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SPECTRAL SEQUENCES VIA LINEAR PRESHEAVES

MURIEL LIVERNET AND SARAH WHITEHOUSE

ABSTRACT. We study homotopy theory of the category of spectral sequences with respect to the class of weak equivalences given by maps which are quasi-isomorphisms on a fixed page. We introduce the category of extended spectral sequences and show that this is bicomplete by analysis of a certain linear presheaf category modelled on discs. We endow the category of extended spectral sequences with various model category structures, restricting to give the almost Brown category structures on spectral sequences of our earlier work. One of these has the property that spectral sequences is a homotopically full subcategory. By results of Meier, this exhibits the category of spectral sequences as a fibrant object in the Barwick-Kan model structure on relative categories, that is, it gives a model for an infinity category of spectral sequences. We also use the presheaf approach to define two décalage functors on spectral sequences, left and right adjoint to a shift functor, thereby clarifying prior use of the term décalage in connection with spectral sequences.

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1. INTRODUCTION

We study the category \mathbf{SpSe} of spectral sequences and its homotopy theory. Since \mathbf{SpSe} is neither complete nor cocomplete, it does not admit model category structures. In [19, Theorem 5.3.1], we established a weaker homotopical framework, that of an *almost Brown category*, and exhibited such a structure, \mathbf{SpSe}_r , on \mathbf{SpSe} for each $r \geq 0$. The class of weak equivalences is given by maps of spectral sequences which are quasi-isomorphisms on page r . Here, we situate \mathbf{SpSe} as a subcategory of the category \mathbf{ESpSe} of *extended spectral sequences* and exhibit various model category structures on this category. This setting provides a new perspective on the category of spectral sequences and its homotopy theory and we deduce consequences for the infinity category of spectral sequences.

To this end, we introduce and study a category of linear presheaves closely related to the category of spectral sequences \mathbf{SpSe} . This is the category \mathbf{LWB} of *linear witness books*, a linear presheaf category built from suitable disc objects. Intermediate between these two categories is the category of extended spectral sequences \mathbf{ESpSe} . In an extended spectral sequence, we require a specified morphism from each page to the homology of the previous one, but we drop the requirement, crucial to spectral sequences, that this should be an isomorphism.

The first parts of the paper are categorical. They are largely motivated by the wish to view \mathbf{SpSe} as a subcategory of a convenient bicomplete category. This is the role played by the category of extended spectral sequences \mathbf{ESpSe} . There is a choice involved here, as we weaken the requirement in spectral sequences that a page is isomorphic to the homology of the previous one. Here we simply require a map, but not that it be an isomorphism. We have chosen that the maps go from a page to the homology of the previous one. In some sense, this choice effectively gives preference to colimits over limits. It fits well with the notion of *witness* cycles and

boundaries appearing in our previous work [7]. Another motivation for our choice is that it is likely to be better behaved in terms of monoidal structure. Although we do not pursue that direction here, we note that Brotherton has shown that, on the category of filtered complexes, model structures which are closely related to spectral sequences are monoidal [5].

It turns out that \mathbf{ESpSe} is bicomplete; see Theorem 5.3.2. Colimits are calculated pagewise, but to understand limits is less straightforward. Here the linear presheaf category of linear witness books plays a vital role, together with a pair of adjoint functors $(\mathcal{Q}, \mathcal{N})$. The terminology of linear witness books is chosen because the objects of this category can be viewed as having pages like those of a spectral sequence, with *witness* maps from a page to the previous one, as well as degeneracy maps in the other direction. This means objects have extra data, compared with spectral sequences, witnessing how elements end up on the r -page.

We establish an adjunction $\mathcal{Q} \dashv \mathcal{N}$ of functors between \mathbf{LWB} and \mathbf{ESpSe} and use its properties to identify subcategories $(\mathbf{LWB})^e$ and $(\mathbf{LWB})^s$ of \mathbf{LWB} equivalent to \mathbf{ESpSe} and \mathbf{SpSe} respectively. As $(\mathbf{LWB})^e$ is a full reflective subcategory of \mathbf{LWB} it has all (small) limits and colimits and thus so does \mathbf{ESpSe} .

This setting offers insight into *décalage* for spectral sequences. We study truncation functors on the underlying category \mathbf{D} on which we take our linear presheaves. The embedding of a suitably truncated version of \mathbf{D} into \mathbf{D} has both a left and a right adjoint. This triple of adjoint functors gives rise to a chain of five adjoint functors on \mathbf{LWB} . Of these, we will see that the leftmost three are internal to $(\mathbf{LWB})^e$ and so there is a corresponding triple of adjoints on \mathbf{ESpSe} . These also restrict to \mathbf{SpSe} . In particular we obtain a shift functor on \mathbf{ESpSe} or \mathbf{SpSe} with both a left adjoint \mathbf{LDec} and a right adjoint \mathbf{Dec} , two versions of *décalage*; see Theorem 6.3.3. The terminology is explained by noting that these functors are suitably compatible with Deligne's functors \mathbf{Dec}^* and \mathbf{Dec} for filtered complexes [10]. *Décalage* has been studied for more general filtered objects; see, for example, [14], where the key idea is attributed to Antieau. Although *décalage* in this sense is always closely connected to the study of spectral sequences, we are not aware of other work defining *décalage* functors directly on the category of spectral sequences, as we do here. Nonetheless, reference is quite often made to *décalage* of a spectral sequence and there are important instances of this relationship. For example, Rognes notes in [26] that it is common to call the Whitehead tower spectral sequence the *décalage* of the Atiyah–Hirzebruch spectral sequence; Levine establishes such a relationship between the Adams–Novikov spectral sequence and Voevodsky's slice tower spectral sequence [18] and Burklund–Hahn–Senger do so for the C_2 -effective slice spectral sequence and the $MU_{\mathbb{R}}$ -based Adams–Novikov spectral sequence [6].

Later parts of the paper are homotopical, extending the study of the homotopy theory of spectral sequences initiated in [19]. For each $r \geq 0$, we study spectral sequences with r -quasi-isomorphisms as a *relative category*, denoted $(\mathbf{SpSe}, \mathcal{E}_r)$. Having situated spectral sequences inside the bicomplete category of extended spectral sequences \mathbf{ESpSe} , we now establish model category structures there, restricting to the relevant structure on spectral sequences.

For each $r \geq 0$, we show the existence of a model category structure on \mathbf{ESpSe} , restricting to recover the corresponding underlying almost Brown category structure on \mathbf{SpSe} . Indeed there are two flavours of such model structures; in both the fibrations are those maps f that are surjective on pages 0 to r . Theorem 7.2.1 gives a structure \mathbf{ESpSe}_r , where the weak equivalences are those maps such that the component on the r -page is a quasi-isomorphism. In Theorem 7.3.6, we obtain \mathbf{ESpSe}'_r , where the weak equivalences are those maps such that the component on the r -page is a quasi-isomorphism and the components on all higher pages are isomorphisms. As relative categories, both \mathbf{ESpSe}_r and \mathbf{ESpSe}'_r restrict to $(\mathbf{SpSe}, \mathcal{E}_r)$.

The methods of proof use the category \mathbf{LWB} . For the first family of structures, we obtain a cofibrantly generated model category structure on \mathbf{LWB} by transfer of a projective-type model structure on the category of r -bigraded complexes and then modify this in order to produce a version \mathbf{LWB}_r which is closely related to the relevant structure on \mathbf{SpSe} . This model structure is then transferred to produce \mathbf{ESpSe}_r . The second family of model structures is established by directly checking the axioms, making use of the existence of the first family.

These model structures have the following relationships to each other. We show in Corollary 7.2.4 that the model categories \mathbf{ESpSe}_r for different r are all Quillen equivalent via shift and décalage functors and indeed, these are all Quillen equivalent to a projective-type model category structure on the category of 0-bigraded complexes. Similarly, the model categories \mathbf{ESpSe}'_r for different r are all Quillen equivalent, see Proposition 7.4.1. Proposition 7.4.3 shows that the identity functor $\mathbf{ESpSe}'_r \rightarrow \mathbf{ESpSe}_r$ is a right Bousfield localization which is not a Quillen equivalence. Thus we provide a right *delocalization* of (a model category Quillen equivalent to) the projective model structure on 0-bigraded complexes, a bigraded version of chain complexes.

The model category \mathbf{ESpSe}'_0 has the important feature that \mathbf{SpSe}_0 is a homotopically full subcategory, in the sense that any object weakly equivalent to a spectral sequence is itself a spectral sequence. In [3] Barwick and Kan provide a model category structure on the category of relative categories, Quillen equivalent to Rezk's complete Segal space model structure on simplicial spaces [24], thus establishing another model for a homotopy theory of homotopy theories. Results of Meier [20, Theorem 4.13] allow us to conclude, in Theorem 8.1.3, that $(\mathbf{SpSe}, \mathcal{E}_0)$ is a fibrant relative category in this model. Our result can therefore be viewed as establishing an infinity-category of spectral sequences. And, via the shift-décalage adjunction, for each r the relative category $(\mathbf{SpSe}, \mathcal{E}_r)$ has $(\mathbf{SpSe}, \mathcal{E}_0)$ as a fibrant replacement.

We note that there is some other recent work on spectral sequences, close in spirit to some of the ideas in this paper. In particular, in [4] the homotopy spectral sequence of a (co)simplicial object in an infinity-category is studied and successive terms are analysed using two types of localization. And [1] studies model structures on the infinity-category of filtered objects in a suitable stable infinity-category, noting that these may be viewed as a categorification of the model structures for filtered complexes of [7].

The paper is organized as follows. Section 2 contains background material on presheaves, model categories and bigraded complexes. In Section 3, we introduce

the category of extended spectral sequences. The category \mathbf{D} of disc objects is presented in Section 4, together with the presheaf category of linear witness books. Section 5 covers the main adjunction relating linear witness books and extended spectral sequences. In Section 6 we study shift and décalage functors. The homotopical parts of the paper are in Section 7, on model category structures, and in Section 8, on an infinity-categorical interpretation. The paper concludes with an appendix of additional material which is helpful in providing a complete story of our presheaf approach, but not essential for the main text. Appendix A contains details of important representable linear witness books. Appendix B provides descriptions of certain adjoint functors on linear witness books. And finally, Appendix C explains some non-existence results for certain model category structures having spectral sequences as fibrant objects.

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2. PRELIMINARIES

Throughout this paper, we let \mathbf{R} denote a commutative ring with unit. The category of \mathbf{R} -modules is denoted $\mathbf{Mod}_{\mathbf{R}}$; it is a closed symmetric monoidal category.

2.1. Linear presheaves. We work with linear categories, that is, categories enriched in $\mathbf{Mod}_{\mathbf{R}}$, and we refer to [17] for our conventions. Given a small linear category \mathbf{B} we denote by $\widehat{\mathbf{B}}$ the category of linear functors from \mathbf{B}^{op} to $\mathbf{Mod}_{\mathbf{R}}$. Such a category is called a category of *linear presheaves*. It is a linear category with linear natural transformations as morphisms. The Yoneda embedding $\mathcal{Y}_{\mathbf{B}}: \mathbf{B} \rightarrow \widehat{\mathbf{B}}$ which associates to b the functor $\text{Hom}_{\mathbf{B}}(-, b)$ is thus an enriched functor. We will denote the Yoneda functor $\mathcal{Y}_{\mathbf{B}}$ simply by \mathcal{Y} when the category \mathbf{B} is clear from the context. We gather here the results in Section 4.1 of [17] that we will use in the paper.

Proposition 2.1.1. *Let \mathbf{B} be a small linear category and let $F: \mathbf{B} \rightarrow \mathbf{C}$ be an enriched functor.*

- (1) *If \mathbf{C} is a cocomplete linear category which is tensored over $\mathbf{Mod}_{\mathbf{R}}$, then the left Kan extension of F along the Yoneda functor $\text{Lan}_{\mathcal{Y}}F: \widehat{\mathbf{B}} \rightarrow \mathbf{C}$ exists and admits the following description as a coend. For $b' \in \widehat{\mathbf{B}}$,*

$$(\text{Lan}_{\mathcal{Y}}F)(b') = \int^{b \in \mathbf{B}} \text{Hom}_{\widehat{\mathbf{B}}}(\mathcal{Y}(b), b') \otimes F(b).$$

It is left adjoint to the functor $\mathbf{C} \rightarrow \widehat{\mathbf{B}}$ which associates to c the functor $\text{Hom}_{\mathbf{C}}(F(-), c)$.

- (2) *If \mathbf{C} is a complete linear category which is cotensored over $\mathbf{Mod}_{\mathbf{R}}$, then the right Kan extension of F along the Yoneda functor $\text{Ran}_{\mathcal{Y}}F: \widehat{\mathbf{B}} \rightarrow \mathbf{C}$ exists*

and admits the following description as an end. For $b' \in \widehat{\mathbf{B}}$,

$$(\text{Ran}_{\mathcal{Y}}F)(b') = \int_{b \in \mathbf{B}} \text{Hom}(\text{Hom}_{\widehat{\mathbf{B}}}(b', \mathcal{Y}(b)), F(b)).$$

It is right adjoint to the functor $\mathbf{C} \rightarrow \widehat{\mathbf{B}}$ which associates to c the functor $\text{Hom}_{\mathbf{C}}(F(-), c)$. \square

The first adjunction corresponds to the usual adjunction between the nerve and the realisation functor.

Given a linear functor $\mathbf{B} \rightarrow \mathbf{C}$, applying the above to the composite with the Yoneda functor $\mathcal{Y}_{\mathbf{C}}: \mathbf{C} \rightarrow \widehat{\mathbf{C}}$ yields the following.

Proposition 2.1.2. *A linear functor $G: \mathbf{B} \rightarrow \mathbf{C}$ gives rise to a linear functor $G^*: \widehat{\mathbf{B}} \rightarrow \widehat{\mathbf{C}}$ where $G^*(b) = b \circ G$. This has a left adjoint $G_!$ and a right adjoint G_* .* \square

2.2. Model category structures. We assume the reader is familiar with the language of model category structures and we will use standard model category theoretic terminology and notation as can be found in [16]. We will use standard recognition results for cofibrantly generated model categories (cf. [15, Theorem 11.3.1] or [16, Theorem 2.1.19]).

We also use standard machinery for right transfer of model category structures, as summarised in [2, Section 4.4.1]. In particular, if we have an adjunction

$$\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \dashv \\ \xleftarrow{R} \end{array} \mathbf{C}' ,$$

with \mathbf{C} a model category, then we have a model category structure on \mathbf{C}' , in which a map is a fibration or weak equivalence precisely if its image under R is such in \mathbf{C} , under either of the following conditions.

- \mathbf{C} is cofibrantly generated, R preserves filtered colimits, and maps in \mathbf{C}' with the left lifting property with respect to fibrations (that is, *anodyne* maps) are weak equivalences. See [22, Proposition 3.1].
- \mathbf{C} is cofibrantly generated, R also has a right adjoint R' and (RL, RR') is a Quillen adjunction. See [11, Theorem 2.3].

We make use of criteria for a Quillen adjunction to be a Quillen equivalence. In particular, if in a Quillen adjunction the right adjoint creates weak equivalences then the adjunction is a Quillen equivalence precisely if for all cofibrant objects the unit is a weak equivalence. See [21, Proposition 2.3].

We also use right Bousfield localization of a model category structure and the corresponding notion of right delocalization [9].

2.3. Bigraded complexes. In this section we let $r \geq 0$ be an integer.

Definition 2.3.1. A *bigraded \mathbf{R} -module* A is a collection of \mathbf{R} -modules $A = \{A^{p,n}\}$ with $p, n \in \mathbb{Z}$.

Definition 2.3.2. An *r -bigraded complex* is a bigraded \mathbf{R} -module $A = \{A^{p,n}\}$ together with maps of \mathbf{R} -modules $d_r: A^{p,n} \rightarrow A^{p-r, n+1-r}$, called differentials, such that

$d_r^2 = 0$. A *morphism of r -bigraded complexes* is a map of bigraded modules commuting with the differentials. We denote by $r\text{-C}$ the category of r -bigraded complexes.

Remark 2.3.3. The grading convention here is chosen to be compatible with our previous work [7, 19]. It will mean that the differential on the r -page of a spectral sequence has bidegree $(-r, 1-r)$. It is straightforward to translate our results to the standard setting of a homological spectral sequence where the corresponding bidegree is $(-r, r-1)$ or that of a cohomological spectral sequence where it is $(r, 1-r)$.

Notation 2.3.4. We denote by $B_r(A)$ the image of d_r , that is the bigraded \mathbf{R} -module of r -boundaries of A , and by $Z_r(A)$ the kernel of d_r , that is, the bigraded \mathbf{R} -module of r -cycles of A . We write $H(A)$ for the homology $Z_r(A)/B_r(A)$.

We note that $r\text{-C}$ can be viewed as a category of linear presheaves as follows.

Definition 2.3.5. For $r \geq 0$, we let $\langle \delta_r \rangle$ be the following linear category. The objects are (p, n) for $p, n \in \mathbb{Z}$. The \mathbf{R} -module of morphisms $\text{Hom}_{\langle \delta_r \rangle}((p, n), (q, m))$ is either 0 or free of rank 1 spanned by $\delta_r^{p,n} : (p-r, n-r+1) \rightarrow (p, n)$ or by the identity. Hence, composition of two non-identity morphisms is 0.

There is an isomorphism of categories

$$(1) \quad \widehat{\langle \delta_r \rangle} \cong r\text{-C}.$$

On objects this is given by the assignment that sends A in $\widehat{\langle \delta_r \rangle}$ to the r -bigraded complex with $A^{p,n} = A(p, n)$ and with $d_r : A^{p,n} \rightarrow A^{p-r, n+1-r}$ given by $A(\delta_r^{p,n})$.

Notation 2.3.6. We write $\mathbf{R}^{p,n}$ for the bigraded module consisting of a free module of rank 1 in bidegree (p, n) and otherwise 0.

Remark 2.3.7. Note that the Yoneda functor $\mathcal{Y} : \langle \delta_r \rangle \rightarrow \widehat{\langle \delta_r \rangle}$ sends (p, n) to the r -bigraded complex having only two non-zero components, each free of rank 1, at bigrading (p, n) and $(p-r, n+1-r)$ with $d_r : \mathbf{R} \rightarrow \mathbf{R}$ the identity map. We have $\mathbf{R}^{p-r, n-r+1} = \text{coker}(\delta_r : \mathcal{Y}(p-2r, n+2-2r) \rightarrow \mathcal{Y}(p-r, n+1-r))$ and since $\delta_r^2 = 0$ we have an induced map $\delta_r^{p,n} : \mathbf{R}^{p-r, n+1-r} \rightarrow \mathcal{Y}(p, n)$.

The homology of an r -bigraded complex is a bigraded \mathbf{R} -module and the category of r -bigraded complexes has a natural class of quasi-isomorphisms, namely morphisms inducing isomorphisms on homology.

Similarly to the treatment of unbounded chain complexes by Hovey in [16, Section 2.3], the r -bigraded complex $\mathcal{Y}(p, q)$ corresponds to the disc object $D^*(\mathbf{R})$, the r -bigraded complex $\mathbf{R}^{p,n}$ to the sphere object $S^*(\mathbf{R})$, and the map $\delta_r : \mathbf{R}^{p-r, n+1-r} \rightarrow \mathcal{Y}(p, n)$ to the injection $S^{*-1}(\mathbf{R}) \rightarrow D^*(\mathbf{R})$. The proposition below provides a model category structure corresponding to the projective model category structure on the category of unbounded chain complexes adapted to the category of r -bigraded complexes.

Proposition 2.3.8. *The category $r\text{-C}$ of r -bigraded complexes has a cofibrantly generated model category structure where*

- *fibrations are bidegreewise surjections,*
- *weak equivalences are quasi-isomorphisms.*

The set of generating acyclic cofibrations is the set $I = \{0 \rightarrow \mathcal{Y}(p, n)\}_{p, n \in \mathbb{Z}}$ and the set of generating cofibrations is the set $J = \{\delta_r^{p, n} : \mathbb{R}^{p-r, n-r+1} \rightarrow \mathcal{Y}(p, n)\}_{p, n \in \mathbb{Z}}$. \square

We will identify $\widehat{\langle \delta_r \rangle}$ with $r\text{-C}$ via the isomorphism (1) and view it as a model category with the above structure, which we refer to as the projective model structure.

We end this section with the r -cone construction that will be useful in the sequel.

Definition 2.3.9. Let A be an r -bigraded complex. The r -cone of A is the r -bigraded complex $(\text{Cone}_r(A), d_r)$ given by

$$\text{Cone}_r(A)^{p, n} = A^{p, n} \oplus A^{p+r, n+r-1} \text{ with } d_r(a, b) = (0, a).$$

It is naturally endowed with a projection $\pi_r : \text{Cone}_r(A) \rightarrow A$ defined as $\pi_r(a, b) = a + d_r^A(b)$.

Note that the assignment $A \mapsto \text{Cone}_r(A)$ is functorial, and that this functor is the composite of the forgetful functor from $r\text{-C}$ to $\mathbf{bgMod}_{\mathbb{R}}$ followed by its left adjoint. Note also that it is isomorphic to the usual cone construction of the identity map in chain complexes. In particular, for every A , $\text{Cone}_r(A)$ is acyclic.

We end this subsection by noting that we can compare the categories of r -bigraded complexes for varying r by using a translation functor.

Definition 2.3.10. For each $r \geq 0$, the translation functor $\mathcal{T} : r\text{-C} \rightarrow (r+1)\text{-C}$ is given by $(\mathcal{T}A)^{p, n} = A^{n, 2n-p}$, with differential that of A , and $(\mathcal{T}f)^{p, n} = f^{n, 2n-p}$.

Note that translation commutes with cone constructions and homology.

