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Refined blow-up behavior for reaction-diffusion equations with non scale invariant exponential nonlinearities

Loth Damagui CHABI ¹

Abstract

We consider positive radial decreasing blow-up solutions of the semilinear heat equation

$$u_t - \Delta u = f(u) := e^u L(e^u), \quad x \in \Omega, \quad t > 0,$$

where $\Omega = \mathbb{R}^n$ or $\Omega = B_R$ and L is a slowly varying function (which includes for instance logarithms and their powers and iterates, as well as some strongly oscillating unbounded functions). We characterize the asymptotic blow-up behavior and obtain the sharp, global blow-up profile in the scale of the original variables (x, t) . Namely, assuming for instance $u_t \geq 0$, we have

$$u(x, t) = G^{-1} \left(T - t + \frac{1}{8} \frac{|x|^2}{|\log|x||} \right) + o(1) \quad \text{as } (x, t) \rightarrow (0, T), \quad \text{where } G(X) = \int_x^\infty \frac{ds}{f(s)} ds.$$

This estimate in particular provides the sharp final space profile and the refined space-time profile. For exponentially growing nonlinearities, such results were up to now available only in the scale invariant case $f(u) = e^u$. Moreover, this displays a universal structure of the global blow-up profile, given by the resolvent G^{-1} of the ODE composed with a fixed time-space building block, which is robust with respect to the factor $L(e^u)$.

Key words: Semilinear heat equation, exponential nonlinearities, asymptotic blowup behavior, blow-up profile, refined space-time behavior, slow variation.

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1 Introduction

This paper is a contribution to the understanding of the precise singularity formation for solutions of superlinear parabolic problems involving *non scale invariant nonlinearities*, in continuation to the works [9, 22, 35, 8, 31, 7] (see below for more details on these works).

Consider the following semilinear heat equation

$$\begin{cases} u_t - \Delta u = f(u), & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is smooth, $u_0 \in L^\infty(\Omega)$ and $f \in C^1([0, \infty))$. For the model cases

$$f(u) = |u|^{p-1}u \quad (p > 1) \quad \text{and} \quad f(u) = e^u,$$

the blowup behavior has been the subject of extensive research and is rather well understood, especially in the Sobolev subcritical power case $1 < p < (n+2)/(n-2)_+$ where a complete classification of blow-up profiles is available. For $f(u) = |u|^{p-1}u$ we refer to [1, 6, 13, 18, 19, 20, 25, 26, 27, 28, 30, 33, 37] and, for $f(u) = e^u$, to [1, 2, 5, 4, 10, 12, 14, 23, 29, 34, 38]. However the analysis of sharp blow-up asymptotics in these works heavily depends on the exact form of the nonlinearities and their scale invariance properties, namely the invariance of the equations under the transformations

$$u_\lambda(x, t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t) \quad (\lambda > 0),$$

and

$$u_\lambda(x, t) = u(\lambda x, \lambda^2 t) + \log \lambda \quad (\lambda > 0)$$

respectively (or at least on an asymptotic homogeneity of the nonlinearities as the variable $u \rightarrow \infty$).

Let us give a brief review of known results on the blowup behavior of solutions of problem (1.1) for non scale-invariant nonlinearities. In the early work [14], blow-up rate estimates were obtained for time increasing positive solutions of (1.1) with rather general nonlinearities. For suitable classes of non scale-invariant nonlinearities, results on blow-up sets were obtained in [15]. In [9, 22], for problem (1.1) with a particular, logarithmically perturbed nonlinearity of the form $f(u) = |u|^{p-1}u \log(2+u^2)$, blowup rate estimates were obtained (without sign or monotonicity assumption), and special solutions with a prescribed blowup profile were constructed (see also [21, 32] for blowup rate estimates for the related nonlinear wave equations). In [7, 8, 31, 35], positive solutions were studied for problem (1.1) with general regularly varying nonlinearities, of the form $f(u) = u^p L(u)$ with L of slow variation (see (2.4) for precise definition). In [31, 35], parabolic Liouville type theorems were obtained, along with applications to universal bounds and blowup rate estimates. An analogue of the classical Giga-Kohn result was obtained in [7], using i.a. the type I blowup estimate from [35]. Moreover, building on the result in [7], the final and space-time blow-up profile of positive radially decreasing solutions was obtained in [8], revealing a *structural universality* of the blow-up profile, robust with respect to the factor $L(u)$.

