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# A purely analytic derivation of Bonnet surfaces

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## Abstract

Bonnet has characterized his surfaces by a geometric condition. What is done here is a characterization of the same surfaces by two analytic conditions: (i) the mean curvature  $H$  of a surface in  $\mathbb{R}^3$  should admit a reduction to an ordinary differential equation; (ii) this latter equation should possess the Painlevé property.

*Keywords:* Bonnet surfaces, Gauss-Codazzi equations, sixth equation of Painlevé.

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## 1. Introduction

Surfaces immersed in the three-dimensional simply connected manifold  $\mathbb{M}^3(c)$  with constant sectional curvature  $\kappa c^2$ , where  $\mathbb{M}^3(c)$  is the sphere  $\mathbb{S}^3(c^2)$  when  $\kappa = 1$ , the Euclidean space  $\mathbb{R}^3$  when  $\kappa = 0$  and the hyperbolic space  $\mathbb{H}^3(c^2)$  when  $\kappa = -1$ , are described by a set of partial differential equations (PDE) known as the Gauss-Codazzi equations<sup>1</sup>

$$\begin{cases} \nu_{z\bar{z}} + \frac{1}{2}(H^2 - c^2)e^\nu - 2|Q|^2e^{-\nu} = 0 \text{ (Gauss),} \\ Q_{\bar{z}} - \frac{1}{2}H_z e^\nu = 0, \bar{Q}_z - \frac{1}{2}H_{\bar{z}} e^\nu = 0 \text{ (Codazzi),} \end{cases} \quad (1)$$

in which  $z, \bar{z}$  are conformal coordinates,  $e^\nu dz d\bar{z}$  the induced metric on the surface,  $H$  the mean curvature ( $\nu$  and  $H$  being both real), and  $Q$  the complex coefficient of the Hopf differential.

By determining all surfaces in  $\mathbb{R}^3$  applicable on a given surface while conserving the two principal radii of curvature, the geometer Pierre-Ossian Bonnet [3] isolated a remarkable class of analytic surfaces, now called Bonnet surfaces, characterized by the third order ordinary differential equation (ODE) [3, Eq. (52) p. 84] [6, Eq. (B16)]

$$(\log h')'' + 2h' - 2 \left( \frac{4a}{\sinh 4a(\xi - \xi_0)} \right)^2 \left( \frac{h' + h^2 - c^2}{h'} \right) = 0, \quad ' = \frac{d}{d\xi}, \quad (2)$$

in which the mean curvature  $H$  is a function of one variable only, denoted  $\xi$ ,  $H(z, \bar{z}) = h(\xi)$ , with  $\xi = \Re(z)$ , and  $a^2$  is an arbitrary real constant. Their metric  $\nu$  and the coefficient  $Q$  are given by (see details in [5, 6])

$$\begin{cases} Q = 2a \coth 2a(z - z_0) - 2a \coth 4a\Re(z - z_0) = \frac{\sinh 2a(\bar{z} - \bar{z}_0)}{\sinh 2a(z - z_0)} \frac{2a}{\sinh 4a\Re(z - z_0)}, \\ \bar{Q} = 2a \coth 2a(\bar{z} - \bar{z}_0) - 2a \coth 4a\Re(z - z_0) = \frac{\sinh 2a(z - z_0)}{\sinh 2a(\bar{z} - \bar{z}_0)} \frac{2a}{\sinh 4a\Re(z - z_0)}, \\ |Q|^2 = \left( \frac{2a}{\sinh 4a\Re(z - z_0)} \right)^2, \\ e^\nu = 4|Q|^2 \frac{d\Re(z)}{dH}. \end{cases} \quad (3)$$

The general solution [1] of the ODE (2), which admits the first integral ([9, p. 48] for  $c = 0$ , [2] for arbitrary  $c$ )

$$K = \left( \frac{h''}{h'} + 8a \coth 4a(\xi - \xi_0) \right)^2 + 8 \left[ \left( \frac{4a}{\sinh 4a(\xi - \xi_0)} \right)^2 \frac{h^2 - c^2}{h'} + h' + 8a \coth 4a(\xi - \xi_0)h \right], \quad (4)$$

is the logarithmic derivative of the tau-function of either (when  $a \neq 0$ ) a sixth Painlevé equation  $P_{VI}$  of codimension two, or (when  $a \rightarrow 0$ ) a fifth Painlevé equation  $P_V$  of codimension three.

<sup>1</sup>The additional parameter  $c$  arises when one replaces the Euclidean space  $\mathbb{R}^3$  by the three-dimensional manifold  $\mathbb{M}^3(c)$  described in the text.

$P_{VI}$  was only to be discovered in 1905 [8] from purely mathematical considerations, but in fact it already occurred in a natural problem of geometry thirty-eight years earlier, this is why Bonnet surfaces are so important.

What we perform here is a straightforward derivation of the third order ODE (2), which is based purely on analysis, and not on geometry like that of Bonnet.