Proposition 2.3.11. For each $r \geq 0$, the categories of r -bigraded complexes and $(r+1)$ -bigraded complexes are isomorphic via the translation functor, $\mathcal{T} : r\text{-C} \rightarrow (r+1)\text{-C}$. This induces a Quillen equivalence between the projective model category structures on $r\text{-C}$ and $(r+1)\text{-C}$. \square

3. THE CATEGORY OF EXTENDED SPECTRAL SEQUENCES

In [19] we studied the category of spectral sequences \mathbf{SpSe} , and provided some examples showing that it is neither complete nor cocomplete. In this section we embed the category \mathbf{SpSe} in the category of extended spectral sequences. We show here that this category is cocomplete and we will see later that it is also complete; see Proposition 5.3.1.

3.1. Spectral sequences and extended spectral sequences.

Definition 3.1.1. An extended spectral sequence (X, φ) is a family of r -bigraded complexes (X_r, d_r) , for $r \geq 0$, together with a family of morphisms of bigraded \mathbb{R} -modules $\varphi_{r+1} : X_{r+1} \rightarrow H(X_r)$ for $r \geq 0$, called *characteristic maps*.

A *morphism of extended spectral sequences* is a family of morphisms $f_r : X_r \rightarrow Y_r$ of r -bigraded complexes, for $r \geq 0$, which is *compatible with characteristic maps*.

We denote by \mathbf{ESpSe} the category of extended spectral sequences.

A *spectral sequence* is an extended spectral sequence such that the characteristic maps are all isomorphisms. We denote by \mathbf{SpSe} the category of spectral sequences, that is, the full subcategory of \mathbf{ESpSe} whose objects are spectral sequences.

We note that the categories \mathbf{SpSe} and \mathbf{ESpSe} implicitly involve the forgetful functors from r -bigraded complexes to bigraded \mathbf{R} -modules, since the characteristic maps are morphisms of bigraded \mathbf{R} -modules.

Proposition 3.1.2. *The category \mathbf{ESpSe} of extended spectral sequences is a linear category with (small) colimits, computed pagewise. It is tensored over $\mathbf{Mod}_{\mathbf{R}}$.*

Proof. That the category is linear is immediate. It is straightforward to see that it is tensored over $\mathbf{Mod}_{\mathbf{R}}$, noting that, for an \mathbf{R} -module M and an extended spectral sequence Y , the characteristic maps of $M \otimes Y$ are given by the composites

$$M \otimes Y_{i+1} \xrightarrow{1_M \otimes \varphi_{i+1}^Y} M \otimes H(Y_i) \xrightarrow{\mu} H(M \otimes Y_i),$$

where $\mu(m \otimes [x]) = [m \otimes x]$.

Let $X: I \rightarrow \mathbf{ESpSe}$ be a functor with I a (small) category. For each $m \geq 0$ denote by

$$\varphi(i)_{m+1}: X(i)_{m+1} \rightarrow H(X(i)_m)$$

the characteristic maps, and by $Y_m = \operatorname{colim}_{i \in I} X(i)_m$ the colimit of the diagram $X_m: I \rightarrow m\text{-}\mathbf{C}_{\mathbf{R}}$ induced by the projection of \mathbf{ESpSe} to $m\text{-}\mathbf{C}_{\mathbf{R}}$. The canonical maps $\rho(i)_m: X(i)_m \rightarrow Y_m$ induce maps of bigraded R -modules

$$H(\rho(i)_m) \circ \varphi(i)_{m+1}: X(i)_{m+1} \rightarrow H(Y_m).$$

Since the forgetful functor from $(m+1)$ -bigraded complexes to bigraded R -modules preserves colimits, there is a (unique) characteristic map $Y_{m+1} \rightarrow H(Y_m)$. Thus the pagewise colimit Y is an object of \mathbf{ESpSe} , with compatible maps $X(i) \rightarrow Y$. For the universal property, if Y' is another such with canonical maps $\rho'(i)_m: X(i)_m \rightarrow Y'_m$, there is a unique map of m -bigraded complexes $\alpha_m: Y_m \rightarrow Y'_m$ for all m , such that $\alpha_m \rho(i)_m = \rho'(i)_m$. And these satisfy compatibility with the characteristic maps by using the universal property of (the underlying bigraded module of) the pagewise colimit. \square

Remark 3.1.3. Note that \mathbf{ESpSe} is not cotensored over $\mathbf{Mod}_{\mathbf{R}}$. This is one consequence of the choice of direction of the characteristic maps.

3.2. Discs in extended spectral sequences. We adapt the notation of Section 5.5 in [19] to the case of extended spectral sequences.

Definition 3.2.1. Let $p, n \in \mathbb{Z}$. For all $r \geq 0$, let $\mathcal{D}_r(p, n)$ be the spectral sequence defined as follows.

$$\begin{cases} \mathcal{D}_r(p, n)_i = \mathbf{R}^{p, n} \oplus \mathbf{R}^{p-r, n+1-r}, & d_i = 0 \quad \text{for } 0 \leq i < r, \\ \mathcal{D}_r(p, n)_r = \mathbf{R}^{p, n} \xrightarrow{1} \mathbf{R}^{p-r, n+1-r} \\ \mathcal{D}_r(p, n)_i = 0 & \text{for } i > r. \end{cases}$$

We denote by $e_{r,i}^{p,n}$ a generator of $\mathcal{D}_r(p, n)_i^{p,n}$ and by $f_{r,i}^{p-r, n+1-r}$ a generator of $\mathcal{D}_r(p, n)_i^{p-r, n+1-r}$, for $0 \leq i \leq r$. The characteristic map

$$\varphi_i: \mathcal{D}_r(p, n)_i \rightarrow H(\mathcal{D}_r(p, n)_{i-1})$$

is given by

$$\begin{aligned}\varphi_i(e_{r,i}^{p,n}) &= [e_{r,i-1}^{p,n}], \\ \varphi_i(f_{r,i}^{p-r,n+1-r}) &= [f_{r,i-1}^{p-r,n+1-r}].\end{aligned}$$

for $0 \leq i \leq r$ and $\varphi_i = 0$ for $i > r$.

Definition 3.2.2. Let (X, φ) be an extended spectral sequence.

- (1) A sequence of elements $(x_0^{p,n}, \dots, x_{m+1}^{p,n})$ with $x_i^{p,n} \in X_i^{p,n}$ is said to be *compatible* if for every $0 \leq i \leq m$, $d_i x_i^{p,n} = 0$ and $\varphi(x_{i+1}^{p,n}) = [x_i^{p,n}]$ where $[x_i^{p,n}]$ is the class of $x_i^{p,n}$ in $H(X_i)$.
- (2) Denote by $\mathcal{N}(X)_r^{p,n}$ the \mathbb{R} -submodule of $X^{\oplus(2r+2)}$ consisting of pairs

$$(x_0^{p,n}, \dots, x_r^{p,n}; y_0^{p-r,n+1-r}, \dots, y_r^{p-r,n+1-r})$$

of compatible sequences satisfying $d_r x_r^{p,n} = y_r^{p-r,n+1-r}$. This yields a functor, denoted by $\mathcal{N}_r^{p,n}: \mathbf{ESpSe} \rightarrow \mathbf{Mod}_{\mathbb{R}}$.

Notation 3.2.3. We denote by $\mathbf{R}(p, n)$ the spectral sequence which at each page has a free module of rank 1 concentrated in bidegree (p, n) and all differentials zero.

The following proposition is a direct consequence of the definitions.

Proposition 3.2.4. *Let (X, φ) be an extended spectral sequence.*

- (1) *There is a one-to-one correspondence between infinite compatible sequences $(x_0^{p,n}, \dots, x_m^{p,n}, \dots)$ and morphisms of extended spectral sequences $\mathbf{R}(p, n) \rightarrow X$.*
- (2) *We have $\mathcal{N}_r^{p,n} = \mathbf{Hom}_{\mathbf{ESpSe}}(\mathcal{D}_r(p, n), -)$, that is, $\mathcal{N}_r^{p,n}$ is represented by $\mathcal{D}_r(p, n)$. \square*

Remark 3.2.5. Later we will see that the functors $\mathcal{N}_r^{p,n}$ assemble to a nerve functor from extended spectral sequences to a presheaf category; see Proposition 5.1.1.

4. THE PRESHEAF CATEGORY OF LINEAR WITNESS BOOKS

In this section we introduce and study a linear presheaf category closely related to spectral sequences.

4.1. The category \mathbf{D} . We define the underlying category on which we will consider presheaves. We show in Proposition 4.1.7 that it can be understood as the full linear subcategory of \mathbf{SpSe} generated by the disc objects $\mathcal{D}_r(p, n)$.

Definition 4.1.1. Let \mathbf{D} be the small linear category defined as follows. Objects of \mathbf{D} are triples (r, p, n) with $r \geq 0$ and $p, n \in \mathbb{Z}$. Morphisms in the category are generated by the following morphisms for all $r \geq 0$ and $p, n \in \mathbb{Z}$.

$$\begin{aligned}\text{co-witness} & \quad \omega_{r+1}^{p,n}: (r, p, n) \rightarrow (r+1, p, n), \\ \text{co-differential} & \quad \delta_r^{p,n}: (r, p-r, n-r+1) \rightarrow (r, p, n), \\ \text{co-degeneracy} & \quad \sigma_r^{p,n}: (r+1, p+1, n+1) \rightarrow (r, p, n),\end{aligned}$$

subject to the following relations for all $r \geq 0$ and $p, n \in \mathbb{Z}$.

$$\sigma_r^{p-1, n-1} \omega_{r+1}^{p,n} = 0, \quad \omega_{r+1}^{p,n} \sigma_r^{p,n} = 0, \quad \sigma_r^{p+r, n+r-1} \delta_{r+1}^{p+r+1, n+r} \omega_{r+1}^{p,n} = \delta_r^{p+r, n+r-1}.$$

Remark 4.1.2. The relations

$$\delta_r^{p+r, n+r-1} \delta_r^{p, n} = \delta_r^{p+r, n+r-1} \sigma_r^{p, n} = \omega_{r+1}^{p, n} \delta_r^{p, n} = 0 \quad \text{for all } r \geq 0, p, n \in \mathbb{Z}$$

also hold in \mathbf{D} as an immediate consequence.

The following lemma is a direct consequence of the definition and gives all the non-zero \mathbf{R} -modules of morphisms in \mathbf{D} .

Lemma 4.1.3. *For every $p, n \in \mathbb{Z}$, $r \geq 0$,*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}}((r-i, p, n), (r, p, n)) &= \mathbf{R}(\omega_r)^i && \text{for } r \geq i \geq 0, \\ \mathrm{Hom}_{\mathbf{D}}((r+j, p+j, n+j), (r, p, n)) &= \mathbf{R}(\sigma_r)^j && \text{for } j > 0, \\ \mathrm{Hom}_{\mathbf{D}}((r, p-r, n-r+1), (r, p, n)) &= \mathbf{R}\delta_r, \\ \mathrm{Hom}_{\mathbf{D}}((r-i, p, n), (r, p+r, n+r-1)) &= \mathbf{R}\delta_r(\omega_r)^i && \text{for } r \geq i > 0, \\ \mathrm{Hom}_{\mathbf{D}}((r+j, p, n), (r, p+r, n+r-1)) &= \mathbf{R}(\sigma_r)^j \delta_{r+j} && \text{for } j > 0. \end{aligned}$$

Here we have omitted the upper indices on the morphisms; they can be determined by the hom modules in which the morphisms live. We denote by $(\omega_r)^i$ the composite $\omega_r \omega_{r-1} \dots \omega_{r-i+1}$ and the same notation is used for the iterated composite of the σ s. For $i = 0$, it denotes the identity map. \square

Remark 4.1.4. The category \mathbf{D} is self-dual. Indeed, for a morphism $\alpha: C \rightarrow C'$ in a category \mathbf{C} we write $\bar{\alpha}$ for the corresponding morphism $C' \rightarrow C$ in \mathbf{C}^{op} . Then an isomorphism $\mathbf{D} \rightarrow \mathbf{D}^{\mathrm{op}}$ is given on objects by $(r, p, n) \mapsto (r, r-p, r-n)$ and on generating morphisms by $\delta_r^{p, n} \mapsto \delta_r^{2r-p, 2r-n-1}$, $\sigma_r^{p, n} \mapsto \omega_{r+1}^{r-p, r-n}$, $\omega_{r+1}^{p, n} \mapsto \sigma_r^{r-p, r-n}$.

We define some maps of spectral sequences between the disc objects $\mathcal{D}_r(p, n)$ of Definition 3.2.1.

Definition 4.1.5. Recall that we denote by $e_{r,i}^{p,n}$ a generator of $\mathcal{D}_r(p, n)_i^{p,n}$ and by $f_{r,i}^{p-r, n+1-r}$ a generator of $\mathcal{D}_r(p, n)_i^{p-r, n+1-r}$, for $0 \leq i \leq r$. We define maps of spectral sequences

$$\begin{aligned} \omega: \mathcal{D}_r(p, n) &\rightarrow \mathcal{D}_{r+1}(p, n) \\ \delta: \mathcal{D}_r(p, n) &\rightarrow \mathcal{D}_r(p+r, n+r-1) \\ \sigma: \mathcal{D}_r(p, n) &\rightarrow \mathcal{D}_{r-1}(p-1, n-1) \end{aligned}$$

determined by, for $0 \leq i \leq r$,

$$\begin{aligned} \omega(e_{r,i}^{p,n}) &= e_{r+1,i}^{p,n}, \\ \omega(f_{r,i}^{p-r, n+1-r}) &= 0, \\ \delta(e_{r,i}^{p,n}) &= f_{r,i}^{p,n}, \\ \delta(f_{r,i}^{p-r, n+1-r}) &= 0, \\ \sigma(e_{r,i}^{p+1, n+1}) &= 0, \end{aligned}$$

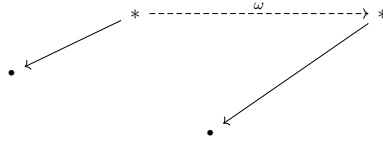
and

$$\begin{aligned}\sigma(f_{r,i}^{p-r,n+1-r}) &= f_{r-1,i}^{p-r,n+1-r} \quad \text{for } 0 \leq i < r, \\ \sigma(f_{r,r}^{p-r,n+1-r}) &= 0.\end{aligned}$$

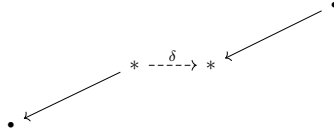
Remark 4.1.6. If we think of the generators e and f as the *source* and *target* respectively, ω maps source to source, δ maps source to target and σ maps target to target.

In the following diagrams both \bullet and $*$ represent \mathbf{R} , a free module of rank 1, the vertex marked $*$ being in bidegree (p, n) . The solid arrows are identity maps within disc objects and the dashed arrows are identity maps between source or target of the disc objects as indicated.

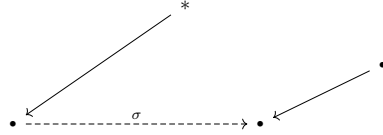
The morphism $\omega: \mathcal{D}_r(p, n) \rightarrow \mathcal{D}_{r+1}(p, n)$ can be depicted as follows.



The corresponding picture for $\delta: \mathcal{D}_r(p, n) \rightarrow \mathcal{D}_r(p+r, n+r-1)$ is:



and for $\sigma: \mathcal{D}_r(p, n) \rightarrow \mathcal{D}_{r-1}(p-1, n-1)$:



Proposition 4.1.7. *The category \mathbf{D} is isomorphic to the full subcategory of \mathbf{SpSe} generated by the objects $\mathcal{D}_r(p, n)$ for $r \geq 0$ and $p, n \in \mathbb{Z}$, via the assignment $(r, p, n) \mapsto \mathcal{D}_r(p, n)$.*

Proof. We send the generating morphisms of \mathbf{D} to the correspondingly named morphisms of spectral sequences given in Definition 4.1.5. It is straightforward to check that the relations of Definition 4.1.1 are satisfied by these morphisms. Thus we have a well-defined functor from \mathbf{D} to \mathbf{SpSe} .

We claim that non-trivial morphisms of spectral sequences $\mathcal{D}_r(p, n) \rightarrow \mathcal{D}_{r'}(p', n')$ correspond exactly to the non-trivial morphisms in \mathbf{D} , as specified in Lemma 4.1.3.

This is checked by direct inspection of morphisms of spectral sequences $\mathcal{D}_r(p, n) \rightarrow \mathcal{D}_{r'}(p', n')$. Briefly, we can have such a morphism of type “source to source” only for $r < r'$ and these are generated by composites of ω s. There is a morphism of type “target to target” only for $r > r'$ and these are generated by composites of σ s. And morphisms of type “source to target” are generated by δ if $r = r'$, by composites of the form $\delta\omega^i$ if $r < r'$ and by composites of the form $\sigma^j\delta$ if $r > r'$. There is no

non-trivial map of spectral sequences of type “target to source” and, together with identity maps, this accounts for all the morphisms. \square

4.2. Linear witness books.

Definition 4.2.1. We write **LWB** for the linear presheaf category $\widehat{\mathbf{D}}$. A *linear witness book* is an object of $\widehat{\mathbf{D}}$ and a *morphism of linear witness books* is a morphism in $\widehat{\mathbf{D}}$.

The following proposition is a direct consequence of the definitions.

Proposition 4.2.2. A linear witness book L is a collection of r -bigraded complexes (L_r, d_r) for $r \geq 0$, endowed with \mathbf{R} -linear witness maps $w_{r+1}: L_{r+1} \rightarrow L_r$ of bidegree $(0, 0)$ and \mathbf{R} -linear degeneracy maps $s_r: L_r \rightarrow L_{r+1}$ of bidegree $(1, 1)$ satisfying the relations

$$w_{r+1}s_r = 0, \quad s_r w_{r+1} = 0 \quad \text{and} \quad d_r = w_{r+1}d_{r+1}s_r, \quad \text{for all } r \geq 0.$$

A morphism of linear witness books $f: K \rightarrow L$ is a collection of morphisms of r -bigraded complexes $f_r: K_r \rightarrow L_r$ for $r \geq 0$, compatible with the witness and degeneracy maps. \square

Definition 4.2.3. The r -page of a linear witness book $L \in \mathbf{LWB}$ is the r -bigraded complex (L_r, d_r) .

We write (L, d_i, w_i, s_i) for an object of **LWB**.

Remark 4.2.4. For $(L, d_i, w_i, s_i) \in \mathbf{LWB}$, we also have the relations $s_r d_r = d_r w_{r+1} = 0$. The relation $d_r^2 = 0$ is redundant, but the connection to spectral sequences is clearer if we explicitly view the r -page as an r -bigraded complex.

It follows directly from the description above that for any linear witness book (L, d_i, w_i, s_i) we have, for all $i \geq 0$,

$$(2) \quad \text{Im } d_i \subseteq \text{Im } w_{i+1} \subseteq \text{Ker } s_i \subseteq \text{Ker } d_i \quad \text{and} \quad \text{Im } s_i \subseteq \text{Ker } w_{i+1}.$$

The combinatorics of the representable object $\mathcal{Y}(r, p, n)$ are treated in Example A.1.1.

Proposition 4.2.5. The embedding $\mathbf{D} \hookrightarrow \mathbf{ESpSe}$ which associates to (r, p, n) the spectral sequence $\mathcal{D}_r(p, n)$ induces an adjunction

$$\mathbf{LWB} \begin{array}{c} \xrightarrow{\mathcal{Q}} \\ \perp \\ \xleftarrow{\mathcal{N}} \end{array} \mathbf{ESpSe}$$

with the right adjoint defined as $\mathcal{N}(X)(r, p, n) = \text{Hom}_{\mathbf{ESpSe}}(\mathcal{D}_r(p, n), X)$.

Proof. Apply Proposition 2.1.1, noting that \mathbf{ESpSe} is a cocomplete linear category tensored over $\text{Mod}_{\mathbf{R}}$ by Proposition 3.1.2. \square

By Proposition 3.2.4, the functor \mathcal{N} is explicitly described in Definition 3.2.2.

5. RELATING LINEAR WITNESS BOOKS AND EXTENDED SPECTRAL SEQUENCES

In this section we explore how the categories of linear witness books and extended spectral sequences are related. We begin by supplying more details of the $(\mathcal{Q}, \mathcal{N})$ adjunction and explaining its properties. This will allow us to identify a full subcategory of LWB that is equivalent to \mathbf{ESpSe} . Importantly, this is a reflective subcategory, allowing us to conclude in Theorem 5.3.2 that \mathbf{ESpSe} is bicomplete. We also provide characterizations of (extended) spectral sequences inside LWB.

5.1. Description and properties of the main adjunction. Though the $(\mathcal{Q}, \mathcal{N})$ adjunction comes from standard category theory, it is important to describe the functors. The functor \mathcal{N} , similar to a nerve functor, is easily defined. The functor \mathcal{Q} , obtained via a coend formula, has a nice description as a quotient by degeneracies.

Proposition 5.1.1. *For the functor $\mathcal{N}: \mathbf{ESpSe} \rightarrow \mathbf{LWB}$, the module $\mathcal{N}(X)_r^{p,n}$ is as described explicitly in part (2) of Definition 3.2.2. In addition, for any pair of compatible sequences $(x; x') = ((x_0, \dots, x_r); (x'_0, \dots, x'_r))$ in $\mathcal{N}(X)_r$ with $d_r^X x_r = x'_r$ we have*

$$d_r(x; x') = (x'; 0), \quad w_r(x; x') = ((x_0, \dots, x_{r-1}); 0), \quad s_r(x; x') = (0; (x'_0, \dots, x'_r, 0)).$$

Proof. We have to show that the maps d_r, w_r, s_r have the correct expression, which is a matter of directly interpreting the expression $\mathcal{N}(X)_r^{p,n} = \mathbf{Hom}_{\mathbf{ESpSe}}(\mathcal{D}_r(p, n), X)$. \square

Notation 5.1.2. For a linear witness book L and $r > 0$, we write $S_r = S_r(L)$ for the sub r -bigraded complex $s_{r-1}L_{r-1} + d_r s_{r-1}L_{r-1}$ of L_r .