In this paper, we are interested in non-scale invariant nonlinearities with exponential growth at infinity. We note that, concerning blowup rate estimates, the results in [14] and in the recent work [15] cover such nonlinearities (under suitable assumptions). Our main goal here is to provide a very precise description of the blowup asymptotics, including space-time and final blowup profiles, for radially decreasing solutions of problem (1.1), for a large class of exponentially growing nonlinearities without scale invariance. Our analysis follows in part that in our previous works [7, 8] for power type regularly varying nonlinearities.

2 Main results

Assume

$$f \in C^1([0, \infty)), \quad f(0) \geq 0, \quad f \text{ is } C^2 \text{ and positive for large } s. \quad (2.1)$$

By standard theory, for $u_0 \in L^\infty(\Omega)$ with $u_0 \geq 0$, problem (1.1) has a unique, nonnegative classical solution, of maximal existence time $T = T(u_0) \in (0, \infty]$, that we will denote by u throughout this paper. If $1/f$ is integrable at infinity and u_0 is suitably large, then u blows up in finite time, i.e $T < \infty$ and

$$\lim_{t \rightarrow T} \|u(t)\|_\infty = \infty.$$

In this case, T is called the blowup time of u . Given $a \in \overline{\Omega} \subset \mathbb{R}^n$, we say that a is a blowup point of u if there exists $(a_j, t_j) \rightarrow (a, T)$ such that $u(a_j, t_j) \rightarrow \infty$ as $j \rightarrow \infty$.

We consider nonlinearities of the form

$$f(s) = e^s L(e^s), \quad s \geq 0. \quad (2.2)$$

The function L will be assumed to satisfy

$$\frac{sL'(s)}{L(s)} = o(\log^{-\alpha}(s)) \quad \text{and} \quad \left(\frac{sL'(s)}{L(s)}\right)' = o\left(\frac{1}{s \log s}\right) \quad \text{as } s \rightarrow \infty, \quad \text{for some } \alpha > \frac{1}{2}. \quad (2.3)$$

Such L include logarithms and their powers and iterates, as well as some strongly oscillating functions (see Remark 3.2 below for typical examples). Note that (2.3) is a subclass of the functions with *slow variation*, i.e

$$\lim_{\lambda \rightarrow \infty} \frac{L(\lambda s)}{L(\lambda)} = 1 \quad \text{for each } s > 0 \quad (2.4)$$

and that, for $L \in C^1$ near infinity, a sufficient condition for (2.4) is $\lim_{s \rightarrow \infty} s \frac{L'(s)}{L(s)} = 0$.

To state our results we introduce the following function

$$G(X) := \int_X^\infty \frac{ds}{f(s)}. \quad (2.5)$$

Note that, under assumptions (2.1)-(2.3), G is well defined and there exists a large $A > 0$ such that $G : [A, \infty) \rightarrow (0, G(A)]$ is decreasing. Moreover,

$$\psi(t) := G^{-1}(T - t)$$

is the positive solution of the corresponding ODE, i.e $y'(t) = f(y)$, which blows up at $t = T$. Concerning u we will focus on positive solutions that are radial decreasing in space and increasing in time. To this end, we will assume

$$\Omega = \mathbb{R}^n \text{ or } \Omega = B_R, \quad u_0 \in L^\infty(\Omega), \quad u_0 \geq 0, \quad (2.6)$$

$$u_0 \text{ radially symmetric, nonincreasing in } r = |x|, \text{ with } u_0 \text{ nonconstant if } \Omega = \mathbb{R}^n. \quad (2.7)$$