## 2. The method

The mean curvature is an important geometric quantity for a surface in a three-dimensional manifold, therefore let us make the first assumption that this mean curvature  $H$  depends on one variable denoted  $\xi = \varphi(z, \bar{z})$  instead of two ( $z$  and  $\bar{z}$ ),

$$H(z, \bar{z}) = h(\xi), \quad \xi = \varphi(z, \bar{z}), \quad (5)$$

and let us then determine the function  $\varphi(z, \bar{z})$  and the ODE for  $h(\xi)$ .

Let us eliminate  $\nu$  and  $H$  between the three equations (1)<sub>1</sub>, (1)<sub>2</sub>, (5) in order to obtain an equation for  $h(\xi)$  whose coefficients depend only on  $Q, \bar{Q}, \varphi$  and their derivatives. The field  $\nu$  is first determined from the first Codazzi equation (1)<sub>2</sub>, then substituted in the two remaining Gauss-Codazzi equations (1). After replacement of the field  $H$  defined by (5), the second Codazzi equation becomes

$$Q_z \varphi_z - Q_{\bar{z}} \varphi_{\bar{z}} = 0, \quad (6)$$

and the Gauss equation is now

$$\left\{ \begin{array}{l} (\log h')'' + F_1 h' + F_2 \frac{h''}{h'} + F_3 \frac{h^2 - c^2}{h'} + F_4 = 0, \\ F_1 = \frac{Q\bar{Q}}{Q_z \varphi_{\bar{z}}}, F_2 = \frac{\varphi_{z\bar{z}}}{\varphi_z \varphi_{\bar{z}}}, F_3 = \frac{Q_{\bar{z}}}{\varphi_z^2 \varphi_{\bar{z}}}, \\ F_4 = \frac{Q_{\bar{z}} Q_{\bar{z}\bar{z}} Q_{z\bar{z}}}{\varphi_z \varphi_{\bar{z}}} - \frac{Q_{z\bar{z}\bar{z}}}{Q_{\bar{z}} \varphi_z \varphi_{\bar{z}}} - \frac{\varphi_{z\bar{z}} \varphi_{z\bar{z}}}{\varphi_z^3 \varphi_{\bar{z}}} + \frac{\varphi_{z\bar{z}\bar{z}}}{\varphi_z^2 \varphi_{\bar{z}}}. \end{array} \right. \quad (7)$$

According to our assumption (5), equation (7)<sub>1</sub> should be an ODE for  $h(\xi)$ , i.e. its coefficients should only depend on  $\xi$ . Let us now make a second assumption, namely that this ODE should possess the Painlevé property [7], i.e. its general solution should have no movable critical singularities.

This double requirement will determine all coefficients  $F_j$  (except  $F_1$ , which will remain arbitrary) in terms of  $\xi$ . Once the  $F_j(\xi)$ 's are known, the last four equations in (7) will determine the functions  $Q(z, \bar{z}), \bar{Q}(z, \bar{z}), \varphi(z, \bar{z})$ .

### 2.1. Painlevé test

For the technical vocabulary, we refer the reader to Ref. [4, p. 160] [7, Chap. 2].

A necessary condition for the absence of movable critical singularities is that (7) pass the Painlevé test. Since  $h^{(3)}(\xi)$  in (7) is singular at  $h' = 0$ , the Painlevé test mainly consists in checking the existence of all Laurent series of both  $h(\xi)$  and  $1/h'(\xi)$  able to locally represent the general solution.

Near a movable singularity  $\chi(\xi) = 0$ , since  $Q$  is nonzero, the field  $h$  has only one family of movable simple poles,

$$h \sim \frac{2}{F_1} \chi^{-1}, \chi \rightarrow 0, \chi' = 1, \quad (8)$$

and the Fuchs indices of this family are  $-1, 1, 2$ . The existence of a Laurent series of  $h$  near this singularity requires that two conditions must be obeyed at the positive Fuchs indices,

$$Q_1 \equiv F_1 F_2 + F_1' = 0, \quad Q_2 \equiv F_1 F_2^2 + 2F_3 - F_1 F_4 + F_1 F_2' = 0. \quad (9)$$

They determine  $F_2$  and  $F_3$  and restrict the ODE (7)<sub>1</sub> to

$$(\log h')'' + F_1 h' + \frac{F_4 h' + (F_1/2)^3 G_4 (h^2 - c^2) - (\log F_1)' h''}{h'} = 0, \quad (10)$$

with the notation

$$G_4 = (2/F_1)^2 [F_4 + (\log F_1)'' - (\log F_1)'^2]. \quad (11)$$

Next, let us build the ODE for  $1/h' = g(\xi)$ . After the substitution  $h' = 1/g(\xi)$  in (10) and its derivative, the elimination of  $h(\xi)$  generates a third order ODE for  $g(\xi)$ , whose degree in the third derivative is two. Its Gauss decomposition then reads,

$$(gS')^2 - 4S = 0, S = \frac{8}{F_1^3 G_4} \left[ \frac{(\log g)''}{g} - \frac{F_1' g'}{g^3} - \frac{F_4}{g} - \frac{F_1}{g^2} \right] + c^2 = h^2. \quad (12)$$

