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BIHARMONIC HYPERSURFACES IN EUCLIDEAN SPACES

HIBA BIBI, MARC SORET AND MARINA VILLE

Abstract

An isometric immersion $X : \Sigma^n \rightarrow \mathbb{E}^{n+1}$ is biharmonic if $\Delta^2 X = 0$, i.e. if $\Delta H = 0$, where Δ and H are the metric Laplacian and the mean curvature vector field of Σ^n respectively. More generally, biconservative hypersurfaces (BCH) are isometric immersions for which only the tangential part of the biharmonic equation vanishes. We study and construct BCH that are holonomic, i.e. for which the principal curvature directions define an integrable net, and we deduce that Σ^n is a holonomic biharmonic hypersurface if and only if it is minimal.

1 Introduction

An isometric immersion in a Euclidean space, $X : \Sigma^n \rightarrow \mathbb{E}^N$, is biharmonic if $\Delta^2 X = 0$, i.e. if $\Delta H = 0$, where Δ and H are the metric Laplacian and the mean curvature vector field of Σ^n respectively.

The immersion X is a critical point of a "bienergy" functional (see [14]). The image of such an immersion is also called a biharmonic submanifold. B.Y. Chen conjectured in [6] that a biharmonic submanifold of the Euclidean space must be minimal. In this direction, a long range argument in [1] proves B.Y. Chen's conjecture for complete, topologically proper, biharmonic submanifolds. The proof uses the maximum principle at infinity on a complete submanifold without singularities applied to the square of the mean curvature function $|H|^2$. We will be more interested in the case of hypersurfaces and we will study small range properties of biharmonic hypersurfaces (BHH), as well as properties of biconservative hypersurfaces (BCH). The BCH hypersurfaces are less restrictive and are defined as critical points of the bienergy functional with respect to tangential deformation, i.e. for which $(\Delta H)^\top = 0$.

Minimal hypersurfaces are trivially BHH, thus we are interested in studying BHH which are not minimal, called proper BHH. Many works were carried out proving the non-existence of proper BHH in Euclidean spaces, hence confirming the B.Y. Chen's conjecture. In low dimensions, the B.Y. Chen's conjecture was proved in \mathbb{E}^3 ([7] and [13]), in \mathbb{E}^4 [12], in \mathbb{E}^5 [11], and recently in \mathbb{E}^6 [10].

Constant mean curvature (CMC) hypersurfaces in a Euclidean space are trivially biconservative. Thus, we look for BCH which are not CMC, called proper BCH. An exhaustive description of all proper BCH in \mathbb{E}^3 was given in [2], and then generalized to the existence of a family of $SO(p)$ -symmetric BCH of \mathbb{E}^p [17]. Also in [17] the authors constructed proper BCH which are $SO(p) \times SO(q)$ -symmetric in \mathbb{E}^{p+q} , and they made a global study on these hypersurfaces.

This paper studies the short range behavior of BCH and BHH that are holonomic, i.e. if the principal curvature directions define an integrable net, i.e. in other words if the principal curvature lines define on an open subdomain a coordinate system (cf. for example [8] and [19]).

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More precisely, we focus on a new construction of local proper BCH. All the known examples so far are locally fibered by spheres or products of spheres, which are isoparametric submanifolds. We also construct proper BCH which are fibered over any given isoparametric submanifold of codimension 2 in \mathbb{E}^{n+1} . We recall that, according to the definition in [20], a submanifold is isoparametric if its normal bundle is flat, and if the principal curvatures in the direction of any parallel normal vector field is constant (see for example [20]).

In this article we prove the following main results:

Theorem 1. *A holonomic proper BCH Σ^n in \mathbb{E}^{n+1} is foliated by the level sets of the mean curvature function h of Σ^n which are isoparametric codimension 2 submanifolds of \mathbb{E}^{n+1} in a neighbourhood of a regular point of h . The symmetries of the leaves extend to symmetries of Σ^n . The integral curves of the gradient field ∇h on Σ^n are congruent planar curves governed by an ODE of order two. Conversely, for any codimension 2 isoparametric submanifold $U_0 \subset \mathbb{E}^{n+1}$ which is holonomic, there locally exists a proper BCH obtained by a local normal evolution of U_0 in the normal bundle of U_0 that preserves the fibers of the normal bundle of U_0 and the symmetry group of U_0 .*

We deduce from Theorem 1

Corollary 1. *Holonomic proper BCH of \mathbb{E}^{n+1} are either foliated by round spheres S^{n-1} , by products of round spheres $S^p \times S^q$, by cylinders $S^p \times \mathbb{R}^q$, where $p + q = n - 1$, or by $S^p \times S^q \times \mathbb{R}^r$, where $p + q + r = n - 1$.*

Theorem 2. *Holonomic BHH of \mathbb{E}^{n+1} are minimal.*

2 Generalities on m -harmonic maps

2.1 Notations

Since our results are of local nature, we can identify isometrically Σ^n with a local open subset Ω of \mathbb{R}^n . Let $X : \Omega \subset \mathbb{R}^n \rightarrow \Sigma^n \subset \mathbb{E}^{n+1}$ be a smooth immersion where the image Σ^n is an open hypersurface of \mathbb{E}^{n+1} and Ω is provided with the pull-back metric.

Let Δ_0 denote the flat Laplacian on Ω : $\Delta_0 = \sum_{i=1}^n \partial_{x_i}^2$.

By definition a function f defined on Ω is harmonic if $\Delta_0 f = 0$, and it is m -harmonic if $\Delta_0^m f = 0$, where

$$\Delta_0^m := \overbrace{\Delta_0 \circ \Delta_0 \circ \dots \circ \Delta_0}^m.$$

More generally, a map X is harmonic if $\Delta_0 X = 0$, i.e. $\Delta_0 X^i = 0$, for all $i = 1, \dots, n + 1$. Thus, X is harmonic if and only if all the coordinate functions that define X are harmonic. By extension, X is m -harmonic if and only if each coordinate function of the position vector X is m -harmonic.

Example 1. *When Σ^n is a graph of a function u over a hyperplane \mathbb{R}^n in \mathbb{R}^{n+1} given by the parametrization*

$$X : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, u),$$

$\Delta_0 X = 0$ if and only if the following $n + 1$ linear PDEs are satisfied

$$\Delta_0 x_1 = 0, \dots, \Delta_0 x_n = 0 \quad \text{and} \quad \Delta_0 u = 0.$$

The first n -equations are always satisfied hence X is harmonic if and only if u is a harmonic function. These remarks extend to the m -harmonic case: X is m -harmonic if and only if $\Delta_0^m u = 0$.

Suppose now that $X = (X^1, \dots, X^{n+1})$ is an isometric immersion. Let $g = g_{ij}$ be the induced metric on Ω and $|g|$ its determinant. The Laplacian of X for this metric, which is denoted by Δ , is equal to the vector-valued function

$$(\Delta X)^i = \Delta X^i = \text{Tr}(\text{Hess}(X^i)) \quad \forall i = 1, \dots, n+1.$$

By definition X is then an m -harmonic isometric immersion if and only if $\Delta^m X = 0$. Equivalently, we say that $\Sigma^n := X(\Omega)$ is an m -harmonic hypersurface of \mathbb{R}^{n+1} .

Let us recall that the Laplacian in terms of the metric "g" of a smooth function $f : \Omega \rightarrow \mathbb{R}$, using Einstein's summation, is given by

$$\Delta f := \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{ij} f_{,i} \right)_{,j} = 0, \quad i, j = 1, \dots, n, \quad (1)$$

where $f_{,i}$ denotes the derivative of f with respect to each coordinate x_i , for all $i = 1, \dots, n$.

2.2 Remarks on the equation of a biharmonic graph

Let us find the PDE equation of a biharmonic graph. Before that, let us recall the 1-harmonic graph equation. Consider, as in the former example, the parametrization

$$X : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, u)$$

hence, $X_{,i} := \frac{\partial X}{\partial x_i} = (0, \dots, 0, 1, \dots, 0, u_{,i})$. The induced metric of a graph on the domain of X is then given by a matrix whose coefficients are in terms of the derivatives of u :

$$g_{ij} = \delta_{ij} + u_{,i} u_{,j} \quad \text{and} \quad g^{ij} = \delta_{ij} - \frac{u_{,i} u_{,j}}{W^2},$$

where (g^{ij}) is the inverse matrix of (g_{ij}) and $W := \sqrt{1 + |\nabla u|^2} = \sqrt{|g|}$. The second fundamental form and the shape operator of X are respectively given by the following matrices :

$$II_{ij} = \frac{u_{,ij}}{W} \quad \text{and} \quad A_{ij} = {}^t g^{-1} II = \frac{u_{,ij}}{W} - \sum_{\alpha=1}^n \frac{u_{,i\alpha} u_{,\alpha} u_{,j}}{W^3},$$

where $i, j = 1, \dots, n$.