This guarantees that $u(x, t) = u(r, t) \geq 0$ with $r = |x|$ and $u_r \leq 0$. As for the property $u_t \geq 0$, we recall that it is satisfied for instance when $u_0 \in BC(\overline{\Omega}) \cap H_{loc}^2(\Omega)$ is such that $\Delta u_0 + f(u_0) \geq 0$ and $u_0 = 0$ on $\partial\Omega$ (see, e.g., [30, Proposition 52.19]).

Our first main result is the following global, refined blowup estimate, valid in the scale of the original variables (x, t) .

Theorem 2.1. *Assume (2.1)-(2.3), (2.6), (2.7), $T = T(u_0) < \infty$ and $u_t \geq 0$. Then*

$$u(x, t) = G^{-1}\left(T - t + \frac{|x|^2}{4|\log|x^2||}\right) + o(1) \quad \text{as } (x, t) \rightarrow (0, T). \quad (2.8)$$

Combining (2.8) with the asymptotic formula of G^{-1} (see Lemma 4.1(ii) below), we obtain the explicit blow-up asymptotics of u , which takes the form

$$u(x, t) = -\log\left(T - t + \frac{|x|^2}{4|\log|x^2||}\right) - \log\left\{L\left(\frac{1}{T - t + \frac{|x|^2}{4|\log|x^2||}}\right)\right\} + o(1) \quad \text{as } (x, t) \rightarrow (0, T).$$

The latter displays the precise influence of the factor L on the profile. For instance for the typical case

$$f(u) = u^q e^u, \quad q \geq 1,$$

corresponding to $L(s) = (\log s)^q$, we obtain

$$u(x, t) = -\log\left(T - t + \frac{|x|^2}{4|\log|x^2||}\right) - q \log\left\{-\log\left(T - t + \frac{|x|^2}{4|\log|x^2||}\right)\right\} + o(1) \quad \text{as } (x, t) \rightarrow (0, T).$$

See Remark 3.2 below for more examples.

On the other hand, we note from (2.8) that u blows up only at $x = 0$ hence, by standard parabolic estimates, the limit $u(x, T) := \lim_{t \rightarrow T} u(x, t)$ exists for all $x \neq 0$. As direct consequence of Theorem 2.1, we obtain the final blow-up profile and the corresponding refined space-time profile of the solution.

Corollary 2.1. *Under the assumptions of Theorem 2.1, the following holds:*

(i) **(Final profile estimate)**

$$u(x, T) = G^{-1}\left(\frac{|x|^2}{4|\log|x|^2|}\right) \quad \text{as } x \rightarrow 0. \quad (2.9)$$

(ii) **(Refined space-time profile estimate)**

$$u(\xi\sqrt{(T-t)|\log(T-t)|}, t) = G^{-1}\left[(T-t)\left(1 + \frac{|\xi|^2}{4}\right)\right] + o(1) \quad \text{as } t \rightarrow T \quad (2.10)$$

uniformly for ξ bounded.

In the special case of the pure exponential, we have $G^{-1}(Y) = -\log Y$ for $Y > 0$ and Corollary 2.1 recovers the known final and refined profiles (cf. [2, 12, 23, 37]). As for the global profile in Theorem 2.1, it seems new even for the pure exponential and completes the upper estimate that was obtained in [34].