Theorem 5.1.3. *The functor $\mathcal{Q}: \mathbf{LWB} \rightarrow \mathbf{ESpSe}$ assigns to $L \in \mathbf{LWB}$ the extended spectral sequence $\mathcal{Q}(L)$, where $\mathcal{Q}(L)_0 = L_0$ and for $r > 0$,*

$$\mathcal{Q}(L)_r = L_r / S_r,$$

with differential induced by that of L_r and with characteristic map $\varphi_r: \mathcal{Q}(L)_r \rightarrow H(\mathcal{Q}(L)_{r-1})$ defined by

$$\varphi_r(\bar{a}) = \overline{[w_r(a)]}.$$

Here bars denote classes in the quotient by S_r and square brackets denote homology classes.

For X an extended spectral sequence, the counit of the $(\mathcal{Q}, \mathcal{N})$ adjunction

$$\epsilon_X: \mathcal{QN}(X) \rightarrow X$$

is induced by the projection $\mathcal{N}(X)_r \rightarrow X_r$ which associates x_r to $(x; x')$.

The counit is a componentwise isomorphism. In particular, \mathcal{N} is fully faithful and \mathcal{Q} is essentially surjective.

For L a linear witness book, the unit of the $(\mathcal{Q}, \mathcal{N})$ adjunction

$$\eta_L: L \rightarrow \mathcal{NQ}(L),$$

is given, for $a \in L_r$, by

$$(\eta_L)_r(a) = \left(\overline{w^r(a)}, \dots, \overline{w(a)}, \bar{a}; \overline{w^r(d_r a)}, \dots, \overline{w(d_r a)}, \overline{d_r a} \right).$$

The unit is componentwise surjective.

Proof. Though the description of \mathcal{Q} has been obtained by the coend formula of Proposition 2.1.1, we prove that $(\mathcal{Q}, \mathcal{N})$ is an adjunction by checking the triangle identities relating the unit and the counit. That ϵ and η are well defined natural transformations is an easy check. We fix X an extended spectral sequence and L a linear witness book. Proposition 5.1.1 implies that

$$S_r(\mathcal{N}(X)) = \{(x; x') \in \mathcal{N}_r(X) \mid x_r = 0\}.$$

In particular, in $\mathcal{QN}(X)_r$, one has $\overline{(x; x')} = \overline{(y; y')}$ if and only if $x_r = y_r$. We claim that this implies the following.

- a) $(\mathcal{Q}\eta_L) \circ \epsilon_{\mathcal{QL}} = 1_{\mathcal{QN}\mathcal{QL}}$.
- b) The counit ϵ_X is componentwise injective.
- c) $(\eta_{\mathcal{NX}}) \circ \mathcal{N}\epsilon_X = 1_{\mathcal{N}\mathcal{QN}X}$.

For a) and b) this is a direct check. Let us prove c). Let $(x; y) \in (\mathcal{N}\mathcal{QN}X)_r$ with $x = (\bar{x}_0, \dots, \bar{x}_r)$ and $y = (\bar{y}_0, \dots, \bar{y}_r)$ where x_i and y_i are in $(\mathcal{NX})_i$. Denote by x_i^i and y_i^i their projections onto X_i , so that $a := \mathcal{N}\epsilon_X(x; y) = (x_0^0, \dots, x_r^r; y_0^0, \dots, y_r^r) \in (\mathcal{NX})_r$. We have, for $i \geq 1$, $w^i(a) = (x_0^0, \dots, x_{r-i}^{r-i}; 0)$ and $w^i d_r a = (y_0^0, \dots, y_{r-i}^{r-i}, 0) \in (\mathcal{NX})_{r-i}$. As a consequence $\overline{w^i a} = \bar{x}_{r-i}$ and $\overline{w^i d_r a} = \bar{y}_{r-i}$ in $(\mathcal{QN}X)_{r-i}$. This shows that $\eta_{\mathcal{NX}}(a) = (x; y)$.

Claims a) and c) imply that the left adjoint of \mathcal{N} is the described functor \mathcal{Q} .

The counit is componentwise surjective because the projection $\mathcal{N}(X)_r \rightarrow X_r$ is. Let us prove that the unit is componentwise surjective.

Let $\alpha = (\bar{a}_0, \dots, \bar{a}_r; \bar{b}_0, \dots, \bar{b}_r) \in \mathcal{N}\mathcal{Q}(L)_r$. We claim that for $j = 0, \dots, r$ there is an element $c_{r-j} \in L_r$ such that $\overline{w^i c_{r-j}} = \bar{a}_{r-i}$ and $\overline{w^i d_r c_{r-j}} = \bar{b}_{r-i}$ for $0 \leq i \leq j$. Then $(\eta_L)_r(c_0) = \alpha$ and $(\eta_L)_r$ is surjective.

We prove the claim by induction on j . For $j = 0$, we can take $c_r = a_r$. Now assume we have c_{r-j} as above. We have $\overline{[w^{j+1} c_{r-j}]} = \overline{[\bar{a}_{r-j-1}]}$ in $H((\mathcal{QL})_{r-j-1})$. So there is some $c' \in L_{r-j-1}$ such that

$$\overline{w^{j+1} c_{r-j}} = \overline{\bar{a}_{r-j-1}} + \overline{d_{r-j-1} c'} = \overline{\bar{a}_{r-j-1}} + \overline{w_{r-j} d_{r-j} s_{r-j-1} c'}$$

and so

$$\overline{w_{r-j} (w^j c_{r-j} - d_{r-j} s_{r-j-1} c')} = \overline{\bar{a}_{r-j-1}}.$$

Similarly, since $\overline{[w^{j+1} d_r c_{r-j}]} = \overline{[\bar{b}_{r-j-1}]}$ in $H((\mathcal{QL})_{r-j-1})$, there is some $c'' \in X_{r-j-1}$ such that

$$\overline{w_{r-j} (w^j d_r c_{r-j} - d_{r-j} s_{r-j-1} c'')} = \overline{\bar{b}_{r-j-1}}.$$

Then it is straightforward to check that $c_{r-j-1} = c_{r-j} - d_r s^{j+1} c' - s^{j+1} c''$ has the required properties. \square

Proposition 5.1.4. *There is a forgetful functor $\mathcal{U}: \text{LWB} \rightarrow \text{ESpSe}$ and the quotient maps $L_r \rightarrow \mathcal{Q}(L)_r$ give rise to a natural transformation of functors from \mathcal{U} to \mathcal{Q} .*

Proof. Let L be a linear witness book. Since $d_{r-1} w_r = 0$, the collection (L_r, d_r) of r -bigraded complexes may be endowed with the characteristic maps $\varphi_r: L_r \rightarrow H(L_{r-1})$ given by $\varphi_r(a) = [w_r(a)]$. This gives the extended spectral sequence $\mathcal{U}(L)$. It is clear that a morphism of linear witness books $f: L \rightarrow K$ gives a morphism of extended spectral sequences $\mathcal{U}(L) \rightarrow \mathcal{U}(K)$.

By Theorem 5.1.3 $\mathcal{Q}(L)_r$ is isomorphic to the quotient L_r/S_r . The description of the characteristic maps of $\mathcal{Q}(L)$ in Theorem 5.1.3 shows that this gives a morphism $\mathcal{U}(L) \rightarrow \mathcal{Q}(L)$ in \mathbf{ESpSe} . \square

The following result will be important for understanding model category structures later.

Proposition 5.1.5. *Let $f: X \rightarrow Y$ be a morphism in \mathbf{ESpSe} . Then f_i is surjective for $0 \leq i \leq r$ if and only if $\mathcal{N}(f)_i$ is surjective for $0 \leq i \leq r$.*

Proof. It is clear that if $\mathcal{N}(f)_r$ is surjective then f_r is surjective. Let us prove that if f_i is surjective for $0 \leq i \leq r$, then so is $\mathcal{N}(f)_i$ for $0 \leq i \leq r$, by induction on r . This is true for $r = 0$. Let X be an extended spectral sequence. We have seen in the proof of Theorem 5.1.3 that Proposition 5.1.1 implies that $S_r(\mathcal{N}X)$ is the kernel of the projection $(\mathcal{N}X)_r \rightarrow X_r$. In addition, $S_r(\mathcal{N}X) = s_{r-1}(\mathcal{N}(X)_{r-1}) \oplus d_r s_{r-1}(\mathcal{N}(X)_{r-1})$. Hence, for $r \geq 1$ and $p, n \in \mathbb{Z}$, there is a natural short exact sequence of \mathbf{R} -modules

$$s_{r-1}(\mathcal{N}(X)_{r-1}^{p-1, n-1}) \oplus d_r s_{r-1}(\mathcal{N}(X)_{r-1}^{p+r-1, n+r-2}) \hookrightarrow \mathcal{N}(X)_r^{p, n} \twoheadrightarrow X_r^{p, n}$$

This implies that if $\mathcal{N}(f)_{r-1}$ is surjective, then f_r is surjective if and only if $\mathcal{N}(f)_r$ is surjective. \square

5.2. Characterizing extended spectral sequences as presheaves. We characterize the essential image of the functor \mathcal{N} , thus giving a subcategory of the category of linear witness books \mathbf{LWB} which is equivalent to the category of extended spectral sequences \mathbf{ESpSe} .

Proposition 5.2.1. *An object (L, d_i, w_i, s_i) of \mathbf{LWB} is in the essential image of \mathcal{N} if and only if $\text{Ker } d_i = \text{Ker } s_i$ for all $i \geq 0$.*

Proof. If L is isomorphic to $\mathcal{N}X$, the description of the maps w, s, d given in Proposition 5.1.1 proves that $\text{Ker } d_i = \text{Ker } s_i$ for all $i \geq 0$.

We have that L is in the essential image of \mathcal{N} if and only if $(\eta_L)_j$ is injective for all j , since it is surjective by Theorem 5.1.3.

Let us assume that $\text{Ker } d_i = \text{Ker } s_i$ for all $i \geq 0$. We prove by induction on r that $\text{Ker}(\eta_L)_r = 0$. Firstly note that $\text{Ker}(\eta_L)_0 = 0$ for all L since $(\mathcal{N}(\mathcal{Q}L))_0$ is isomorphic to L_0 . Hence the result is valid for $r = 0$. Let $r \geq 0$ and assume the statement holds for every $k \leq r$. Let $a \in L_{r+1}$ be such that $(\eta_L)_{r+1}(a) = 0$. Thus $(\eta_L)_r(w_{r+1}a) = 0$ and $(\eta_L)_r(w_{r+1}d_{r+1}a) = 0$, and so $w_{r+1}a = 0$ and $w_{r+1}d_{r+1}a = 0$ since $(\eta_L)_r$ is injective. By assumption, the image of a in $(\mathcal{Q}L)_{r+1}$ is zero, thus there exist α, β such that $a = s_r\alpha + d_{r+1}s_r\beta$. Applying w_{r+1} we get $0 = w_{r+1}a = d_r\beta$. So $\beta \in \text{Ker } d_r \subseteq \text{Ker } s_r$ and hence $s_r\beta = 0$. Applying $w_{r+1}d_{r+1}$ we get $0 = w_{r+1}d_{r+1}a = d_r\alpha$, hence $\alpha \in \text{Ker } d_r \subseteq \text{Ker } s_r$ and so $s_r\alpha = 0$. In conclusion $a = 0$. \square

Definition 5.2.2. We denote by $(\mathbf{LWB})^e$ the full subcategory of \mathbf{LWB} on objects satisfying $\text{Ker } d_i = \text{Ker } s_i$ for all $i \geq 0$.

Theorem 5.2.3. *The fixed point equivalence associated to the adjunction $(\mathcal{Q}, \mathcal{N})$ gives an equivalence of categories $(\mathbf{LWB})^e \sim \mathbf{ESpSe}$.*

Proof. It follows from Theorem 5.1.3 and Proposition 5.2.1 that the adjoint functors \mathcal{Q} and \mathcal{N} restrict to give an equivalence:

$$(\text{LWB})^e \begin{array}{c} \xrightarrow{\mathcal{Q}} \\ \xleftarrow[\mathcal{N}]{\sim} \end{array} \text{ESpSe} . \quad \square$$

We now note some properties of the category $(\text{LWB})^e$. In Proposition A.2.3, we explain how to characterize $(\text{LWB})^e$ via a right lifting property. And in Remark A.2.4, we note that it can be viewed as a certain subcategory of local objects.

- Proposition 5.2.4.** (1) For L in $(\text{LWB})^e$, the r -bigraded complex $S_r(L)$ is the direct sum $s_{r-1}L_{r-1} \oplus d_r s_{r-1}L_{r-1}$ and $S_r(L)$ is acyclic for all r .
(2) For L in $(\text{LWB})^e$, the quotient map $L_r \rightarrow \mathcal{Q}(L)_r$ is a quasi-isomorphism for all r .
(3) For X an extended spectral sequence, the projection $\rho_r: (\mathcal{N}X)_r \rightarrow X_r$ is a quasi-isomorphism for all r .
(4) For X an extended spectral sequence satisfying $d_i = 0$ for $0 \leq i \leq r-1$ and $\varphi_i = 1$ for $1 \leq i \leq r$, the projection $\rho_r: (\mathcal{N}X)_r \rightarrow X_r$ is an isomorphism.

Proof. Let $a \in L$ with $s_{r-1}a = d_r s_{r-1}b$. Then applying w_r we get $d_{r-1}b = 0$ hence $s_{r-1}b = 0$ and so $s_{r-1}a = 0$ too. So we see that $S_r(L)$ is a direct sum. So it is the cone in r -bigraded complexes of the identity map of $s_{r-1}L_{r-1}$ and so acyclic. Since the kernel of the quotient map $L_r \rightarrow \mathcal{Q}(L)_r$ is $S_r(L)$, we deduce that the quotient map is a quasi-isomorphism for all r . The projection $\mathcal{N}X \rightarrow X$ is the composition of the quotient map $\mathcal{N}X \rightarrow \mathcal{Q}\mathcal{N}X$ followed by the counit ϵ_X of the adjunction $(\mathcal{Q}, \mathcal{N})$, which is an isomorphism. Since $\mathcal{N}X \in (\text{LWB})^e$, the third statement follows. The final statement can be seen directly from the description of $(\mathcal{N}X)_r$. \square

5.3. Limits and colimits in extended spectral sequences.

Proposition 5.3.1. *The category $(\text{LWB})^e$ is a full reflective subcategory of LWB . It has all small limits and colimits, with limits computed pagewise and colimits via the reflector $\mathcal{N}\mathcal{Q}$.*

Proof. By Theorem 5.1.3, the counit of the adjunction $(\mathcal{Q}, \mathcal{N})$ is a natural isomorphism. It follows that the essential image of the right adjoint functor \mathcal{N} is a full reflective subcategory of LWB , with reflector $\mathcal{N}\mathcal{Q}$ left adjoint to the inclusion. Proposition 5.2.1 identifies the essential image of \mathcal{N} as $(\text{LWB})^e$.

Note that LWB is bicomplete with small limits and colimits, calculated componentwise. It follows that the inclusion creates limits and the full subcategory also has all colimits, obtained by applying the reflector to colimits in LWB [25, Proposition 4.5.15]. \square

Theorem 5.3.2. *The category ESpSe has all small limits and colimits, with colimits computed pagewise.*

Proof. The existence of small limits and colimits follows from the equivalence of categories of Theorem 5.2.3. In Proposition 3.1.2, we saw that colimits are page-wise. If $G: I \rightarrow \text{ESpSe}$ is a functor from a small category I , then $\lim_i G(i) = \mathcal{Q}(\lim_i \mathcal{N}(G(i)))$. \square

As we noted earlier, the category \mathbf{ESpSe} is not cotensored over $\mathbf{Mod}_{\mathbf{R}}$. So, although it is complete, one should not expect \mathcal{N} to have a right adjoint. Indeed, \mathcal{N} does not preserve all colimits, but we do have the following result which will be useful for transfer of model structures later.

Proposition 5.3.3. *The functor $\mathcal{N}: \mathbf{ESpSe} \rightarrow \mathbf{LWB}$ preserves filtered colimits.*

Proof. We recall that colimits are computed pagewise in \mathbf{ESpSe} and in \mathbf{LWB} . Suppose that $i: \mathcal{C} \rightarrow \mathcal{C}'$ is the inclusion of a full subcategory, that $\text{colim}_J(i \circ X)$ exists in \mathcal{C}' and is (isomorphic to) an object of \mathcal{C} . Then $\text{colim}_J X$ exists and is given by the same object. Applying this to the inclusion of the essential image of \mathcal{N} into \mathbf{LWB} , we therefore need to show that for filtered colimits in $(\mathbf{LWB})^e$ the pagewise colimit lies in $(\mathbf{LWB})^e$.

Let $L: J \rightarrow (\mathbf{LWB})^e$ be a diagram in $(\mathbf{LWB})^e$ with J a filtered category. The pagewise colimit is

$$C := \text{colim}_J L(j) = (\oplus_{j \in J} L(j)) / \sim$$

where $a \sim a'$ for $a \in L(j), a' \in L(j')$ if there exist $f: j \rightarrow k, f': j' \rightarrow k$ such that $L(f)(a) = L(f')(a')$. Let $[a] = [(a_j)_j] \in C$, where finitely many a_j are non-zero, say for $j \in J'$, a finite subset of J . Suppose $d_i[a] = 0$ for some $i \geq 0$. Now since J is filtered there exists $k \in J$ and $f_j: j \rightarrow k$ in J for all $j \in J'$. Let $b = \sum_{j \in J'} L(f_j)(a_j) \in L(k)$. Then $a = \sum_{j \in J'} a_j \sim b$. Now $0 = d_i[a] = d_i[b] = [d_i b]$. So $d_i b \sim 0$ and there exists $f: k \rightarrow k' \in J$ such that $d_i L(f)(b) = L(f)(d_i b) = 0$. Thus $L(f)(b) \in \text{Ker } d_i = \text{Ker } s_i$ in $L(k')$. And since $a \sim b \sim L(f)(b)$, we see that $s_i[a] = 0$. Thus C satisfies $\text{Ker } d_i = \text{Ker } s_i$ for all $i \geq 0$ and so C lies in $(\mathbf{LWB})^e$. \square

5.4. Characterizing spectral sequences as presheaves.

Definition 5.4.1. We denote by $(\mathbf{LWB})^s$ the subcategory of \mathbf{LWB} given by the essential image under \mathcal{N} of the category of spectral sequences \mathbf{SpSe} .

By restricting our adjunction we obtain an equivalence of categories $(\mathbf{LWB})^s \sim \mathbf{SpSe}$. Equivalently, $(\mathbf{LWB})^s$ is the full subcategory on those L in $(\mathbf{LWB})^e$ such that QL is a spectral sequence.

The following diagram gives an overview of the relationships between these categories.

$$\begin{array}{ccc}
 \mathbf{LWB} & \xrightarrow{Q} & \mathbf{ESpSe} \\
 \uparrow & \xleftarrow{\mathcal{N}} & \uparrow \\
 (\mathbf{LWB})^e & \xleftarrow{\sim} & \mathbf{SpSe} \\
 \uparrow & & \uparrow \\
 (\mathbf{LWB})^s & \xleftarrow{\sim} & \mathbf{SpSe}
 \end{array}$$

Proposition 5.4.2. *Let (L, d_i, w_i, s_i) be an object of $(\mathbf{LWB})^e$ and let $i \geq 0$.*

- (1) *The characteristic map φ_{i+1} of QL is surjective if and only if in L we have $\text{Im } w_{i+1} = \text{Ker } d_i = \text{Ker } s_i$.*

(2) The characteristic map φ_{i+1} of $\mathcal{Q}L$ is injective if and only if in L we have $\text{Ker } w_{i+1} = \text{Im } s_i$.

Thus an object (L, d_i, w_i, s_i) of LWB lies in $(\text{LWB})^s$ if and only if $\text{Im } w_{i+1} = \text{Ker } d_i = \text{Ker } s_i$ and $\text{Ker } w_{i+1} = \text{Im } s_i$ for all $i \geq 0$.

Proof. We may consider L as $L = \mathcal{N}X$ with X an extended spectral sequence. In particular $\mathcal{Q}L$ is isomorphic to X by Theorem 5.1.3. We recall that $\text{Im } w_{i+1} \subseteq \text{Ker } s_i$ and $\text{Im } s_i \subseteq \text{Ker } w_{i+1}$. Let $(x; y)$ in $(\mathcal{N}X)_i$. We have

- $(x; y) \in \text{Im } w_{i+1}$ if and only if $y = 0$ (and thus $d_i x_i = 0$) and there exists $x_{i+1} \in X_{i+1}$ such that $\varphi_{i+1}(x_{i+1}) = [x_i]$.
- $(x; y) \in \text{Ker } s_i$ if and only if $y = 0$ (and thus $d_i x_i = 0$).

In consequence the characteristic map φ_{i+1} of $\mathcal{Q}L$ is surjective if and only if in L we have $\text{Im } w_{i+1} = \text{Ker } s_i = \text{Ker } d_i$.

Let $(x; y)$ in $(\mathcal{N}X)_{i+1}$. We have

- $(x; y) \in \text{Im } s_i$ if and only if $x = 0$ (and thus $x_{i+1} = 0$).
- $(x; y) \in \text{Ker } w_{i+1}$ if and only if $(x_0, \dots, x_i) = (0, \dots, 0)$.

