L} (-1)^{p+1} \gamma_{2(p+k-l_0)-1}^{\ell,L}$$

$$+ \sum_{\substack{p,q=0, \\ p \neq q}}^{l_0} \alpha_{p,\ell} \alpha_{q,L} (-i)^{p+q} \left( R^{p+1,q}(c^{L,\ell}) - R^{q,p+1}(c^{\ell,L}) \right).$$

For all  $p \in \llbracket 0, l_0-1 \rrbracket$ , for all  $q \in \llbracket p+1, l_0 \rrbracket$ , Corollary A.5 with  $q-p-1 \rightarrow \nu$  and  $p+k-l \rightarrow p$  gives

$$R^{p+1,q}(c^{L,\ell}) - R^{q,p+1}(c^{\ell,L}) \in \text{Span} \left( \gamma_{2(p+k-l_0)+r}^{\ell,L}, r \in \llbracket 0, q-p-1 \rrbracket \right) \subset \mathbb{R}.$$

Moreover, as  $2(p+k-l_0) + r \leq 2k-2$ , using  $(\mathbf{H})_{\text{null}}$ , this sum is zero. Finally, for all  $p \in \llbracket 1, l_0 \rrbracket$ , for all  $q \in \llbracket 0, p-1 \rrbracket$ , Corollary A.5 with  $p-q+1 \rightarrow \nu$  and  $q+k-l-1 \rightarrow p$  gives

$$R^{p+1,q}(c^{L,\ell}) - R^{q,p+1}(c^{\ell,L}) \in \text{Span} \left( \gamma_{2(q+k-l_0-1)+r}^{\ell,L}, r \in \llbracket 0, p-q+1 \rrbracket \right).$$

As  $2(q+k-l_0-1) + r \leq 2k-2$ , using  $(\mathbf{H})_{\text{null}}$ , this sum is zero. Finally,

$$C_{\ell,L} - D_{\ell,L} = \alpha_{l_0,L} \alpha_{l_0,\ell} (-1)^{l_0+1} \gamma_{2k-1}^{\ell,L}. \quad (2.11)$$

With similar computations, we prove that for all  $1 \leq L \leq \ell \leq r$ ,

$$C_{\ell,L} - D_{\ell,L} = \alpha_{l_0,\ell} \alpha_{l_0,L} (-1)^{l_0+1} \gamma_{2k-1}^{L,\ell}. \quad (2.12)$$

As  $C = D = 0$ ,  $C - D = 0$  and the equations (2.11) and (2.12) lead to

$$\sum_{\ell=1}^r \gamma_{2k-1}^{\ell} \frac{\alpha_{l_0, \ell}^2}{2} + \sum_{1 \leq \ell < L \leq r} \gamma_{2k-1}^{\ell, L} \alpha_{l_0, \ell} \alpha_{l_0, L} = 0, \quad \text{i.e.} \quad q(\alpha_{l_0, 1}, \dots, \alpha_{l_0, r}) = 0.$$

Using  $(\mathbf{H})_{\text{pos}}$ ,  $q$  is a definite quadratic form on  $\mathbb{R}^r$  so  $(\alpha_{l_0, \ell})_{\ell \in \llbracket 1, r \rrbracket} = 0$ . This is a contradiction with the choice of  $l_0$ . We then obtain the result.  $\square$

**Remark 2.6.** *The proof of Lemma 2.5 may seem a little mysterious at first sight, but there is a strict analogy with the result proved in the finite-dimensional case – see [34, Lemma 3.12]. Note that, if  $\mu \in C_c^\infty((0, 1), \mathbb{R})^r$ , then using [35, Lemma A.9],*

$$\forall 1 \leq \ell \leq r, 0 \leq p \leq k-1, \quad (\langle \mu_\ell \varphi_1, \varphi_j \rangle (-i\omega_j)^p)_{j \geq 1} = i^p (\langle \underline{\text{ad}}_A^p(\mu_\ell) \varphi_1, \varphi_j \rangle)_{j \geq 1}.$$

