



HAL
open science

Equivalences among parabolicity, comparison principle and capacity on complete Riemannian manifolds

A. Aiolfi, L. Bonorino, J. Ripoll, M. Soret, M. Ville

► **To cite this version:**

A. Aiolfi, L. Bonorino, J. Ripoll, M. Soret, M. Ville. Equivalences among parabolicity, comparison principle and capacity on complete Riemannian manifolds. 2025. hal-04953680

HAL Id: hal-04953680

<https://hal.science/hal-04953680v1>

Preprint submitted on 18 Feb 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Equivalences among parabolicity, comparison principle and capacity on complete Riemannian manifolds

A. Aiolfi, L. Bonorino, J. Ripoll, M. Soret, and M. Ville

Abstract

In this work we establish new equivalences for the concept of p -parabolic Riemannian manifolds. We define a concept of comparison principle for elliptic PDE's on exterior domains of a complete Riemannian manifold M and prove that M is p -parabolic if and only if this comparison principle holds for the p -Laplace equation. We show also that the p -parabolicity of M implies the validity of this principle for more general elliptic PDS's and, in some cases, these results can be extended for non p -parabolic manifolds or unbounded solutions, provided that some growth of these solutions are assumed.

1 Introduction

We recall that a complete Riemannian manifold M is p -parabolic, $p > 1$, if M does not admit a Green function that is, a positive real function $G(x, y)$ defined for $x, y \in M$ with $x \neq y$ such that $G(\cdot, y)$ is of C^1 class in $M \setminus \{y\}$ for any fixed $y \in M$, and

$$-\operatorname{div}(\|\nabla G(\cdot, y)\|^{p-2} \nabla G(\cdot, y)) = \delta_y, \quad y \in M.$$

This equality is in the sense of distributions that is,

$$\int_M \langle \|\nabla G(\cdot, y)\|^{p-2} \nabla G(\cdot, y), \nabla \varphi \rangle = \varphi(y)$$

*This work was partially supported by the ‘‘Brazilian-French Network in Mathematics’’, a project sponsored by CNPq, from Brazil, and by Fondation des Sciences Mathématiques de Paris (FSMP), from France.

for all $\varphi \in C_c^1(M)$.

A well known subject of research is to find characterizations for the p -parabolicity of complete Riemannian manifolds. Gathering together results which have been proved in the last decades, see Grigor'yan [5] and Holopainen [7, 8, 9, 10] for instance, we have (the notions used in the theorem are all defined below):

Theorem 1.1. *Let M be a complete Riemannian manifold. Then the following alternatives are equivalent:*

- (a) M is p -parabolic;
- (b) the p -capacity of any compact subset of M is zero;
- (c) the p -capacity of some precompact open subset of M is zero;
- (d) any bounded from below supersolution for the p -Laplacian operator is constant.

The case $p = 2$ is more classical, with a longer history, and there are other equivalences (see [5, 6]).

For a domain $\Omega \subset M$, we define the p -capacity of a set $E \subset \Omega$ with respect to Ω by

$$\text{cap}_p(E; \Omega) := \inf_{v \in \mathcal{F}_{E, \Omega}} \int_{\Omega} |\nabla v|^p dx, \quad (1)$$

where the infimum is taken over all the functions v that belongs to

$$\mathcal{F}_{E, \Omega} = \{v \in C_c^\infty(M) : v = 1 \text{ in } E \text{ and } \text{supp}(v) \subset \Omega\}. \quad (2)$$

Observe that the p -capacity of a bounded subset $E \subset M$ with respect to M can be obtained by taking a sequence of geodesic balls $B(o, R_k)$ centered at a given point $o \in M$, with $R_k \rightarrow +\infty$, and taking the limit

$$\text{Cap}_p(E) := \lim_{k \rightarrow +\infty} \text{cap}_p(E; B(o, R_k)). \quad (3)$$

It is not difficult to prove that the limit always exists, that does depend not on the sequence of balls and that $\text{Cap}_p(E) = \text{cap}_p(E; M)$.

Now we observe that as a result of the equivalence between (b) and (c) of Theorem 1.1 (see [7, 8, 9, 10]), we obtain the following result which is an important tool to the study of p -parabolicity and comparison principle:

Corollary 1.2 (Dichotomy). *Let M be a complete noncompact Riemannian manifold and $p > 1$. If $\text{Cap}_p(B_0) > 0$ for some ball $B_0 \subset M$, then*

$$\text{Cap}_p(B) > 0$$

for any ball $B \subset M$. More generally, if $\text{Cap}_p(E) > 0$ for some bounded set E , then $\text{Cap}_p(F) > 0$ for any set F that contains some interior point.

Recall that the p -Laplacian operator Δ_p is defined by

$$\Delta_p u = \text{div} (\|\nabla u\|^{p-2} \nabla u), \quad u \in C^1(M).$$

A function $w \in C^1(M)$ is a supersolution of Δ_p if $\Delta_p w \leq 0$ in the weak sense.

In the main result of our paper we add two other equivalences in Theorem 1.1. The main one relates p -parabolicity with another very classical notion, the comparison principle, which is likely the most fundamental tool on PDE (see [15], Abstract). In the second one we use the p -capacity to show in an explicit way that the p -parabolicity of a Riemannian manifold depends on its behaviour at infinity.

We first define that M satisfies the comparison principle for the p -Laplace operator (or the p -comparison principle for short) if, given any exterior domain Ω of M ($\Omega = M \setminus K$, where K is a compact subset), if $v, w \in C^0(\overline{\Omega})$ are bounded sub and super solutions for Δ_p then, whenever $v|_{\partial\Omega} \leq w|_{\partial\Omega}$, it follows that $v \leq w$ in Ω . (In this case, we also say that the comparison principle for exterior domains in M holds for the p -Laplace operator.) We prove:

Theorem 1.3. *Let M be a complete Riemannian manifold. Then the following alternatives are equivalent:*

- (a) M is p -parabolic
- (b) M satisfies the comparison principle for Δ_p
- (c) Given $o \in M$ there are sequences R_1^k, R_2^k satisfying $R_1^k < R_2^k$ and

$$\lim_{k \rightarrow \infty} R_1^k = \lim_{k \rightarrow \infty} R_2^k = \infty$$

such that

$$\sup_k \text{cap}_p(B(o, R_1^k); B(o, R_2^k)) < +\infty$$

The existence of many different conditions implying p -parabolicity is well known and hence, where the p -comparison principle holds (see [12], [11], [18], [4], [8], [13], [17], [10] and also references therein). Among them, there are several works giving conditions in terms of the volume growth of geodesic balls. We prove here the following result:

Theorem 1.4. *Let M be a complete noncompact Riemannian manifold and $p > 1$. Suppose that, for a fixed point $o \in M$, there exist two increasing sequences r_k and s_k such that $s_k > r_k \rightarrow +\infty$ and*

$$\sup_k \left(\frac{2}{s_k - r_k} \right)^p \text{Vol}(B(o, s_k) \setminus B(o, r_k)) < +\infty. \quad (4)$$

Then M is p -parabolic.

We note that condition (4) is satisfied if there exists some sequence $r_k \rightarrow +\infty$ such that

$$\text{Vol}(B(o, r_k)) \leq C r_k^q \quad \text{for any } k,$$

where $q \leq p$ and $C > 0$ is a constant that depends only on M .

2 A general comparison result

In this section, we prove that a comparison result holds for a general class of equations which contains the p -Laplacian provided that the p -capacity of a sequence of annuli in the manifold has some decay. This section is of independent interest, but it will be important for the next one.

We will study a class of second order elliptic PDE's of the form

$$\text{div} \left(\frac{A(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 \quad \text{in } M \setminus K, \quad (5)$$

where $K \subset M$ is a compact set and $A \in C[0, +\infty) \cap C^1(0, +\infty)$ satisfies, for some $\alpha \in (0, 1]$ and $\beta \geq 1$, the following conditions:

$$\bullet A(0) = 0 \text{ and } A(s) > 0 \text{ for } s > 0; \quad (6)$$

$$\bullet sA'(s) \geq \alpha A(s) \text{ for } s \geq 0; \quad (7)$$

$$\bullet sA'(s) \leq \beta A(s) \text{ for } s \geq 0; \quad (8)$$

There exist D_2 and D_1 positive such that

$$\bullet A(s) \geq D_1 s^{p-1} \text{ for any } s \geq 0; \quad (9)$$

$$\bullet A(s) \leq D_2 s^{p-1} \text{ for any } s \geq 0. \quad (10)$$

Related to the operator in (5), define $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\mathcal{A}(0) = 0 \quad \text{and} \quad \mathcal{A}(\mathbf{v}) = \frac{A(|\mathbf{v}|)}{|\mathbf{v}|} \mathbf{v} \quad \text{for } \mathbf{v} \neq 0.$$

Hence equation (5) is equivalent to

$$\operatorname{div} \mathcal{A}(\nabla u) = 0 \quad \text{in } M \setminus K.$$

Remind that $u \in C^1(M \setminus K)$ is said to be a weak solution to this equation if

$$\int_{M \setminus K} \nabla \varphi \cdot \mathcal{A}(\nabla u) \, dx = 0$$

for any $\varphi \in C_c^1(M \setminus K)$. By an approximating argument, this holds for any $\varphi \in W_0^{1,p}(M \setminus K)$.