In consequence $\text{Ker } w_{i+1} = \text{Im } s_i$ if and only if, for all $x_{i+1} \in X_{i+1}$, $\varphi_{i+1}(x_{i+1}) = 0 \Rightarrow x_{i+1} = 0$. So the characteristic map φ_{i+1} of $\mathcal{Q}L$ is injective if and only if in L we have $\text{Ker } w_{i+1} = \text{Im } s_i$.

The final statement follows since $L \in (\text{LWB})^s$ if and only if the characteristic map φ_{i+1} of $\mathcal{Q}L$ is an isomorphism for all $i \geq 0$. \square

In Proposition A.2.5, we explain how to characterize $(\text{LWB})^s$ via a right lifting property.

6. TRUNCATION, SHIFT AND DÉCALAGE

In this section we introduce various truncated versions of the categories \mathbf{D} , LWB and ESpSe and set up functors and adjunctions relating these. Indeed all of these categories contain nested copies of themselves giving rise to these adjunctions.

In the first part we consider truncated versions of \mathbf{D} , where we take objects (i, p, n) with $r \leq i \leq t$. We study the embedding of this truncated version into \mathbf{D} , and the associated triple of adjoint functors at the level of linear witness books. The special case $r = t$ will be used for transfer of model category structures in Section 7.1.

In the second part we consider the case $t = \infty$. In this case, the embedding into \mathbf{D} has both a left and a right adjoint. This triple of adjoint functors gives rise to a sequence of five adjoint functors on LWB . Of these, we will see that to the leftmost three there is a corresponding triple of adjoints on ESpSe and on SpSe . When $r = 1$ we use the term *décalage* for the functor on ESpSe or SpSe corresponding to the embedding and denote it by Dec . Its left adjoint is called the *shift* functor and the left adjoint of the shift is a variant of *décalage*, which we denote LDec . We explain our use of the term *décalage* by noting compatibility with Deligne's notions for filtered complexes. The presheaf approach gives important insight in determining the correct form of the *décalage* functor for (extended) spectral sequences.

This section is quite technical, involving many adjoint functors. These arise formally as left or right Kan extensions as in Proposition 2.1.2. In several cases we

give explicit descriptions of these functors on objects, omitting the corresponding descriptions for morphisms and omitting proofs which are routine. We provide full details in the most delicate case of Proposition 6.3.6.

6.1. Truncation. We begin by considering truncated versions of the category \mathbf{D} and the corresponding truncations of the category of linear witness books \mathbf{LWB} , where we have only pages r to t , for some $0 \leq r \leq t \leq \infty$. Truncation of \mathbf{D} may be thought of as analogous to truncation of the simplex category, giving rise to (co)skeleta functors on simplicial sets.

Notation 6.1.1. For $0 \leq r \leq t \leq \infty$, we denote by $\mathrm{tr}_{[r,t]} \mathbf{D}$ the full subcategory of \mathbf{D} generated by the objects (i, p, n) for $r \leq i \leq t$. We write $\mathrm{tr}_{\geq r} \mathbf{D}$ for $\mathrm{tr}_{[r,\infty]} \mathbf{D}$ and $\mathrm{tr}_{\leq r} \mathbf{D}$ for $\mathrm{tr}_{[0,r]} \mathbf{D}$ when convenient. Note that $\mathrm{tr}_{[r,r]} \mathbf{D} = \langle \delta_r \rangle$.

We use the corresponding notation for \mathbf{LWB} . Indeed the category $\mathrm{tr}_{[r,t]} \mathbf{LWB}$ has for objects families of i -bigraded complexes (L_i, d_i) with $r \leq i \leq t$ together with linear maps $w_{i+1} : L_{i+1} \rightarrow L_i$ of bidegree $(0, 0)$ and $s_i : L_i \rightarrow L_{i+1}$ of bidegree $(1, 1)$ for $r \leq i < t$ subject to the same relations as those for linear witness books. Note that we have $\mathrm{tr}_{[r,r]} \mathbf{LWB} = \widehat{\langle \delta_r \rangle}$.

We denote by $\iota_{[r,t]} : \mathrm{tr}_{[r,t]} \mathbf{D} \rightarrow \mathbf{D}$ the embedding.

The functor $\iota_{[r,t]}$ induces adjunctions on the associated linear presheaf categories; see Proposition 2.1.2.

Definition 6.1.2. We define

$$\mathcal{U}_{[r,t]} = \iota_{[r,t]}^*, \quad \mathcal{F}_{[r,t]} = (\iota_{[r,t]})! \quad \text{and} \quad \mathcal{R}_{[r,t]} = (\iota_{[r,t]})_*.$$

So we have the adjunctions:

$$\begin{array}{ccc} & \mathcal{F}_{[r,t]} & \\ & \curvearrowright & \\ & \perp & \\ \mathrm{tr}_{[r,t]} \mathbf{LWB} & \xleftarrow{\mathcal{U}_{[r,t]}} & \mathbf{LWB} \\ & \perp & \\ & \curvearrowleft & \\ & \mathcal{R}_{[r,t]} & \end{array}$$

We note that $\mathcal{U}_{[r,t]} : \mathbf{LWB} \rightarrow \mathrm{tr}_{[r,t]} \mathbf{LWB}$ is the forgetful functor. The following proposition describes the left adjoint functor $\mathcal{F}_{[r,t]}$ explicitly. For the interested reader, the description of the right adjoint $\mathcal{R}_{[r,t]}$ is given in Proposition B.1.1. The latter functor does not behave as nicely as the former functor with respect to $(\mathbf{LWB})^e$ and $(\mathbf{LWB})^s$. We omit proofs since the arguments are routine.

Proposition 6.1.3. *Let (L, d_i, w_i, s_i) be an object of $\mathrm{tr}_{[r,t]} \mathbf{LWB}$. We have*

$$\mathcal{F}_{[r,t]}(L)_i^{p,n} = \begin{cases} L_r^{p,n} & \text{for } 0 \leq i < r, \\ L_i^{p,n} & \text{for } r \leq i \leq t, \\ \mathrm{Cone}_i(L_t/B_t(L_t))^{p+t-i, n+t-i} & \text{for } i > t, \end{cases}$$

with the maps $d_i^{\mathcal{F}}, w_i^{\mathcal{F}}, s_i^{\mathcal{F}}$ defined as

$$\begin{aligned} d_i^{\mathcal{F}} &= 0 & w_{i+1}^{\mathcal{F}} &= 1 & s_i^{\mathcal{F}} &= 0 & \text{for } 0 \leq i < r, \\ d_i^{\mathcal{F}} &= d_i & w_{i+1}^{\mathcal{F}} &= w_{i+1} & s_i^{\mathcal{F}} &= s_i & \text{for } r \leq i < t, \\ d_i^{\mathcal{F}} &= d_t & w_{i+1}^{\mathcal{F}} &= (0 \quad d_t) & s_i^{\mathcal{F}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } i = t, \\ d_i^{\mathcal{F}} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & w_{i+1}^{\mathcal{F}} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & s_i^{\mathcal{F}} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } i > t. \end{aligned}$$

The unit of the adjunction $(\mathcal{F}_{[r,t]}, \mathcal{U}_{[r,t]})$ is the identity. The counit of the adjunction $\mathcal{F}_{[r,t]} \mathcal{U}_{[r,t]}(L) \rightarrow L$ takes the following form.

- For $0 \leq i < r$ it corresponds to $w^{r-i}: L_r \rightarrow L_i$.
- It is the identity at page i for $r \leq i \leq t$.
- For $i > t$ it is the composite of the projection onto the first component $\text{Cone}_i(L_t/B_t(L_t)) \rightarrow L_t/B_t(L_t)$ followed by $s^{i-t}: L_t/B_t(L_t) \rightarrow L_i$. \square

The translation functor of Definition 2.3.10 interacts nicely with the truncations of the category LWB.

Proposition 6.1.4. *The translation functor applied to each page of a (truncated) linear witness book induces isomorphisms of categories*

$$\mathcal{T}: \text{tr}_{[r,s]} \text{LWB} \rightarrow \text{tr}_{[r+1,s+1]} \text{LWB}, \quad \mathcal{T}: \text{LWB} \rightarrow \text{tr}_{\geq 1} \text{LWB}$$

and thus an isomorphism of categories

$$\mathcal{T}^r: \text{LWB} \rightarrow \text{tr}_{\geq r} \text{LWB}. \quad \square$$

6.2. Special case of truncation. Now we consider some extra properties that are satisfied by truncation $\iota_{[r,t]}: \text{tr}_{[r,t]} \mathbf{D} \rightarrow \mathbf{D}$ in the case $t = \infty$. Recall that we write $\iota_{\geq r}$ for the embedding in this case. This embedding functor $\iota_{\geq r}: \text{tr}_{\geq r} \mathbf{D} \rightarrow \mathbf{D}$ has both a left and a right adjoint, so that $\text{tr}_{\geq r} \mathbf{D}$ is a reflective and coreflective subcategory of \mathbf{D} . We therefore obtain via Proposition 2.1.2 a chain of five adjoint functors at the level of the presheaf category LWB and its truncations. Most of our focus will be on the leftmost three of these five functors, since these are the ones that give rise to interesting functors on (extended) spectral sequences.

Proposition 6.2.1. *The embedding $\iota_{\geq r}: \text{tr}_{\geq r} \mathbf{D} \rightarrow \mathbf{D}$ admits a left and a right adjoint, denoted respectively $\mathcal{W}_{\geq r}$ and $\mathcal{S}_{\geq r}$ from \mathbf{D} to $\text{tr}_{\geq r} \mathbf{D}$. The left adjoint $\mathcal{W}_{\geq r}$ is given on objects by*

$$\begin{aligned} \mathcal{W}_{\geq r}(i, p, n) &= (r, p, n) & \text{if } i < r, \\ \mathcal{W}_{\geq r}(i, p, n) &= (i, p, n) & \text{if } i \geq r, \end{aligned}$$

and on generating morphisms by, for $i < r$,

$$\mathcal{W}_{\geq r}(\omega_{i+1}^{p,n}) = 1, \quad \mathcal{W}_{\geq r}(\delta_i^{p,n}) = 0, \quad \mathcal{W}_{\geq r}(\sigma_i^{p,n}) = 0,$$

with $\mathcal{W}_{\geq r}$ acting as the identity on all other generating morphisms.

The right adjoint $\mathcal{S}_{\geq r}$ is given on objects by

$$\begin{aligned}\mathcal{S}_{\geq r}(i, p, n) &= (r, p + r - i, n + r - i) && \text{if } i < r, \\ \mathcal{S}_{\geq r}(i, p, n) &= (i, p, n) && \text{if } i \geq r,\end{aligned}$$

and on generating morphisms by, for $i < r$,

$$\mathcal{S}_{\geq r}(\omega_{i+1}^{p,n}) = 0, \quad \mathcal{S}_{\geq r}(\delta_i^{p,n}) = 0, \quad \mathcal{S}_{\geq r}(\sigma_i^{p,n}) = 1,$$

with $\mathcal{S}_{\geq r}$ acting as the identity on all other generating morphisms.

We have $\mathcal{W}_{\geq r} \iota_{\geq r} = \mathcal{S}_{\geq r} \iota_{\geq r} = \text{Id}_{\text{tr}_{\geq r} \mathcal{D}}$.

The counit of the adjunction $(\mathcal{W}_{\geq r}, \iota_{\geq r})$ is the identity. The unit of the adjunction $\eta_{(i,p,n)}^{\mathcal{W}} : (i, p, n) \rightarrow \mathcal{W}_{\geq r}(i, p, n)$ has the following form.

$$\begin{aligned}\eta_{(i,p,n)}^{\mathcal{W}} &= \omega^{r-i} : (i, p, n) \rightarrow (r, p, n) && \text{if } i < r, \\ \eta_{(i,p,n)}^{\mathcal{W}} &= 1 && \text{if } i \geq r.\end{aligned}$$

The unit of the adjunction $(\iota_{\geq r}, \mathcal{S}_{\geq r})$ is the identity. The counit of the adjunction $\epsilon_{(i,p,n)}^{\mathcal{S}} : \mathcal{S}_{\geq r}(i, p, n) \rightarrow (i, p, n)$ has the following form.

$$\begin{aligned}\epsilon_{(i,p,n)}^{\mathcal{S}} &= \sigma^{r-i} : (r, p + r - i, n + r - i) \rightarrow (i, p, n) && \text{if } i < r, \\ \epsilon_{(i,p,n)}^{\mathcal{S}} &= 1 && \text{if } i \geq r.\end{aligned}$$

Proof. That the functors are well defined is immediate. To prove that $\mathcal{W}_{\geq r}$ is left adjoint to $\iota_{\geq r}$, we only need to show that $\mathcal{W}_{\geq r}(\eta_{(i,p,n)}^{\mathcal{W}})$ and $\eta_{\iota_{\geq r}(i,p,n)}^{\mathcal{W}}$ are the identity maps, which is the case. To prove that $\mathcal{S}_{\geq r}$ is right adjoint to $\iota_{\geq r}$, we only need to show that $\mathcal{S}_{\geq r}(\epsilon_{(i,p,n)}^{\mathcal{S}})$ and $\epsilon_{\iota_{\geq r}(i,p,n)}^{\mathcal{S}}$ are the identity maps, which is the case. \square

Thus we have a chain of five adjoint functors on (truncated) linear witness books

$$\begin{array}{ccccccccc}(\mathcal{W}_{\geq r})! & \dashv & \mathcal{W}_{\geq r}^* & \dashv & \iota_{\geq r}^* & \dashv & \mathcal{S}_{\geq r}^* & \dashv & (\mathcal{S}_{\geq r})_* \\ & & \parallel & & \parallel & & \parallel & & \\ & & \mathcal{F}_{\geq r} & & \mathcal{U}_{\geq r} & & \mathcal{R}_{\geq r} & & \end{array}$$

The explicit description of the functor $(\mathcal{W}_{\geq r})!$ is given in Proposition B.1.3.

6.3. Shift and décalage for (extended) spectral sequences. Next we consider how these adjoint functors behave with respect to the subcategory $(\text{LWB})^e$ and its truncations. We show that the leftmost three of the five adjoints restrict to this setting and thus give rise, via the $(\mathcal{Q}, \mathcal{N})$ adjunction, to corresponding adjoint functors on ESpSe . We will show that they also restrict to SpSe .

The translation functor $\mathcal{T}^r : \text{ESpSe} \rightarrow \text{tr}_{\geq r} \text{ESpSe}$ is an isomorphism of categories. One can then define a pair of adjoint functors $(\mathcal{Q}_{\geq r}, \mathcal{N}_{\geq r})$ between the categories $\text{tr}_{\geq r} \text{LWB}$ and $\text{tr}_{\geq r} \text{ESpSe}$, transferring the adjunction $(\mathcal{Q}, \mathcal{N})$ along the isomorphism \mathcal{T}^r . In particular, the essential image of $\mathcal{N}_{\geq r}$ can be described as in Proposition 5.2.1; we denote this by $\text{tr}_{\geq r} (\text{LWB})^e$. This category has for objects the truncated linear witness books satisfying $\text{Ker } s_i = \text{Ker } d_i$, for all $i \geq r$.

Proposition 6.3.1. *The functors $\mathcal{U}_{\geq r}$ and $\mathcal{F}_{\geq r}$ restrict to the categories $(\text{LWB})^e$ and $\text{tr}_{\geq r} (\text{LWB})^e$ and to the categories $(\text{LWB})^s$ and $\text{tr}_{\geq r} (\text{LWB})^s$.*

Proof. The functor $\mathcal{U}_{\geq r} : \text{LWB} \rightarrow \text{tr}_{\geq r} \text{LWB}$ is the forgetful functor, hence it sends $(\text{LWB})^e$ to $\text{tr}_{\geq r}(\text{LWB})^e$ and $(\text{LWB})^s$ to $\text{tr}_{\geq r}(\text{LWB})^s$. Let L be a truncated linear witness book. For $0 \leq i < r$ we have, in $\mathcal{F}_{\geq r}L$, $\text{Ker } d_i = \text{Ker } s_i = \text{Im } w_{i+1} = L_r$ and $\text{Im } s_i = \text{Ker } w_{i+1} = 0$. For $i \geq r$ we have, in $\mathcal{F}_{\geq r}L$, all the structural morphisms coincide with those in L . In conclusion, if $L \in \text{tr}_{\geq r}(\text{LWB})^e$ or in $\text{tr}_{\geq r}(\text{LWB})^s$ then $\mathcal{F}_{\geq r}L$ is in $(\text{LWB})^e$ or in $(\text{LWB})^s$, by Propositions 5.2.1 and 5.4.2. \square

The corresponding statement for the functor $(\mathcal{W}_{\geq r})!$ can be found in Proposition B.1.4

Next we introduce notation for the triple of adjoint functors we are interested in on ESpSe . These correspond to the functors $(\mathcal{W}_{\geq r})!, \mathcal{F}_{\geq r}$ and $\mathcal{U}_{\geq r}$, transferred via the $(\mathcal{Q}, \mathcal{N})$ adjunction and composed with suitable translations in order to give functors $\text{ESpSe} \rightarrow \text{ESpSe}$.

Notation 6.3.2. We introduce the following functors $\text{ESpSe} \rightarrow \text{ESpSe}$:

$$\begin{aligned} \text{LDec} &= \mathcal{T}^{-1} \circ \mathcal{Q}_{\geq 1} \circ (\mathcal{W}_{\geq 1})! \circ \mathcal{N}, \\ \text{Shift} &= \mathcal{Q} \circ \mathcal{F}_{\geq 1} \circ \mathcal{N}_{\geq 1} \circ \mathcal{T}, \\ \text{Dec} &= \mathcal{T}^{-1} \circ \mathcal{Q}_{\geq 1} \circ \mathcal{U}_{\geq 1} \circ \mathcal{N}. \end{aligned}$$

We denote by $\text{tr}_{\geq r} \text{ESpSe}$ the category whose objects are collections of i -bigraded complexes (X_i, d_i) for $i \geq r$ endowed with characteristic maps $\varphi_{i+1} : X_{i+1} \rightarrow H(X_i)$ for $i \geq r$. And $\text{tr}_{\geq r} \text{SpSe}$ denotes the corresponding truncation of SpSe .

Theorem 6.3.3. *On the category ESpSe we have adjunctions*

$$\text{LDec} \dashv \text{Shift} \dashv \text{Dec}.$$

Proof. We give the details for the $\text{Shift} \dashv \text{Dec}$ adjunction. Using that the unit of the adjunction $L \rightarrow \mathcal{N}_{\geq 1} \mathcal{Q}_{\geq 1}(L)$ is an isomorphism for $L \in \text{tr}_{\geq 1}(\text{LWB})^e$, that $\mathcal{U}_{\geq 1}$ sends $(\text{LWB})^e$ to $\text{tr}_{\geq 1}(\text{LWB})^e$ and that $\mathcal{N}_{\geq 1}$ is fully faithful, we obtain the following sequence of bijections:

$$\begin{aligned} \text{Hom}_{\text{ESpSe}}(\text{Shift}(X), Y) &\cong \text{Hom}_{\text{LWB}}(\mathcal{F}_{\geq 1} \mathcal{N}_{\geq 1} \mathcal{T}(X), \mathcal{N}(Y)) \\ &\cong \text{Hom}_{\text{tr}_{\geq 1} \text{LWB}}(\mathcal{N}_{\geq 1} \mathcal{T}(X), \mathcal{U}_{\geq 1} \mathcal{N}(Y)) \\ &\cong \text{Hom}_{\text{tr}_{\geq 1} \text{LWB}}(\mathcal{N}_{\geq 1} \mathcal{T}(X), \mathcal{N}_{\geq 1} \mathcal{Q}_{\geq 1} \mathcal{U}_{\geq 1} \mathcal{N}(Y)) \\ &\cong \text{Hom}_{\text{tr}_{\geq 1} \text{ESpSe}}(\mathcal{T}(X), \mathcal{Q}_{\geq 1} \mathcal{U}_{\geq 1} \mathcal{N}(Y)) \\ &\cong \text{Hom}_{\text{tr}_{\geq 1} \text{ESpSe}}(\mathcal{T}(X), \mathcal{T} \text{Dec}(Y)) \\ &\cong \text{Hom}_{\text{ESpSe}}(X, \text{Dec}(Y)) \end{aligned}$$

The same arguments prove the other adjunction, using that $\mathcal{F}_{\geq 1}$ sends $\text{tr}_{\geq 1}(\text{LWB})^e$ to $(\text{LWB})^e$. \square

We write F^r for the r -fold iterated composite $F \circ F \circ F \cdots \circ F$ of a functor F . The following proposition gives explicit descriptions of the iterated composites Shift^r and Dec^r .

Proposition 6.3.4. *For X a truncated extended spectral sequence in $\mathrm{tr}_{\geq r} \mathrm{ESpSe}$, we have*

$$(\mathrm{Shift}^r \mathcal{T}^{-r} X)_i = \begin{cases} (X_r, 0) & \text{for } 0 \leq i < r, \\ (X_i, d_i) & \text{for } r \leq i. \end{cases}$$

The characteristic maps φ_i are the identity morphism for $1 \leq i \leq r$ and coincide with φ_i^X for $r+1 \leq i$. If X is in SpSe so is $\mathrm{Shift}^r(X)$.

For X an extended spectral sequence in ESpSe , we have in $\mathrm{tr}_{\geq r} \mathrm{ESpSe}$

$$(\mathcal{T}^r \mathrm{Dec}^r X)_i = \begin{cases} (\mathcal{N}X)_r & \text{for } i = r, \\ (X_i, d_i) & \text{for } r < i. \end{cases}$$

The characteristic maps φ_i coincide with φ_i^X for $r+1 < i$ and φ_{r+1} is the following composite

$$X_{r+1} \xrightarrow{\varphi_{r+1}^X} H(X_r) \xrightarrow{H(\rho_r)^{-1}} H((\mathcal{N}X)_r).$$

If X is in SpSe then so is $\mathrm{Dec}^r(X)$.