Consequently, we proved that the family  $(\underline{\text{ad}}_A^p(\mu_\ell) \varphi_1)_{(p, \ell) \in \llbracket 0, k-1 \rrbracket \times \llbracket 1, r \rrbracket}$  is linearly independent. This is the same family as in the finite-dimensional case. In [34], we consider  $(\alpha_{p, \ell})_{(p, \ell) \in \llbracket 0, k-1 \rrbracket \times \llbracket 1, r \rrbracket} \in \mathbb{R}^{k \times r}$  scalars, not all zero, such that

$$B\varphi_1 := \sum_{p \in \llbracket 0, k-1 \rrbracket} \sum_{\ell \in \llbracket 1, r \rrbracket} \alpha_{p, \ell} i^p \underline{\text{ad}}_A^p(\mu_\ell) \varphi_1 = 0.$$

Then, we define  $l_0 := \max\{p \in \llbracket 0, k-1 \rrbracket; (\alpha_{p, 1}, \dots, \alpha_{p, r}) \neq 0\}$ . As  $B\varphi_1 = 0$ , one has  $\left[ \underline{\text{ad}}_A^{k-l_0-1}(B), \underline{\text{ad}}_A^{k-l_0}(B) \right] \varphi_1 = 0$ , and we expand this term. Actually, this is exactly what we did in the proof of Lemma 2.5, in a disguised way, by not assuming regularity on  $\mu_\ell$  since

$$\left\langle \underline{\text{ad}}_A^{k-l_0-1} \left( \underline{\text{ad}}_A^{k-l_0}(B) \right) \varphi_1, \varphi_K \right\rangle = -C, \quad \left\langle \underline{\text{ad}}_A^{k-l_0} \left( \underline{\text{ad}}_A^{k-l_0-1}(B) \right) \varphi_1, \varphi_K \right\rangle = -D,$$

where  $C$  and  $D$  have been introduced in the proof of Lemma 2.5.

## 2.4 Closed-loop estimates

Using Lemma 2.5, we are now able to estimate the boundary terms. This is what we need in the following statement.

**Proposition 2.7.** *Assume that  $(\mathbf{H})_{\text{conv}}$ ,  $(\mathbf{H})_{\text{reg}}$ ,  $(\mathbf{H})_{\text{null}}$  and  $(\mathbf{H})_{\text{pos}}$  hold. Then, as  $\|u\|_{L^2} \rightarrow 0$ ,*

$$\sum_{p \in \llbracket 1, k \rrbracket} \sum_{\ell \in \llbracket 1, r \rrbracket} |u_p^\ell(T)| = \mathcal{O} \left( \|u\|_{L^2(0, T)}^2 + \sqrt{T} \|u_k\|_{L^2(0, T)} + \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2(0, 1)} \right).$$

*Proof.* By Lemma 2.5, there exist  $j_1, \dots, j_{kr} \in \mathbb{N}^*$ , pairwise distinct such that

$$M := (\langle \mu_\ell \varphi_1, \varphi_{j_n} \rangle (-i\omega_{j_n})^p)_{((p, \ell), n) \in \llbracket 0, k-1 \rrbracket \times \llbracket 1, r \rrbracket \times \llbracket 1, kr \rrbracket} \in GL_{kr}(\mathbb{C}).$$

Then, using the remainder estimate (2.6) and the expansion of the linearized system given by (2.3), one has for all  $j \geq 1$ , as  $\|u\|_{L^2} \rightarrow 0$ ,

$$\langle \psi(T; u, \varphi_1) - \psi_1(T), \varphi_j e^{-i\lambda_1 T} \rangle = i \sum_{\ell=1}^r \langle \mu_\ell \varphi_1, \varphi_j \rangle \int_0^T u^\ell(t) e^{i\omega_j(t-T)} dt + \mathcal{O} \left( \|u\|_{L^2(0, T)}^2 \right).$$