Observe that conditions (7) and (9) imply that

$$\sum_{i,j=1}^n \frac{\partial \mathcal{A}_j(\mathbf{v})}{\partial \mathbf{v}_i} \xi_i \xi_j \geq D_1 \alpha |\mathbf{v}|^{p-2} |\xi|^2 \quad (11)$$

and, from (8) and (10), we have that

$$\sum_{i,j=1}^n \left| \frac{\partial \mathcal{A}_j(\mathbf{v})}{\partial \mathbf{v}_i} \right| \leq n^2 D_2 (\beta + 2) |\mathbf{v}|^{p-2} \quad (12)$$

for any $\xi \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$, where \mathcal{A}_j is the j th coordinate functions of \mathcal{A} . Moreover, $\mathcal{A} \in C(\mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ and $\mathcal{A}(0) = 0$. Hence, we have the following result, proved in [2] (Lemma 2.1):

Lemma 2.1. *There exist positive constants $c_1 = c_1(n, p, \alpha)$ and $c_2 = c_2(n, p, \beta)$ such that*

$$(\mathcal{A}(\mathbf{v}_2) - \mathcal{A}(\mathbf{v}_1)) \cdot (\mathbf{v}_2 - \mathbf{v}_1) \geq c_1 (|\mathbf{v}_2| + |\mathbf{v}_1|)^{p-2} |\mathbf{v}_2 - \mathbf{v}_1|^2 \quad (13)$$

$$|\mathcal{A}(\mathbf{v}_2) - \mathcal{A}(\mathbf{v}_1)| \leq c_2 (|\mathbf{v}_2| + |\mathbf{v}_1|)^{p-2} |\mathbf{v}_2 - \mathbf{v}_1| \quad (14)$$

for $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ that satisfy $|\mathbf{v}_1| + |\mathbf{v}_2| \neq 0$. If $p \geq 2$, then

$$(\mathcal{A}(\mathbf{v}_2) - \mathcal{A}(\mathbf{v}_1)) \cdot (\mathbf{v}_2 - \mathbf{v}_1) \geq c_1 |\mathbf{v}_2 - \mathbf{v}_1|^p \quad (15)$$

for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$. If $1 < p \leq 2$, then

$$|\mathcal{A}(\mathbf{v}_2) - \mathcal{A}(\mathbf{v}_1)| \leq c_2 |\mathbf{v}_2 - \mathbf{v}_1|^{p-1} \quad (16)$$

for $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$.

We can also conclude directly, from (10) and (9), that

$$|\mathcal{A}(\mathbf{v})| \leq D_2 |\mathbf{v}|^{p-1} \quad \text{and} \quad |\mathcal{A}(\mathbf{v})| \geq D_1 |\mathbf{v}|^{p-1} \quad \text{for } \mathbf{v} \in \mathbb{R}^n. \quad (17)$$

We need the following result:

Lemma 2.2. *Let K be a compact set of M and $o \in M$ be a fixed point. Suppose that U and V are bounded domains of M that satisfy $K \subset V \subset\subset U$. For $R_2 > R_1 > 0$ such that the ball $B(o, R_1)$ contains \bar{U} , there exists a function $\eta = \eta_{R_1, R_2} \in W_0^{1,p}(M \setminus K)$ such that*

- $0 \leq \eta \leq 1$;
- $\eta(x) = 1$ for $x \in B(o, R_1) \setminus U$;
- $\eta(x) = 0$ if $x \in V$ or $d(x, o) \geq R_2$;
- $|D\eta(x)| \leq m$ for $x \in U \setminus V$, where m is a constant that depends on U , V and M , but not on R_1 and R_2 ;
- $\Delta_p \eta(x) = 0$ if $R_1 < d(x, o) < R_2$ in the weak sense.

Proof. First minimize the functional

$$J(v) := \int_{B(o, R_2)} |\nabla v|^p dx$$

in the convex set $\{v \in W_0^{1,p}(B(o, R_2)) : v = 1 \text{ in } B(o, R_1)\}$. From the classical theory, there exists a unique minimizer v_0 in this convex set such that $v_0 = 1$ in $B(o, R_1)$ and $\Delta_p v_0 = 0$ in $B(o, R_2) \setminus B(o, R_1)$ in the weak sense. Now, for some neighborhood V_ε of V such that $V \subset\subset V_\varepsilon \subset\subset U$, let $v_1 \in C^\infty(U)$ satisfying $0 \leq v_1 \leq 1$ in U , $v_1 = 0$ in V and $v_1 = 1$ in $U \setminus V_\varepsilon$. Defining η by $\eta := v_1$ in U , $\eta := v_0$ in $B(o, R_2) \setminus U$ and $\eta := 0$ in $M \setminus B(o, R_2)$, it follows the result. \square

Now we fix some notation before stating the main result. For a fixed point $o \in M$, we denote the capacity of $B(o, R_1)$ with respect to $B(o, R_2)$, for $R_2 > R_1$, by

$$\text{cap}_p(R_1, R_2, o) := \text{cap}_p(B(o, R_1); B(o, R_2)).$$

If it is clear that the center of the balls is o , we simply denote by $\text{cap}_p(R_1, R_2)$.

The annulus $B(o, R_2) \setminus \overline{B(o, R_1)}$, centered at o , is denoted by

$$\mathcal{A}_{R_1, R_2} = \mathcal{A}_{R_1, R_2, o} := B(o, R_2) \setminus \overline{B(o, R_1)} \quad \text{for } R_2 > R_1$$

and the oscillation of a function v in this annulus is defined by

$$\operatorname{osc}_{\mathcal{A}_{R_1, R_2}} v := \sup_{\mathcal{A}_{R_1, R_2}} v - \inf_{\mathcal{A}_{R_1, R_2}} v. \quad (18)$$

The next proposition is a variant of the result that the derivatives of global solutions are in L^p if the manifold is p -parabolic.

Proposition 2.3. *Let $u \in C^1(M \setminus K)$ be a weak solution of (5) in $M \setminus K$. Suppose that A satisfies (6) - (10) and $U \subset M$ is an open bounded set such that $K \subset U$.*

(a) *If u is bounded and*

$$\sup_k \left(\operatorname{osc}_{\mathcal{A}_{R_1^k, R_2^k}} u \right)^p \cdot \operatorname{cap}_p(R_1^k, R_2^k) < +\infty,$$

where $B(o, R_1^k)$ and $B(o, R_2^k)$ are increasing sequences of balls that contain U and such that $R_2^k > R_1^k \rightarrow \infty$, then $|Du| \in L^p(M \setminus U)$ and there exist positive constants C_1 and C_2 , such that

$$\|Du\|_{L^p(M \setminus U)} \leq C_2 + C_1 \limsup_{k \rightarrow \infty} \left(\operatorname{osc}_{\mathcal{A}_{R_1^k, R_2^k}} u \right)^p \cdot \operatorname{cap}_p(R_1^k, R_2^k).$$

(b) *In the case that u is not necessarily bounded, if we assume*

$$\sup_k \left(\max_{\mathcal{A}_{R_1^k, R_2^k}} |u| \right)^p \cdot \operatorname{cap}_p(R_1^k, R_2^k) < +\infty,$$

then $|Du| \in L^p(M \setminus U)$ and

$$\|Du\|_{L^p(M \setminus U)} \leq C_2 + C_1 \limsup_{k \rightarrow \infty} \left(\max_{\mathcal{A}_{R_1^k, R_2^k}} |u| \right)^p \cdot \operatorname{cap}_p(R_1^k, R_2^k).$$

(In both cases, C_1 depends on p, c_1, D_2 and C_2 depends on p, c_1, D_2, K, U , and u .)