For example, in dimension 2, the metric and the second fundamental form are

$$g = \begin{pmatrix} 1 + u_{,x}^2 & u_{,x} u_{,y} \\ u_{,x} u_{,y} & 1 + u_{,y}^2 \end{pmatrix} \quad \text{and} \quad II = \frac{1}{W} \begin{pmatrix} u_{,xx} & u_{,xy} \\ u_{,xy} & u_{,yy} \end{pmatrix}. \quad (2)$$

The mean curvature vector field of the graph

$$H = \frac{\Delta X}{n} = \frac{1}{n} (\Delta x_1, \dots, \Delta x_n, \Delta u)$$

decomposes into a horizontal part: $nH_{hor} = (\Delta x_1, \dots, \Delta x_n, 0)$ and a vertical part: $nH_{ver} = (0, \dots, 0, \Delta u)$. A harmonic graph satisfies $H_{hor} = 0$ and $H_{ver} = 0$. Using equation (1), for $H_{hor} = 0$ and $H_{ver} = 0$, we obtain, using Einstein's summation, the following order two quasi-linear PDEs:

$$\begin{cases} nH_{hor}^l = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,k} = 0, & l = 1, \dots, n \\ nH_{ver}^{n+1} = \Delta u = g^{kl} u_{,kl} = 0. \end{cases} \quad (3)$$

If we substitute the value of g^{ij} in terms of u in the second equation we obtain the so-called minimal graph equation which can also be expressed as :

$$\left(\frac{u_{,i}}{W}\right)_{,i} = 0 = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

For example, in dimension 2

$$(1 + u_{,y}^2)u_{,xx} - 2u_{,x}u_{,y}u_{,xy} + (1 + u_{,x}^2)u_{,yy} = 0.$$

In other terms a minimal surface is the image of an isometric immersion which is harmonic. We similarly proceed for biharmonic graphs. First, we deduce from (1) that the bilaplacian applied to a function f is:

$$\begin{aligned} \Delta^2 f &= g^{ij} \left(g^{kl} f_{,kl} + \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,k} f_{,l} \right)_{,ij} + \\ &\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{ij} \right)_{,i} \left(g^{kl} f_{,kl} + \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,k} f_{,l} \right)_{,j}. \end{aligned} \quad (4)$$

The biharmonic equation $\Delta^2 X = \Delta H = 0$ decomposes into a horizontal part: $n(\Delta H)_{hor} = (\Delta^2 x_1, \dots, \Delta^2 x_n, 0)$ and a vertical part: $n(\Delta H)_{ver} = (0, \dots, 0, \Delta^2 u)$. A straightforward computation yields the following fourth order PDEs:

$$n(\Delta H)_{hor}^m = g^{ij} \left(\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{km} \right)_{,k} \right)_{,ij} + \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{ij} \right)_{,i} \left(\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{km} \right)_{,k} \right)_{,j} = 0 \quad (5)$$

for $m = 1, \dots, n$. Using (5) we obtain

$$\begin{aligned} n(\Delta H)_{ver}^{n+1} &= g^{ij} \left((g^{kl} u_{,kl})_{,ij} + \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,k} u_{,lij} \right) + \\ &2g^{ij} \left(\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,ik} u_{,lj} \right) + \\ &\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{ij} \right)_{,i} \left(g^{kl} u_{,klj} + \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{kl} \right)_{,k} u_{,jl} \right) = 0, \end{aligned} \quad (6)$$

which is a semi-linear PDE of order 4 (its highest order term being $g^{ij} g^{kl} u_{,klj}$).

3 Biharmonic Hypersurfaces in \mathbb{E}^{n+1}

Let us determine the PDEs that characterize biharmonic hypersurfaces in \mathbb{E}^{n+1} (cf. for example [5] for more general cases).

Let X be any isometric immersion :

$$X : \Omega \subset \mathbb{R}^n \longrightarrow X(\Omega) = \Sigma^n \subset \mathbb{E}^{n+1}.$$

We denote the mean curvature vector field by H , the mean curvature function by h , the unit normal vector field by N , the shape operator by A , and the Laplacian operator by Δ .

It is easy to show that the position vector field X and the mean curvature vector field H , where $H = hN$, are related by the following Beltrami equation

$$\Delta X = nH. \quad (7)$$

One can similarly show that

$$\Delta N = -|A|^2 N - \nabla \text{Tr} A, \quad (8)$$

where $\text{Tr} A = nh$. This equation decomposes the Laplacian of the unit normal vector field with respect to its normal component ($|A|^2 N$) and its tangential component ($\nabla \text{Tr} A$).

X is biharmonic if and only if

$$\Delta H = \Delta(hN) = 0,$$

thus

$$(\Delta h)N + h\Delta N + 2 \sum_{i=1}^n \nabla_{e_i} h \nabla_{e_i} N = (\Delta h)N + h\Delta N + 2\nabla_{\nabla h} N = 0. \quad (9)$$

Now, from equations (7), (8) and (9) we deduce that Σ^n is biharmonic if and only if it satisfies the following two semi-linear PDEs of orders 3 and 4, respectively:

$$\begin{cases} (\Delta H)^\top := 2\nabla_{\nabla h} N - nh\nabla h = 0 \\ (\Delta H)^\perp := \Delta h - |A|^2 h = 0. \end{cases} \quad (10)$$

The solutions of the first equation ($(\Delta H)^\top = 0$) correspond to critical points of the bienergy functional for compact deformations that are tangent to Σ^n .

Terminology 1. A hypersurface satisfying $(\Delta H)^\top = 0$ will be denoted by BCH which stands for "Biconservative Hypersurface", and a hypersurface satisfying both equations in (10) will be denoted by BHH which stands for "Biharmonic Hypersurface".

Terminology 2. If Σ^n has constant mean curvature (CMC), then Σ^n is a BCH. Furthermore, a CMC hypersurface satisfies the normal equation in (10) if and only if $h = 0$. Thus, any minimal hypersurface Σ^n is a BHH. We will later denote by **proper BCH** (resp. **proper BHH**) a BCH which is not CMC [17] (resp. a BHH which is not minimal [16]).

3.1 Biconservative equation in a principal curvature frame

At a point $p \in \Sigma^n$, we have s distinct eigenvalues and s corresponding eigenspaces of the shape operator A_N . This defines s plane distributions T_i , where $T_i = \ker(A_N(p) - \lambda_i \text{Id})$ and λ_i is the corresponding eigenvalue, or principal curvature, of multiplicity $\text{rk}(T_i) = m_i$ such that $T_p \Sigma^n = \bigoplus_{i=1}^s T_i$, where $\sum_{i=1}^s m_i = n$. We will select an open subdomain where the multiplicities m_i , $i = 1, \dots, s$, of the corresponding eigenvalues are constant. We will obtain a splitting of the tangent bundle $T\Sigma^n$ into a family of integrable distributions $T_i \subset T\Sigma^n$.

We now express the biharmonicity in terms of a principal curvature frame.

Proposition 1. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field of principal directions on Σ^n . Then the BCH equation

$$2\nabla_{\nabla h} N - nh\nabla h = 0$$

is equivalent to the following system of algebraic-differential equations:

$$\nabla_{e_i} h = 0 \quad \text{or} \quad nh + 2\lambda_i = 0, \quad i = 1, \dots, n \quad (11)$$

where $\lambda_i, i = 1, \dots, n$ are the principal curvatures.

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field of Σ^n where e_i is the principal direction of principal curvature $\lambda_i, i = 1, \dots, n$. Then

$$2\nabla_{(\nabla_{e_i} h)e_i} N - nh(\nabla_{e_i} h)e_i = -2\nabla_{e_i} h (2\lambda_i + nh) e_i = 0.$$

By continuity and by reducing the domain, the system given in (11) holds, thus we conclude. \square

Definition 1. Let Σ^n be a BCH in \mathbb{E}^{n+1} . Σ^n is of rank k if $rk(T\Sigma^n(-\frac{nh}{2})) = k$, where $T\Sigma^n(-\frac{nh}{2})$ is the eigenspace of the shape operator of eigenvalue $-\frac{nh}{2}$ in $T\Sigma^n$.

∇h is null when projected onto $T\Sigma^n(-\frac{nh}{2})^\perp$. As we will see later, the rank of Σ^n is either 0 or 1.

4 Reducible biconservative and biharmonic hypersurfaces

There is a simple automatic way to construct BCH (or BHH) from BCH (BHH) submanifolds of lower dimensions by simply constructing cylinders with BCH (or BHH) basis.