Using integrations by parts and Cauchy–Schwarz’s inequality, one gets, for all  $j \geq 1$ ,

$$\begin{aligned} \langle \psi(T; u, \varphi_1) - \psi_1(T), \varphi_j e^{-i\lambda_1 T} \rangle &= i \sum_{p \in \llbracket 0, k-1 \rrbracket} \sum_{\ell \in \llbracket 1, r \rrbracket} \langle \mu_\ell \varphi_1, \varphi_j \rangle (-i\omega_j)^p u_{p+1}^\ell(T) \\ &\quad + \mathcal{O} \left( \|u\|_{L^2(0, T)}^2 + \sqrt{T} \|u_k\|_{L^2(0, T)} \right). \end{aligned}$$

Focusing on  $j \in \{j_1, \dots, j_{kr}\}$ , we finally obtain

$$M (u_p^\ell(T))_{\substack{p \in \llbracket 1, k \rrbracket \\ \ell \in \llbracket 1, r \rrbracket}} = \mathcal{O} \left( \|u\|_{L^2(0, T)}^2 + \sqrt{T} \|u_k\|_{L^2(0, T)} + \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2(0, 1)} \right).$$

As  $M$  is invertible, one obtains the desired result.  $\square$

## 2.5 Interpolation inequality

In this paper, we expand the solution to the Schrödinger equation (1.1) to the second-order term. The remainder term is given by (2.7) and is estimated as  $\mathcal{O}\left(\|u\|_{L^2(0,T)}^3\right)$ . The purpose of the following proposition is to compare this error term with the drift size  $\|u_k\|_{L^2(0,T)}^2$ .

**Proposition 2.8.** *There exists  $C > 0$  s.t. for all  $T > 0$ ,  $f \in H^{2k}((0, T), \mathbb{R})$ ,*

$$\|f\|_{L^2}^3 \leq C(1 + T^{-2k}) \|f\|_{H^{2k}} \|f_k\|_{L^2}^2.$$

*Proof.* We apply the Gagliardo–Nirenberg interpolation inequalities given in Proposition A.7 with  $j = k$ ,  $l = 3k$ ,  $\alpha = \frac{1}{3}$ ,  $p = q = r = s = 2$  et  $\varphi = f_k$  to obtain

$$\|f\|_{L^2}^3 \leq C \left\| f^{(2k)} \right\|_{L^2} \|f_k\|_{L^2}^2 + CT^{-3k} \|f_k\|_{L^2}^3.$$

Moreover,

$$\|f_k\|_{L^2} \leq T^k \|f\|_{L^2} \leq T^k \|f\|_{H^{2k}}.$$

Thus, we obtain the desired result.  $\square$

## 2.6 Proof of the drift

We prove Theorem 1.10 as a consequence of the following more precise statement.

**Theorem 2.9.** *Let  $k, K, r \in \mathbb{N}^*$ ,  $\mu_1, \dots, \mu_r$  be functions satisfying  $(\mathbf{H})_{\text{conv}}$ ,  $(\mathbf{H})_{\text{reg}}$ ,  $(\mathbf{H})_{\text{lin}}$ ,  $(\mathbf{H})_{\text{null}}$  and  $(\mathbf{H})_{\text{pos}}$ . There exist  $C, T^* > 0$  such that for all  $T \in (0, T^*)$ , there exists  $\eta > 0$  such that for all  $u \in H^{2k}((0, T), \mathbb{R})^r$  with  $\|u\|_{H^{2k}} \leq \eta$ , the solution  $\psi(\cdot; u, \varphi_1)$  of (1.1) satisfies*

$$(-1)^{k+1} \text{sgn}(\gamma_{2k-1}^1) \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle \geq C \|u_k\|_{L^2}^2 - C \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2}^2. \quad (2.13)$$

**Remark 2.10.** *Theorem 2.9 shows that there exists  $0 < R < 1$  such that the following targets cannot be reached by the solution to (1.1)*

$$\forall \delta \in (0, R), \quad \psi_f := \left( \sqrt{1 - \delta^2} \varphi_1 + i(-1)^k \text{sgn}(\gamma_{2k-1}^1) \delta \varphi_K \right) e^{-i\lambda_1 T}.$$