Proof. **(a)** Let V be an open set and $R_0 > 0$ s.t. $K \subset V \subset\subset U \subset\subset B(o, R_0)$. For $R_1 > R_0$ and $R_2 > R_0$, consider the function $\eta = \eta_{R_1, R_2}$ as in Lemma 2.2 associated to V, U, R_1 and R_2 . Note that the function

$$\varphi := \eta^p(u - I), \quad \text{where } I = \inf_{\mathcal{A}_{R_1, R_2}} u,$$

belongs to $W_0^{1,p}(M \setminus K)$. Hence, using that u is a weak solution of (5), we obtain

$$\int_F (u - I) \nabla \eta^p \cdot \mathcal{A}(\nabla u) dx + \int_F \eta^p \nabla u \cdot \mathcal{A}(\nabla u) dx = 0,$$

where $F \subset B(o, R_2) \setminus V$ is the compact support of η . Hence, (13) and (17) imply that

$$\begin{aligned} c_1 \int_F \eta^p |\nabla u|^p dx &\leq \int_F \eta^p \nabla u \cdot \mathcal{A}(\nabla u) dx \\ &= - \int_F (u - I) \nabla \eta^p \cdot \mathcal{A}(\nabla u) dx \\ &\leq \int_F p |u - I| \eta^{p-1} |\nabla \eta| |\mathcal{A}(\nabla u)| dx \\ &\leq p D_2 \int_F |u - I| \eta^{p-1} |\nabla \eta| |\nabla u|^{p-1} dx. \end{aligned}$$

From Hölder inequality,

$$c_1 \int_F \eta^p |\nabla u|^p dx \leq p D_2 \left(\int_F \eta^p |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left(\int_F |u - I|^p |\nabla \eta|^p dx \right)^{\frac{1}{p}}, \quad (19)$$

that is,

$$\int_F \eta^p |\nabla u|^p dx \leq \left(\frac{p D_2}{c_1} \right)^p \int_F |u - I|^p |\nabla \eta|^p dx. \quad (20)$$

Using the hypotheses on the derivative of η , we obtain

$$\begin{aligned} \int_F |u - I|^p |\nabla \eta|^p dx &= \int_{U \setminus V} |u - I|^p |\nabla \eta|^p dx + \int_{\mathcal{A}_{R_1, R_2}} |u - I|^p |\nabla \eta|^p dx \\ &\leq 2^p S_0^p m^p \text{Vol}(U \setminus V) + \left(\text{osc}_{\mathcal{A}_{R_1, R_2}} u \right)^p \int_{\mathcal{A}_{R_1, R_2}} |\nabla \eta|^p dx, \end{aligned} \quad (21)$$

where $S_0 = \sup_{M \setminus V} |u|$. Since η is p -harmonic in \mathcal{A}_{R_1, R_2} , $\eta = 1$ on $\partial B(o, R_1)$ and $\eta = 0$ on $\partial B(o, R_2)$, then

$$\int_{\mathcal{A}_{R_1, R_2}} |\nabla \eta|^p dx = \inf_{v \in \mathcal{F}_{B(o, R_1), B(o, R_2)}} \int_M |\nabla v|^p dx = \text{cap}_p(R_1, R_2).$$

Hence, from (20) and (21), we conclude that

$$\int_F \eta^p |\nabla u|^p dx \leq C_1 2^p S_0^p m^p \text{Vol}(U \setminus V) + C_1 \left(\text{osc}_{\mathcal{A}_{R_1, R_2}} u \right)^p \text{cap}_p(R_1, R_2),$$

where $C_1 = (p D_2 / c_1)^p$. Since $B(o, R_1) \setminus U \subset F$ and $\eta = 1$ in $B(o, R_1) \setminus U$,

$$\int_{B(o, R_1) \setminus U} |\nabla u|^p dx \leq C_2 + C_1 \left(\text{osc}_{\mathcal{A}_{R_1, R_2}} u \right)^p \text{cap}_p(R_1, R_2),$$

where $C_2 = C_1 2^p S_0^p m^p \text{Vol}(U \setminus V)$. In particular, this holds for R_1^k and R_2^k . Then, according to the hypotheses, the right-hand side is bounded, proving (a).

(b) Taking the test function $\varphi := \eta^p u$ and applying the same argument as in (a), we conclude (b). The main difference in relation to (a) is that we have to replace S_0 by $\sup_{U \setminus V} |u|$ and $\text{osc}_{\mathcal{A}_{R_1, R_2}} u$ by $\max_{\mathcal{A}_{R_1^k, R_2^k}} |u|$. \square

The idea of the proof of the next theorem is based on the proof of Theorem 2 of [1].

Theorem 2.4. *Let $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be weak solutions of (5) in $M \setminus K$, where A satisfies (6) - (10) and $p > 1$. Assume also that*

$$\sup_k \left(\max_{\mathcal{A}_{R_1^k, R_2^k}} |u| \right)^p \text{cap}_p(R_1^k, R_2^k) \quad \text{and} \quad \sup_k \left(\max_{\mathcal{A}_{R_1^k, R_2^k}} |v| \right)^p \text{cap}_p(R_1^k, R_2^k) \quad (22)$$

are finite for some sequences (R_1^k) and (R_2^k) such that $R_2^k > R_1^k \rightarrow \infty$. If $u \leq v$ on ∂K , then

$$u \leq v \quad \text{in} \quad M \setminus K.$$

Proof. Let (ε_k) be a sequence of positive numbers such that $\varepsilon_k \downarrow 0$ and

$$\lim_{k \rightarrow \infty} \varepsilon_k^p \text{cap}_p(R_1^k, R_2^k) = 0. \quad (23)$$

Observe that $v - u + \varepsilon_k \geq \varepsilon_k$ on ∂K for any k . Then, using that $v - u + \varepsilon_k$ is continuous, we conclude that there exists a bounded open set $U = U_k \supset K$ such that $v - u + \varepsilon_k > 0$ in $U \setminus K$. We can suppose that U_k is a decreasing sequence and $\bigcap_{k=1}^{\infty} U_k = K$. As in the previous proposition, let $V = V_k$ be

an open set and $R_0 \geq 1$ such that $K \subset V_k \subset U_k \subset U_1 \subset B(o, R_0)$. For $R_2^k > R_1^k > R_0$, let $\eta = \eta_{R_1^k, R_2^k}$ as described in Lemma 2.2. Observe that

$$(v - u + \varepsilon_k)^- = \max\{-(v - u + \varepsilon_k), 0\} = 0 \quad \text{in } U \setminus K.$$

Then $\varphi := \eta^p (v - u + \varepsilon_k)^-$ has a compact support

$$F_k \subset B(o, R_2^k) \setminus U_k \subset M \setminus K.$$

Hence, using that $v - u + \varepsilon_k \in C^1(M \setminus K)$, we have that $\varphi \in W_0^{1,p}(M \setminus K)$. Since u and v are weak solutions of (5) and $\varphi \in W_0^{1,p}(M \setminus K)$, we get

$$\int_{M \setminus K} \nabla \varphi \cdot [\mathcal{A}(\nabla v) - \mathcal{A}(\nabla u)] dM = 0. \quad (24)$$

Observe that $F_k \subset \{x \in M \setminus K : v(x) - u(x) + \varepsilon_k \leq 0\}$ and, therefore,

$$\nabla \varphi = \eta^p \chi_{F_k} \nabla(u - v) + p \eta^{p-1} (v - u + \varepsilon_k)^- \nabla \eta \quad \text{a.e. in } M \setminus K,$$

where χ_{F_k} is the characteristic function of the set F_k . Hence, it follows that

$$\begin{aligned} \int_{F_k} \eta^p \nabla(v - u) \cdot [\mathcal{A}(\nabla v) - \mathcal{A}(\nabla u)] dM &= \\ \int_{\mathcal{A}_{R_1^k, R_2^k}} p \eta^{p-1} (v - u + \varepsilon_k)^- \nabla \eta \cdot [\mathcal{A}(\nabla v) - \mathcal{A}(\nabla u)] dM. & \quad (25) \end{aligned}$$

Replacing F_k by $G_k := F_k \cap \{x \in M \setminus K : |\nabla u(x)| + |\nabla v(x)| \neq 0\}$, the left-hand side of (25) does not change. Then, from (13) and (17), we have

$$\begin{aligned} c_1 \int_{G_k} \eta^p (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v - u)|^2 dM &\leq \\ \leq \int_{\mathcal{A}_{R_1^k, R_2^k}} p \eta^{p-1} (v - u + \varepsilon_k)^- \nabla \eta \cdot [\mathcal{A}(\nabla v) - \mathcal{A}(\nabla u)] dM. & \quad (26) \end{aligned}$$

$$\leq D_2 p \int_{\mathcal{A}_{R_1^k, R_2^k}} \eta^{p-1} S_k |\nabla \eta| (|\nabla v|^{p-1} + |\nabla u|^{p-1}) dM, \quad (27)$$

where

$$S_k = \max_{\mathcal{A}_{R_1^k, R_2^k}} (v - u + \varepsilon_k)^- \leq \max_{\mathcal{A}_{R_1^k, R_2^k}} |u| + \max_{\mathcal{A}_{R_1^k, R_2^k}} |v| + \varepsilon_k < \infty.$$