Proposition 2. If $\Sigma^n \subset \mathbb{E}^{n+1}$ is a BCH (resp. a BHH) of rank k then the Riemann product $\tilde{\Sigma} := \Sigma^n \times \mathbb{E}^l \subset \mathbb{E}^{n+l+1}$ is a BCH (resp. a BHH) of rank k .

Proof. If $h = 0$ then Σ^n is minimal and so is $\Sigma^n \times \mathbb{E}^l$. Suppose that $h \neq 0$, then the Riemann product $\tilde{\Sigma}$ has a mean curvature $\tilde{h} = \frac{n}{n+l}h$ and $rk(T\tilde{\Sigma}(-\frac{(n+l)\tilde{h}}{2})) = rk(T\tilde{\Sigma}(-\frac{nh}{2})) = rk(T\Sigma^n(-\frac{nh}{2})) = k$. Furthermore, ∇h is null on $T\Sigma^n(-\frac{nh}{2})^\perp \oplus \mathbb{E}^l$. Hence, it is biconservative from Proposition 1 and of rank k . If Σ^n is a BHH, then the second equation of the system holds also for $\tilde{\Sigma}$ since $|A|^2 = |\tilde{A}|^2$. \square

BCH (resp. BHH) which are Riemannian products will be called *reducible*. Thus, from now on we will mainly investigate irreducible BCH.

5 Geometry of proper holonomic BCH in \mathbb{E}^{n+1}

BCH are locally solutions of the system of algebraic-differential equations which is given by Proposition 1. In fact, we will show that BCH are either of rank (i.e. CMC) or of rank 1. In the latter case, a BCH is foliated by level hypersurfaces defined by the mean curvature function h . We will then study the extrinsic geometry of these leaves and that of the integral curves which are orthogonal to the leaves.

Notice that Lemmas 1, 2 and 3, are similar to Lemmas 2.2, 2.3 and 2.5 of [12], but our proofs are different and based on the holonomic condition. This condition in turn allows us to prove lemmas 4 and 3 which were not known. Also, the fact that $rk(\lambda_n) = 1$, in Lemma 1, was already proven in [9].

5.1 Foliation of a BCH by the mean curvature function

Lemma 1. A BCH $\Sigma^n \subset \mathbb{E}^{n+1}$ is either any CMC, i.e. of rank 0, or it is a proper BCH of rank 1. In the later case, Σ^n is in a neighbourhood of regular points of h , foliated by leaves, $U_t = h^{-1}(t)$, which are the level hypersurfaces of the mean curvature function h on Σ^n . These U_t are, locally, codimension 2 submanifolds of \mathbb{E}^{n+1} with flat normal bundle. The orthogonal curves to U_t , which are integral curves of the vector field ∇h on Σ^n , are curvature lines of principal curvature $\lambda_n = -\frac{nh}{2}$. Any other principal curvature satisfies $\lambda_i \neq -\frac{nh}{2}$, $i = 1, \dots, n-1$.

Proof. When the rank of the BCH is null, h is trivially constant. Hence, the BCH is CMC and from the normal equation of (4) a corresponding BHH automatically satisfies $h = 0$, i.e., Σ^n is minimal.

We suppose now that the rank k of Σ^n is at least two. Σ^n is, in a neighbourhood of regular points of h , foliated by hypersurfaces of Σ^n which are codimension 2 submanifolds of \mathbb{E}^{n+1} , and which are given by

$$U_t := \{p \in \Sigma^n : h(p) = t\}.$$

From Proposition 1, $T\Sigma^n(-\frac{nh}{2})^\perp \subset TU_t \subset T\Sigma^n$, hence $TU_t^\perp \subset T\Sigma^n(-\frac{nh}{2}) \subset T\Sigma^n$. Thus, the normal curves to the foliation on Σ^n are integral curves of

$$e_n := \frac{\nabla h}{|\nabla h|}$$

and are curvature lines of the principal curvature $\lambda_n = -\frac{nh}{2}$.

When the rank $k \geq 2$, the distribution defined by the k -planes, $T\Sigma^n(-\frac{nh}{2})$, is integrable into a submanifold of dimension k that has constant principal curvature $\lambda_n = -\frac{nh}{2}$ in the normal direction N (see Appendix 9). In particular λ_n is constant along the ∇h integral curves of ∇h . and so is h ; but h is also constant on the U_t , hence h is constant everywhere which contradicts the hypothesis. Thus, we conclude that $rk(T\Sigma^n(-\frac{nh}{2})) = 1$.

Let us show that the normal bundle $NU_t \subset \mathbb{E}^{n+1}$ is flat. Along the principal directions $\{e_i\}_{i=1}^{n-1}$, in TU_t , we have

$$\langle D_{e_i}N, e_n \rangle = -\langle \lambda_i e_i, e_n \rangle = 0 \quad \text{and} \quad \langle D_{e_i}e_n, N \rangle = -\langle e_n, \nabla_{e_i}N \rangle = 0,$$

where D is the connection in \mathbb{E}^{n+1} . Hence, the normal bundle is flat and its normal curvature is null and using the Ricci equation we get

$$\langle R(X, Y)\xi, \tau \rangle = \langle [A_\xi, A_\tau]X, Y \rangle = 0, \quad \forall X, Y \in TU_t, \quad \forall \xi, \tau \in NU_t.$$

So the shape operators A_N and A_{e_n} commute. Hence A_{e_n} is diagonalised in the same frame $\{e_i\}_{i=1}^{n-1}$, i.e. there exist principal curvatures $\{\mu_i\}_{i=1}^{n-1}$ in the e_n -direction such that

$$\nabla_{e_i}e_n = -\mu_i e_i \tag{12}$$

for all $i = 1, \dots, n-1$. □

5.2 Holonomic hypersurfaces of the Euclidean space

For more information on holonomic coordinates, we refer to [8], and for more details on twisted products we refer to [19] and [15]. At each point $p \in \Sigma^n$, the tangent space $T_p\Sigma^n$ decomposes into an orthogonal sum of the eigenspaces T_i of the symmetric shape operator $A(p)$ of respective distinct eigenvalues λ_i , $i = 1, \dots, s$, i.e. $T_p\Sigma^n = \bigoplus_{i=1}^s T_i$.

Let us consider a neighborhood of a point p where the planes $\{T_i\}_{i=1}^s$ define a smooth distribution. From Codazzi equations (cf. Appendix 8) one shows that the distribution T_i is integrable for each $i = 1, \dots, s$.

Then, again, using Codazzi equations, the leaves of the distribution are totally umbilical. Recall that a submanifold is totally umbilical if the second fundamental form equals $\langle X, Y \rangle H$, where H is *de facto* the mean curvature vector field of the submanifold. When $rk(T_i) \geq 2$, the leaves of the distribution T_i are $rk(T_i)$ -spheres of \mathbb{E}^{n+1} (see for example Appendix 9).

The set of the s orthogonal foliations $\{\mathcal{T}_i\}_{i=1}^s$, obtained by the integration of the T_i -distributions, defines an orthogonal net on Σ^n which is the principal curvature net (cf. [19] and [15]).

Σ^n is *holonomic* if locally Σ^n is diffeomorphic to the product manifold $P := \prod_{i=1}^s \mathcal{T}_i$ and there exists an isometric parametrization of $X : \Omega \subset P \rightarrow \Sigma^n$ and the pullback twisted metric g , which in terms of elements $(x_i)_{i=1, \dots, n} \in P$, can be expressed by the following form

$$g = \sum_{i=1}^s v_i^2(x_1, \dots, x_n) d\sigma_i^2 = \sum_{i=1}^n \tilde{v}_i^2(x_1, \dots, x_n) dx_i^2, \quad (13)$$

where $d\sigma_i^2$ are line metrics or spherical metrics of each \mathcal{T}_i , $i = 1, \dots, s$.

5.3 Integral curves of ∇h

We now assume that the BCHs are holonomic.

Lemma 2. *Let Σ^n be a holonomic proper BCH. Let U_t be a level hypersurface on Σ^n of mean curvature function h . Then Σ^n is locally fibered over U_t by the integral curves of ∇h . These fibers are planar lines in NU_t , i.e. for each $p \in U_t$, $N_p U_t \cap \Sigma^n$ is a curvature line of principal curvature λ_n and a geodesic of Σ^n .*

Proof. Consider γ a perpendicular curve to U_0 , starting at a point $p \in U_0$ and denote the unit tangent vector field along the curve by $T(s)$ which, at $s = 0$, is $e_n := \frac{\nabla h}{|\nabla h|}$. First, the curves γ are geodesics of Σ^n . Indeed, by the Codazzi equations and since λ_n is constant on U_0 , we have (cf. Appendix 8):

$$(\lambda_n - \lambda_i) \Gamma_{ni}^i = (\lambda_n - \lambda_i) (\log v_n)_{,i} = 0,$$

so, as $\lambda_n \neq \lambda_i$, for all $i = 1, \dots, n-1$

$$v_{n,i} = 0, \quad i = 1, \dots, n-1.$$

Hence, the coefficient of dx_n^2 in the expression of the metric g , which is v_n , depends only on x_n . By a possible change of coordinates along the x_n -curvature line the metric $g = \sum_{i=1}^{n-1} v_i^2 d\sigma_i^2 + d\tilde{x}_n^2$. It is then clear that the \tilde{x}_n -curves are geodesics.