*Indeed, if there exist controls  $u \in H^{2k}(0, T)^r$  such that  $\psi(T; u, \varphi_1) = \psi_f$ , the equation (2.13) leads to*

$$-\delta \geq K \|u_k\|_{L^2}^2 - 2K(1 - \sqrt{1 - \delta^2}) \geq -2K\delta^2.$$

*This is impossible when  $\delta \rightarrow 0$ . Thus, we obtain Theorem 1.10.*

*Proof of Theorem 2.9.* Using  $(\mathbf{H})_{\text{lin}}$  and the remainder estimate (2.7), the quadratic expansion of the solution gives, as  $\|u\|_{L^2} \rightarrow 0$ ,

$$\Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle = \Im \langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle + \mathcal{O}\left(\|u\|_{L^2}^3\right).$$

Using (2.5) and Proposition 2.3, one gets

$$\begin{aligned} \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle &= \mathcal{O} \left( \sum_{p \in [1, k]} \sum_{\ell \in [1, r]} |u_p^\ell(T)|^2 + T \|u_k\|_{L^2}^2 + \|u\|_{L^2}^3 \right) \\ &+ (-1)^{k+1} \int_0^T \left( \sum_{\ell=1}^r \gamma_{2k-1}^\ell \frac{u_k^\ell(t)^2}{2} + \sum_{1 \leq \ell < L \leq r} \gamma_{2k-1}^{\ell, L} u_k^\ell(t) u_k^L(t) \right) \cos(\omega_K(t - T)) dt \end{aligned}$$

We use Proposition 2.7 to estimate the boundary terms  $u_p^\ell(T)$  and the interpolation inequality proved in Proposition 2.8 to obtain, as  $\|u\|_{L^2} \rightarrow 0$ ,

$$\begin{aligned} \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle &= (-1)^{k+1} \int_0^T q(u_k^1, \dots, u_k^r)(t) \cos(\omega_K(t-T)) dt \\ &\quad + \mathcal{O} \left( (T + (1 + T^{-2k}) \|u\|_{H^{2k}}) \|u_k\|_{L^2}^2 + \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2}^2 \right). \end{aligned}$$

Then, there exist  $C_1, T_1 > 0$  such that, for all  $T \in (0, T_1)$ , there exists  $\eta_1 > 0$  such that, for all  $u \in H^{2k}(0, T)$  satisfying  $\|u\|_{L^2} < \eta_1$ ,

$$\begin{aligned} &\left| \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle - (-1)^{k+1} \int_0^T q(u_k^1, \dots, u_k^r)(t) \cos(\omega_K(t-T)) dt \right| \\ &\leq C_1 \left( (T + (1 + T^{-2k}) \|u\|_{H^{2k}}) \|u_k\|_{L^2}^2 + \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2}^2 \right). \end{aligned} \quad (2.14)$$

Let  $T^* := \frac{\pi}{3\omega_K}$  if  $K \neq 1$ ,  $T^* = +\infty$  else. Let  $T \in (0, T^*)$ . By  $(\mathbf{H})_{\text{pos}}$  and Remark 1.8, there exists  $C_2 > 0$  such that  $\text{sgn}(\gamma_{2k-1}^1)q(a) \geq 4C_2 \|a\|^2$ , for every  $a \in \mathbb{R}^r$ . Thus,

$$\text{sgn}(\gamma_{2k-1}^1) \int_0^T q(u_k^1, \dots, u_k^r)(t) \cos(\omega_K(t-T)) dt \geq 2C_2 \|u_k\|_{L^2}^2. \quad (2.15)$$