In the course of the proof of Theorem 2.1, we establish the following expansion of $u(y\sqrt{T-t}, t) - \psi(t)$ with second key order term for bounded y . This is of independent interest since it gives more precise expansion of the solution than (2.8), but only in the more restricted range $|x| = O(\sqrt{T-t})$. In the special case $f(u) = e^u$ this result was proved in [2, 12, 23]. Here and in the rest of this paper, we denote

$$L_\rho^2 := \left\{v \in L_{loc}^2(\mathbb{R}^n); \int_{\mathbb{R}^n} v^2(y)\rho(y)dy < \infty\right\}, \quad H_\rho^1 := \{f \in L_\rho^2; \nabla f \in L_\rho^2\}, \quad \rho(y) := e^{-|y|^2/4}, \quad (2.11)$$

and u is defined to be 0 outside Ω in case $\Omega = B_R$.

Theorem 2.2. *Under assumptions of Theorem 2.1, we have*

$$u(y\sqrt{T-t}, t) - \psi(t) = \frac{2n - |y|^2 + o(1)}{4|\log(T-t)|}, \quad \text{as } t \rightarrow T, \quad (2.12)$$

with convergence in $H_\rho^1(\mathbb{R}^n)$ and uniformly for bounded y .

3 Discussion and outline of proofs

3.1 Discussion

Remark 3.1. (Structure of the profile) *The global blow-up profile (2.8) possesses a universal structure, given by the “resolvent” G^{-1} of the ODE composed with the universal time-space building block*

$$T - t + \frac{|x|^2}{4|\log|x|^2|}, \quad (3.1)$$

and the latter is not affected by the slowly varying factor L of the nonlinearity. At principal order, the effect of the latter on the profile is reflected only in the application of the resolvent G^{-1} to this building block.

A similar observation was made in [8, Remark 2.1] in the case of nonlinearities of the form $f(u) = u^p L(u)$. In turn, we remark that the time-space building block (3.1) is the limit of that in [8, formula (2.10)] (in the power case) as $p \rightarrow \infty$.

The above also applies for the final and refined space-time profiles and for the sharp behavior in backward space-time parabolas (cf. Corollary 2.1 and Theorem 2.2).

Remark 3.2. (Examples) *Typical examples of nonlinearities $f(s)$ to which Theorems 2.1-2.2 apply are given by:*

$$\left\{ \begin{array}{l} \bullet (s+K)^q e^s \text{ for } K > 0 \text{ and } q \in \mathbb{R}, \text{ or } K = 0 \text{ and } q \geq 1, \\ \bullet [\log^q(s+K)]e^s \text{ for } K > 1 \text{ and } q \in \mathbb{R}, \text{ or } K = 1 \text{ and } q \geq 1 \\ \bullet \exp(s \pm (s+1)^\nu) \text{ with } \nu \in (0, 1/2), \\ \bullet \text{ the strongly oscillating functions} \\ \quad (s+1)^{\sin[\log(s+1)]} e^s \quad \text{and} \quad \exp[s + (s+1)^\nu \cos((s+1)^\gamma)], \quad \nu, \gamma > 0, \nu + \gamma < 1/2, \\ \bullet [1 + a \sin((s+1)^\nu)]e^s \text{ with } \nu \in (0, 1/2) \text{ and } |a| < 1. \end{array} \right. \quad (3.2)$$

Indeed, by tedious but elementary computations, one can check that the corresponding functions L satisfy condition (2.3). Our results thus cover a large class of exponentially growing, non scale invariant nonlinearities. However, it so far remains an open problem what is the largest possible class of f for which the conclusions of Theorems 2.1-2.2 hold.

Remark 3.3. (Alternative type I assumption) For $n \leq 2$, Theorems 2.1-2.2 remain valid (see Remark 7.1 below) if we replace the time increasing assumption $u_t \geq 0$ by the type I blow-up assumption, namely

$$\|u(\cdot, t)\|_\infty \leq G^{-1}(T - t) + M, \quad T - \delta < t < T, \quad (3.3)$$

for some $M, \delta > 0$.

In any dimension, property (3.3) is in particular guaranteed if $u_t \geq 0$ (see Lemma 4.6, which follows from arguments in [14, Section 4]). However, in higher dimensions, the conclusions of Theorems 2.1-2.2 are not true in general under the weaker assumption (3.3) (see Remark 3.4(i)).