Second, geodesics that are curvature lines must be planar:

$$\frac{D}{ds}(T \wedge N) = \frac{D}{ds}T \wedge N + T \wedge \frac{D}{ds}N.$$

As γ is a geodesic of Σ^n , $\frac{D}{ds}T = \kappa N$, and as $\frac{D}{ds}N = -\lambda_n T$, we obtain

$$\frac{D}{ds}(T \wedge N) = 0. \quad (14)$$

Therefore, the integral curves of ∇h are planar and contained in the fixed plane $T_0 \wedge N_0$. □

5.4 Parametrization of a holonomic BCH

Lemma 3. *Let Σ^n be a holonomic proper BCH. Let N_0 be the unit normal vector field of Σ^n restricted to U_0 and let $e_n = \frac{\nabla h}{|\nabla h|}$. Then, locally, there exists in a neighborhood of U_0 a parametrization of Σ^n , in terms of the curvature line coordinates x_1, \dots, x_n , given by*

$$X(x_1, \dots, x_n) = Y(x_1, \dots, x_{n-1}) + \alpha(x_n)N_0(x_1, \dots, x_{n-1}) + x_n e_n(x_1, \dots, x_{n-1}), \quad (15)$$

where Y parametrizes U_0 , and α is a smooth function such that $(x_n, \alpha(x_n))$ parametrizes the planar x_n -curves.

Proof. By Lemma 2, there is a local parametrization of Σ^n by curvature line coordinates

$$\begin{aligned} X : \quad \Omega \subset \mathbb{R}^n &\longrightarrow \Sigma^n \subset \mathbb{R}^{n+1} \\ (x_1, \dots, x_n) &\mapsto X(x_1, \dots, x_n) \end{aligned} \quad (16)$$

Since the principal curvature coordinates form an orthogonal net in Subsection 5.2 and from the orthogonal decomposition of its metric (13), U_0 is parametrized by

$$Y(x_1, \dots, x_{n-1}) := X(x_1, \dots, x_{n-1}, 0).$$

Again, using lemma 2, for each point $p = Y(x_1, \dots, x_{n-1})$, the integral curve of ∇h passing through p which is a x_n -curvature line, is a planar curve in the affine plane P_p passing through p and generated by the normal at p to Σ^n , N_0 and $e_n = \frac{\nabla h}{|\nabla h|}$. Locally near U_0 this curvature line is a graph in P_p over the \mathbb{R}_{e_n} -axis. Thus, there exist two functions $\alpha(x_1, \dots, x_n)$ and $\beta(x_n)$ such that :

$$X(x_1, \dots, x_n) = Y(x_1, \dots, x_{n-1}) + \alpha(x_1, x_2, \dots, x_n)N_0(x_1, \dots, x_{n-1}) + \beta(x_n)e_n(x_1, \dots, x_{n-1}). \quad (17)$$

Note that, if (x_1, \dots, x_n) is locally a curvature line coordinate system (or orthogonal coordinate system), so is the new coordinate system $(x_1, \dots, x_{n-1}, \beta_n(x_n))$ for a smooth local one-to-one function β_n . So by abuse of notation, we write the following

$$X(x_1, \dots, x_n) = Y(x_1, \dots, x_{n-1}) + \alpha(x_1, x_2, \dots, x_n)N_0(x_1, \dots, x_{n-1}) + x_n e_n(x_1, \dots, x_{n-1}). \quad (18)$$

Now, by orthogonality of the curvature coordinates :

$$\langle X_i, X_n \rangle = 0 \quad (19)$$

for all $i \in \{1, \dots, n-1\}$. Let λ_{0i} , $i \in \{1, \dots, n-1\}$, be the principal curvatures λ_i on U_0 , then differentiating X with respect to x_i we obtain

$$X_i = (1 - \alpha\lambda_{0i})Y_i + x_n e_{n,i} + \alpha_i N_0 = (1 - \alpha\lambda_{0i} - x_n \mu_{0i})Y_i + \alpha_i N_0, \quad i \in \{1, \dots, n-1\}, \quad (20)$$

and

$$X_n = \alpha_n N_0 + e_n. \quad (21)$$

From equations (19), (20) and (21) one obtains $\alpha_n \cdot \alpha_i = 0$. If α_n is null, then the x_n -curves are straight, i.e. $\lambda_n = 0$. This implies $\lambda_n = \frac{-nh}{2} = 0$, i.e. Σ^n is minimal, which contradicts the properness of Σ^n . Consequently $\alpha_i = 0$. Hence, α is a function of x_n only. \square

5.5 Second order ODE governing the integral curves of ∇h

We saw in the previous lemma that the function α depends only on the parameter x_n . We now show that α is the unique solution of an ODE with the initial conditions $\alpha(0) = \alpha'(0) = 0$, which implies that the level sets of h are isoparametric (Lemma 3).

Denote by α' the derivative of $\alpha(x_n)$ with respect to x_n .

Lemma 4. *Using the notations of Lemma 3 the function α locally satisfies a second order ODE of the form $\alpha'' = R(x_n, \alpha, \alpha')$ where $R(x, y, z)$ is a rational function which is smooth in a neighborhood of 0. The solution is locally unique with the initial condition $\alpha(0) = \alpha'(0) = 0$.*

Proof. From Lemma 3 the BCH Σ^n , locally, admits a parametrization of the form :

$$X(x_1, \dots, x_n) = Y(x_1, \dots, x_{n-1}) + \alpha(x_n)N_0(x_1, \dots, x_{n-1}) + x_n e_n(x_1, \dots, x_{n-1}). \quad (22)$$

Differentiating X , we obtain

$$X_{,i} = (1 - \alpha\lambda_{0i})Y_{,i} + x_n e_{n,i}, \quad i \in \{1, \dots, n-1\},$$

and

$$X_{,n} = \alpha' N_0 + e_n.$$

From the Codazzi equation for Σ^n (41) there exists a principal curvature μ_{0i} on U_0 such that

$$e_{n,i} = -\mu_{0i} Y_{,i},$$

where $Y_{,i} \in TU_0$, for all $i \in \{1, \dots, n-1\}$.

Equivalently, from the Codazzi equation for Σ^n (40), and by holonomicity

$$[e_i, e_n] = \nabla_{e_i} e_n - \nabla_{e_n} e_i = 0,$$

then

$$\nabla_{e_i}(A)(e_n) - \nabla_{e_n}(A)(e_i) = A([e_i, e_n]) = 0.$$

As $\nabla_{e_i}(A)(e_n) = \nabla_{e_i}(Ae_n) - A(\nabla_{e_i} e_n)$ and $\nabla_{e_n}(A)(e_i) = \nabla_{e_n}(Ae_i) - A(\nabla_{e_n} e_i)$, we get

$$\nabla_{e_i}(\lambda_n e_n) - \nabla_{e_n}(\lambda_i e_i) = 0.$$

And since $\nabla_{e_i}(\lambda_n) = 0$, we obtain

$$\nabla_{e_i} e_n = -\frac{\nabla_{e_n} \lambda_i}{\lambda_i - \lambda_n} e_i$$

which is equation (41).

Thus, we deduce that

$$\begin{aligned} X_{,i} &= (1 - \alpha\lambda_{0i} - x_n \mu_{0i}) Y_{,i} := \beta_i Y_{,i} \quad i \in \{1, \dots, n-1\}, \\ X_{,n} &= \alpha' N_0 + e_n. \end{aligned} \quad (23)$$

The coefficients of the metric (13) of Σ^n are thus equal to

$$v_i^2 = \langle X_{,i}, X_{,i} \rangle = v_{0i}^2 (1 - \alpha(x_n) \lambda_{0i} - x_n \mu_{0i})^2 \quad \text{if } i \neq n$$

and

$$v_n^2 = 1 + \alpha'^2$$

which will be also denoted by $g_{ii} = \beta_i^2 v_{0i}^2$, $i \in \{1, \dots, n-1\}$, $g_{nn} = 1 + \alpha'^2 = \beta_n^2$.