Let  $T_f := \min(T_1, T^*, \frac{C_2}{3C_1})$ . For all  $T \in (0, T_f)$ , we define  $\eta := \min(\eta_1, \frac{C_2}{3C_1}, \frac{C_2}{3C_1} T^{2k})$ . Then, for all  $u \in H^{2k}(0, T)^r$  with  $\|u\|_{H^{2k}} \leq \eta$ , the estimate (2.14) leads to

$$\begin{aligned} &\left| \Im \langle \psi(T; u, \varphi_1), \varphi_K e^{-i\lambda_1 T} \rangle - (-1)^{k+1} \int_0^T q(u_k^1, \dots, u_k^r)(t) \cos(\omega_K(t-T)) dt \right| \\ &\leq C_2 \|u_k\|_{L^2}^2 + C_1 \|\psi(T; u, \varphi_1) - \psi_1(T)\|_{L^2}^2. \end{aligned}$$

This equation together with (2.15) conclude the proof.  $\square$

## A Postponed proofs

### A.1 Existence of $\mu_1, \dots, \mu_r$ verifying the hypotheses

**Theorem A.1.** *Let  $k, K, r \in \mathbb{N}^*$ . There exist  $\mu_1, \dots, \mu_r$  satisfying  $(\mathbf{H})_{\text{reg}}$ ,  $(\mathbf{H})_{\text{conv}}$ ,  $(\mathbf{H})_{\text{lin}}$ ,  $(\mathbf{H})_{\text{null}}$  and  $(\mathbf{H})_{\text{pos}}$ .*

We use arguments very similar to those developed in [16, 18, 35]. That is why we will only give a proof skeleton for  $r = 2$ .

*Ideas of proof.* We prove more precisely the existence of  $\mu_1, \mu_2 \in C_c^\infty(0, 1)$  such that the previous assumptions are satisfied. If  $\mu_1, \mu_2 \in C_c^\infty(0, 1)$ ,  $(\mathbf{H})_{\text{reg}}$  is already verified. This is also the case for  $(\mathbf{H})_{\text{conv}}$  by Remark 1.5. In [35], using a trick that divides function supports into two parts, we have explained how to guarantee the existence of functions  $\mu_1, \mu_2 \in C_c^\infty(0, 1)$  so that the following equalities hold

$$\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0, \quad (\text{A.1})$$

$$\forall 1 \leq p \leq 2k-2, \forall 1 \leq \ell \leq L \leq 2, \quad \gamma_p^{\ell, L} = 0, \quad (\text{A.2})$$

$$\gamma_{2k-1}^{1,2} = 0 \quad (\text{A.3})$$

As a consequence,  $(\mathbf{H})_{\text{lin}}$  and  $(\mathbf{H})_{\text{null}}$  are satisfied. Moreover using Remark 1.9, as  $\gamma_{2k-1}^{1,2} = 0$ , the assumption  $(\mathbf{H})_{\text{pos}}$  becomes

$$\gamma_{2k-1}^1 \gamma_{2k-1}^2 > 0.$$

Then, we want to guarantee that (A.1), (A.2), (A.3),  $\gamma_{2k-1}^1 > 0$  and  $\gamma_{2k-1}^2 > 0$  can be satisfied simultaneously to conclude the proof. This is what Bournoissou did in [18, Theorem A.4]. By adapting her method to our setting, we obtain the claimed result.  $\square$

## A.2 Some series expansions

We recall that  $\omega_j$  and  $\nu_j$  are defined in (2.1).

**Definition A.2.** Let  $a := (a_j)_{j \geq 1}, b := (b_j)_{j \geq 1}$  be sequences of real numbers and  $l \in \mathbb{N}$  s.t.

$$(a_j j^{2l})_{j \geq 1}, (b_j j^{2l})_{j \geq 1} \in \ell^1(\mathbb{N}^*). \text{ We define } \gamma_l(a, b) := \sum_{j=1}^{+\infty} \left( a_j \omega_j^{\lfloor \frac{l}{2} \rfloor} \nu_j^{\lfloor \frac{l+1}{2} \rfloor} - b_j \omega_j^{\lfloor \frac{l+1}{2} \rfloor} \nu_j^{\lfloor \frac{l}{2} \rfloor} \right).$$