Using Hölder's inequality, we have

$$\begin{aligned} c_1 \int_{G_k} \eta^p (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v - u)|^2 dM \\ \leq D_2 p \int_{\mathcal{A}_{R_1^k, R_2^k}} S_k |\nabla \eta| \eta^{p-1} 2 \max\{|\nabla u|, |\nabla v|\}^{p-1} dM \\ \leq 2D_2 p \left(\int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^p |\nabla \eta|^p dM \right)^{\frac{1}{p}} \left(\int_{\mathcal{A}_{R_1^k, R_2^k}} \eta^p (|\nabla u|^p + |\nabla v|^p) dM \right)^{\frac{p-1}{p}}. \end{aligned}$$

Therefore, since $|\eta| \leq 1$ and $p > 1$,

$$\begin{aligned} \int_{G_k} \eta^p (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v - u)|^2 dM \leq \frac{2D_2 p}{c_1} \left(\int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^p |\nabla \eta|^p dM \right)^{\frac{1}{p}} \\ \times \left(\|\nabla u\|_{L^p(\mathcal{A}_{R_1^k, R_2^k})}^{p-1} + \|\nabla v\|_{L^p(\mathcal{A}_{R_1^k, R_2^k})}^{p-1} \right). \quad (28) \end{aligned}$$

Observe now that

$$\begin{aligned} \int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^p |\nabla \eta|^p dM \leq \left[\max_{\mathcal{A}_{R_1^k, R_2^k}} (v - u + \varepsilon_k)^- \right]^p \int_{\mathcal{A}_{R_1^k, R_2^k}} |\nabla \eta|^p dM \\ \leq 3^p \left[\max_{\mathcal{A}_{R_1^k, R_2^k}} |v|^p + \max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^p + \varepsilon_k^p \right] \text{cap}_p(R_1^k, R_2^k) \quad (29) \end{aligned}$$

for all k . From this and (28), we conclude that

$$\begin{aligned} \int_{G_k} \eta^p (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v - u)|^2 dM \leq \frac{2D_2 p 3^p}{c_1} \left[\max_{\mathcal{A}_{R_1^k, R_2^k}} |v|^p + \right. \\ \left. \max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^p + \varepsilon_k^p \right] \text{cap}_p(R_1^k, R_2^k) \left(\|\nabla u\|_{L^p(\mathcal{A}_{R_1^k, R_2^k})}^{p-1} + \|\nabla v\|_{L^p(\mathcal{A}_{R_1^k, R_2^k})}^{p-1} \right). \quad (30) \end{aligned}$$

Note that the last two terms in this inequality converges to zero, according to Proposition 2.3. Moreover,

$$\max_{A_{R_1^k, R_2^k}} |v|^p \operatorname{cap}_p(R_1^k, R_2^k) \quad \text{and} \quad \max_{A_{R_1^k, R_2^k}} |u|^p \operatorname{cap}_p(R_1^k, R_2^k)$$

are bounded from hypothesis and $\varepsilon_k^p \operatorname{cap}_p(R_1^k, R_2^k)$ is bounded from (23). Hence, the right-hand side of (30) converges to 0 as $k \rightarrow \infty$. On the other hand,

$$\int_{G_k} \eta_{R_1^k, R_2^k}^p (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v-u)|^2 dM \rightarrow \int_{G_0} (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v-u)|^2 dM$$

as $k \rightarrow \infty$, where

$$G_0 = \{x \in M \setminus K : v(x) < u(x)\} \cap \{x \in M \setminus K : |\nabla u(x)| + |\nabla v(x)| \neq 0\}$$

since $\eta_{R_1^k, R_2^k} \rightarrow 1$ in $M \setminus K$ as $k \rightarrow \infty$ and G_k is an increasing sequence (F_k is an increasing sequence) of sets such that $\bigcup G_k = G_0$. Then

$$\int_{G_0} (|\nabla v| + |\nabla u|)^{p-2} |\nabla(v-u)|^2 dM = 0. \quad (31)$$

Therefore, $\chi_{G_0} \nabla(v-u) = 0$. Since

$$\nabla(v-u)^- = \chi_{\{x \in M \setminus K : v(x) < u(x)\}} \nabla(u-v) = \chi_{G_0} \nabla(u-v) \quad \text{a.e. in } M \setminus K,$$

it follows that $\nabla(v-u)^- = 0$ a.e. in $M \setminus K$. Using this and that $(v-u)^- = 0$ on ∂K , we conclude that $(v-u)^- = 0$ in $M \setminus K$. Therefore,

$$u(x) \leq v(x) \quad \text{for } x \in M \setminus K,$$

proving the result. \square

In particular, if u and v are bounded, the control of $\operatorname{cap}_p(R_1^k, R_2^k)$ guarantees the comparison principle for exterior domains:

Corollary 2.5. *Let M be a complete noncompact Riemannian manifold. Let A be a function that satisfies (6) - (10) and $p > 1$. If*

$$\sup_k \operatorname{cap}_p(R_1^k, R_2^k) < +\infty,$$

then the comparison principle holds for the exterior problem (5).

Remark 2.6. For $p = 2$, Theorem 2.4 holds even if the terms in (22) are not necessarily bounded, provided that

$$\max_{\mathcal{A}_{R_1^k, R_2^k}} |v - u|^p \operatorname{cap}_p(R_1^k, R_2^k) \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (32)$$

Indeed, according to (25), (15), (16) and the definition of F_k , for $p = 2$,

$$\begin{aligned} c_1 \int_{F_k} \eta^2 |\nabla(v - u)|^2 dM &\leq 2 c_2 \int_{\mathcal{A}_{R_1^k, R_2^k} \cap F_k} \eta S_k |\nabla \eta| |\nabla v - \nabla u| dM \\ &\leq \left(\int_{F_k} \eta^2 |\nabla(v - u)|^2 dM \right)^{\frac{1}{2}} \left(\int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^2 |\nabla \eta|^2 dM \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\int_{F_k} \eta^2 |\nabla(v - u)|^2 dM \leq \left(\frac{2 c_2}{c_1} \right)^2 \int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^2 |\nabla \eta|^2 dM.$$

Doing the same computation as in (29), we have

$$\int_{\mathcal{A}_{R_1^k, R_2^k}} S_k^2 |\nabla \eta|^2 dM \leq 2^2 \left[\sup_{\mathcal{A}_{R_1^k, R_2^k}} |v - u|^p + \varepsilon_k^p \right] \operatorname{cap}_p(R_1^k, R_2^k)$$

and, therefore,

$$\int_{F_k} \eta^2 |\nabla(v - u)|^2 dM \leq \left(\frac{2 c_2}{c_1} \right)^2 4 \left[\sup_{\mathcal{A}_{R_1^k, R_2^k}} |v - u|^p + \varepsilon_k^p \right] \operatorname{cap}_p(R_1^k, R_2^k).$$

Hence, using (32) and (23), we have that

$$\int_{F_k} \eta_{R_1^k, R_2^k}^2 |\nabla(v - u)|^2 dM \rightarrow 0.$$

Then, as in the theorem, relation (31) holds and following the same argument as before, we conclude that $u \leq v$.

3 Equivalence between the p -parabolicity and the p -comparison principle

In this section we prove that the p -comparison principle, as defined previously, holds for the exterior problem (5) if and only if M is p -parabolic.

First note that

$$\text{Cap}_p(E) \leq \text{cap}_p(E; \Omega) \quad \text{and} \quad \text{Cap}_p(E) \leq \text{Cap}_p(F) \quad (33)$$

for any $\Omega \subset M$ and $E \subset F \subset M$, since $\mathcal{F}_{E,\Omega} \subset \mathcal{F}_{E,M}$ and $\mathcal{F}_{E,M} \supset \mathcal{F}_{F,M}$ in this case.

We also need the next result, that is an extension of the one established in Corollary 4.6 of [5] to the case $p = 2$.