Let us investigate the evolution of the principal curvature along an integral curve of ∇h defined by α . Since, from equation (14), the normal planes to U_0 are constant along integral curves, it is straightforward that

$$N = (N_0 - \alpha' e_n) \frac{1}{\sqrt{1 + \alpha'^2}}. \quad (24)$$

Along curvature lines of U_0 and by definition of the principal curvatures λ_{0i} of U_0 :

$$N_{0,i} = -\lambda_{0i} Y_{,i} \quad i \in \{1, \dots, n-1\}.$$

Plug these identities into the derivatives of (24) along curvature lines :

$$N_{,i} = N_{0,i} \frac{1}{\sqrt{1+\alpha'^2}} - \alpha' \mu_{0i} \frac{1}{\sqrt{1+\alpha'^2}} Y_{,i} = -(\lambda_{0i} + \alpha' \mu_{0i}) Y_{,i} \frac{1}{\sqrt{1+\alpha'^2}}.$$

Since the variation of the normal of Σ^n along the x_i -curvature lines for $i \in \{1, \dots, n-1\}$ is given by the identities $N_{,i} = -\lambda_i X_{,i} = -\lambda_i Y_{,i} \beta_i$, (see equations (23)) we deduce that the variation of the principal curvature along the x_n -curvature lines are :

$$\lambda_i = (\lambda_{0i} + \alpha' \mu_{0i}) \frac{\gamma}{\beta_i}, \quad i \in \{1, \dots, n-1\}, \text{ where } \gamma = \frac{1}{\sqrt{1+\alpha'^2}}. \quad (25)$$

First, we obtain

$$\sum_{i=1}^{n-1} \lambda_i = \frac{1}{\sqrt{1+\alpha'^2}} \sum_{i=1}^{n-1} \frac{\lambda_{0i} + \alpha' \mu_{0i}}{1 - \alpha \lambda_{0i} - x_n \mu_{0i}}. \quad (26)$$

Also differentiating along the x_n -curvature line

$$N_{,n} = \left(\frac{1}{\sqrt{1+\alpha'^2}} \right)' N_0 - e_n \left(\frac{\alpha'}{\sqrt{1+\alpha'^2}} \right)' = -\lambda_n X_{,n} = -\lambda_n (\alpha' N_0 + e_n)$$

we simply obtain the standard equation:

$$\lambda_n = \frac{\alpha''}{(1+\alpha'^2)^{3/2}}. \quad (27)$$

Since $nh + 2\lambda_n = 0$, i.e. $\sum_{i \neq n} \lambda_i + 3\lambda_n = 0$, and from equations (26) and (27) α satisfies the following ODE of order 2

$$\frac{\alpha''}{1+\alpha'^2} = -\frac{1}{3} \sum_{i=1}^{n-1} \lambda_i = \sum_{i=1}^{n-1} \frac{\lambda_{0i} + \alpha' \mu_{0i}}{1 - \alpha \lambda_{0i} - x_n \mu_{0i}}. \quad (28)$$

The 2nd order ODE is of the form $\alpha'' = R(x_n, \alpha, \alpha')$, where the rational function R equals

$$R(x, y, z) = -\frac{(1+z^2)}{3} \sum_{i=1}^{n-1} \frac{\lambda_{0i} + z \mu_{0i}}{1 - \lambda_{0i} y - x \mu_{0i}}.$$

R is a smooth function when (x, y) lies in a neighborhood of $(0, 0)$. Hence, α is locally the unique solution with the initial condition $\alpha(0) = 0$ by construction, and $\alpha'(0) = 0$ since the α -curve is tangent to e_n at 0. From equation (28) α is analytic. □

5.6 Level sets of h

Proposition 3. *The level sets of h are isoparametric and the integral curves of ∇h are congruent planar curves.*

Proof. We are going to show that the principal curvatures of the h -level set U_0 are constant. Indeed From Lemma 4 the α -curve is a function of only the x_n -coordinate. Hence all the α -curves are congruent. In particular, from equation (28), from Lemma 1, and as h is constant on U_0 , for any two points $s, s' \in U_0$ and $x_n = t$ in a small neighborhood of 0, we have

$$\tilde{R}(s, t) := \sum_{i=1}^{n-1} \frac{\lambda_{0i}(s) + \alpha'(t) \mu_{0i}(s)}{1 - \lambda_{0i}(s) \alpha(t) - t \mu_{0i}(s)} = \tilde{R}(s', t) := \sum_{i=1}^{n-1} \frac{\lambda_{0i}(s') + \alpha'(t) \mu_{0i}(s')}{1 - \lambda_{0i}(s') \alpha(t) - t \mu_{0i}(s')}. \quad (29)$$

Let $c_i(s) := \frac{1}{\lambda_{0i}}(s)$ and $b_i(s) = \frac{\mu_{0i}}{\lambda_{0i}}(s)$. First, we show the uniqueness of the coefficients (b_i, c_i) in the case where α, α' and t are algebraically independent.

Let us see α as a (convergent) power series of $\mathbb{C}[[t]]$. From its properties, $\alpha = h.t^2$ and $\alpha' = k.t$, where h and k are invertible elements of $\mathbb{C}[[t]]$ since their zero order coefficient is non-zero (cf. equation (27)). Then equation (29) becomes an equality in the

$$\sum_{i=1}^{n-1} \frac{tkb_i(s) + 1}{t^2h + tb_i(s) - c_i(s)} = \sum_{i=1}^{n-1} \frac{tkb_i(s') + 1}{t^2h + tb_i(s') - c_i(s')} = \frac{f}{g}. \quad (30)$$

$\frac{f}{g}$ is a rational function of t in the extension field $K = \mathbb{C}(h, k)$ where f and g are polynomials in t with coefficients in the field K and $\deg(f) < \deg(g)$. Equation (30) then provides two decompositions of the same $\frac{f}{g}$.

Lemma 5. *If there are two points $s_1, s_2 \in U_0$ such that $\tilde{R}(s_1, t) = \tilde{R}(s_2, t)$ and there is no non-zero polynomial P such that $P(t, k, h) = 0$, then $(b_i(s_1), c_i(s_1)) = (b_{\sigma(i)}(s_2), c_{\sigma(i)}(s_2))$ for $i = 1, \dots, n-1$ and some permutation $\sigma \in \mathfrak{S}_{n-1}$.*

Proof. Let us consider the first decomposition, for example, the LHS. Consider any two polynomials of the form $ht^2 + bt - c$ and $ht^2 + b't - c'$ that are different, i.e. such that $(b, c) \neq (b', c')$:

1. If $b = b'$, then $c \neq c'$ and the polynomials are coprime.
2. If $b \neq b'$, then the Euclidean algorithm yields at the second step for remainder

$$r = -c' + \frac{c - c'}{(b - b')} \left(b' + \frac{(c - c')h}{(b - b')} \right).$$

- (a) If $c = c'$, then $r = c'$ which is non-zero.
- (b) If $c \neq c'$, then the h coefficient is non-zero; hence $r \neq 0$.

In all these cases, the remainder is a non-zero element of K ; hence the denominators are coprime. In conclusion, any two polynomials of the form $ht^2 + bt - c$ are either equal or coprime. Second, we consider a term of the LHS:

$$\frac{tkb + 1}{t^2h + tb - c'} \quad (31)$$

then, either the fraction is irreducible, or $h = b^2k(1 + kc)$ and the fraction reduces to the irreducible fraction

$$\frac{kb}{ht - kbc}. \quad (32)$$

Third, any considered denominator is either irreducible in K or a product of 2 linear forms with non equal roots because the discriminant $b^2 + 4ch \neq 0$ since $c \neq 0$. The corresponding fraction can then be explicitly and uniquely split as a sum of two fractions of the form:

$$\frac{tkb + 1}{t^2h + tb - c} = \frac{\delta}{t - \rho_1} + \frac{\epsilon}{t - \rho_2} \quad \delta, \epsilon, \rho_1, \rho_2 \in K. \quad (33)$$

Grouping the fractions having identical denominators, we obtain a partial fraction decomposition of $\frac{f}{g}$, i.e. a sum of irreducible fractions with coprime denominators. By the uniqueness of the

partial fraction decomposition (cf. for example [21]) the partial fraction decomposition of $\frac{f}{g}$ thus obtained by the LHS is identical to the partial fraction decomposition given by the RHS. \square

In general, suppose the principal curvatures of U_0 are non-constant. Beforehand, one represents the $\tilde{R}(s, t)$ as follows:

$$\tilde{R}(s, t) = \left(\sum_{i=1}^{n-1} \frac{1}{\alpha(t) - l_i(s)(t)} \right) + \left(\sum_{i=1}^{n-1} \frac{b_i(s)}{\alpha(t) - l_i(s)(t)} \right) \alpha' = A_s(t, \alpha) + B_s(t, \alpha) \alpha'$$

where the linear forms $l_i(s)(t) = c_i(s) - b_i(s)t, i = 1, \dots, n-1$. Without reference to the parameter s , we define

$$A(t, \alpha) = \sum_{i=1}^{n-1} \frac{1}{\alpha(t) - l_i(t)}$$

and

$$B(t, \alpha) = \sum_{i=1}^{n-1} \frac{b_i}{\alpha(t) - l_i(t)}.$$

Then, equations (29) or (30) are true for a continuum of s (parametrizing for example a curve on U_0), with equal $\tilde{R}(s, t)$, but different $\{(b_i(s), c_i(s))\}_{i=1, \dots, n-1}$. Hence

$$\alpha' = Q_s(t, \alpha)$$

for distinct rational functions $Q_s(x, y)$ such that $\frac{d}{ds}|_{s=s_1} Q_s(x, y) \neq 0$ in $\mathbb{C}(x, y)$.