**Remark A.3.** We recall that  $(\gamma_p^{\ell, L})$  are specified in Definition 1.6. If  $(\mathbf{H})_{\text{conv}}$  holds,

$$\forall 0 \leq p \leq 2k-1, 1 \leq \ell \leq L \leq r, \quad \gamma_p^{\ell, L} = \gamma_p(c^{\ell, L}, c^{L, \ell}).$$

**Lemma A.4.** Let  $\nu \in \mathbb{N}$ . There exist coefficients  $(\beta_l^\nu)_{l \in \llbracket 0, \nu \rrbracket} \in \mathbb{Z}^{\nu+1}$  such that, for every  $p \in \mathbb{N}$  and  $a := (a_j)_{j \geq 1}, b := (b_j)_{j \geq 1} \in \mathbb{R}^{\mathbb{N}^*}$  verifying  $\sum_{j=1}^{+\infty} (|a_j| + |b_j|) j^{2(2p+\nu)} < +\infty$ ,

$$\sum_{j=1}^{+\infty} (a_j \omega_j^{p+\nu} \nu_j^p - b_j \omega_j^p \nu_j^{p+\nu}) = \sum_{l=0}^{\nu} \beta_l^\nu (-1)^l \gamma_{2p+l}(a, b) \omega_K^{\nu-l}.$$

*Proof.* We prove this statement by induction on  $\nu \in \mathbb{N}$ . *Initialisation.* For  $\nu = 0$ , the desired equality is true by definition with  $\beta_0^0 = 1$ . For  $\nu = 1$ , one can notice that  $\omega_K - \nu_j = \omega_j$ . Then, by definition,

$$\begin{aligned} \sum_{j=1}^{+\infty} (a_j \omega_j^{p+1} \nu_j^p - b_j \omega_j^p \nu_j^{p+1}) &= \sum_{j=1}^{+\infty} a_j \omega_j^p \nu_j^p (\omega_K - \nu_j) - \sum_{j=1}^{+\infty} b_j \omega_j^p \nu_j^p (\omega_K - \omega_j) \\ &= \omega_K \gamma_{2p}(a, b) - \gamma_{2p+1}(a, b). \end{aligned}$$

We obtain the result with  $\beta_0^1 = \beta_1^1 = 1$ . *Induction step:* assume that the result holds for  $\nu$  and  $\nu + 1$ . Let  $p \in \mathbb{N}, (a_j)_{j \geq 1}, (b_j)_{j \geq 1} \in \mathbb{R}^{\mathbb{N}^*}$  be such that  $\sum_{j=1}^{+\infty} (|a_j| + |b_j|) j^{2(2p+\nu+2)} < +\infty$ .

Using the same strategy,

$$\begin{aligned} \sum_{j=1}^{+\infty} (a_j \omega_j^{p+\nu+2} \nu_j^p - b_j \omega_j^p \nu_j^{p+\nu+2}) &= \omega_K \sum_{j=1}^{+\infty} (a_j \omega_j^{p+\nu+1} \nu_j^p - b_j \omega_j^p \nu_j^{p+\nu+1}) \\ &\quad - \sum_{j=1}^{+\infty} ((a_j \omega_j \nu_j) \omega_j^{p+\nu} \nu_j^p - (b_j \omega_j \nu_j) \omega_j^p \nu_j^{p+\nu}). \end{aligned}$$

We use the equality  $\gamma_{2p+l}((a_j \omega_j \nu_j)_{j \geq 1}, (b_j \omega_j \nu_j)_{j \geq 1}) = \gamma_{2(p+1)+l}(a, b)$  and the induction hypothesis to obtain the result, with  $\beta_l^{\nu+2} = \beta_l^{\nu+1} - \beta_{l-2}^\nu$ .  $\square$