Lemma 3.1. *If $U \subset M$ is a bounded domain with a C^2 boundary and $\text{Cap}_p(U) > 0$, then there exists some non-constant p -harmonic function u such that $u = 1$ on ∂U and $0 < u < 1$ in $M \setminus U$.*

Proof. Let (W_k) be an increasing sequence of bounded domains with C^2 boundary such that

$$\bigcup_{k=1}^{\infty} W_k = M \quad \text{and} \quad U \subset\subset W_k \subset\subset W_{k+1} \quad \text{for any } k \in \mathbb{N}.$$

(If there exists an increasing sequence of balls $B(o, R_k)$ with C^2 boundary, where $R_k \rightarrow +\infty$, we can take $W_k = B(o, R_k)$.) From the theory for PDE, there exists a function $u_k \in \mathcal{F}_{U, W_k}$ such that

$$\int_M |\nabla u_k|^p dx = \inf_{v \in \mathcal{F}_{U, W_k}} \int_M |\nabla v|^p dx = \text{cap}_p(U; W_k). \quad (34)$$

Moreover, u_k is the p -harmonic function in the $A_k = A_{U, W_k} := W_k \setminus \bar{U}$ such that

$$u_k = 1 \quad \text{on } \partial U \quad \text{and} \quad u_k = 0 \quad \text{on } \partial W_k \quad \text{in the trace sense.}$$

Due to smoothness of ∂U and ∂W_k , the theory of regularity implies that $u_k \in C^1(\bar{A}_k)$ (for instance, see [11]). From the strong maximum principle

$0 < u_k < 1$ in A_k . Hence $u_k > 0$ on ∂W_{k-1} for any $k \geq 2$, since $\partial W_{k-1} \subset A_k = W_k \setminus \overline{U}$. Therefore, using the comparison principle, we conclude that

$$u_{k-1} < u_k \quad \text{in} \quad A_{k-1}. \quad (35)$$

Observe also that $u_k = 1$ in \overline{U} and $u_k = 0$ in $M \setminus W_k$, since $u_k \in \mathcal{F}_{U, W_k}$. (This implies that u_k is continuous in M due to the fact that $u_k \in C^1(\overline{A}_k)$.) Then, from (35), we have that

$$u_{k-1} \leq u_k \quad \text{in} \quad M \quad \text{for} \quad k \in \{2, 3, \dots\}.$$

Hence, using that $0 \leq u_k \leq 1$ for any k , we have that u_k converges to some function u defined in M satisfying $0 \leq u \leq 1$. In particular, $u = 1$ in \overline{U} . Note also that $u \geq u_k > 0$ in W_k for any k . Thus, $u > 0$ in M .

• *Statement 1:* u is p -harmonic. Observe that for any bounded domain $V \subset\subset M \setminus \overline{U}$, we have that $\overline{V} \subset A_k$ for large k . Then, starting from some large k , (u_k) is a uniformly bounded sequence of p -harmonic functions in V . Hence, according to Theorem 1.1 of [19], there exists some constant $C > 0$ that depends on V , M , n and p such that $|\nabla u_k| \leq C$ in \overline{V} . Thus, u_k is uniformly bounded in $W^{1,p}(V)$ and, therefore, up to a subsequence, we have that

$$u_k \rightharpoonup v_0 \quad \text{in} \quad W^{1,p}(V) \quad \text{and} \quad u_k \rightarrow v_0 \quad \text{in} \quad L^p(V)$$

for some $v_0 \in W^{1,p}(V)$, due to the reflexivity of $W^{1,p}(V)$ and the Rellich-Kondrachov theorem. Indeed, $v_0 = u$ since $u_k \rightarrow u$ pointwise. Hence, using that u_k is p -harmonic and the same argument as in [3] (see Theorem 3 of page 495), we conclude that

$$0 = \lim_{k \rightarrow \infty} \int_V |\nabla u_k|^{p-2} \nabla u_k \nabla \varphi \, dx = \int_V |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx,$$

for any $\varphi \in C_c^\infty(V)$, that is, u is p -harmonic in V , proving the statement.

• *Statement 2:* u is non-constant and $0 < u < 1$ in $M \setminus \overline{U}$. Since we already proved that $0 < u \leq 1$ in $M \setminus \overline{U}$, from the maximum principle, we just need to show that u is non-constant. Suppose that u is constant. Let $B^* = B(o, R_0)$ be an open ball such that $\overline{U} \subset\subset B^*$ and $\rho \in C^\infty(M)$ be a function that satisfies $0 \leq \rho \leq 1$, $\rho = 0$ in some neighborhood of \overline{U} and $\rho = 1$

in $M \setminus \overline{B^*}$. (We can suppose w.l.g. that $\overline{B^*} \subset W_k$ for any k .) Since u_k is p -harmonic in A_k and $\varphi = \rho^p u_k \in W_0^{1,p}(A_k)$, we have

$$- \int_{A_k} p u_k \rho^{p-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx = \int_{A_k} \rho^p |\nabla u_k|^p \, dx \geq \int_{M \setminus \overline{B^*}} |\nabla u_k|^p \, dx.$$

Hence, using (34) and (33), we conclude that

$$\begin{aligned} - \int_{A_k} p u_k \rho^{p-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx + \int_{B^*} |\nabla u_k|^p \, dx \\ \geq \int_{M \setminus \overline{B^*}} |\nabla u_k|^p \, dx + \int_{B^*} |\nabla u_k|^p \, dx \\ = \text{cap}_p(U; W_k) \geq \text{Cap}_p(U) > 0. \end{aligned}$$

Then, from the fact that $\nabla \rho = 0$ in $M \setminus \overline{B^*}$ and $\nabla u_k = 0$ in U , we have

$$- \int_{B^* \setminus U} p u_k \rho^{p-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx + \int_{B^* \setminus U} |\nabla u_k|^p \, dx \geq \text{Cap}_p(U). \quad (36)$$

Now the idea is to show that the left-hand side goes to zero as $k \rightarrow +\infty$, leading a contradiction. For that, observe that since u_1 is $C^1(\overline{A_1})$, $u_1 = 1$ on ∂U , ∂U is C^2 and $0 < u_1 < 1$ in A_1 , there exists $c_1 > 0$ such that

$$1 - c_1 \text{dist}(x, \partial U) \leq u_1(x) < 1 \quad \text{for } x \in A_1.$$

Using that $0 < u_1 \leq u_k < 1$ in A_1 for any k , we have

$$0 < 1 - u_k(x) \leq \min\{1, c_1 \text{dist}(x, \partial U)\} \quad \text{for } x \in A_1. \quad (37)$$

Furthermore, the inclusion $\overline{B^*} \subset W_1$ implies that there exists $r_0 > 0$ such that $B(x, r_0) \subset W_1$ for any $x \in \overline{B^*}$. Therefore, for any $x \in \overline{B^*} \setminus \overline{U}$ it follows that

$$B(x, r) \subset W_1 \setminus \overline{U} = A_1 \quad \text{if } r \leq r_1 := \min\{r_0, \text{dist}(x, \partial U)\}.$$

Hence, using that $1 - u_k$ is positive and p -harmonic in A_1 , Theorem 1.1 of [19] and (37), we have

$$|\nabla u_k(x)| \leq \tilde{C} \frac{(1 - u_k(x))}{r_1} \leq \tilde{C} \frac{\min\{1, c_1 \text{dist}(x, \partial U)\}}{r_1} \leq \tilde{C} \max\{c_1, 1/r_0\},$$

for $x \in \overline{B^*} \setminus \overline{U}$, where $\tilde{C} > 0$ is a constant that depends on n, p, M and A_1 . Therefore, the sequence $|\nabla u_k|$ is uniformly bounded in $\overline{B^*} \setminus \overline{U}$. Hence, if we prove that $|\nabla u_k|$ converges to zero pointwise in $\overline{B^*} \setminus \overline{U}$, then the bounded convergence theorem implies that the left-hand side of (36) converges to zero generating a contradiction. For that, observe that $u = 1$ in M , since we are assuming that u is a constant and $u = 1$ in U . From the fact that $u_k \rightarrow u$, we conclude that $1 - u_k \rightarrow 0$. Then, using that $B(x, r_1) \subset A_1$ for any $x \in \overline{B^*} \setminus \overline{U}$ and

$$|\nabla u_k(x)| \leq \tilde{C} \frac{(1 - u_k(x))}{r_1}$$

as before, it follows that $|\nabla u_k(x)| \rightarrow 0$ as $k \rightarrow +\infty$ for $x \in \overline{B^*} \setminus \overline{U}$. Therefore, from the bounded convergence theorem we have that left-hand side of (36) converges to zero, contradicting $\text{Cap}_p(U) > 0$. \square

The following result is a consequence of Corollary 1.2.