The unit of the adjunction $(\mathrm{Shift}^r, \mathrm{Dec}^r)$ is an isomorphism.

Proof. Obtaining the descriptions of the functors is straightforward. The statements about restriction to SpSe follow from the descriptions of the characteristic maps. That the unit η is an isomorphism follows from part (4) of Proposition 5.2.4, with $(\eta_X)_0 = (\rho_r)^{-1}$ and $(\eta_X)_i = 1$ for $i > 0$. \square

In order to describe LDec^r explicitly, we introduce some notation.

Notation 6.3.5. Let X be in ESpSe . Denote by $q_r: (\mathcal{N}X)_r^{p+r, n+r-1} \rightarrow \mathrm{Cone}_r(X_0)^{p, n}$ the map defined by

$$q_r(x_0, \dots, x_r; y_0, \dots, y_r) = (-y_0, x_0).$$

For an r -bigraded complex A we denote by ΣA the r -bigraded complex defined as $(\Sigma A)^{p, n} = (\Sigma A)^{p+r, n+r-1}$ with differential $d^{\Sigma A} = -d^A$. The map q_r induces a short exact sequence in r -C

$$0 \rightarrow \Sigma(\mathcal{N}X)_r / \mathrm{Ker} q_r \rightarrow \mathrm{Cone}_r(X_0) \rightarrow \mathrm{Cone}_r(X_0) / \mathrm{Im} q_r \rightarrow 0$$

which yields a long exact sequence in homology and thus an isomorphism

$$\partial_r: H(\mathrm{Cone}_r(X_0) / \mathrm{Im} q_r) \rightarrow H((\mathcal{N}X)_r / \mathrm{Ker} q_r).$$

Proposition 6.3.6. *For X in ESpSe , we have in $\mathrm{tr}_{\geq r} \mathrm{ESpSe}$*

$$(\mathcal{T}^r \mathrm{LDec}^r X)_i = \begin{cases} \mathrm{Cone}_r(X_0) / \mathrm{Im} q_r & \text{for } i = r, \\ (X_i, d_i) & \text{for } r < i. \end{cases}$$

The characteristic maps φ_i coincide with φ_i^X for $r+1 < i$ and φ_{r+1} is the following composite

$$X_{r+1} \xrightarrow{\varphi_{r+1}^X} H(X_r) \xrightarrow{H(\rho_r)^{-1}} H((\mathcal{N}X)_r) \rightarrow H((\mathcal{N}X)_r / \mathrm{Ker} q_r) \xrightarrow{\partial_r^{-1}} H(\mathrm{Cone}_r(X_0) / \mathrm{Im} q_r),$$

where the unlabelled arrow is induced by the projection.

If X is in SpSe then so is $\mathrm{LDec}^r(X)$.

The counit of the adjunction $(\text{LDec}^r, \text{Shift}^r)$ is an isomorphism.

Proof. We prove that the functor $\mathcal{LD} : \text{ESpSe} \rightarrow \text{tr}_{\geq r} \text{ESpSe}$ defined by the formulas of the statement is left adjoint to the functor $\text{Shift}^r \mathcal{T}^{-r}$. For the proof, we denote by $\text{Hom}, \text{Hom}_{[r,t]}$ and $\text{Hom}_{\geq r}$ the \mathbb{R} -modules of morphisms in ESpSe and in the corresponding truncated categories. Any $X \in \text{ESpSe}$ can be viewed as an object in $\text{tr}_{[r,t]} \text{ESpSe}$ by forgetting some pages. By abuse of notation, we also denote it by X .

For $X', Y \in \text{tr}_{\geq r} \text{ESpSe}$, we have

$$\text{Hom}_{\geq r}(X', Y) \cong \text{Hom}_{\geq r+1}(X', Y) \quad \times \quad \text{Hom}_{[r,r+1]}(X', Y)$$

$$\text{Hom}_{(r+1)\text{-C}}(X', Y)$$

and similarly, for $X, Y' \in \text{ESpSe}$, we have

$$\text{Hom}(X, Y') \cong \text{Hom}_{\geq r+1}(X, Y') \quad \times \quad \text{Hom}_{[0,r+1]}(X, Y')$$

$$\text{Hom}_{(r+1)\text{-C}}(X, Y')$$

For $X \in \text{ESpSe}$ and $Y \in \text{tr}_{\geq r} \text{ESpSe}$, we have

$$\text{Hom}_{\geq r+1}(\mathcal{LD}(X), Y) = \text{Hom}_{\geq r+1}(X, Y) = \text{Hom}_{\geq r+1}(X, \text{Shift}^r \mathcal{T}^{-r} Y)$$

and

$$\text{Hom}_{(r+1)\text{-C}}(\mathcal{LD}(X), Y) = \text{Hom}_{(r+1)\text{-C}}(X, Y) = \text{Hom}_{(r+1)\text{-C}}(X, \text{Shift}^r \mathcal{T}^{-r} Y).$$

In conclusion, proving that

$$\text{Hom}_{\geq r}(\mathcal{LD}(X), Y) \cong \text{Hom}(X, \text{Shift}^r \mathcal{T}^{-r} Y)$$

is equivalent to proving that

$$\text{Hom}_{[r,r+1]}(\mathcal{LD}(X), Y) \cong \text{Hom}_{[0,r+1]}(X, \text{Shift}^r \mathcal{T}^{-r} Y).$$

From now on we set $X' = \mathcal{LD}(X)$ and $Y' = \text{Shift}^r \mathcal{T}^{-r} Y$.

The following diagrams are pullback diagrams.

$$\begin{array}{ccc} \text{Hom}_{[r,r+1]}(X', Y) & \longrightarrow & \text{Hom}_{r\text{-C}}(X'_r, Y_r) \\ \downarrow & & \downarrow H(-) \circ \varphi_{r+1}^{X'} \\ \text{Hom}_{(r+1)\text{-C}}(X'_{r+1}, Y_{r+1}) & \xrightarrow{(\varphi_{r+1}^{Y'})^*} & \text{Hom}_{\text{bgMod}_{\mathbb{R}}}(X'_{r+1}, H(Y_r)) \end{array}$$

$$\begin{array}{ccc} \text{Hom}_{[0,r+1]}(X, Y') & \longrightarrow & \text{Hom}_{[0,r]}(X, Y') \\ \downarrow & & \downarrow H(-) \circ \varphi_{r+1}^X \\ \text{Hom}_{(r+1)\text{-C}}(X_{r+1}, Y'_{r+1}) & \xrightarrow{(\varphi_{r+1}^{Y'})^*} & \text{Hom}_{\text{bgMod}_{\mathbb{R}}}(X_{r+1}, H(Y_r)) \end{array}$$

The bottom horizontal maps in the two diagrams coincide. It remains to show that $\text{Hom}_{r\text{-C}}(X'_r, Y_r) \cong \text{Hom}_{[0,r]}(X, Y')$ and that the two right vertical maps coincide up to this isomorphism. Since $X'_r = \text{Coker } q_r$, we have

$$\text{Hom}_{r\text{-C}}(X'_r, Y_r) \cong \text{Ker}(q_r^* : \text{Hom}_{r\text{-C}}(C_r(X_0), Y_r) \rightarrow \text{Hom}_{r\text{-C}}(\Sigma \mathcal{N}(X)_r, Y_r)).$$

The above isomorphism tells us that giving a morphism of r -bigraded complexes $\alpha : X'_r \rightarrow Y_r$ amounts to giving a morphism of bigraded modules $\alpha_0 : X_0 \rightarrow Y_r$ satisfying for every $(x_0, \dots, x_r; y_0, \dots, y_r) \in (\mathcal{N}X)_r$, $\alpha_0(y_0) = d_r^Y \alpha_0(x_0)$. Let $\alpha_0 \in$

$\text{Hom}_{r\text{-C}}(X'_r, Y_r)$ and $0 \leq i \leq r$.

We claim that for $x_i \in X_i$ and a compatible sequence (x_0, \dots, x_i) (see Definition 3.2.2), the element $\alpha_0(x_0)$ does not depend on the choice of the compatible sequence. We also claim that if $i < r$ and $x_i = d_i x'_i$ for some $x'_i \in X_i$, then $\alpha_0(x_0) = 0$. Both claims are proved by induction on i . Note that $(0, \dots, 0; d_0 x'_0, \dots, 0) \in (\mathcal{N}X)_r$ implies $\alpha_0(d_0 x'_0) = 0$ so that the case $i = 0$ is proved. Assume we have proved the claims for $0 \leq k < i$. Let $(x_0, \dots, x_{i-1}, x_i)$ and $(x'_0, \dots, x'_{i-1}, x_i)$ be two compatible sequences, so that there exists u_{i-1} such that $x'_{i-1} - x_{i-1} = d_{i-1} u_{i-1}$. Then $(x'_0 - x_0, \dots, x'_{i-1} - x_{i-1})$ is a compatible sequence starting at $d_{i-1} u_{i-1}$ and thus $\alpha_0(x'_0 - x_0) = 0$. If $i < r$, a compatible sequence $(y_0, \dots, y_{i-1}, d_i x_i)$ yields an element of the form $(0, \dots, 0; y_0, \dots, y_{i-1}, d_i x_i, 0, \dots, 0) \in (\mathcal{N}X)_r$ so that $\alpha_0(y_0) = d_r \alpha_0(0) = 0$.

In conclusion, to any $\alpha : X'_r \rightarrow Y_r$ is associated a collection of well defined morphisms $\hat{\alpha}_i : X_i \rightarrow Y_r$ for $0 \leq i \leq r$ defined by $\hat{\alpha}_i(x_i) = \alpha_0(x_0)$ with (x_0, \dots, x_i) a compatible sequence. The two claims ensure that $\hat{\alpha} \in \text{Hom}_{[0,r]}(X, Y')$. This construction provides us with a map

$$\text{Hom}_{r\text{-C}}(X'_r, Y_r) \rightarrow \text{Hom}_{[0,r]}(X, Y')$$

which is clearly a bijection.

The right vertical map of the first pullback diagram sends $\alpha : X'_r \rightarrow Y_r$ to the map $X'_{r+1} = X_{r+1} \rightarrow H(Y_r)$ which assigns to $x_{r+1} \in X_{r+1}$ the class of $\alpha_0(x_0)$ where (x_0, \dots, x_{r+1}) is a compatible sequence. Indeed, we have defined the characteristic map $X'_{r+1} \rightarrow H(X_r)$ as the composite

$$\begin{array}{ccccccc} X_{r+1} & \xrightarrow{\varphi_{r+1}^X} & H(X_r) & \xrightarrow{H(\rho_r)^{-1}} & H((\mathcal{N}X)_r) & \longrightarrow & H(\text{Cone}_r(X_0)/\text{Im } q_r) \\ x_{r+1} & \longmapsto & \varphi_{r+1}^X x_{r+1} & \longmapsto & [(x_0, \dots, x_r; 0, \dots, 0)] & \longmapsto & [\overline{(x_0, 0)}]. \end{array}$$

The right vertical map of the second diagram sends $\hat{\alpha} \in \text{Hom}_{[0,r]}(X, Y')$ to the map which assigns to $x \in X_{r+1}$ the element $[\hat{\alpha}_r(x_r)]$ with x_r a representative of $\varphi_{r+1}^X(x_{r+1})$. By definition of $\hat{\alpha}$, we have $\hat{\alpha}_r(x_r) = \alpha_0(x_0)$. Hence the adjunction is proved.

Now we show that if X is a spectral sequence then $\text{LDec}^r(X)$ is a spectral sequence. Note that if X is a spectral sequence then $\text{Ker } q_r = 0$. An element of $\text{Ker } q_r$ is of the form $u = (0, x_1, \dots, x_r; 0, y_1, \dots, y_r) \in \mathcal{N}(X)_r$, so that $\varphi_1(x_1) = [0]$ and $x_1 = 0$ since φ_1 is an isomorphism. Inductively we obtain that $u = 0$. It follows that the characteristic map $\varphi_{r+1} : X_{r+1} \rightarrow H(\text{Cone}_r(X_0)/\text{Im } q_r)$ is an isomorphism. The other characteristic maps are isomorphisms since they coincide with those of X . Finally, let $Y \in \text{tr}_{\geq r} \text{ESpSe}$ and set $Y' = \text{Shift}^r \mathcal{T}^{-r} Y$. The map $Y_r \rightarrow (\mathcal{N}Y')_r$ which assigns to y_r the element $(y_r, \dots, y_r; d_r y_r, \dots, d_r y_r)$ is an isomorphism, so $q_r : (\mathcal{N}Y')_r \rightarrow \text{Cone}_r(Y'_0)$ is isomorphic to the map $Y_r \rightarrow \text{Cone}_r(Y_r)$ which sends y_r to $(-d_r y_r, y_r)$. This proves that $\text{Cone}_r(Y_r)/\text{Im } q_r$ is isomorphic to Y_r . In particular the counit of the adjunction $\mathcal{T}^r \circ \text{LDec}^r \circ \text{Shift}^r \circ \mathcal{T}^{-r} \rightarrow \text{Id}_{\text{tr}_{\geq r} \text{ESpSe}}$ is an isomorphism at the r -page. It is clear that it is an isomorphism at pages $i > r$. \square

Theorem 6.3.7. *On the category \mathbf{ESpSe} , and on the category \mathbf{SpSe} , we have adjunctions*

$$\mathbf{LDec}^r \dashv \mathbf{Shift}^r \dashv \mathbf{Dec}^r,$$

with isomorphisms:

$$\begin{aligned} \mathbf{LDec}^r &\cong \mathcal{T}^{-r} \circ \mathcal{Q}_{\geq r} \circ (\mathcal{W}_{\geq r})! \circ \mathcal{N}, \\ \mathbf{Shift}^r &\cong \mathcal{Q} \circ \mathcal{F}_{\geq r} \circ \mathcal{N}_{\geq r} \circ \mathcal{T}^r, \\ \mathbf{Dec}^r &\cong \mathcal{T}^{-r} \circ \mathcal{Q}_{\geq r} \circ \mathcal{U}_{\geq r} \circ \mathcal{N}. \end{aligned}$$

The unit of the adjunction $(\mathbf{Shift}^r, \mathbf{Dec}^r)$ is an isomorphism and the counit of the adjunction $(\mathbf{LDec}^r, \mathbf{Shift}^r)$ is an isomorphism.

Proof. The first part is immediate from Theorem 6.3.3, together with the fact that the functors restrict to \mathbf{SpSe} . The expression for \mathbf{Shift}^r as a composite is easy to check directly with the explicit description of Proposition 6.3.4. Then the composite expressions for \mathbf{LDec}^r and \mathbf{Dec}^r give respectively left and right adjoints to \mathbf{Shift}^r by the same argument as in the proof of Theorem 6.3.3, thus establishing the other isomorphisms. The statements about the unit and counit were proved in Proposition 6.3.4 and Proposition 6.3.6 respectively. \square

6.4. Compatibility with shift and décalage for filtered complexes. In this section we look at compatibility of the functors \mathbf{LDec} , \mathbf{Shift} and \mathbf{Dec} with the functors \mathbf{Dec}^* , S and \mathbf{Dec} on the category of filtered complexes \mathbf{FC}_R . These were introduced by Deligne in [10, Section 1.3]; a useful exposition of their properties is given in [8, Section 2.3]. We write E for the functor associating a spectral sequence to a filtered complex and we will use the notation of [7, Section 2] for constructions related to the spectral sequence of a filtered complex.

Proposition 6.4.1. *We have a commutative diagram*

$$\begin{array}{ccc} \mathbf{FC}_R & \xrightarrow{E} & \mathbf{SpSe} \\ S \downarrow & & \downarrow \mathbf{Shift} \\ \mathbf{FC}_R & \xrightarrow{E} & \mathbf{SpSe} \end{array}$$

Proof. This is a matter of direct calculation with the definitions. Let C be a filtered complex. For $i \geq 1$, we have $E(SC)_i = \mathcal{T}(E(C)_{i-1})$ and for $i = 0$ we obtain $E(SC)_0 = \mathcal{T}(E(C)_0)$ with zero differential. \square

Proposition 6.4.2. *Let C be a filtered complex. There is a natural isomorphism of spectral sequences $E(\mathbf{Dec}(C)) \rightarrow \mathbf{Dec}(E(C))$.*

Proof. Firstly consider the functor $\mathcal{T}^{-1} \circ \mathrm{tr}_{\geq 1} : \mathbf{SpSe} \rightarrow \mathbf{SpSe}$ which just shifts down the pages, differing from \mathbf{Dec} only by information related to the 0-page. From [10, Proposition 1.3.4], we have a canonical morphism of spectral sequences

$$u : E(\mathbf{Dec}(C)) \rightarrow \mathcal{T}^{-1} \circ \mathrm{tr}_{\geq 1}(E(C))$$

which is a surjection on the 0-page and an isomorphism from the 1-page onwards. In fact $u_0: E(\text{Dec}(C))_0 \rightarrow \mathcal{T}^{-1} \circ \text{tr}_{\geq 1}(E(C))_0$ factors via $\text{Dec}(E(C))_0$ and we use this to produce the required isomorphism. Recall that

$$E(\text{Dec}(C))_0^{p,p+n} = \mathcal{T}^{-1}(Z_1^{p,p+n}(C)/Z_1^{p-1,p+n-1}(C))$$

with notation as in [7, Section 2]. Define

$$\tilde{u}_0: E(\text{Dec}(C))_0 \rightarrow \text{Dec}(E(C))_0 = \mathcal{T}^{-1}(\mathcal{N}E(C))_1$$

by $[x] \mapsto ([x]_0, [x]_1; [dx]_0, [dx]_1)$. Direct checks show that \tilde{u}_0 is a well-defined isomorphism of 0-bigraded complexes. Furthermore $u_0 = \rho_1 \tilde{u}_0$ and we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{T}^{-1}H(E(C))_1 & \xleftarrow{\mathcal{T}^{-1}\varphi_2^{E(C)}} & \mathcal{T}^{-1}E(C)_2 \\ \mathcal{T}^{-1}H(\rho_1) \uparrow & \swarrow \mathcal{T}^{-1}(H(\rho_1)^{-1}\varphi_2^{E(C)}) & \uparrow u_1 \\ \mathcal{T}^{-1}H(\mathcal{N}(E(C))_1) & & E(\text{Dec}(C))_1 \\ H(\tilde{u}_0) \uparrow & \xleftarrow{\varphi_1^{\text{Dec}(C)}} & \\ H(E(\text{Dec}(C))_0) & & \end{array}$$

Thus $\tilde{u} = (\tilde{u}_0, u_1, u_2, \dots): E(\text{Dec}(C)) \rightarrow \text{Dec}(E(C))$ is an isomorphism of spectral sequences. \square

Proposition 6.4.3. *Let C be a filtered complex. There is a natural isomorphism of spectral sequences $E(\text{Dec}^*(C)) \rightarrow \text{LDec}(E(C))$.*

Proof. From [10, 1.3.5], we have a canonical morphism of spectral sequences

$$v: \mathcal{T}^{-1} \circ \text{tr}_{\geq 1}(E(C)) \rightarrow E(\text{Dec}^*(C))$$

which is an injection on the 0-page and an isomorphism from the 1-page onwards.

In fact $v_0: \mathcal{T}^{-1} \circ \text{tr}_{\geq 1}(E(C))_0 \rightarrow E(\text{Dec}^*(C))_0$ factors via $\text{LDec}(E(C))_0$ and we use this to produce the required isomorphism. Recall that

$$E(\text{Dec}^*(C))_0^{p,p+n} = \mathcal{T}^{-1}(B_1^{p,p+n}(C)/B_1^{p-1,p+n-1}(C))$$

with notation as in [7, Section 2]. It is routine to check that there is a well-defined map

$$\tilde{v}_0: \text{LDec}(E(C))_0 = \mathcal{T}^{-1}(\text{Cone}_1(E(C))_0 / \text{Im } q_1) \rightarrow E(\text{Dec}^*(C))_0$$

given by

$$\overline{(a, b)} \mapsto [a + db].$$

Furthermore, v_0 is the composite of the map sending a to $\overline{(a, 0)}$ and \tilde{v}_0 . It is straightforward to check that $\tilde{v} = (\tilde{v}_0, v_1, v_2, \dots): E(\text{Dec}^*(C)) \rightarrow \text{LDec}(E(C))$ is an isomorphism of spectral sequences. \square

This compatibility with décalage for filtered complexes clarifies the relation of our construction to other uses of the term décalage in connection with spectral sequences, such as in [6, 18]. Sometimes the term décalage of a spectral sequence is used to refer to décalage of some underlying filtered object. Other times it refers to the corresponding process of *turning the page* of the spectral sequence, up to reindexing,

generally only specifying the resulting spectral sequence from the 1-page onwards. Here we have provided the two versions of décalage as functors on the category of spectral sequences, including the (not completely obvious) descriptions of the 0-page of each.

7. MODEL CATEGORY STRUCTURES ON EXTENDED SPECTRAL SEQUENCES

In this section we establish two different families of model category structures on ESpSe . In the first part, we work with model category structures on LWB . The main result here is Theorem 7.1.5, establishing such a structure LWB_r for each $r \geq 0$. Then we move to extended spectral sequences and we first use right transfer from the model category structure LWB_r , resulting in the model structure ESpSe_r of Theorem 7.2.1, which we show is Quillen equivalent to the projective model structure on 0-bigraded complexes. Then in Theorem 7.3.6 we give a different model category structure on ESpSe , denoted ESpSe'_r , where weak equivalences $\mathcal{E}'_r \subset \mathcal{E}_r$ are those maps that are quasi-isomorphisms at page r and isomorphisms at pages $i > r$. The advantage of this class of morphisms is that it more accurately reflects the weak equivalences in spectral sequences. Also we have a hierarchy of weak equivalences: $\mathcal{E}'_r \subset \mathcal{E}'_{r+1}$ for all r .