From the preceding lemma, we can suppose that there is a non-trivial polynomial $P(x, y, z)$ such that $P(t, \alpha, \alpha') = 0$.

Lemma 6. *Suppose that the principal curvatures of U_0 are non-constant and that there is a non-zero polynomial P such that $P(t, \alpha, \alpha') = 0$, then α extends to a non-constant multi-valued holomorphic algebraic function on \mathbb{C} .*

Proof. Let us show that there is a non-trivial polynomial \tilde{P} such that, $\tilde{P}(t, \alpha) = 0$. If $\deg_z P(x, y, z) = 0$, then $P(x, y, z) = P(x, y) \neq 0$, by hypothesis, so we can choose $\tilde{P} = P$. Suppose now that $\deg_z P(x, y, z) = 1$, then $P(x, y, z) = P_0(x, y) + P_1(x, y)z$ and by substitution we get

$$P(t, \alpha, \alpha') = P_0(t, \alpha) + P_1(t, \alpha) \alpha' = P_0(t, \alpha) + P_1(t, \alpha) Q_s(t, \alpha) = 0.$$

Differentiating with respect to s , we see that $P_1(t, \alpha) = 0$ with P_1 non-trivial (by iteration, the result extends for any degree of P with respect to z). Furthermore, the existence of a non-trivial P such that $P(t, \alpha) = 0$ implies that α is an algebraic function which extends to an m -valued holomorphic algebraic function on \mathbb{C} . This, in turn, lifts to a holomorphic function defined on a compact Riemann surface \mathfrak{R} and with values in the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$. The analytic continuation of α , also denoted by α , covers S^2 d -times, where $d \geq 1$ (α is assumed not to be constant). \square

Let us now derive a contradiction in case the hypotheses of the preceding lemma hold. Suppose

$$A_{s_1}(t, \alpha) + B_{s_1}(t, \alpha) \alpha' = A_{s_2}(t, \alpha) + B_{s_2}(t, \alpha) \alpha' \quad (34)$$

but with non identical polynomials $A_{s_i}(x, y)$ or $B_{s_i}(x, y), i = 1, 2$. Hence, there is a linear form $l_i(s_1)$ different from the $l_j(s_2), j = 1, \dots, n-1$. A zero of the function $\alpha - l_i$, for some $i \in \{1, \dots, n-1\}$,

is a pole of $A_{s_1}(t, \alpha) + \alpha' B_{s_1}(t, \alpha)$. Since all values are taken by α , there is at least a $t_0 \in \mathbb{C}$ such that $\alpha(t_0) - l_i(t_0) = 0$. By equation (34), t_0 is also a pole of the RHS, i.e. a zero of $\alpha - l_i(s_2)$ (up to a possible reindexing of i), so

$$t_0(b_i(s_2) - b_i(s_1)) = c_i(s_2) - c_i(s_1). \quad (35)$$

A local Laurent series expansion around t_0 shows that the orders of the corresponding poles are equal, i.e. $n_i = n'_i$, and that $\alpha'(t_0) + b_i = \alpha'(t_0) + b'_i$. Hence, necessarily, $b_i = b'_i$ and from (35), $c_i = c'_i$ ($t_0 \neq 0$). This contradicts the hypothesis on $l_i(s_1)$ and $l_i(s_2)$. Consequently $(b_i(s_1), c_i(s_1)) = (b_{\sigma(i)}(s_2), c_{\sigma(i)}(s_2))$ for $i = 1, \dots, n-1$ and for some permutation $\sigma \in \mathfrak{S}_{n-1}$.

We conclude that the principal curvatures of the h -level set U_0 must be constant, and since its normal bundle is flat from Lemma 1, U_0 is isoparametric (cf. introduction). \square

5.7 Proof of the first part of Theorem 1

Lemmas 1 till 4 prove Theorem 1 except for the symmetries of Σ^n . Let G be the subgroup (possibly trivial) of the ambient isometries of \mathbb{E}^{n+1} that preserve U_0 . Let $g \in G$ and $p \in \Sigma^n$. Then p belongs to an integral curve of ∇h that cuts U_0 at point $p_0 \in U_0$. Let $q_0 := g(p_0) \in U_0$. The normal plane $N_{p_0}U_0$ is sent by the isometry g to $g(N_{p_0}U_0) = N_{q_0}U_0$, and the normalized mean curvature at p_0 , $N(p_0)$, is sent to $N(q_0)$. Hence we have also $g(e_n(p_0)) = e_n(q_0)$. By uniqueness of the ODE (28) with given initial conditions and by Lemma 3, the integral curve $N_{p_0}U_0 \cap \Sigma^n$ is sent to $g(N_{p_0}U_0 \cap \Sigma^n) = N_{q_0}U_0 \cap \Sigma^n$. Consequently, $g(\Sigma^n) = \Sigma^n$.

6 Proof of the last part of Theorem 1 and Theorem 2

Conversely, Subsection 5.5 gives a method to generate proper BCH by evolution of a given isoparametric codimension 2 submanifold $U_0 \subset \mathbb{E}^{n+1}$ in the direction of a unit normal vector field to U_0 . First let us illustrate the method in case the isoparametric submanifold $U_0 \subset \mathbb{E}^n \subset \mathbb{E}^{n+1}$.

Proposition 4. *Let U_0 be an isoparametric and holonomic hypersurface of $\mathbb{E}^n \subset \mathbb{E}^{n+1}$. There exists a proper holonomic BCH containing U_0 and whose n -plane containing U_0 is a plane of symmetry.*

Proof. We refer to notations of Subsection 5.5. We consider the parametrization (18) where e_n is a unit normal vector field to the n -plane containing U_0 . The principal curvatures μ_i , i.e. the eigenvalues of the shape operator A_{e_n} , are zero. Hence, the second order ODE (28) which characterizes a BCH becomes:

$$\frac{\alpha''}{1 + \alpha'^2} = -\frac{1}{3} \sum_{i=1}^{n-1} \frac{\lambda_{0i}}{1 - \lambda_{0i}\alpha} = R(\alpha). \quad (36)$$

The solution can be explicitly given by double integration.

Let $\alpha' := u$ then

$$\alpha'' = \frac{du}{dx} = \frac{du}{d\alpha} \frac{d\alpha}{dx} = (1 + u^2)R(\alpha).$$

Hence,

$$\frac{u du}{(1 + u^2)} = R(\alpha) d\alpha \text{ and } \log \sqrt{1 + u^2} = \int^\alpha R(t) dt + C_1$$

Replacing R by its expression, we obtain

$$1 + \alpha'^2 = C_2 \prod_{i=1}^{n-1} (1 - \lambda_{0i}\alpha)^{2/3}.$$

The initial conditions $\alpha(0) = \alpha'(0) = 0$ implies $C_2 = 1$. Choose the orientation of the normal N such that $h < 0$. Then $\prod_{i=1}^{n-1}(1 - \lambda_{0i}\alpha)^{2/3} - 1$ is nonnegative for α positive and

$$\alpha' = \pm \sqrt{\prod_{i=1}^{n-1}(1 - \lambda_{0i}\alpha)^{2/3} - 1}, \quad (37)$$