Lemma 3.2. *Let M be a complete noncompact Riemannian manifold, $p > 1$ and $B_0 = B(o, R_0)$ some open ball in M . Then $\text{Cap}_p(B_0) = 0$ if and only if there exist two sequences R_1^k and R_2^k such that $R_2^k > R_1^k \rightarrow +\infty$ and $\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$.*

Proof. Suppose that $\text{Cap}_p(B_0) = 0$. Let (R_1^k) be an increasing sequence such that $R_1^k \rightarrow +\infty$. Then, from Corollary 1.2, $\text{Cap}_p(B(o, R_1^k)) = 0$ for any k . Moreover, for a fixed k and any sequence (R_j) satisfying $R_j \rightarrow +\infty$, (3) implies that

$$0 = \text{Cap}_p(B(o, R_1^k)) = \lim_{j \rightarrow +\infty} \text{cap}_p(B(o, R_1^k), B(o, R_j))$$

Hence, there exists some $R_2^k > R_1^k$ such that

$$\text{cap}_p(B(o, R_1^k), B(o, R_2^k)) < 1/k,$$

proving that $\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$. Reciprocally, assume that $\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$, where $R_2^k > R_1^k \rightarrow +\infty$. From (33), for $R_1^k > R_1^1$, we have

$$\text{Cap}_p(B(o, R_1^1)) \leq \text{Cap}_p(B(o, R_1^k)) \leq \text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0.$$

Therefore, Corollary 1.2 implies that $\text{Cap}_p(B_0) = 0$. \square

Combining this lemma with the results of the previous section, we obtain a comparison principle for exterior problems where the operator involved is more general than the p -laplacian operator. Since the comparison principle is an important issue in the PDE theory, the following result that holds for a larger class of operators might be interesting by itself.

Theorem 3.3. *Let M be a complete noncompact Riemannian manifold, A be a function that satisfies (6) - (10) and $p > 1$. If M is p -parabolic, then the comparison principle holds for the exterior problem (5).*

Proof. Let $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be bounded weak solutions of (5) such that $u \leq v$ on ∂K , where $K \subset M$ is a compact set. Consider a ball $B_0 = B(o, R_0)$ such that $K \subset\subset B_0$. Then $\text{Cap}_p(B_0) = 0$, since M is p -parabolic. Therefore, Lemma 3.2 implies that there exist two sequences R_1^k and R_2^k such that $R_2^k > R_1^k \rightarrow +\infty$ and $\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$. Hence, using that u and v are bounded, we have that

$$\max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^p \text{cap}_p(R_1^k, R_2^k) \quad \text{and} \quad \max_{\mathcal{A}_{R_1^k, R_2^k}} |v|^p \text{cap}_p(R_1^k, R_2^k)$$

are bounded. Then, from Theorem 2.4 we conclude that $u \leq v$ in $M \setminus K$. \square

Now we obtain the equivalence between (a) and (b) of Theorem 1.3 in the following result:

Theorem 3.4. *Let M be a complete noncompact Riemannian manifold and $p > 1$. The following are equivalents*

- (i) M is p -parabolic;
- (ii) the comparison principle holds for the exterior problem

$$\Delta_p v = 0 \quad \text{in} \quad M \setminus K, \tag{38}$$

for every compact set K of M ;

- (iii) the comparison principle holds for the exterior problem (38) for some compact set $K_0 = \overline{U_0}$, where $U_0 \neq \emptyset$ is some bounded open set with C^2 boundary.

Proof. (iii) \Rightarrow (i) : Suppose that M is not p -parabolic. Then, $\text{Cap}_p(E) > 0$ for some compact set E . Since, we are assuming (iii), the comparison principle holds for some compact $K_0 = \overline{U_0}$, where $U_0 \neq \emptyset$ is open. Then, Corollary 1.2 implies that $\text{Cap}_p(K_0) > 0$. Remind also that ∂U_0 is C^2 .

Therefore, from Lemma 3.1, there exists a p -harmonic function w such that $w = 1$ on ∂K_0 and $0 < w < 1$ in $M \setminus K_0$. Let u be defined by $u = 1$ in M . Thus u and w are bounded p -harmonic functions, $u \leq w$ on ∂K_0 , but $w < 1 = u$ in $M \setminus K_0$. That is, the comparison principle does not hold for (38) for $K_0 = \overline{U}_0$, contradicting the hypothesis. Hence M is p -parabolic.

(i) \Rightarrow (ii) : Assuming that M is p -parabolic, the comparison principle for the exterior domains with the p -laplacian operator is a consequence of Theorem 3.3.

(ii) \Rightarrow (iii) : Trivial. □

From the previous results we obtain the following property about the p -capacity, that corresponds the equivalence between (a) and (c) of Theorem 1.3:

Corollary 3.5. *Let M be a complete noncompact Riemannian manifold and $p > 1$. There exist two sequences R_1^k and R_2^k such that $R_2^k > R_1^k \rightarrow +\infty$ and*

$$\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$$

if and only if there exist two sequences \tilde{R}_1^k and \tilde{R}_2^k such that $\tilde{R}_2^k > \tilde{R}_1^k \rightarrow +\infty$ and

$$\sup_k \text{cap}_p(\tilde{R}_1^k, \tilde{R}_2^k, o) < +\infty.$$

Moreover, M is p -parabolic if and only if some of these two conditions holds.

Proof. If M is p -parabolic, then Lemma 3.2 guarantees that there exist two sequences R_1^k and R_2^k such that $R_2^k > R_1^k \rightarrow +\infty$ and $\text{cap}_p(R_1^k, R_2^k, o) \rightarrow 0$. Therefore, it is trivial that $\sup_k \text{cap}_p(R_1^k, R_2^k, o) < +\infty$.

Hence, if M is p -parabolic then the first condition holds which implies the second one.

Now suppose that

$$\sup_k \text{cap}_p(\tilde{R}_1^k, \tilde{R}_2^k, o) < +\infty.$$

Hence, from Corollary 2.5, the comparison principle holds for the exterior problem (38). Then, Theorem 3.4 implies that M is p -parabolic, concluding the proof. □

4 Comparison Principle under volume growth conditions

In [10], Holopainen proves that M is p -parabolic assuming the following condition on the volume growth of geodesic balls:

$$\int^{+\infty} \left(\frac{R}{V(R)} \right)^{\frac{1}{p-1}} dR = +\infty \quad \text{or} \quad \int^{+\infty} \left(\frac{1}{V'(R)} \right)^{\frac{1}{p-1}} dR = +\infty, \quad (39)$$

for $p > 1$, where $V(R) = \text{Vol}(B(o, R))$. Therefore, according to Theorem 1.3, the p -comparison principle for exterior domains in M holds if those conditions are satisfied.

In this section we follow a different kind of assumptions, using Theorem 2.4 to obtain comparison principles. One advantage is that we can prove results even for solutions that are not bounded a priori and for manifolds that are not p -parabolic, provided there is some relation between the growth of the solution and the volume of the geodesic balls.

We have to observe the following:

Remark 4.1. *For $R_2 > R_1 > 0$ and $o \in M$, there exists a function*

$$w_0 \in \mathcal{F}_{B(o, R_1), B(o, R_2)}$$

such that $|\nabla w_0(x)| \leq 2/(R_2 - R_1)$ for any $x \in \mathcal{A}_{R_1, R_2} = B(o, R_2) \setminus \overline{B(o, R_1)}$. For instance, take w_0 as a mollification of

$$w_1(x) = \begin{cases} 1 & \text{if } \text{dist}(x, o) \leq R_1 \\ \frac{R_2 - \text{dist}(x, o)}{R_2 - R_1} & \text{if } R_1 < \text{dist}(x, o) < R_2 \\ 0 & \text{if } \text{dist}(x, o) \geq R_2. \end{cases}$$

Then

$$\text{cap}_p(R_1, R_2, o) \leq \int_{\mathcal{A}_{R_1, R_2}} |\nabla w_0|^p dx \leq \left(\frac{2}{R_2 - R_1} \right)^p \text{Vol}(\mathcal{A}_{R_1, R_2}). \quad (40)$$

As a consequence of Remark 4.1 and Theorem 2.4, we have the following result that can be applied for possibly unbounded solutions or non p -parabolic manifolds:

Theorem 4.2. Let $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be weak solutions of (5) in $M \setminus K$, where A satisfies (6) - (10) and $p > 1$. Assume also that

$$\max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^p \frac{\text{Vol}(\mathcal{A}_{R_1^k, R_2^k})}{(R_2^k - R_1^k)^p} \quad \text{and} \quad \max_{\mathcal{A}_{R_1^k, R_2^k}} |v|^p \frac{\text{Vol}(\mathcal{A}_{R_1^k, R_2^k})}{(R_2^k - R_1^k)^p} \quad (41)$$

are bounded sequences for some (R_1^k) and (R_2^k) such that $R_2^k > R_1^k \rightarrow \infty$. If $u \leq v$ on ∂K , then

$$u \leq v \quad \text{in} \quad M \setminus K.$$

Corollary 4.3. Assume the same hypotheses as in the previous theorem. Then the solutions u and v are bounded.