7.1. Model category structures from transfer on linear witness books.

For each $r \geq 0$, we present a model category structure on the category LWB of linear witness books, closely related to the projective model structure on r -bigraded complexes, utilising a criterion for right transfer due to Drummond-Cole and Hackney [11]. Such a model structure can be directly obtained by a right transfer and then we explain how to modify the transferred structure to give the model structure of Theorem 7.1.5.

We consider the special case $r = t$ of the adjunctions of Section 6.1 for linear witness books and abbreviate to $\mathcal{F}_{[r]}$, $\mathcal{U}_{[r]}$ the functors $\mathcal{F}_{[r,r]}$, $\mathcal{U}_{[r,r]}$. Recalling that $\text{tr}_{[r,r]} \text{LWB} = \widehat{\langle \delta_r \rangle}$, we focus on the adjunction:

$$\widehat{\langle \delta_r \rangle} \begin{array}{c} \xrightarrow{\mathcal{F}_{[r]}} \\ \perp \\ \xleftarrow{\mathcal{U}_{[r]}} \end{array} \text{LWB}$$

The following is immediate from the description of $\mathcal{F}_{[r]}$, see Proposition 6.1.3.

Proposition 7.1.1. *Let C be an r -bigraded complex. The homology of the linear witness book $\mathcal{F}_{[r]}(C)$ is given by*

$$H((\mathcal{F}_{[r]}(C))_m) = \begin{cases} C & \text{if } 0 \leq m \leq r-1, \\ H(C) & \text{if } m = r, \\ 0 & \text{if } m > r. \end{cases}$$

□

Proposition 7.1.2. *Let C be an r -bigraded complex. The linear witness book $\mathcal{F}_{[r]}(C)$ is an object of $(\text{LWB})^e$ if and only if $H(C) = 0$ if and only if it is an object of $(\text{LWB})^s$.*

Proof. Note that for every $i \neq r$ we have $\text{Ker } s_i = \text{Ker } d_i$. For $i = r$, we have $\text{Ker } s_r = B_r C$ and $\text{Ker } d_r = Z_r C$. Hence the first equivalence follows from Proposition 5.2.1. In addition, we have $\text{Im } w_{i+1} = \text{Ker } s_i$ for all i and $\text{Im } s_i = \text{Ker } w_{i+1}$ for all $i \neq r$. We have $(s_r(C))^{pq} = (C/B_r C)^{p-1, q-1}$ and $(\text{Ker } w_{r+1})^{pq} = (C/B_r C)^{p-1, q-1} \oplus (Z_r C/B_r C)^{p+r, q+r-1}$. Thus the second equivalence follows from Proposition 5.4.2. \square

We write $\mathcal{Z}(r, p, n) \in \text{LWB}$ for the object representing $\text{Ker } d_r$; see A.2.1 for more details. Since $\delta_r^2 = 0$ we have a natural map $\delta_r: \mathcal{Z}(r, p, n) \rightarrow \mathcal{Y}(r, p+r, n+1-r)$. Note that because $\mathcal{F}_{[r]}$ preserves colimits, we have $\mathcal{Z}(r, p, n) = \mathcal{F}_{[r]}(\mathbb{R}^{p, n})$. In particular Proposition 7.1.2 shows that $\mathcal{Z}(r, p, n)$ does not lie in $(\text{LWB})^e$; see also Lemma A.2.2.

Notation 7.1.3. We introduce notation for various sets of morphisms of LWB.

$$\begin{aligned} I_r &:= \{\delta_r: \mathcal{Z}(r, p, n) \rightarrow \mathcal{Y}(r, p+r, n+1-r)\}_{p, n \in \mathbb{Z}}, \\ J_r &:= \{0 \longrightarrow \mathcal{Y}(r, p, n)\}_{p, n \in \mathbb{Z}}, \\ I_{\leq r} &:= I_r \cup \bigcup_{i=0}^{r-1} J_i, \\ J_{\leq r} &:= \bigcup_{i=0}^r J_i. \end{aligned}$$

Recall the model structure on r -bigraded complexes of Proposition 2.3.8, with generating sets I, J and note that we have $\mathcal{F}_{[r]}(I) = I_r$ and $\mathcal{F}_{[r]}(J) = J_r$.

Definition 7.1.4. Let $K, L \in \text{LWB}$. A morphism $f: K \rightarrow L$ is an r -fibration if f_i is bidegreewise surjective for $0 \leq i \leq r$. It is an r -weak equivalence if $H(f_r)$ is an isomorphism. We denote by Fib_r the class of r -fibrations and by \mathcal{E}_r that of r -weak equivalences.

Theorem 7.1.5. *There is a cofibrantly generated model structure with generating sets $I_{\leq r}, J_{\leq r}$ on LWB, denoted LWB_r . This has fibrations Fib_r and weak equivalences \mathcal{E}_r .*

Proof. We first show that there is a model category structure cofibrantly generated by the sets I_r and J_r . For this, note that the functor $\mathcal{U}_{[r]}$ admits a left adjoint $\mathcal{F}_{[r]}$ and a right adjoint $\mathcal{R}_{[r]}$ (described in Proposition B.1.1). Moreover $\mathcal{U}_{[r]}\mathcal{F}_{[r]}$ and $\mathcal{U}_{[r]}\mathcal{R}_{[r]}$ are both the identity functor and hence form a Quillen adjunction. By [11, Theorem 2.3], there is a cofibrantly generated model structure, where the generating classes of cofibrations and acyclic cofibrations are $I_r = \mathcal{F}_{[r]}(I)$ and $J_r = \mathcal{F}_{[r]}(J)$ respectively. In this model structure, the class of weak equivalences is \mathcal{E}_r and the class of fibrations consists of those maps $f: K \rightarrow L$ such that f_r is surjective.

Now we modify this model structure. We claim that $I_{\leq r}$ and $J_{\leq r}$ are the generating cofibrations and generating acyclic cofibrations respectively of a model category structure as in the statement. We follow the treatment in [16]. A morphism has the right lifting property with respect to $J_{\leq r}$ if and only if f_i is surjective for $0 \leq i \leq r$. It is clear that \mathcal{E}_r satisfies the two-out-of-three property and is closed under retracts. The smallness conditions are satisfied. We have $I_{\leq r}\text{-inj} = \mathcal{E}_r \cap J_{\leq r}\text{-inj}$ just as in the proof of [7, Theorem 3.16]. It remains to prove that the class of maps having the left lifting property with respect to r -fibrations is in \mathcal{E}_r . The technique we use here

is that of [13]. Let $f: L \rightarrow M$ be such a map. Consider the diagram

$$\begin{array}{ccc} L & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & L \oplus M' \\ f \downarrow & & \downarrow (f;p) \\ M & \xrightarrow{=} & M \end{array}$$

where $M' = \bigoplus_{i=0}^r \mathcal{F}_{[i]} \text{Cone}_i \mathcal{U}_{[i]} M$ (see Definition 2.3.9 for the cone) and $p = p_0 + \dots + p_r$ with

$$p_i: \mathcal{F}_{[i]} \text{Cone}_i \mathcal{U}_{[i]} M \rightarrow \mathcal{F}_{[i]} \mathcal{U}_{[i]} M \rightarrow M$$

induced by the projection π_i in $i\text{-C}$ followed by the counit of the adjunction $(\mathcal{F}_{[i]}, \mathcal{U}_{[i]})$. From Proposition 7.1.1 and the fact that in r -bigraded complexes the r -cone is acyclic we have

$$H((\mathcal{F}_{[i]} \text{Cone}_i \mathcal{U}_{[i]} M)_m) = \begin{cases} \text{Cone}_i(M_i) & \text{if } 0 \leq m \leq i-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular the r -bigraded complex M'_r is acyclic.

From the descriptions of the functor $\mathcal{F}_{[i]}$ and the counit of the adjunction in Proposition 6.1.3 we have that $(\mathcal{F}_{[i]} \mathcal{U}_{[i]} M)_i \rightarrow M_i$ is the identity. Hence the i -page of p_i is the projection from the i -cone of $\mathcal{U}_{[i]} M$ to $\mathcal{U}_{[i]} M$. It is bidegreewise surjective. It follows that $p_0 + \dots + p_r$ is an r -fibration. Thus (f, p) is an r -fibration and a lift exists in the diagram, say $(g, h): M \rightarrow L \oplus M'$. Considering the homology on the r -page in the diagram we see that $H(f_r)$ is an isomorphism with inverse $H(g_r)$. \square

Theorem 7.1.6. *The following adjunction is a Quillen equivalence.*

$$\widehat{\langle \delta_r \rangle} \begin{array}{c} \xrightarrow{\mathcal{F}_{[r]}} \\ \perp \\ \xleftarrow{\mathcal{U}_{[r]}} \end{array} \text{LWB}_r$$

In addition, the adjunction $\mathcal{F}_{\geq 1} \circ \mathcal{T} \dashv \mathcal{T}^{-1} \circ \mathcal{U}_{\geq 1}$ gives a Quillen equivalence between LWB_r and LWB_{r+1} for all $r \geq 0$.

Proof. It is clear that $\mathcal{U}_{[r]}$ preserves fibrations and acyclic fibrations, so it is a Quillen adjunction and we have seen in Proposition 6.1.3 that the unit of the adjunction is the identity. Furthermore the right adjoint $\mathcal{U}_{[r]}$ creates weak equivalences, so this is a Quillen equivalence.

For $\mathcal{F}_{\geq 1} \circ \mathcal{T} \dashv \mathcal{T}^{-1} \circ \mathcal{U}_{\geq 1}$, we have that $\mathcal{T}^{-1} \circ \mathcal{U}_{\geq 1}: \text{LWB}_{r+1} \rightarrow \text{LWB}_r$ preserves fibrations and weak equivalences. The unit is a natural isomorphism and the counit of the adjunction $\epsilon: \mathcal{F}_{\geq 1} \circ \mathcal{U}_{\geq 1} \rightarrow \text{Id}_{\text{LWB}}$, given by $\epsilon_L = (w_1^L, 1, 1, \dots)$, is in \mathcal{E}_i for all $i \geq 1$. \square

So all the model structures on LWB in this section are Quillen equivalent to the projective model structure on 0-bigraded complexes.

7.2. Model category structures from transfer on extended spectral sequences. We will use right transfer to obtain a model category structure on ESpSe , denoted ESpSe_r , from the model category LWB_r described in Theorem 7.1.5.

Theorem 7.2.1. *There is a cofibrantly generated model category structure on ESpSe denoted ESpSe_r with fibrations $\mathcal{F}ib_r$, the class of maps $f : X \rightarrow Y$ such that f_i is bidegreewise surjective for $0 \leq i \leq r$, and weak equivalences \mathcal{E}_r , the class of maps $f : X \rightarrow Y$ such that f_r is a quasi-isomorphism. This model category structure is Quillen equivalent to LWB_r .*

Proof. We use right transfer of the cofibrantly generated model category structure LWB_r by the adjunction

$$\text{LWB} \begin{array}{c} \xrightarrow{\mathcal{Q}} \\ \perp \\ \xleftarrow{\mathcal{N}} \end{array} \text{ESpSe}$$

The transferred weak equivalences are maps $f : X \rightarrow Y$ such that $N(f) \in \mathcal{E}_r$, which is equivalent to $f \in \mathcal{E}_r$ by Proposition 5.2.4. The fibrations are those maps $f : X \rightarrow Y \in \text{ESpSe}$ such that $\mathcal{N}(f)_i$ is surjective for $0 \leq i \leq r$, or equivalently f_i is surjective for $0 \leq i \leq r$ by Proposition 5.1.5. By Proposition 5.3.3, \mathcal{N} preserves filtered colimits. It then remains to show that any map $f : X \rightarrow Y$ having the left lifting property with respect to maps in $\mathcal{F}ib_r$ is in \mathcal{E}_r . Consider the following diagram in ESpSe

$$\begin{array}{ccc} X & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & X \oplus Y' \\ f \downarrow & & \downarrow (f,p) \\ Y & \xrightarrow{=} & Y \end{array}$$

where Y' is the following extended spectral sequence:

$$Y'_m = \begin{cases} Y_m & \text{if } m < r, \\ \text{Cone}_r(Y_r) & \text{if } m = r, \\ 0 & \text{if } m > r, \end{cases}$$

with $\varphi_r^{Y'} = \varphi_r^Y \circ \pi_r$ where $\pi_r : \text{Cone}_r(Y_r) \rightarrow Y_r$ is the projection of Definition 2.3.9, and where p is the identity on pages $i < r$, it is π_r on page r and then it is zero. So $(f, p) \in \mathcal{F}ib_r$ and a lift exists in the diagram, proving that $H(f_r)$ is an isomorphism. Note that, because $Y'_m = 0$ for $m > r$, the existence of the lift also implies that f_m is an isomorphism for $m > r$. In conclusion, we have a sequence of Quillen adjunctions

$$\widehat{\langle \delta_r \rangle} \begin{array}{c} \xrightarrow{\mathcal{F}_{[r]}} \\ \perp \\ \xleftarrow{\mathcal{U}_{[r]}} \end{array} \text{LWB}_r \begin{array}{c} \xrightarrow{\mathcal{Q}} \\ \perp \\ \xleftarrow{\mathcal{N}} \end{array} \text{ESpSe}_r$$

where the left adjunction is a Quillen equivalence. By the two-out-of-three property, proving that the right adjunction is a Quillen equivalence is equivalent to proving

that the adjunction $(\mathcal{K}_r, \mathcal{N}_r)$ with $\mathcal{K}_r = \mathcal{Q} \circ \mathcal{F}_{[r]}$ and $\mathcal{N}_r = \mathcal{U}_{[r]} \circ \mathcal{N}$

$$\widehat{\langle \delta_r \rangle} \begin{array}{c} \xrightarrow{\mathcal{K}_r} \\ \perp \\ \xleftarrow{\mathcal{N}_r} \end{array} \text{ESpSe}_r$$

is a Quillen equivalence. Explicitly, for every r -bigraded complex C , we have $\mathcal{K}_r(C)_i = C$ if $0 \leq i \leq r$ and 0 if $i > r$. The differential on each page is 0 except at page r where it coincides with that of C . The characteristic maps are either 1_C or 0. By the definition of weak equivalences in ESpSe_r we have that the composite \mathcal{N}_r creates weak equivalences. Hence it is enough to prove that the unit of the adjunction $C \rightarrow \mathcal{N}_r(\mathcal{K}_r C)$ is a quasi-isomorphism. Indeed it is an isomorphism. \square

Remark 7.2.2. The proof of the above theorem shows that any map f having the left lifting property with respect to r -fibrations, in addition to satisfying that f_r is a quasi-isomorphism, also satisfies that f_i is an isomorphism for $i > r$.

Corollary 7.2.3. *If L is a cofibrant linear witness book in LWB_r , then the projection $L_r \rightarrow \mathcal{Q}(L)_r = L_r/S_r(L)$ is a quasi-isomorphism.*

Proof. Since the adjunction $(\mathcal{Q}, \mathcal{N})$ is a Quillen equivalence, for every cofibrant linear witness book L in LWB_r and for any (fibrant) extended spectral sequence X if $\alpha : \mathcal{Q}(L) \rightarrow X$ is a weak equivalence in ESpSe_r , then $\mathcal{N}(\alpha) \circ \eta_L$ is a weak equivalence in LWB_r . Since every object X is fibrant in ESpSe_r , one can pick α to be the identity of $\mathcal{Q}(L)$. Thus η_L is a weak equivalence. The projection $L_r \rightarrow \mathcal{Q}(L)_r = L_r/S_r(L)$ is the r -page of the composite of η_L with the projection $\mathcal{N}\mathcal{Q}L \rightarrow \mathcal{Q}(L)$ which is a quasi-isomorphism by Proposition 5.2.4, thus a quasi-isomorphism. \square

The next corollary is a résumé of the model category structures we have encountered so far and the Quillen equivalences between them.

Corollary 7.2.4. *In the diagram of model categories below, every adjunction is a Quillen equivalence.*

$$\begin{array}{ccccc} & \xleftarrow{\text{LDec}^r} & & \xleftarrow{\text{LDec}} & \\ \text{ESpSe}_0 & \xrightarrow{\text{Shift}^r} & \text{ESpSe}_r & \xrightarrow{\text{Shift}} & \text{ESpSe}_{r+1} \\ & \xleftarrow{\text{Dec}^r} & & \xleftarrow{\text{Dec}} & \\ \uparrow \mathcal{Q} \downarrow \mathcal{N} & & \uparrow \mathcal{Q} \downarrow \mathcal{N} & & \uparrow \mathcal{Q} \downarrow \mathcal{N} \\ \text{LWB}_0 & \xrightarrow{(\mathcal{F}_{\geq 1} \circ \mathcal{T})^r} & \text{LWB}_r & \xrightarrow{\mathcal{F}_{\geq 1} \circ \mathcal{T}} & \text{LWB}_{r+1} \\ & \xleftarrow{(\mathcal{T}^{-1} \circ \mathcal{U}_{\geq 1})^r} & & \xleftarrow{\mathcal{T}^{-1} \circ \mathcal{U}_{\geq 1}} & \\ \uparrow \mathcal{F}_{[0]} \downarrow \mathcal{U}_{[0]} & & \uparrow \mathcal{F}_{[r]} \downarrow \mathcal{U}_{[r]} & & \uparrow \mathcal{F}_{[r+1]} \downarrow \mathcal{U}_{[r+1]} \\ 0\text{-C} & \xrightarrow[\sim]{\mathcal{T}^r} & r\text{-C} & \xrightarrow[\sim]{\mathcal{T}} & (r+1)\text{-C} \\ & \xleftarrow{(\mathcal{T})^{-r}} & & \xleftarrow{\mathcal{T}^{-1}} & \end{array}$$

Proof. The model structures and equivalences of the bottom horizontal line are by Proposition 2.3.11. The model structures of the middle horizontal line are by Theorem 7.1.5. The Quillen equivalences on the middle horizontal line, as well as the lower vertical Quillen equivalences are by Theorem 7.1.6. The model structures of the top line and the upper vertical Quillen equivalences are by Theorem 7.2.1.

We check that the equivalences on the top horizontal line are Quillen equivalences. For $r \geq 1$, it is clear that $\text{Dec}(\mathcal{E}_{r+1}) = \mathcal{E}_r$. This also holds for $r = 0$, using that $H(\rho_1): H((\mathcal{N}X)_1) \rightarrow H(X_1)$ is an isomorphism, by Proposition 5.2.4. If f is an $(r+1)$ -fibration, $\text{Dec}(f)$ is an r -fibration using Proposition 5.1.5. And the unit of the adjunction $\eta_X: X \rightarrow \text{Dec} \circ \text{Shift}(X)$ is in \mathcal{E}_i for all $i \geq 0$, since it is an isomorphism. So the adjunction $(\text{Shift}, \text{Dec})$ is a Quillen equivalence.

We now show that $(\text{LDec}, \text{Shift})$ is a Quillen equivalence. We have adjunctions

$$\text{ESpSe}_r \begin{array}{c} \xrightarrow{\text{Shift}} \\ \perp \\ \xleftarrow{\text{Dec}} \end{array} \text{ESpSe}_{r+1} \begin{array}{c} \xleftarrow{\text{LDec}} \\ \perp \\ \xrightarrow{\text{Shift}} \end{array} \text{ESpSe}_r$$

where the left adjunction is a Quillen equivalence. The right adjunction is a Quillen adjunction since Shift preserves fibrations and weak equivalences. Since we have $\text{Dec} \circ \text{Shift} \cong 1$ and $\text{LDec} \circ \text{Shift} \cong 1$, the composite adjunction is an adjoint equivalence and thus a Quillen equivalence. So the right adjunction is a Quillen equivalence by the two-out-of-three property. \square

7.3. Main model category structure on extended spectral sequences. In this section, for each $r \geq 0$, we prove the existence of a different model structure on extended spectral sequences, with a stronger notion of weak equivalence, but such that restriction gives the same weak equivalences on spectral sequences. This structure has the important feature that, in the case $r = 0$, spectral sequences sit inside extended spectral sequences as a homotopically full subcategory. We will explain and exploit this property in Section 8. The class of fibrations, \mathcal{Fib}_r , is unchanged and so the previous model structure ESpSe_r is a localization of the one established here.

Definition 7.3.1. We denote by \mathcal{E}'_r the class of maps $f: X \rightarrow Y$ of extended spectral sequences satisfying f_r is a quasi-isomorphism and f_s is an isomorphism for $s > r$. Let Cof'_r be the class of morphisms having the left lifting property with respect to maps in $\mathcal{E}'_r \cap \mathcal{Fib}_r$.

Remark 7.2.2 shows that any map having the left lifting property with respect to \mathcal{Fib}_r is in \mathcal{E}'_r . We will show directly that there is a model category with these new equivalences. The proof uses the existence of the model structures ESpSe_r of Theorem 7.2.1.

We write $\mathcal{I}so_{\leq r}$ for the class of morphisms f in ESpSe such that f_i is an isomorphism for $0 \leq i \leq r$.

Proposition 7.3.2. *Morphisms in $\mathcal{I}so_{\leq r}$ have the left lifting property with respect to those in \mathcal{E}'_r and vice versa.*

Proof. If $u \in \mathcal{I}so_{\leq r}$ and $p \in \mathcal{E}'_r$, there is a lift $h: Z \rightarrow Y$ in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ u \downarrow & & \downarrow p \\ Z & \xrightarrow{\beta} & W \end{array}$$

given by

$$h_s = \begin{cases} \alpha_s u_s^{-1} & 0 \leq s \leq r, \\ p_s^{-1} \beta_s & s \geq r+1, \end{cases}$$

which is compatible with characteristic maps using that $H(p_r)$ is an isomorphism.