so

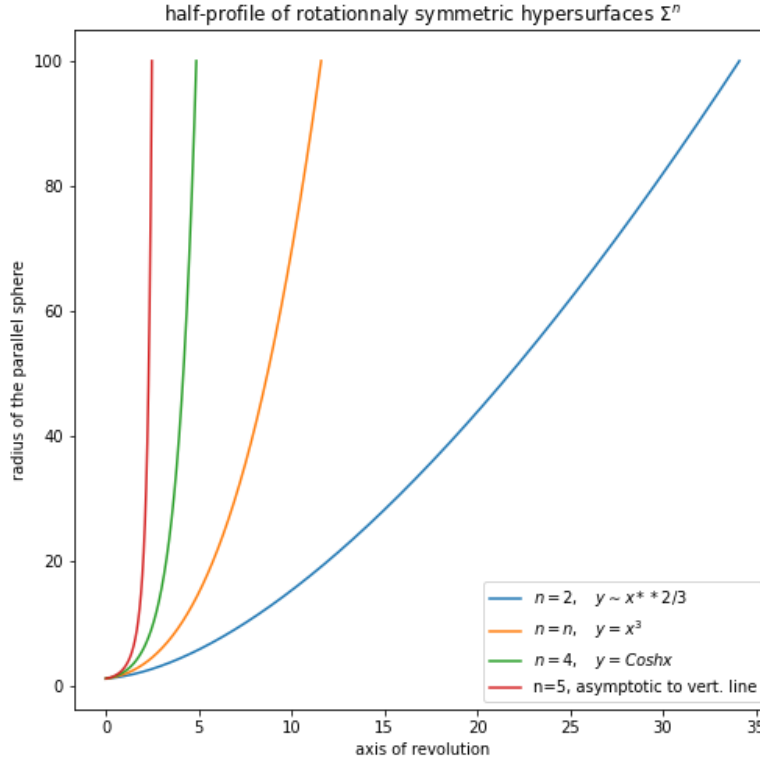
$$\frac{d\alpha}{\sqrt{\prod_{i=1}^{n-1}(1 - \lambda_{0i}\alpha)^{2/3} - 1}} = \pm dx.$$

A second integration yields an expression of the inverse of the function $x = f(\alpha)$ and the value of λ_n from equation (27):

$$x = C_3 \pm \int_0^\alpha \frac{dt}{\sqrt{\prod_{i=1}^{n-1}(1 - \lambda_{0i}t)^{2/3} - 1}} \text{ and } \lambda_n = \frac{1}{3} \left(\sum_{i=1}^{n-1} \frac{1}{1 - \lambda_{0i}\alpha} \right) \prod_{i=1}^{n-1} (1 - \alpha\lambda_{0i})^{-1/3}. \quad (38)$$

The initial condition $\alpha(0) = 0$ implies $C_3 = 0$. Notice that the solution x of equation (38) is an integral which is equivalent to $\int_0^\alpha \frac{dt}{\sqrt{t}}$, hence, integrable if $\sum_1^{n-1} \lambda_i \neq 0$ ($\sum_1^{n-1} \lambda_i = 0$ then $\lambda_n = 0$ and Σ^n would be minimal). Notice also that the function α is similar to an elliptic function in the sense that its inverse is an integral of an algebraic function. \square

In the case where U_0 is a $(n - 1)$ -sphere invariant by $O(n)$, the principal curvatures λ_{0i} 's are all equal and we obtain BCHs that are stable by $O(n)$, that are of catenoidal type and that are described in details in [17] (see also [18]).



Similarly, if U_0 is a product of spheres $\mathbb{S}^p(r_1) \times \mathbb{S}^q(r_2)$, $p + q = n - 1$ then $U_0 \subset \mathbb{S}^n \left(\sqrt{r_1^2 + r_2^2} \right) \subset \mathbb{E}^{n+1}$ is invariant by $O(p + 1) \times O(q + 1)$, hence Σ^n is also invariant by $O(p + 1) \times O(q + 1)$. The parametrization in (18), where Y parametrizes U_0 , N_0 is the normalized mean curvature vector field of U_0 -which is the unit normal vector field to $\mathbb{S}^n \left(\sqrt{r_1^2 + r_2^2} \right)$, e_n is orthonormal to N_0 in NU_0 and α satisfies the solvable ODE (28), determines the proper BCHs that were already described in [17].

These are the only cases to be considered as we shall see now. The fact that the level sets of the mean curvature function h of a holonomic BCH are isoparametric, is very restrictive. We refer to [4] and [20] for the following assertions on isoparametric submanifolds of the Euclidean space. Let l be the number of distinct principal curvatures of a codimension 2 isoparametric submanifold $U_0 \subset \mathbb{E}^{n+1}$. If $l = 1$, then U_0 is a piece of a sphere or a plane. If $l = 2$, then U_0 is a product of spheres or spheres and planes. In both cases, we recover the ones of Section 6 already described in [17]. If $l \geq 3$, then U_0 is not holonomic (cf. for example [8]). Hence, from Theorem 1 all the proper holonomic BCHs have been described, but none of them is a proper BBH because the ODE (28) and the normal biharmonic equation (10) are not compatible. This was already proved in [17] thus Theorem 2 follows.

7 A question

Although all known examples of proper BCH are holonomic, the holonomic hypothesis is strong. So the question is: can we extend the method of construction of proper BCH described above to non-holonomic isoparametric codimension 2 submanifolds? The simplest candidate is a projective tube in S^4 i.e. a tubular neighbourhood of an embedding of $\mathbb{R}P^2 \subset S^4$, that is a non-holonomic isoparametric codimension 2 submanifold of \mathbb{E}^5 that we describe now.

Remark 1. The embedding of $\mathbb{R}P^2$ is given in [3] without proof so let us give an explicit description. $\mathbb{R}P^2$ is embedded into a round sphere of radius A , by $X : \mathbb{R}P^2 \longrightarrow S^4(A) \subset \mathbb{E}^5$, a Veronese-type embedding with unknown coefficients a, b, c, d, e, f and A where

$$X : (x, y, z) \mapsto (yz, xz, xy, ax^2 + by^2 + cz^2, dx^2 + ey^2 + fz^2).$$

Since the codomain of X is $S^4(A)$ then necessarily $a = A \cos \alpha, d = A \sin \alpha, b = A \cos \beta, e = A \sin \beta, c = A \cos \gamma, f = A \sin \gamma$ with $\alpha = 0, \beta = -2\pi/3, \gamma = 2\pi/3$. This forces $A = 1/\sqrt{3}$ and

yields

$$X(x, y, z) = (yz, xz, xy, \frac{1}{\sqrt{3}}(x^2 - (y^2 + z^2)/2), \frac{1}{2}(-y^2 + z^2)).$$

Parametrize with $x = \cos \theta, y = \sin \theta \cos \phi$ and $z = \sin \theta \sin \phi$, we obtain:

$$X(\theta, \phi) = \frac{\sqrt{3} \sin^2 \theta}{2} \begin{pmatrix} e^{-2i\phi} \\ 0 \\ 0 \end{pmatrix} + \frac{\sqrt{3} \sin 2\theta}{2} \begin{pmatrix} 0 \\ e^{i\phi} \\ 0 \end{pmatrix} + \frac{(3 \cos^2 \theta - 1)}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (39)$$

8 Appendix: Codazzi's equations and curvature line coordinates

One deduces directly from Codazzi's equations two key properties related to the principal curvatures of a hypersurface $\Sigma^n \subset \mathbb{E}^{n+1}$. Recall that the shape operator in the normal direction N , A_N , decomposes $T\Sigma^n$ into distributions T_i , where $T_i = \ker(A_N(p) - \lambda_i Id)$ and λ_i is the corresponding eigenvalue -or principal curvature- of multiplicity $rk(T_i) = m_i$. We will select an open subdomain where the multiplicities $m_i, i = 1, \dots, s$, of the corresponding eigenvalues are constant. Then we can split the tangent bundle $T\Sigma^n$ into a family of integrable distributions $T_i \subset T\Sigma^n$. Let $X, Y \in T\Sigma^n$ two tangent vector fields and let A be the shape operator of Σ^n , then Codazzi's equation is

$$\nabla_X(A)Y = \nabla_Y(A)X,$$

where $\nabla_X(A)Y = \nabla_X(AY) - A(\nabla_X Y)$.

1. Let T_λ be the distribution which corresponds to the eigenvalue λ of the shape operator A of $\Sigma^n \subset \mathbb{E}^{n+1}$ of locally constant rank larger than 2, then it is integrable and defines a curvature surface $S(\lambda)$ on which λ is constant.

Indeed, let $X, Y \in T_\lambda$ be linearly independent vector fields in the given distribution T_λ . Then from the Codazzi equation:

$$\nabla_X(A)Y = \nabla_Y(A)X, \quad (40)$$

so

$$(X\lambda)Y - (Y\lambda)X = (A - \lambda Id)[X, Y].$$

As the LHS belongs to $\ker(A - \lambda Id)$ and RHS belongs to $Im(A - \lambda Id)$, $X(\lambda) = Y(\lambda) = 0$ and $[X, Y] \in T_\lambda$. Thus, from Frobenius, the distribution associated to the principal eigenvalue λ is integrable into a curvature surface $S(\lambda)$ onto which λ is constant.