In the special case $f(u) = e^u$ with u_0 satisfying (2.6), (2.7), property (3.3) is also known to be true when $n = 1$ (see [23, p.133]) or $n \in [3, 9]$ (see [12, Theorem 1] and also [36]).

3.2 Outline of proofs

The route map of the proofs is organized in the following order, where each step uses the previous ones:

- **Step 1.** Convergence to the ODE solution in the scale $x = O(\sqrt{T - t})$
- **Step 2.** Upper part of Theorem 2.1 and lower estimate of $|u_r|$
- **Step 3.** Theorem 2.2
- **Step 4.** Lower part of Theorem 2.1
- **Step 5.** Corollary 2.1

Step 1 (Section 4) is based on the following quasiconvergence property of u . To this end we introduce the similarity profile equation

$$z'' + \left(\frac{n-1}{r} - \frac{r}{2}\right)z' + e^z - 1 = 0, \quad r > 0, \quad \text{with } z'(0) = 0 \quad (3.4)$$

and the relevant subset of equilibria

$$\mathcal{S} := \left\{ z \in C^2([0, \infty)); z \text{ solution of (3.4) such that } z' \in L^\infty(0, \infty), z' \leq 0 \text{ and } z(0) \geq 0 \right\}. \quad (3.5)$$

Proposition 3.1. Let $f \in C^1([0, \infty))$, with $f(0) \geq 0$ and $f(s) = e^s L(e^s)$, where L satisfies

$$\frac{sL'(s)}{L(s)} = O(\log^{-\alpha}(s)) \text{ as } s \rightarrow \infty, \text{ for some } \alpha > \frac{1}{2}. \quad (3.6)$$

Assume (2.6), (2.7), $T = T(u_0) < \infty$ and (3.3) for some $M, \delta > 0$. Then, for any sequence $t_j \rightarrow T$, there exists a subsequence (still denoted t_j) and $\phi \in \mathcal{S}$ such that

$$\lim_{j \rightarrow \infty} u(y\sqrt{T - t_j}, t_j) - \psi(t_j) = \phi(|y|), \text{ uniformly for } y \text{ in compact sets of } \mathbb{R}^n. \quad (3.7)$$

Proposition 3.1 is proved by modifying arguments from [7], which rely on transformation of the equation by similarity variables and ODE renormalization, along with a perturbed energy functional. We recall that an extension to non scale invariant, regularly varying nonlinearities of the classical Giga-Kohn analysis [18, 20] for the pure power nonlinearity was done in [7, Section 3]. We note that the similarity profile equation (3.4) is the same as for the pure exponential. This is due to the fact that, under assumption (3.6), after appropriate ODE renormalization, the terms coming from the slowly varying factor L disappear in the limiting procedure (see Lemma 4.3). We then use the assumption $u_t \geq 0$ of Theorem 2.1 to eliminate all possible limits $\phi \in \mathcal{S}$ except 0 (see Lemma 4.5). Since this assumption also guarantees type I blowup (see Lemma 4.6), this leads to the following consequence of Proposition 3.1.

Proposition 3.2. Under the assumptions of Theorem 2.1, we have

$$\lim_{t \rightarrow T} u(y\sqrt{T - t}, t) - G^{-1}(T - t) = 0 \quad \text{uniformly for } y \text{ bounded}. \quad (3.8)$$

In Step 2, given in Section 5, we establish the upper part of Theorem 2.1, along with a lower estimate of $|u_r|$. This will be derived as a consequence of [8, Theorem 3.1] combined with Proposition 3.2. The former, which follows from a maximum principle argument for a suitable auxiliary function, provides a precise upper space-time estimate of u in terms of $u(0, t)$, as well as a lower estimate of $|u_r|$, for radial decreasing solutions of (1.1) with general nonlinearities $f(u)$, under appropriate assumptions on f . Our task will be to check that these assumptions are satisfied for the exponential type nonlinearities in Theorem 2.1 and that the resulting estimates can be recast under the form required by Theorem 2.1.