It has one family of movable double poles with an arbitrary leading coefficient,

$$g \sim g_0(\xi) \chi^{-2}, \chi \rightarrow 0, \chi' = 1. \quad (13)$$

Its Fuchs indices are  $-1, 0, 2$ , and the index 2 generates a necessary condition on  $G_4, F_1$  for the Laurent series to exist,

$$Q_2 \equiv \left[ \left( \frac{2G_4'}{F_1 G_4} \right)^2 + 2G_4 \right]' = 0, \quad (14)$$

thus defining the value of  $G_4$

$$G_4 = 2(4a)^2 \left[ 1 - \coth^2 \left( 4a \int (F_1(\xi)/2) d\xi \right) \right], a \text{ arbitrary constant.} \quad (15)$$

After substitution of this value (15) in (10), the change of independent variable  $\xi \rightarrow \int (F_1(\xi)/2) d\xi$  performs the identification of the ODE (10) to the ODE (4) for the mean curvature of the Bonnet surfaces.

Therefore, the unique ODE resulting from the two above mentioned assumptions is the ODE characterizing the Bonnet surfaces.

## 2.2. Determination of the variables $\varphi, Q, \bar{Q}, \nu$

Given the values  $F_j(\xi)$  obtained in section 2.1, let us now determine the reduced variable  $\xi = \varphi(z, \bar{z})$ , the Hopf coefficients  $Q(z, \bar{z}), \bar{Q}(z, \bar{z})$  and the metric function  $\nu(z, \bar{z})$  by integrating the last four equations in (7) and the remaining Gauss-Codazzi equations (1).

Of course, by the results of Bonnet, one knows that the expressions (3) together with  $\varphi = (z + \bar{z})/2$  are indeed a solution of (7)<sub>2:5</sub>, therefore the question is to prove the unicity of this solution.

Without loss of generality, let us first make the change of independent variable  $\xi \rightarrow \int (F_1(\xi)/2) d\xi$ , i.e. consider (7)<sub>2:5</sub> with the values given by the conditions (9), (11),

$$F_1 = 2, F_2 = 0, F_3 = F_4 = G_4 = 2(4a)^2 \left[ 1 - \coth^2(4a\xi) \right], \xi = \varphi(z, \bar{z}). \quad (16)$$

The equation involving  $F_2$  in (7) first yields the function  $\varphi$  as a sum of two arbitrary functions of one variable,

$$\xi = \varphi = \frac{F_a(z) + F_b(\bar{z})}{2}. \quad (17)$$

The remaining equations (7)<sub>2:5</sub> are then equivalent to

$$\begin{cases} Q_z = -\frac{1}{8} F_a'^2 F_b' 2(4a)^2 \left[ 1 - \coth^2 \left( 4a \frac{F_a(z) + F_b(\bar{z})}{2} \right) \right], \\ \bar{Q}_z = -\frac{1}{8} F_a' F_b'^2 2(4a)^2 \left[ 1 - \coth^2 \left( 4a \frac{F_a(z) + F_b(\bar{z})}{2} \right) \right], \\ Q\bar{Q} = -\frac{1}{8} F_a'^2 F_b'^2 2(4a)^2 \left[ 1 - \coth^2 \left( 4a \frac{F_a(z) + F_b(\bar{z})}{2} \right) \right]. \end{cases} \quad (18)$$

Two quadratures provide  $Q$  and  $\bar{Q}$  up to an additive arbitrary function of respectively  $z$  and  $\bar{z}$ ,

$$\begin{cases} Q = -2F_a'(z)^2 a \coth \left( 4a \frac{F_a(z) + F_b(\bar{z})}{2} \right) + f_1(z), \\ \bar{Q} = -2F_b'(\bar{z})^2 a \coth \left( 4a \frac{F_a(z) + F_b(\bar{z})}{2} \right) + f_2(\bar{z}). \end{cases} \quad (19)$$

The existence of an addition formula for  $\coth$  suggests the natural change of these two arbitrary functions by

$$f_1(z) = 2F_a'(z)^2 a \coth(2a\tilde{f}_1(z)), f_2(\bar{z}) = 2F_b'(\bar{z})^2 a \coth(2a\tilde{f}_2(\bar{z})). \quad (20)$$

The constraint that the product of these  $Q$  and  $\bar{Q}$  be equal to (18)<sub>3</sub> is then easily solved as

$$\tilde{f}_1(z) = F_a(z), \tilde{f}_2(\bar{z}) = F_b(\bar{z}). \quad (21)$$

This provides values of  $Q, \bar{Q}, |Q|^2$  identical to the Bonnet values (3) upon replacement of, respectively,  $F_a(z)$  by  $z - z_0$  and  $F_b(\bar{z})$  by  $\bar{z} - \bar{z}_0$ , i.e. by a conformal transformation. Finally, the value of the metric function  $\nu$  follows directly from the Gauss-Codazzi equations.

## 3. Conclusion

This direct analytic method could be applied to other problems of differential geometry in which one variable is clearly privileged.

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