2. $S(\lambda)$ is totally umbilical, i.e. the second fundamental form of $S(\lambda)$ is equal to

$$II(X, Y) = \langle X, Y \rangle H, \forall X, Y \in TS(\lambda),$$

where H is the mean curvature vector field of $S(\lambda)$.

Indeed, let X be a vector field in $TS(\lambda)$ and let η be a vector field normal to $S(\lambda)$ such that $A\eta = \mu\eta$ and $\mu \neq \lambda$. Then

$$\nabla_X(A)\eta = \nabla_\eta(A)X.$$

Expanding

$$(\mu Id - A)(\nabla_X\eta) = (\eta \cdot \lambda)X - (A - \lambda Id)(\nabla_\eta X) - (X \cdot \mu)\eta.$$

The projection operator onto $TS(\lambda)$ commutes with A hence

$$(\mu Id - A)(\nabla_X\eta)^{TS(\lambda)} = (\eta \cdot \lambda)X - (A - \lambda Id)(\nabla_\eta X)^{TS(\lambda)} = (\eta \cdot \lambda)X.$$

Hence,

$$(\nabla_X\eta)^{TS(\lambda)} = \frac{(\eta \cdot \lambda)X}{\mu - \lambda}. \quad (41)$$

Let $\{e_k\}_{k=1, \dots, l}$ be an orthonormal basis of $TS(\lambda)^\perp \cap T\Sigma^n$ in the principal curvature directions, so that $(\nabla_{e_k}N)^{TS(\lambda)} = -\lambda_k e_k$, where N is a unit normal vector field to $T\Sigma^n$, then the second fundamental form of $S(\lambda) \subset \mathbb{E}^{n+1}$, according to equation (41), is equal to

$$II(X, Y) = \sum_{k=1}^n -\langle \nabla_X e_k, Y \rangle e_k - \langle \nabla_X N, Y \rangle N = \sum_{k=1}^n \frac{e_k \cdot \lambda}{\lambda - \lambda_k} \langle X, Y \rangle e_k - \lambda \langle X, Y \rangle N.$$

Consequently,

$$II(X, Y) = \langle X, Y \rangle H, \quad H = \sum_{i=1}^l \frac{(e_i \cdot \lambda)}{\lambda_i - \lambda} e_i - \lambda N.$$

where H is the mean vector field of $S(\lambda) \subset \mathbb{E}^{n+1}$. $S(\lambda)$ is thus totally umbilical (see next Appendix 9).

Let Σ^n be parametrized by an immersion

$$X : \quad \Omega \subset \mathbb{R}^n \longrightarrow \Sigma^n \subset \mathbb{R}^{n+1} \\ (x_1, \dots, x_n) \mapsto X(x_1, \dots, x_n), \quad (42)$$

where Ω is an open domain, and (x_1, \dots, x_n) form a system of orthogonal coordinates. Then the metric g in these coordinates is diagonalized as follows:

$$g_{ij} = \delta_{ij} E_i = \delta_{ij} v_i^2. \quad (43)$$

The Christoffel Symbols of the metric g are:

$$\Gamma_{jk}^i = \frac{g^{ii}}{2} (g_{ki,j} + g_{ij,k} - g_{jk,i}) = \frac{1}{2E_i} (\delta_{ki} E_{k,j} + \delta_{ij} E_{i,k} - \delta_{jk} E_{j,i}).$$

If $i \neq j \neq k$ then $\Gamma_{jk}^i = 0$.

Hence, the only possible nonzero Christoffel symbols up to permutation on lower indices are:

- If $i = j \neq k$, then $\Gamma_{ik}^i = \frac{E_{i,k}}{2E_i} = (\log v_i)_k$.
- If $j = k \neq i$, then $\Gamma_{jj}^i = -\frac{E_{ji}}{2E_i} = -(\log v_j)_{,i} \frac{v_j^2}{v_i^2}$.
- If $j = k = i$, then $\Gamma_{ii}^i = \frac{E_{ii}}{2E_i} = (\log v_i)_{,i}$.

In the frame of this coordinate system where $e_i = \partial_i$, $i = 1, \dots, n$ the connection on $T\Sigma^n$ is given by the Cristoffel symbols

$$\nabla_{e_i} e_k = \Gamma_{ik}^\alpha e_\alpha = \Gamma_{ik}^i e_i + \Gamma_{ik}^k e_k. \quad (44)$$

In the holonomic case, when the curvature lines form a coordinate system, the shape operator A is diagonalized and related to the second fundamental form II as follows

$$A_i^j = \delta_i^j \lambda_i \text{ and } II_{ij} = A_i^\alpha g_{\alpha j} = \delta_{ij} \mu_i,$$

where $\{\lambda_i\}_{i=1}^n$ are the principal curvatures and $\mu_i = \lambda_i v_i^2$. The Codazzi equations are equivalent to the following system of $n(n-1)$ equations:

$$\lambda_{i,j} - (\lambda_j - \lambda_i) \Gamma_{ij}^i = 0, \quad \forall i \neq j \quad i, j \in \{1, \dots, n\}. \quad (45)$$

9 Appendix : totally umbilical submanifolds of \mathbb{E}^{n+1}

Let us recall that a submanifold Σ^n is umbilical at $p \in \Sigma^n$ if there is a unit normal field $N(p) \in N_p \Sigma^n$ and curvature $\lambda \in \mathbb{R}$ such that:

$$II(p)(X, Y) = \langle X, Y \rangle \lambda(p) N(p) \quad \forall X, Y \in T_p \Sigma^n.$$

By definition, Σ^n is totally umbilical if it is umbilical at each point of Σ^n .

Lemma 7. *A totally umbilical hypersurface $\Sigma^n \subset \mathbb{E}^{n+1}$ is a piece of a hypersphere or a hyperplane.*

Proof. Let N be a unit normal field to Σ^n . Let $\gamma(s)$ be a curve in Σ^n and T be the unit vector field tangent to $\gamma(s)$ then as $\frac{DN}{ds}$ and $\nabla_T N$ are both orthogonal to N , we obtain

$$D_T N = \frac{DN}{ds} = \nabla_T N \quad (46)$$

and since any direction in $T\Sigma^n$ is a principal curvature direction,

$$\nabla_T N = -\lambda T. \quad (47)$$

From Appendix 8 1, λ is constant and integrating Equation (47),

$$N(s) - N(0) = -\lambda \int \frac{DX}{ds} ds = -\lambda (X(s) - X(0)).$$

Hence, if $\lambda \neq 0$ we obtain

$$X(s) = X(0) - \frac{1}{\lambda} (N(s) - N(0)) := C(0) - \frac{1}{\lambda} N(s).$$

Thus $\Sigma^n \subset S^n(C(0), \frac{1}{\lambda})$. If $\lambda = 0$, Σ^n is a piece of a hyperplane. □

More generally

Lemma 8. *A totally umbilical submanifold $\Sigma^k \subset \mathbb{E}^{n+1}$, $k > 1$ with the extra condition that the mean curvature is parallel is a piece of a round sphere.*

Proof. Let (T_1, \dots, T_k) be an orthonormal basis at some point extended to orthonormal frames around some point. The unit normal vector field $N = \frac{H}{h}$ seen as a section of the normal bundle of Σ^k is parallel, hence for any $i = 1, \dots, k$, $(D_{T_i}N)^\perp = 0$ from which we deduce equation (47) and, as in Lemma 7, that $\Sigma^k \subset S^n(C(0), \frac{1}{\lambda})$.

On the other hand, consider the $(k+1)$ -vector field $W := T_1 \wedge \dots \wedge T_k \wedge N$; then

$$D_{T_i}W = \sum_{\alpha=1}^k T_1 \wedge \dots \wedge D_{T_i}T_\alpha \wedge \dots \wedge T_k \wedge N + T_1 \wedge \dots \wedge T_k \wedge D_{T_i}N, \quad i \in \{1, \dots, k\}.$$

The last term of the RHS is zero because $D_{T_i}N = -\lambda T_i$, and the first term is also zero since $D_{T_i}T_j \perp T_j$. Hence $D_{T_i}W = 0$ and W is constant on Σ^k which means Σ^k is in a $(k+1)$ -plane P . Finally Σ^k is in $S^n \cap P$ which is a round k -sphere. \square

Furthermore, it is shown (cf. for example Prop. 1.19 in [8]) that the mean curvature vector field of any umbilical submanifold of dimension at least two is parallel.

We then deduce from Lemma 8:

Corollary 2. *A totally umbilical submanifold $\Sigma^k \subset \mathbb{R}^{n+1}$, $k > 1$ is a piece of a sphere or a plane.*

Note that a helix with constant curvature and nonzero torsion is umbilical, but the mean curvature vector is not parallel and the curve is not a circle unless the torsion is null, i.e. unless the normal bundle is flat.

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