If $u \in \mathcal{E}'_r$ and $p \in \mathcal{I}so_{\leq r}$, there is a lift $h': Z \rightarrow Y$ given by

$$h_s = \begin{cases} p_s^{-1} \beta_s & 0 \leq s \leq r, \\ \alpha_s u_s^{-1} & s \geq r+1, \end{cases}$$

which is compatible with characteristic maps using that $H(u_r)$ is an isomorphism. \square

Note that the lifts are functorial.

Proposition 7.3.3. *Let $f \in \mathcal{C}of_r$ and let $g = uf$ where $u \in \mathcal{I}so_{\leq r}$. Then $g \in \mathcal{C}of'_r$.*

Proof. Consider the left hand diagram with $p \in \mathcal{F}ib_r \cap \mathcal{E}'_r$:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ g \downarrow & & \downarrow p \\ Z & \xrightarrow{\beta} & W \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ f \downarrow & \nearrow \tilde{h} & \downarrow p \\ Z' & \xrightarrow{u} & Z \xrightarrow{\beta} W \end{array}$$

The factorization of g as $g = uf$ gives the outer diagram on the right. Since $f \in \mathcal{C}of_r$ and $p \in \mathcal{F}ib_r \cap \mathcal{E}'_r \subset \mathcal{F}ib_r \cap \mathcal{E}_r$, we have a lift $\tilde{h}: Z' \rightarrow Y$ and then, since $u \in \mathcal{I}so_{\leq r}$, we have a lift $h: Z \rightarrow Y$ by Proposition 7.3.2. \square

Proposition 7.3.4. *In $\mathcal{E}SpSe$ any map $f: X \rightarrow Y$ with $f \in \mathcal{E}_r$ has a functorial factorization $f = \tilde{f}u$ where $u \in \mathcal{I}so_{\leq r}$ and $\tilde{f} \in \mathcal{E}'_r$. If $f \in \mathcal{F}ib_r$, so is \tilde{f} .*

Proof. Define Y' in $\mathcal{E}SpSe$ by

$$Y'_m = \begin{cases} X_m & \text{for } m \leq r, \\ Y_m & \text{for } m \geq r+1, \end{cases}$$

with characteristic maps

$$\varphi_m^{Y'} = \begin{cases} \varphi_m^X & \text{for } m \leq r, \\ (Hf_r)^{-1} \varphi_{r+1}^Y & \text{for } m = r+1, \\ \varphi_m^Y & \text{for } m \geq r+2. \end{cases}$$

Then $u: X \rightarrow Y'$ and $\tilde{f}: Y' \rightarrow Y$ given by

$$u_m = \begin{cases} 1 & \text{for } m \leq r, \\ f_m & \text{for } m \geq r+1, \end{cases} \quad \tilde{f}_m = \begin{cases} f_m & \text{for } m \leq r, \\ 1 & \text{for } m \geq r+1, \end{cases}$$

are compatible with characteristic maps and it is clear that $u \in \mathcal{I}so_{\leq r}$ and $\tilde{f} \in \mathcal{E}'_r$. Since $\tilde{f}_m = f_m$ for $0 \leq m \leq r$, it is clear that if $f \in \mathcal{F}ib_r$, so is \tilde{f} . \square

Proposition 7.3.5. *Any map $f: X \rightarrow Z$ in \mathbf{ESpSe} has a functorial factorization $f = pi$ where $i \in \mathcal{C}of'_r$ and $p \in \mathcal{F}ib_r \cap \mathcal{E}'_r$.*

Proof. Use the model structure \mathbf{ESpSe}_r to factorize f as $f = g\iota$ where $\iota: X \rightarrow Y$ is in $\mathcal{C}of_r$ and $g: Y \rightarrow Z$ is in $\mathcal{F}ib_r \cap \mathcal{E}_r$. Now factorize g as $g = \tilde{g}u$ by Proposition 7.3.4, where $u \in \mathcal{I}so_{\leq r}$ and $\tilde{g} \in \mathcal{F}ib_r \cap \mathcal{E}'_r$. Then $f = \tilde{g}(u)$ and $u \in \mathcal{C}of'_r$ by Proposition 7.3.3. \square

Theorem 7.3.6. *There is a model structure on \mathbf{ESpSe} denoted \mathbf{ESpSe}'_r with fibrations $\mathcal{F}ib_r$, the class of maps $f: X \rightarrow Y$ such that f_i is bidegree-wise surjective for $0 \leq i \leq r$, and weak equivalences \mathcal{E}'_r , the class of maps $f: X \rightarrow Y$ such that f_r is a quasi-isomorphism and f_i is an isomorphism for $i > r$.*

Proof. We give a direct proof, checking the axioms as given in [2, Definition 2.1.3]. Firstly it is clear that the weak equivalences, fibrations and cofibrations are closed under composition. Axiom (MC1) is the existence of all small limits and colimits, which holds by Theorem 5.3.2. (MC2) is the two-out-of-three property for weak equivalences, which is clear.

Axiom (MC3) is that each of the three classes is closed under retracts. This is clear for fibrations and weak equivalences and it holds for cofibrations as they are defined by the left lifting property.

We consider axiom (MC5) next. This comprises the two factorization properties: for $f: X \rightarrow Y$ in \mathbf{ESpSe} , we need to be able to factorize (1) $f = pi$ where $i \in \mathcal{C}of'_r$ and $p \in \mathcal{E}'_r \cap \mathcal{F}ib_r$ and (2) $f = p'i'$ where $i' \in \mathcal{E}'_r \cap \mathcal{C}of'_r$ and $p' \in \mathcal{F}ib_r$.

The first factorization property holds by Proposition 7.3.5. Since $\mathcal{C}of_r \subseteq \mathcal{C}of'_r$, it follows from Remark 7.2.2 that we have $\mathcal{E}_r \cap \mathcal{C}of_r \subseteq \mathcal{E}'_r \cap \mathcal{C}of'_r$. So the second factorization property holds because it holds in the model structure \mathbf{ESpSe}_r .

Finally, axiom (MC4) comprises the two lifting properties: given a diagram in \mathbf{ESpSe}

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow p \\ Z & \xrightarrow{g} & W \end{array}$$

there must be a lift (1) when $i \in \mathcal{C}of'_r$ and $p \in \mathcal{E}'_r \cap \mathcal{F}ib_r$ and (2) when $i \in \mathcal{E}'_r \cap \mathcal{C}of'_r$ and $p \in \mathcal{F}ib_r$. We have defined $\mathcal{C}of'_r$ precisely so that (1) holds. Lifting property (2) holds by a standard retract argument: factorize i in \mathbf{ESpSe}_r as $i = \rho\iota$ where $\iota \in \mathcal{E}_r \cap \mathcal{C}of_r$ and $\rho \in \mathcal{F}ib_r$. Then $\iota \in \mathcal{E}'_r$ by Remark 7.2.2, so $\rho \in \mathcal{E}'_r$ by the two-out-of-three property. Thus i left lifts against ρ and so i is a retract of ι . Hence $i \in \mathcal{E}'_r \cap \mathcal{C}of_r$ and i left lifts against p . \square

7.4. Comparison of model structures.

Proposition 7.4.1. *The adjunction $\text{Shift} \dashv \text{Dec}$ gives a Quillen equivalence between \mathbf{ESpSe}'_r and \mathbf{ESpSe}'_{r+1} for all $r \geq 0$. And the adjunction $\text{LDec} \dashv \text{Shift}$ gives a Quillen equivalence between \mathbf{ESpSe}'_{r+1} and \mathbf{ESpSe}'_r for all $r \geq 0$.*

Proof. The proof is essentially the same as that of Corollary 7.2.4, noting for the first statement that $\text{Dec}(\mathcal{E}'_{r+1}) = \mathcal{E}'_r$ for all $r \geq 0$ and that the unit of the adjunction $\eta_X: X \rightarrow \text{Dec} \circ \text{Shift}(X)$ is in \mathcal{E}'_i for all $i \geq 0$, since it is an isomorphism. For the second statement, the adjunction is a Quillen adjunction since Shift preserves fibrations and weak equivalences. And the same two-out-of-three argument as for Corollary 7.2.4 concludes the proof. \square

Remark 7.4.2. Note that $\text{LDec}(\mathcal{E}'_{r+1}) \subset \mathcal{E}'_r$ for $r \geq 1$. For $r = 0$ this is not true in general. Indeed, let $f: X \rightarrow Y$ be in \mathcal{E}'_1 and consider, using Notation 6.3.5,

$$\text{LDec}(f)_0: \mathcal{T}^{-1} \text{Cone}_1(X_0)/\text{Im } q_1^X \rightarrow \mathcal{T}^{-1} \text{Cone}_1(Y_0)/\text{Im } q_1^Y.$$

We see that $\text{LDec}(f)_0$ is a quasi-isomorphism if and only if the induced map $\text{Ker } q_1^X \rightarrow \text{Ker } q_1^Y$ is a quasi-isomorphism. We know this is true for X and Y cofibrant in ESpSe'_1 , because LDec is a left Quillen functor. It is also true if X and Y are spectral sequences because then $\text{Ker } q_1^X = \text{Ker } q_1^Y = 0$.

The following result explains the relationship between the new model structures and those obtained in the previous section.

Proposition 7.4.3. *The identity functor $\text{ESpSe}'_r \rightarrow \text{ESpSe}_r$ is a right Bousfield localization which is not a Quillen equivalence.*

Proof. The first part is immediate since we have established the two model category structures on ESpSe with the same fibrations and $\mathcal{E}'_r \subset \mathcal{E}_r$.

Let X be a cofibrant object of ESpSe'_r and Y any object in ESpSe . If this were a Quillen equivalence, we would have that $f: X \rightarrow Y$ is in \mathcal{E}_r if and only if $f: X \rightarrow Y$ is in \mathcal{E}'_r . Since 0 is cofibrant this implies that any extended spectral sequence X such that $H(X_r) = 0$ satisfies $X_i = 0$ for all $i > r$ which is certainly not true. \square

So all our model structures are Quillen equivalent to one of either ESpSe_0 or ESpSe'_0 . Since ESpSe_0 is Quillen equivalent to the projective model structure on 0-bigraded complexes, Proposition 7.4.3 can be phrased as saying that ESpSe'_0 is a right *delocalization* of (a model category Quillen equivalent to) the projective model structure on 0-bigraded complexes.

7.5. Cofibrations in ESpSe'_0 . In this section we characterize the cofibrations of the model category ESpSe'_0 .

Lemma 7.5.1. *Let $f: X \rightarrow Z$ be a map in ESpSe . Any commutative diagram as on the left*

$$(*) \quad \begin{array}{ccc} X_0 & \xrightarrow{\alpha} & Y \\ f_0 \downarrow & & \downarrow p \\ Z_0 & \xrightarrow{\beta} & W \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\bar{\alpha}} & \bar{Y} \\ f \downarrow & & \downarrow \bar{p} \\ Z & \xrightarrow{\bar{\beta}} & \bar{W} \end{array}$$

in $\widehat{\langle \delta_0 \rangle}$ is obtained as the 0-page of a commutative diagram in ESpSe as on the right. Furthermore, if p is a surjective quasi-isomorphism, then $\bar{p} \in \text{Fib}_0 \cap \mathcal{E}_0$.

Proof. A diagram with the required properties is obtained by setting

$$\begin{aligned} \overline{Y}_m &= \begin{cases} Y & \text{for } m = 0, \\ X_m & \text{for } m \geq 1, \end{cases} & \varphi_m^{\overline{Y}} &= \begin{cases} (H\alpha)\varphi_1^X & \text{for } m = 1, \\ \varphi_m^X & \text{for } m \geq 2, \end{cases} \\ \overline{W}_m &= \begin{cases} W & \text{for } m = 0, \\ Z_m & \text{for } m \geq 1, \end{cases} & \varphi_m^{\overline{W}} &= \begin{cases} (H\beta)\varphi_1^Z & \text{for } m = 1, \\ \varphi_m^Z & \text{for } m \geq 2, \end{cases} \\ \overline{\alpha}_m &= \begin{cases} \alpha & \text{for } m = 0, \\ 1 & \text{for } m \geq 1, \end{cases} & \overline{\beta}_m &= \begin{cases} \beta & \text{for } m = 0, \\ 1 & \text{for } m \geq 1, \end{cases} & \overline{p}_m &= \begin{cases} p & \text{for } m = 0, \\ f_m & \text{for } m \geq 1. \end{cases} \end{aligned}$$

It is routine to check that $\overline{\alpha}, \overline{\beta}, \overline{p}$ are maps of extended spectral sequences. It is clear that the diagram commutes and that taking the 0-page recovers the original diagram. The final statement is immediate since $\overline{p}_0 = p$. \square

Proposition 7.5.2. *The cofibrations \mathcal{Cof}'_0 in \mathbf{ESpSe}'_0 are those $f: X \rightarrow Y$ such that f_0 is a cofibration of 0-bigraded complexes in the model category structure on 0-bC given in Proposition 2.3.8. That is, $f \in \mathcal{Cof}'_0$ if and only if f_0 has the left lifting property with respect to surjective quasi-isomorphisms of 0-bigraded complexes.*

Proof. Suppose that f is such that f_0 has the left lifting property with respect to surjective quasi-isomorphisms of 0-bigraded complexes. Consider the diagram in \mathbf{ESpSe}

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Z \\ f \downarrow & & \downarrow p \\ Y & \xrightarrow{\beta} & W \end{array}$$

where $p \in \mathcal{Fib}_0 \cap \mathcal{E}'_0$. There is a lift $h_0: Y_0 \rightarrow Z_0$ on the 0-page by assumption and then

$$h_s = \begin{cases} h_0 & s = 0, \\ p_s^{-1}\beta_s & s \geq 1, \end{cases}$$

gives a lift, using that $H(p_0)$ is an isomorphism for compatibility with characteristic maps.

Conversely, let $f \in \mathcal{Cof}'_0$ and let p be a surjective quasi-isomorphism of 0-bigraded complexes. We need to show there is a lift in the diagram appearing on the left of (*) in the statement of Lemma 7.5.1 when p is a surjective quasi-isomorphism. By the lemma we have the diagram on the right of (*) and $\overline{p} \in \mathcal{Fib}_0 \cap \mathcal{E}_0$.

Now apply Lemma 7.3.4 to factorize \overline{p} as $\overline{p} = \tilde{p}u$ where $u: \overline{Z} \rightarrow Z$ is in $\mathcal{Iso}_{\leq 0}$ and $\tilde{p}: Z \rightarrow \overline{W}$ is in $\mathcal{Fib}_0 \cap \mathcal{E}'_0$. Since $f \in \mathcal{Cof}'_0$ there is a lift $h: Y \rightarrow Z$

$$\begin{array}{ccccc} X & \xrightarrow{\overline{\alpha}} & \overline{Z} & \xrightarrow{u} & Z \\ f \downarrow & & \nearrow h & & \downarrow \tilde{p} \\ Y & \xrightarrow{\beta} & & & \overline{W} \end{array}$$

and then $u_0^{-1}h_0$ provides the required lift. \square

8. INFINITY-CATEGORICAL INTERPRETATION

In this section we will interpret our results in the context of relative categories in the setting of Barwick-Kan [3], as well as explaining the relationship to our previous work [19].

In [19, Theorem 5.3.1] we established the existence of structures on the category of spectral sequences \mathbf{SpSe} closely related to the model category structures on \mathbf{ESpSe} presented here. The relevant structure has a class of fibrations and a class of weak equivalences and is called an *almost Brown category with functorial factorization*. Given $r \geq 0$, there is such a structure, denoted \mathbf{SpSe}_r , with fibrations the class \mathcal{Fib}_r of morphisms of spectral sequences f such that f_i is surjective for $0 \leq i \leq r$ and weak equivalences f such that $H(f_r)$ is an isomorphism.

Definition 8.1.1. A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$ where \mathcal{C} is a category and \mathcal{W} is a subcategory containing all the objects of \mathcal{C} ; the maps in \mathcal{W} are called *weak equivalences*. We write \mathbf{RelCat} for the category of relative categories endowed with the model category structure of Barwick-Kan [3].

We study the relative category $(\mathbf{ESpSe}, \mathcal{E}'_r)$ and its restriction $(\mathbf{SpSe}, \mathcal{E}_r)$.

We have seen that for $f: X \rightarrow Y$ a morphism in \mathbf{ESpSe} we have $f \in \mathcal{E}'_{r+1}$ if and only if $\text{Dec } f \in \mathcal{E}'_r$. And it is easy to see that $f \in \mathcal{E}'_r$ if and only if $\text{Shift } f \in \mathcal{E}'_{r+1}$.

We will use ideas and results of Pascaleff [23] and the main theorem of Meier [20].

Definition 8.1.2. [23, Definition 2.1] A *Dwyer-Kan adjunction* between relative categories is an adjunction $(L, R): (\mathcal{C}_1, \mathcal{W}_1) \rightarrow (\mathcal{C}_2, \mathcal{W}_2)$ such that L and R preserve the class of weak equivalences and such that the unit of the adjunction lies in \mathcal{W}_1 and the counit of the adjunction in \mathcal{W}_2 .

As noted by Pascaleff, a result of Bousfield-Kan [12, Corollary 3.6] shows that if two relative categories are related by a Dwyer-Kan adjunction, then they are Barwick-Kan equivalent in \mathbf{RelCat} .

Consider the following commutative diagram of relative categories.

$$\begin{array}{ccccc}
 (\mathbf{ESpSe}, \mathcal{E}'_0) & \xleftarrow[\text{Dec}^r]{\text{Shift}^r} & (\mathbf{ESpSe}, \mathcal{E}'_r) & \xleftarrow[\text{Dec}]{\text{Shift}} & (\mathbf{ESpSe}, \mathcal{E}'_{r+1}) \\
 \uparrow & & \uparrow & & \uparrow \\
 (\mathbf{SpSe}, \mathcal{E}_0) & \xleftarrow[\text{Dec}^r]{\text{Shift}^r} & (\mathbf{SpSe}, \mathcal{E}_r) & \xleftarrow[\text{Dec}]{\text{Shift}} & (\mathbf{SpSe}, \mathcal{E}_{r+1}) \\
 & \xleftarrow[\text{Dec}^r]{\text{LDec}^r} & & \xleftarrow[\text{Dec}]{\text{LDec}} & \\
 & & & &
 \end{array}$$

Theorem 8.1.3. *The adjunctions in the diagram above are Dwyer-Kan adjunctions. In particular the rows are Barwick-Kan equivalences in \mathbf{RelCat} . In addition the relative categories $(\mathbf{SpSe}, \mathcal{E}_0)$ and $(\mathbf{ESpSe}, \mathcal{E}'_r)$ for $r \geq 0$ are fibrant relative categories in the Barwick-Kan model structure on \mathbf{RelCat} .*

Proof. We consider the $(\text{Shift}, \text{Dec})$ adjunction. The unit is an isomorphism. The counit $\epsilon: \text{Shift} \circ \text{Dec} \rightarrow \text{Id}_{\mathbf{ESpSe}}$ is the identity at page $i \geq 2$ and a quasi-isomorphism at page 1, using Proposition 5.2.4. Hence the counit lives in \mathcal{E}'_r for every $r \geq 1$. The category \mathbf{ESpSe}'_r is a model category and so $(\mathbf{ESpSe}, \mathcal{E}'_r)$ is fibrant by [20, Theorem

4.13]. Furthermore if $f: X \rightarrow Y$ is in \mathcal{E}'_0 where one of X or Y is a spectral sequence, then so is the other. Hence $(\mathbf{SpSe}_0, \mathcal{E}_0)$ is a relative category which is homotopically full in $(\mathbf{ESpSe}, \mathcal{E}'_0)$, in the sense of Meier and so it is fibrant by [20, Theorem 4.13].

Now we consider the $(\mathbf{LDec}, \mathbf{Shift})$ adjunction on \mathbf{SpSe} . We have seen that \mathbf{Shift} preserves weak equivalences and by Remark 7.4.2, $\mathbf{LDec}: (\mathbf{SpSe}, \mathcal{E}_{r+1}) \rightarrow (\mathbf{SpSe}, \mathcal{E}_r)$ preserves weak equivalences for all r . The counit of the adjunction is an isomorphism. The unit of the adjunction is the identity at pages $i \geq 2$ and, by the same reasoning as in Remark 7.4.2, it is a quasi-isomorphism at page 1 on \mathbf{SpSe} (but not in general on \mathbf{ESpSe}). \square

Corollary 8.1.4. *For each $r \geq 1$, the relative category $(\mathbf{SpSe}, \mathcal{E}_r)$ has $(\mathbf{SpSe}, \mathcal{E}_0)$ as a fibrant replacement in the Barwick-Kan model structure on \mathbf{RelCat} , via the functor \mathbf{Dec}^r , or the functor \mathbf{LDec}^r . \square*

APPENDIX A. REPRESENTABLE LINEAR WITNESS BOOKS

A.1. Description of the representable linear witness book $\mathcal{Y}(r, p, n)$.

Example A.1.1. We treat the combinatorics of the representable object $\mathcal{Y}(r, p, n)$ of LWB. This is the free linear witness book generated by an element $\alpha_r^{p,n}$, corresponding to the identity map in $\text{Hom}_{\mathbb{D}}((r, p, n), (r, p, n))$.