**Corollary A.5.** Assume that  $(\mathbf{H})_{\text{conv}}$  holds. Let  $\nu \in \mathbb{N}$ , there exist  $(\beta_l^\nu)_{l \in \llbracket 0, \nu \rrbracket}, (\delta_l^\nu)_{l \in \llbracket 0, \nu \rrbracket} \in \mathbb{Z}^{\nu+1}$  such that, for every  $p \in \mathbb{N}$  satisfying  $2p + \nu \leq 2k - 1$ , for every  $1 \leq \ell \leq L \leq r$ ,

$$\begin{aligned} \sum_{j=1}^{+\infty} (c_j^{\ell, L} \omega_j^{p+\nu} \nu_j^p - c_j^{L, \ell} \omega_j^p \nu_j^{p+\nu}) &= \sum_{l=0}^{\nu} \beta_l^\nu (-1)^l \gamma_{2p+l}^{\ell, L} \omega_K^{\nu-l}, \\ \sum_{j=1}^{+\infty} (c_j^{L, \ell} \omega_j^{p+\nu} \nu_j^p - c_j^{\ell, L} \omega_j^p \nu_j^{p+\nu}) &= \sum_{l=0}^{\nu} \delta_l^\nu (-1)^l \gamma_{2p+l}^{\ell, L} \omega_K^{\nu-l}. \end{aligned}$$

*Proof.* The first point is a direct consequence of Lemma A.4 with  $a = c^{\ell, L}$  and  $b = c^{L, \ell}$ . The second one can be proved in the same way (induction).  $\square$



**Remark A.6.** Assume that  $\mu_1, \dots, \mu_r \in C_c^\infty(0, 1)$  so that  $(\mathbf{H})_{\text{conv}}$  holds by Remark 1.5. Then, for all  $p, \nu \in \mathbb{N}$  with  $2p + \nu \leq 2k - 1$ , for all  $1 \leq \ell \leq L \leq r$ , a bracket computation gives

$$\langle [\underline{\text{ad}}_A^p(\mu_\ell), \underline{\text{ad}}_A^{p+\nu}(\mu_L)]\varphi_1, \varphi_K \rangle = (-1)^\nu \sum_{j=1}^{+\infty} \left( c_j^{\ell, L} \omega_j^{p+\nu} \nu_j^p - c_j^{L, \ell} \omega_j^p \nu_j^{p+\nu} \right). \quad (\text{A.4})$$

Using (A.4) and [35, Proposition A.11], the first expansion of Corollary A.5 can be written

$$\begin{aligned} & \langle [\underline{\text{ad}}_A^p(\mu_\ell), \underline{\text{ad}}_A^{p+\nu}(\mu_L)]\varphi_1, \varphi_K \rangle \\ &= \sum_{l=0}^{\nu} \beta_l^\nu (-1)^l \left\langle \underline{\text{ad}}_A^{\nu-l} \left( \left[ \underline{\text{ad}}_A^{p+l\lfloor \frac{l+1}{2} \rfloor}(\mu_\ell), \underline{\text{ad}}_A^{p+l\lfloor \frac{l}{2} \rfloor}(\mu_L) \right] \right) \varphi_1, \varphi_K \right\rangle. \end{aligned}$$

Thus, Corollary A.5 is the equivalent of [34, Lemma A.1] in the infinite-dimensional case. Moreover, the sequences defined in Corollary A.5 and [34, Lemma A.1] are the same.

### A.3 Gagliardo–Nirenberg interpolation inequalities

We recall the Gagliardo–Nirenberg interpolation inequalities proved in [33, 42].

**Proposition A.7.** Let  $p, q, r, s \in [1, +\infty]$ ,  $0 \leq j < l \in \mathbb{N}$  and  $\alpha \in (0, 1)$  be such that

$$\frac{j}{l} \leq \alpha \quad \text{and} \quad \frac{1}{p} = j + \left( \frac{1}{r} - l \right) \alpha + \frac{1 - \alpha}{q}.$$

There exists  $C > 0$  such that, for every  $t > 0$  and  $\varphi \in C^\infty([0, t], \mathbb{R})$ ,

$$\|D^j \varphi\|_{L^p} \leq C \|D^l \varphi\|_{L^r}^\alpha \|\varphi\|_{L^q}^{1-\alpha} + C t^{\frac{1}{p} - j - \frac{1}{s}} \|\varphi\|_{L^s}.$$

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