Proof. First Case: The sequence of quotients

$$Q_k := \frac{\text{Vol}(\mathcal{A}_{R_1^k, R_2^k})}{(R_2^k - R_1^k)^p} \quad (42)$$

is also bounded for the same (R_1^k) and (R_2^k) given by Theorem 4.2. Observe that the constant function $w = \max_{\partial K} v$ is a solution (5). Since the sequence in (42) is bounded, then the sequence of (41) with u replaced by w is also bounded. Then we can apply the previous theorem to w and v . Hence $v \leq w = \max_{\partial K} v < +\infty$. Similarly, v is bounded from below by $\min_{\partial K} v$. The argument for u is the same.

Second case: The sequence of ratios Q_k is not bounded.

Then there exists some subsequence Q_{k_j} such that $Q_{k_j} \rightarrow +\infty$. Hence, from the boundedness of the sequences given in (41), we have

$$\max_{\mathcal{A}_{R_1^{k_j}, R_2^{k_j}}} |u|^p \rightarrow 0.$$

In particular, this sequence is bounded by some constant $C > 0$. Therefore, $|u| \leq C^{1/p}$ on the spheres $\partial B(o, R_1^{k_j})$ for any j . Since u satisfies the maximum principle, we get

$$|u| \leq D := \max\{C^{1/p}, \max_{\partial K} |u|\} < +\infty \quad \text{in} \quad B(o, R_1^{k_j}) \setminus K \quad \text{for any } j.$$

Using that $R_1^{k_j} \rightarrow +\infty$, we conclude that $|u| \leq D$ in $M \setminus K$. The same holds for v . \square

Corollary 4.4. *Assume the same hypotheses as in Theorem 4.2. If $u = v$ on ∂K , then $u = v$ in $M \setminus K$. Moreover u is bounded.*

Observe that condition (41) holds, for example, if there exist $C > 0$, $q > 0$ and some sequence $R_2^k \rightarrow +\infty$ such that

$$\max_{B(o, R_2^k)} |u| \leq C(R_2^k)^{(p-q)/p} \quad , \quad \max_{B(o, R_2^k)} |v| \leq C(R_2^k)^{(p-q)/p}$$

and

$$\text{Vol}(B(o, R_2^k)) \leq C(R_2^k)^q.$$

In particular, condition (41) is satisfied if we assume that

$$\max_{B(o, R)} |u|, \max_{B(o, R)} |v| \leq CR^{(p-q)/p} \quad \text{and} \quad \text{Vol}(B(o, R)) \leq CR^q, \quad (43)$$

for any $R \geq R_0$, where $R_0 > 0$ is any fixed positive. Hence we have the following result:

Corollary 4.5. *Let $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be weak solutions of (5) in $M \setminus K$, where A satisfies (6) - (10) and $p > 1$. If (43) holds and $u \leq v$ on ∂K , then $u \leq v$ in $M \setminus K$. Moreover u and v are bounded.*

From this corollary, for $p > q$, we do not need to assume that the solutions are bounded to have a comparison principle. It is sufficient that they satisfy (43). Anyway we conclude that they are bounded from Corollary 4.3. On the other hand, if $p < q$, we cannot guarantee that M is p -parabolic, since the quotient given by (42) may diverge to infinity. Still, we have some comparison principle provided we assume that u and v goes to zero at infinity with some speed.

Remark 4.6. *If the sequence of Q_k , defined in (42), is bounded, then the sequence of capacities $\text{cap}_p(R_1^k, R_2^k, o)$ is also bounded, according to (40). Then, from (c) of Theorem 1.3, we conclude that M is p -parabolic. Therefore, from Theorem 3.3, it holds the comparison principle for bounded solutions.*

This implies the uniqueness of bounded solution for a Dirichlet problem in exterior domains. Moreover, from the fact that $v \equiv \text{const.}$ is a bounded solution of (5), we have the following extension of Liouville's result to exterior domains:

Corollary 4.7. *Assume the same hypotheses as in Theorem 4.2 about M , K , p and A . Suppose that the quotient Q_k of (42) is bounded (or simply, M is p -parabolic). If u is a bounded weak solution of (5) and u is constant on ∂K , then u is constant. More generally, given a continuous function ϕ on ∂K , there exists at most one bounded solution of (5) such that $u = \phi$ on ∂K .*

5 Comparison principle for rotationally symmetric manifolds

According to Milnor's lemma in [14], a complete rotationally symmetric 2-dimensional Riemannian manifold with the metric $ds^2 = dr^2 + f^2(r)d\theta$ is parabolic if and only if

$$\int_a^\infty \frac{1}{f(r)} dr = +\infty$$

for some $a > 0$. For higher dimension and $p > 1$, a complete rotationally symmetric n -dimensional Riemannian manifold $M = \mathbb{R}^+ \times S^{n-1}$ with respect to a point $o \in M$ endowed with the metric

$$ds^2 = dr^2 + f^2(r)d\omega^2,$$

where $r = \text{dist}(x, o)$, $d\omega^2$ is the standard metric of S^{n-1} and f is a C^1 positive function in $(0, +\infty)$, is p -parabolic if and only if

$$\int_a^\infty f^{-\frac{n-1}{p-1}}(r) dr = +\infty \quad \text{for some } a > 0. \quad (44)$$

Indeed, that the divergence of this integral implies the p -parabolicity is a consequence of the second integral condition in (39) (or Ilkka's condition) and the fact that the volume of the ball $B(o, r)$ is $V(r) = n\omega_n \int_0^r (f(s))^{n-1} ds$, where ω_n is the volume of the unit ball in \mathbb{R}^n . The converse is true since the convergence of this integral implies the existence of a nonconstant bounded p -superharmonic function: for instance, consider the function $\eta_{a,+\infty}$ defined by $\eta_{a,+\infty}(x) = 1$ if $\text{dist}(x, o) \leq a$ and

$$\eta_{a,+\infty}(x) = \frac{\int_{\text{dist}(x,o)}^{+\infty} f^{\frac{1-n}{p-1}}(s) ds}{\int_a^{+\infty} f^{\frac{1-n}{p-1}}(s) ds} \quad \text{if } \text{dist}(x, o) > a,$$

that is p -harmonic outside the ball $B(o, a)$ as we will see later and, therefore, p -superharmonic in M . Hence, from Theorems 3.4 and 3.3, we conclude respectively the following results:

Corollary 5.1. *Let $M = \mathbb{R}^+ \times S^{n-1}$ be a complete rotationally symmetric n -dimensional Riemannian manifold with respect to a point $o \in M$ endowed with the metric*

$$ds^2 = dr^2 + f^2(r)d\omega^2.$$

Then the comparison principle holds for the exterior problem $\Delta_p v = 0$ in $M \setminus K$, where K is any compact set, if and only if condition (44) holds.

Corollary 5.2. *Let (M, ds^2) be as in the previous corollary and assume A satisfies (6) - (10) for $p > 1$. Suppose also that $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ are bounded weak solutions of (5) in $M \setminus K$, where K is a compact set of M , and that condition (44) is satisfied. If $u \leq v$ on ∂K , then*

$$u \leq v \quad \text{in } M \setminus K.$$

In particular, if u is constant on ∂K , then u is constant in $M \setminus K$.