We describe it in terms of Proposition 4.2.2. For each i , the i -bigraded complex $(\mathcal{Y}(r, p, n))_i$ has two bigraded components free of rank one, and the other components are 0. More precisely we have

$$\begin{aligned}\mathcal{Y}(r, p, n)_i &= \mathbb{R} (w_r)^{r-i} d_r \alpha_r^{p,n} \oplus \mathbb{R} (w_r)^{r-i} \alpha_r^{p,n} \quad \text{for } 0 \leq i < r, \\ \mathcal{Y}(r, p, n)_r &= \mathbb{R} d_r \alpha_r^{p,n} \oplus \mathbb{R} \alpha_r^{p,n}, \\ \mathcal{Y}(r, p, n)_{r+k} &= \mathbb{R} d_{r+k} (s_r)^k \alpha_r^{p,n} \oplus \mathbb{R} (s_r)^k \alpha_r^{p,n} \quad \text{for } k \geq 0.\end{aligned}$$

And the actions of d, s, w are clearly identified knowing the relations of Proposition 4.2.2 (compare also with Lemma 4.1.3). Here we denote by $(w_r)^j$ the composite $w_{r-j+1} \dots w_{r+1} w_r$ and the same notation is used for the iterated composite of the ss.

In addition, any $\psi: (r, p, n) \rightarrow (r', p', n')$ induces a morphism of linear witness books $\mathcal{Y}(r, p, n) \rightarrow \mathcal{Y}(r', p', n')$ sending the identity morphism to the morphism ψ in $\mathcal{Y}(r', p', n')$. More concretely we have

$$\begin{array}{ccc} \mathcal{Y}(r, p, n) \xrightarrow{\delta_r} \mathcal{Y}(r, p+r, n+r-1) & & \mathcal{Y}(r, p, n) \xrightarrow{\omega} \mathcal{Y}(r+1, p, n) \\ \varphi \alpha_r^{p,n} \longmapsto \varphi d_r \alpha_r^{p+r, n+r-1} & & \varphi \alpha_r^{p,n} \longmapsto \varphi w_{r+1} \alpha_{r+1}^{p,n} \\ \\ \mathcal{Y}(r, p, n) \xrightarrow{\sigma} \mathcal{Y}(r-1, p-1, n-1) & & \\ \varphi \alpha_r^{p,n} \longmapsto \varphi s_{r-1} \alpha_{r-1}^{p-1, n-1}. & & \end{array}$$

Of course, $\text{Hom}_{\text{LWB}}(\mathcal{Y}(r, p, n), A) \cong A(r, p, n)$.

A.2. Characterizations of (LWB)^e and (LWB)^s via lifting properties.

Notation A.2.1. Let $\mathcal{Z}(r, p, n)$ (resp. $\mathfrak{S}(r, p, n)$) \in LWB be the representative of $\text{Ker } d_r$ (resp. $\text{Ker } s_r$). These objects are defined as the following cokernels.

$$\begin{aligned}\mathcal{Z}(r, p, n) &= \text{coker}(\delta_r: \mathcal{Y}(r, p-r, n+1-r) \rightarrow \mathcal{Y}(r, p, n)), \\ \mathfrak{S}(r, p, n) &= \text{coker}(\sigma_r: \mathcal{Y}(r+1, p+1, n+1) \rightarrow \mathcal{Y}(r, p, n)).\end{aligned}$$

Since any linear witness book satisfies $\text{Ker } s_r \subseteq \text{Ker } d_r$, there is a (surjective) morphism $\tau_r: \mathcal{Z}(r, p, n) \rightarrow \mathfrak{S}(r, p, n)$. Let M^r denote the set of morphisms of LWB given by

$$M^r := \{\tau_r: \mathcal{Z}(r, p, n) \rightarrow \mathfrak{S}(r, p, n)\}_{r \geq 0, p, n \in \mathbb{Z}}.$$

We denote by $\mathbb{R}_{\leq r}(p, n)$ the extended spectral sequence having \mathbb{R} concentrated in bidegree (p, n) at pages i such that $0 \leq i \leq r$ and 0 elsewhere, with characteristic maps being the identity or zero.

Lemma A.2.2. *The linear witness book $\mathfrak{S}(r, p, n)$ lies in $(\text{LWB})^e$, but the linear witness book $\mathfrak{Z}(r, p, n)$ does not. We have the following isomorphisms in $(\text{LWB})^e$:*

$$\mathfrak{S}(r, p, n) \cong \mathcal{N}\mathcal{Q}\mathfrak{S}(r, p, n) = \mathcal{N}\mathcal{Q}\mathfrak{Z}(r, p, n) \cong \mathcal{N}(\mathbb{R}_{\leq r}(p, n)).$$

Proof. We first compute $\mathfrak{S}(r, p, n)$ and $\mathfrak{Z}(r, p, n)$ in LWB , thanks to Example A.1.1. On the one hand we see that at page $i \leq r$, $\mathfrak{S}(r, p, n)_i = \mathfrak{Z}(r, p, n)_i = \mathbb{R}w^{r-i}\alpha_r^{p,n}$, with $d_i = 0$, w_i is the identity and $s_i = 0$, for $i < r$. At page $i > r$ we have $\mathfrak{S}(r, p, n)_i = 0$. This implies $\text{Ker } d_i = \text{Ker } s_i$ in $\mathfrak{S}(r, p, n)_i$, for all $i \geq 0$, so that $\mathfrak{S}(r, p, n) \in (\text{LWB})^e$. At page $i > r$ we have $\mathfrak{Z}(r, p, n)_i = \mathbb{R}d_i s^{i-r}\alpha_r^{p,n} \oplus \mathbb{R}s^{i-r}\alpha_r^{p,n}$. In particular $\text{Ker } s_r = 0$ while $\text{Ker } d_r = \mathbb{R}\alpha_r^{p,n}$. In addition we have $\mathcal{Q}\mathfrak{S}(r, p, n) = \mathcal{Q}\mathfrak{Z}(r, p, n) = \mathbb{R}_{\leq r}(p, n)$. \square

The following characterization of $(\text{LWB})^e$ via right lifting properties is an immediate consequence of the descriptions.

Proposition A.2.3. (1) *A map $f: K \rightarrow L$ has the right lifting property with respect to all morphisms in M^τ if and only if whenever $a \in \text{Ker } d_r$ and $f(a) \in \text{Ker } s_r$ then $a \in \text{Ker } s_r$.*

(2) *A linear witness book L is in $(\text{LWB})^e$ if and only if $L \rightarrow 0$ has the right lifting property with respect to morphisms in M^τ .* \square

Remark A.2.4. Equivalently, we can characterize $(\text{LWB})^e$ as the subcategory of M^τ -local objects in LWB , in the ordinary categorical sense, see for example [25, p143]. That is, an M^τ -local object L is one such that for all morphisms $\tau_r: \mathfrak{Z} \rightarrow \mathfrak{S}$ in M^τ , the induced map $\text{LWB}(\mathfrak{S}, L) \rightarrow \text{LWB}(\mathfrak{Z}, L)$ is a bijection. Surjectivity is by Proposition A.2.3 and injectivity holds because of surjectivity of the maps τ_r in M^τ .

The characterization of objects of $(\text{LWB})^s$ in Proposition 5.4.2 can also be described by means of right lifting properties. Namely, for $r \geq 1$, let

$$\mathcal{W}(r, p, n) = \text{Coker}(\omega_r: \mathcal{Y}(r-1, p, n) \rightarrow \mathcal{Y}(r, p, n)),$$

representing $\text{Ker } \omega_r$. Note that $\mathcal{W}(r, p, n)$ is in $(\text{LWB})^e$ and that the extended spectral sequence $\mathcal{Q}\mathcal{W}(r, p, n)$ takes the following form:

$$\mathcal{Q}\mathcal{W}(r, p, n)_i = \begin{cases} \mathbb{R}w^{r-i}d_r\alpha_r^{p,n} & \text{if } 0 \leq i < r, \\ \mathbb{R}d_r\alpha_r^{p,n} \oplus \mathbb{R}\alpha_r^{p,n} & \text{if } i = r, \\ 0 & \text{if } i > r. \end{cases}$$

The relations $\omega_{r+1}\sigma_r = 0$ and $\sigma_r\omega_{r+1} = 0$ imply that there are well defined sets of morphisms of LWB :

$$\begin{aligned} M^{\sigma, \omega} &:= \{\sigma_r: \mathcal{W}(r+1, p+1, n+1) \rightarrow \mathcal{Y}(r, p, n)\}_{r \geq 0, p, n \in \mathbb{Z}}, \\ M^{\omega, \sigma} &:= \{\omega_{r+1}: \mathfrak{S}(r, p, n) \rightarrow \mathcal{Y}(r+1, p, n)\}_{r \geq 0, p, n \in \mathbb{Z}}. \end{aligned}$$

The following proposition is a reformulation of Proposition 5.4.2.

Proposition A.2.5. *Let L be an object of $(\text{LWB})^e$.*

(1) *The characteristic maps φ_i of $\mathcal{Q}L$ are surjective for all i if and only if $L \rightarrow 0$ has the right lifting property with respect to $M^{\omega, \sigma}$.*

(2) *The characteristic maps φ_i of $\mathcal{Q}L$ are injective for all i if and only if $L \rightarrow 0$ has the right lifting property with respect to $M^{\sigma, \omega}$.* \square

We collect information about key objects in LWB in the following table.

TABLE 1. Key objects in LWB

object	represents	homology	
$\mathcal{Y}(r, p, n)$	element	$\mathbf{R}^{p,n} \oplus \mathbf{R}^{p-r, n+1-r}$	on pages $0 \leq i < r$
		0	on pages $i \geq r$
$\mathcal{Z}(r, p, n)$	$\text{Ker } d_r$	$\mathbf{R}^{p,n}$	on pages $0 \leq i \leq r$
		0	on pages $i > r$
$\mathcal{G}(r, p, n)$	$\text{Ker } s_r$	$\mathbf{R}^{p,n}$	on pages $0 \leq i \leq r$
		0	on pages $i > r$
$\mathcal{W}(r, p, n)$	$\text{Ker } \omega_r$	$\mathbf{R}^{p-r, n+1-r}$	on pages $0 \leq i < r$
		0	on pages $i \geq r$

APPENDIX B. FURTHER ADJOINT FUNCTORS ON LWB

In this section we give the descriptions of the right adjoint functor $\mathcal{R}_{[r,t]}$ to the forgetful functor $\mathcal{U}_{[r,t]}: \text{LWB} \rightarrow \text{tr}_{[r,t]} \text{LWB}$ and of the functor $(\mathcal{W}_{\geq r})_!$ left adjoint to the functor $\mathcal{F}_{\geq r}: \text{LWB}_{\geq r} \rightarrow \text{LWB}$. These descriptions have been obtained by applying the (co)end formulas of Proposition 2.1.1.

Proposition B.1.1. *Let (L, d_i, w_i, s_i) be an object of $\text{tr}_{[r,t]} \text{LWB}$. We have*

$$\mathcal{R}_{[r,t]}(L)_i^{p,n} = \begin{cases} L_r^{p+r-i, n+r-i} & \text{for } 0 \leq i < r, \\ L_i^{p,n} & \text{for } r \leq i \leq t, \\ \text{Cone}_i(Z_t(L_t))^{p-i, n-i+1} & \text{for } i > t, \end{cases}$$

with the maps $d_i^{\mathcal{R}}, w_i^{\mathcal{R}}, s_i^{\mathcal{R}}$ defined as

$$\begin{aligned} d_i^{\mathcal{R}} &= 0 & w_{i+1}^{\mathcal{R}} &= 0 & s_i^{\mathcal{R}} &= 1 & \text{for } 0 \leq i < r, \\ d_i^{\mathcal{R}} &= d_i & w_{i+1}^{\mathcal{R}} &= w_{i+1} & s_i^{\mathcal{R}} &= s_i & \text{for } r \leq i < t, \\ d_t^{\mathcal{R}} &= d_t^L & w_{t+1}^{\mathcal{R}} &= \begin{pmatrix} 0 & 1 \end{pmatrix} & s_t^{\mathcal{R}} &= \begin{pmatrix} d_t \\ 0 \end{pmatrix} & \text{for } i = t, \\ d_i^{\mathcal{R}} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & w_{i+1}^{\mathcal{R}} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & s_i^{\mathcal{R}} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } i > t. \end{aligned}$$

The counit of the adjunction $(\mathcal{U}_{[r,t]}, \mathcal{R}_{[r,t]})$ is the identity. The unit of the adjunction $L \rightarrow \mathcal{R}_{[r,t]} \mathcal{U}_{[r,t]} L$ takes the following form.

- For $0 \leq i < r$ it corresponds to $s^{r-i}: L_i^{p,n} \rightarrow L_r^{p+r-i, n+r-i}$.
- It is the identity at the i -page, for $r \leq i \leq t$.
- For $i > t$ the unit corresponds to the map

$$\begin{pmatrix} 0 \\ w^{i-t} \end{pmatrix}: L_i^{p,n} \rightarrow Z_t(L_t)^{p-i, n+1-i} \oplus Z_t(L_t)^{p,n}. \quad \square$$

Remark B.1.2. If $r > 0$ then $\mathcal{R}_{\geq r}$ does not restrict to a functor from $\text{tr}_{\geq r}(\text{LWB})^e$ to $(\text{LWB})^e$, since $\text{Ker } d_i \neq \text{Ker } s_i$ for $i < r$.

Proposition B.1.3. *The left adjoint functor $(\mathcal{W}_{\geq r})_! : \text{LWB} \rightarrow \text{tr}_{\geq r} \text{LWB}$ to the functor $\mathcal{F}_{\geq r}$ has the following form. For (L, d_i, w_i, s_i) a linear witness book, and for $i \geq r$,*

$$(\mathcal{W}_{\geq r})_!(L)_i^{p,n} = (L_0^{p+r-i, n+r-i} \oplus L_0^{p+r; n+r-1} \oplus L_i^{p,n}) / \sim$$

where \sim is the equivalence relation generated by, for all $x \in L_r$,

$$\begin{aligned} (0, 0, s^{i-r}x) &\sim (w^r x, 0, 0), \\ (0, 0, d_i s^{i-r}x) &\sim (0, w^r x, 0). \end{aligned}$$

The structure maps $d_i^!, s_i^!, w_{i+1}^!$ are defined for $i \geq r$ by

$$d_i^! = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & d_i \end{pmatrix}, \quad w_{i+1}^! = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w_{i+1} \end{pmatrix}, \quad s_i^! = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s_i \end{pmatrix}.$$

The counit of the adjunction is an isomorphism at each page $i \geq r$ and the unit of the adjunction $\eta_L : L \rightarrow \mathcal{F}_{\geq r}(\mathcal{W}_{\geq r})_!L$ has the following description.

- For $x \in L_i$ and $0 \leq i < r$

$$\eta_L(x) = \overline{(w^i x, 0, 0)} \in (\mathcal{W}_{\geq r})_!(L)_r.$$

- For $x \in L_i$ and $i \geq r$

$$\eta_L(x) = \overline{(0, 0, x)} \in (\mathcal{W}_{\geq r})_!(L)_i.$$

Using the characterization of $(\text{LWB})^e$ in Proposition 5.2.1 and that of $(\text{LWB})^s$ in Proposition 5.4.2 it is immediate to prove the next statement.

Proposition B.1.4. *The functor $(\mathcal{W}_{\geq r})_!$ restricts to a functor $(\text{LWB})^e \rightarrow \text{tr}_{\geq r} (\text{LWB})^e$ and to a functor $(\text{LWB})^s \rightarrow \text{tr}_{\geq r} (\text{LWB})^s$. \square*

APPENDIX C. NON-EXISTENCE OF CERTAIN MODEL CATEGORIES

In this section, we prove that there is no model category structure on either LWB or on $(\text{LWB})^e$ such that $(\text{LWB})^s$ is the subcategory of fibrant objects, where the class of weak equivalences between fibrant objects is the class \mathcal{E}_r (or equivalently \mathcal{E}'_r).

We characterize morphisms $p: L \rightarrow M$ in LWB with L, M in $(\text{LWB})^s$, such that $\text{Ker } p$ lives in $(\text{LWB})^s$. Recall that $\text{Ker } p$ is computed in LWB , hence pagewise. Since limits in $(\text{LWB})^e$ are computed pagewise, we have $\text{Ker } p \in (\text{LWB})^e$. We use the characterization to exhibit a counterexample to stability of fibrations under pullback along any map.

Lemma C.1.1. *Let $p: L \rightarrow M$ be a morphism in LWB with $L = (L_i, d_i^L, s_i^L, w_i^L)$ and $M = (M_i, d_i^M, s_i^M, w_i^M)$ in $(\text{LWB})^s$. Then $\text{Ker } p$ lies in $(\text{LWB})^s$ if and only if for all $i \geq 0$ the following conditions hold*

$$\begin{aligned} (C_\sigma^i) \quad \forall a \in L_i, \quad s_i^M p a = 0 &\implies \exists a' \in L_i, p a' = p a \text{ and } s_i^L a' = 0, \\ (C_\omega^{i+1}) \quad \forall a \in L_{i+1}, \quad w_{i+1}^M p a = 0 &\implies \exists a' \in L_{i+1}, p a' = p a \text{ and } w_{i+1}^L a' = 0. \end{aligned}$$

Proof. We use the characterization from Proposition 5.4.2 for a linear witness book $K \in (\text{LWB})^e$ to be in $(\text{LWB})^s$, that is: for all i , $\text{Ker } w_{i+1}^K = \text{Im } s_i^K$ (which will

correspond to the condition C_σ^i) and $\text{Ker } s_i^K = \text{Im } w_{i+1}^K$ (which will correspond to condition C_ω^{i+1}).

We let $K = \text{Ker } p$, and $w^K = w^L$, $s^K = s^L$.

Assume we have $\text{Ker } w_{i+1}^K = \text{Im } s_i^K$. In particular, for every $u \in L_{i+1}$, such that $pu = 0$ and $w_{i+1}^L u = 0$, there exists $v \in L_i$ such that $pv = 0$ and $s_i^L v = u$. Let $a \in L_i$ with $s_i^M pa = ps_i^L a = 0$, and set $u = s_i^L a$. We have $pu = 0$ and $w_{i+1}^L u = 0$, hence there exists $v \in L_i$ such that $pv = 0$ and $s_i^L v = u$. Let $a' = a - v$. Then $pa' = pa$ and $s_i^L a' = s_i^L a - s_i^L v = u - u = 0$. Hence condition C_σ^i holds.

Conversely assume C_σ^i holds. Let $u \in L_{i+1}$, such that $pu = 0$ and $w_{i+1}^L u = 0$. Since $L \in (\text{LWB})^s$, there exists $v \in L_i$ such that $s_i^L v = u$. In particular $s_i^M pv = 0$. Hence there exists $v' \in L_i$ with $pv' = pv$ and $s_i^L v' = 0$. In particular $s_i(v - v') = u$ and $p(v - v') = 0$.

The proof that condition C_ω^{i+1} is equivalent to $\text{Ker } s_i^K = \text{Im } w_{i+1}^K$ proceeds in exactly the same way, exchanging the roles of s and w . \square

Proposition C.1.2. *There is no model category structure on LWB or on $(\text{LWB})^e$ such that the category of fibrant objects is the category $(\text{LWB})^s$ with weak equivalences \mathcal{E}_r .*

Proof. The proof is by contradiction. If such a model category structure exists with fibrant objects $(\text{LWB})^s$, then any map $f: L \rightarrow M$ in LWB between two objects $L, M \in (\text{LWB})^s$ factors as $f = pi$ with $i: L \rightarrow K$ an acyclic cofibration and $p: K \rightarrow M$ a fibration. Since M is fibrant and p is a fibration, K is fibrant. Since fibrations are stable under pullbacks, p should satisfy conditions C_σ^k and C_ω^k for every k .

We provide the following counterexample. Let $f = \mathcal{N}(\pi)$ where $\pi: \mathcal{D}_r(0,0) \rightarrow \mathbf{R}(0,0)$ is the projection in SpSe . Recall that $\mathcal{N}\mathcal{D}_r(0,0) = \mathcal{Y}(r,0,0)$ with generator in $\mathcal{Y}(r,0,0)_r^{0,0}$ denoted $\alpha_r^{0,0}$. We have for all k , $(\mathcal{N}\mathbf{R}(0,0))_k = \mathbf{R}^{0,0}$ with generator denoted $\beta_k^{0,0}$. We have $s_k^{\mathcal{N}\mathbf{R}(0,0)} = 0$. For every $0 \leq k \leq r$, we have π_k is surjective, hence f_k is surjective by Proposition 5.1.5, and $f_r(\alpha_r^{0,0}) = \beta_r^{0,0}$. Finally $H(\mathcal{Y}(r,0,0)_r) = 0$.

Let us write $f = pi$, with $i: \mathcal{Y}(r,0,0) \rightarrow K \in \mathcal{E}_r$, so that $H(K_r) = 0$. Since f_r is surjective, $p_r: K_r \rightarrow \mathbf{R}^{0,0}$ is surjective and there exists $z \in K_r$ such that $p_r z = \beta_r^{0,0}$. In particular $s_r p_r z = 0$. By condition C_σ^r , there exists $z' \in K_r$ such that $p_r z' = \beta_r^{0,0}$ and $s_r^K z' = 0$. But $\text{Ker } s_r^K = \text{Ker } d_r^K = \text{Im } d_r^K$ because $K \in (\text{LWB})^s \subset (\text{LWB})^e$ and $H(K_r) = 0$. Let $y \in K_r$ be such that $d_r^K y = z'$. We have $p_r z' = \beta_r^{0,0} = d_r p_r y$, a contradiction.

Note that this proof by contradiction also works for $(\text{LWB})^e$. \square

Corollary C.1.3. *There is no model category structure on ESpSe such that the category of fibrant objects is the category SpSe with weak equivalences \mathcal{E}_r .*

Proof. This follows from the equivalences of categories with weak equivalences between $(\text{SpSe}, \mathcal{E}_r)$ and $((\text{LWB})^s, \mathcal{E}_r)$ and between $(\text{ESpSe}, \mathcal{E}_r)$ and $((\text{LWB})^e, \mathcal{E}_r)$. \square

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