Now we show some comparison result that holds for non bounded solutions or hyperbolic manifolds. Taking the point o as the reference, let $\eta = \eta_{R_1, R_2}$ be the function defined in Lemma 2.2 for $R_2 > R_1 > 0$. Since η is p -harmonic in \mathcal{A}_{R_1, R_2} , it is radially symmetric due to the symmetry of M with respect to o . Then η is a function of r in \mathcal{A}_{R_1, R_2} and satisfies the equation

$$\frac{\Delta_p \eta}{|\nabla \eta|^{p-2}} = (p-1)\eta''(r) + (n-1)\frac{f'(r)}{f(r)}\eta'(r) = 0 \quad \text{for } r \in (R_1, R_2),$$

where the prime denotes the derivative with respect to r . Moreover, $\eta(R_1) = 1$ and $\eta(R_2) = 0$. Hence

$$\eta(r) = \frac{\int_r^{R_2} f^{\frac{1-n}{p-1}}(s) ds}{\int_{R_1}^{R_2} f^{\frac{1-n}{p-1}}(s) ds}.$$

Then, using that the element of volume is $dx = n\omega_n f^{n-1}(r)dr$, we have

$$\int_{\mathcal{A}_{R_1, R_2}} |\nabla \eta|^p dx = n\omega_n \int_{R_1}^{R_2} (\eta')^p f^{n-1}(r) dr = \frac{n\omega_n}{\left(\int_{R_1}^{R_2} f^{\frac{1-n}{p-1}}(s) ds \right)^{p-1}}.$$

Observe that the capacity $\text{cap}_p(R_1, R_2) := \text{cap}_p(B(o, R_1); B(o, R_2))$ is attained at η_{R_1, R_2} , since it is harmonic and, therefore, minimizes the Dirichlet integral over \mathcal{A}_{R_1, R_2} in the set

$$\{v \in C^0(\overline{\mathcal{A}_{R_1, R_2}}) \cap C^1(\mathcal{A}_{R_1, R_2}) : v = 1 \text{ on } \partial B(o, R_1) \text{ and } v = 0 \text{ on } \partial B(o, R_2)\}.$$

Therefore, from the last equation,

$$\left(\max_{\mathcal{A}_{R_1, R_2}} |u| \right)^p \text{cap}_p(R_1, R_2) = n\omega_n \left(\max_{\mathcal{A}_{R_1, R_2}} |u| \right)^p \left(\int_{R_1}^{R_2} f^{-\frac{n-1}{p-1}}(s) ds \right)^{-(p-1)}$$

From this expression, Theorem 2.4 and Remark 2.6, we have the following:

Corollary 5.3. *Assume the same hypotheses as in the previous corollary about M , K and A . Let $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be weak solutions of (5) in $M \setminus K$. Suppose also that*

$$\frac{\max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^p}{\left(\int_{R_1^k}^{R_2^k} f^{-\frac{n-1}{p-1}}(s) ds \right)^{p-1}} \quad \text{and} \quad \frac{\max_{\mathcal{A}_{R_1^k, R_2^k}} |v|^p}{\left(\int_{R_1^k}^{R_2^k} f^{-\frac{n-1}{p-1}}(s) ds \right)^{p-1}} \quad \text{are bounded,} \quad (45)$$

where R_1^k and R_2^k are sequences such that $R_2^k > R_1^k \rightarrow +\infty$. For $p = 2$, this condition can be replaced by

$$\frac{\max_{\mathcal{A}_{R_1^k, R_2^k}} |v - u|^2}{\int_{R_1^k}^{R_2^k} f^{1-n}(s) ds} \rightarrow 0. \quad (46)$$

If $u \leq v$ on ∂K , then

$$u \leq v \quad \text{in} \quad M \setminus K.$$

As an application we present a result where the functions u and v can go to infinity, provided that their growth are bounded by some specific function.

Corollary 5.4. *Let (M, d, s^2) be as in the previous corollary, where $0 < f(r) \leq E_1 r$ for some $E_1 > 0$ and any large r . Suppose that A satisfies (6)*

- (10) with $p = n$. Let $K \subset M$ be a compact set and $u, v \in C(\overline{M \setminus K}) \cap C^1(M \setminus K)$ be weak solutions of (5) in $M \setminus K$ such that

$$|u(r)| \leq E_2 (\ln r)^{\frac{n-1}{n}} \quad \text{and} \quad |v(r)| \leq E_2 (\ln r)^{\frac{n-1}{n}} \quad (47)$$

for some $E_2 > 0$ and any large r . If $u \leq v$ on ∂K , then

$$u \leq v \quad \text{in} \quad M \setminus K.$$

Proof. By hypothesis, there exists $R_0 > 0$ such that $f(r) \leq E_1 r$ for $r \geq R_0$. Then,

$$\int_{R_1^k}^{R_2^k} \frac{1}{f(s)} ds \geq \int_{R_1^k}^{R_2^k} \frac{1}{E_1 s} ds = \frac{1}{E_1} \ln \left(\frac{R_2^k}{R_1^k} \right) \quad \text{for} \quad R_2^k > R_1^k > R_0.$$

Let (R_k) be a sequence that goes to infinity. Take

$$R_1^k = R_k \quad \text{and} \quad R_2^k = (R_1^k)^2 = (R_k)^2.$$

Hence, using the last inequality, we have

$$\int_{R_1^k}^{R_2^k} \frac{1}{f(s)} ds \geq \frac{1}{E_1} \ln R_k.$$

Therefore, using the growth hypothesis about u and v , we have

$$\frac{\max_{\mathcal{A}_{R_1^k, R_2^k}} |u|^n}{\left(\int_{R_1^k}^{R_2^k} (f(s))^{-1} ds \right)^{n-1}} \leq \frac{E_1^{n-1} E_2^n (\ln R_2^k)^{n-1}}{(\ln R_k)^{n-1}} = (2E_1)^{n-1} E_2^n$$

and a similar estimate for v . Then, from Corollary 5.3, we conclude the result. \square

Remark 5.5. *This corollary holds, for instance, for any complete noncompact rotationally symmetric Riemannian manifold such that the sectional curvature is nonnegative, since $f(r) \leq r$ in this case.*

It can be applied also for some Hadamard manifolds provided the curvature goes to zero sufficiently fast.

References

- [1] L.P. Bonorino., A.R. Silva, and P.R.A. Zingano, *Liouville's theorem and comparison results for solutions of degenerate elliptic equations in exterior domains*, J. Math. Anal. Appl, vol. **463**, no. 2 (2018), 794-809.
- [2] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. Henri Poincaré, vol. **15**, no. 4 (1998), 493-516.
- [3] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Graduate studies in Mathematics **19**, 1998.
- [4] A. Esteve and V. Palmer, *On the characterization of parabolicity in Real SpaceForms*, Preprint 2009
- [5] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999) 135-249.
- [6] A. Grigor'yan, *Isoperimetric inequalities and capacities on Riemannian manifolds*. In: Rossmann J., Takác P., Wildenhain G. (eds) *The Maz'ya Anniversary Collection. Operator Theory: Advances and Applications*, vol 109. Birkhäuser (1999), 139-153.
- [7] I. Holopainen, *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds*, Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. **74** (1990), 1-45.
- [8] I. Holopainen, *Positive solutions of quasilinear elliptic equations on Riemannian manifolds*, Proc. London Math. Soc. (3) **65** (1992), 651-672.
- [9] I. Holopainen and S. Rickman, *Classification of Riemannian manifolds in nonlinear potential theory*, Potential Anal. **2** (1993), 37-66.
- [10] I. Holopainen, *Quasiregular mappings and the p -Laplace operator*, Contemp. Math. **338** (2003), 219-239.
- [11] I. Holopainen, S. Markvorsen, and V. Palmer, *p -Capacity and p -Hyperbolicity of Submanifolds*, Rev. Mat. Iberoamericana **25** (2009), no. 2, 709-738.

- [12] A. Hurtado and V. Pamer: A Note on the p -Parabolicity of Submanifolds, *Potential Analysis* 34(2), 2009.
- [13] P. Li and L.F. Tam, Green's functions, harmonic functions, and volume comparison, *J. Differential Geom.* 41 (1995), 277-318. MR 96f:53054
- [14] J. Milnor, *On deciding whether a surface is parabolic or hyperbolic*, *The American Mathematical Monthly*, vol. **84**, no. 1 (1977), 43-46.
- [15] P. Pucci and J. Serrin, The strong maximum principle revisited, *J. Differential Equations* 196 (2004) 1–66
- [16] J. B. Ripoll and F. Tomi, Notes on the Dirichlet problem of a class of second order elliptic partial differential equations on a Riemannian manifold. *Ensaos Matemáticos*, vol. 32, pp. 1–64. Sociedade Brasileira de Matemática (2018).
- [17] C.-J. Sung, A note on the existence of positive Green's function, *J. Funct. Anal.* 156 (1998), 199–207
- [18] M. Troyanov, Parabolicity of Manifolds, *Siberian Advances in Mathematics* 9(1999) 125–150
- [19] X. Wang and L. Zhang, *Local gradient estimate for p -harmonic functions on Riemannian manifolds*, *communications in analysis and geometry*, vol. 19, no. 4 (2011), 759-771.

Ari Aiolfi
Universidade Federal de Santa Maria
Brazil
ari.aiolfi@ufsm.br

Leonardo Bonorino
Universidade Federal do Rio Grande do Sul
Brazil
leonardo.bonorino@ufrgs.br

Jaime Ripoll
Universidade Federal do Rio Grande do Sul
Brazil
jaime.ripoll@ufrgs.br

Marc Soret
Université de Tours
France
marc.soret@idpoisson.fr

Marina Ville
Univ Paris Est Creteil, CNRS, LAMA, F-94010 Creteil
France
villemarina@yahoo.fr