Step 3 (cf. Section 6) is the proof of Theorem 2.2. To this end, we further use similarity variables and ODE normalization and linearize around the 0 solution so as to describe more accurately the convergence in Proposition 3.1. For this, owing to the lower estimate of $|u_r|$ from Step 2 and since the linearized equation has the same form as in the case of nonlinearities $f(u) = u^p L(u)$, we can rely on a result from [8] for the linearized equation (see Proposition 6.1 below). We note that the latter was based on modifications of the classical center manifold type analysis introduced in [13] (see also [23, 37, 2, 24]) and simplified in [33] in the case of radially decreasing solutions.

Steps 4 and 5 are given in Section 7. We first prove the lower estimate of Theorem 2.1. To this end, for t_0 suitably close to T , we use the lower estimate on $u(t_0)$ provided by Theorem 2.2 in the region $|x| = O(\sqrt{T - t_0})$ and propagate it forward in time via a suitable comparison argument. Lastly, the final and refined space-time profiles in Corollary 2.1 are easily deduced from the global profile in Theorem 2.1.

Remark 3.4. (i) *The result in Proposition 3.1 is of independent interest since it is valid for more general solutions of (1.1) (in particular, only type I blow-up rate is needed instead of $u_t \geq 0$), as compared with Theorem 2.2. But of course the latter gives more precise information on the asymptotic behavior of the solution. Note that in dimensions $3 \leq n \leq 9$, for the type I blowup solutions without the property $u_t \geq 0$, the conclusions of Theorems 2.1-2.2 may fail. Indeed, it is well known that \mathcal{S} is not reduced to $\{0\}$ for $3 \leq n \leq 9$ (it actually contains infinitely many elements, see e.g [11, 12]), and such nontrivial $\phi \in \mathcal{S}$ provide radial decreasing, backward self-similar solutions of the equation $u_t - \Delta u = e^u$ in $\mathbb{R}^n \times (0, T)$, of the form*

$$u(x, t) = \phi\left(\frac{x}{\sqrt{T-t}}\right) - \log(T-t),$$

which exhibit different final profiles from those in Theorem 2.1 and Corollary 2.1.

(ii) *For $f(s) = e^s$, the result of Proposition 3.2 was obtained in [3]. The proof therein was based on similarity variables and energy arguments inspired from [18] (without perturbation), combined with an intersection-comparison analysis with the singular steady-state U^* of $\Delta u + e^u = 0$. More precisely, the property $u_t \geq 0$ is used in [3, Section 3] to show that the solution (recast in similarity variables) intersects U^* exactly once for t close to T . It is then shown in [3, Section 4] by delicate ODE analysis that 0 is the only element of \mathcal{S} with such intersection property, and this allows to eliminate the nonzero limits $\phi \in \mathcal{S}$. Here, by directly using the observation that the property $u_t \geq 0$ yields the additional constraint $r\phi' + 2 \geq 0$ on the possible limits $\phi \in \mathcal{S}$ in (3.7) (see (4.44)-(4.45)), it turns out that the nonzero limits can be eliminated by using only a small part of the ODE analysis from [3] (see Lemma 4.5 below). We thus avoid any use of intersection-comparison arguments, which leads to a simpler proof even in the special case $f(s) = e^s$.*

4 Proof of Propositions 3.1 and 3.2

Here and later on we will need various asymptotic properties of the ODE solution and of related quantities. We start with the following. In the rest of paper, the asymptotic notation \sim means that the quotient of the two functions converges to 1.

Lemma 4.1. *Assume (2.2), where L is C^1 and positive for large s , and let G be as in (2.5).*

(i) *If*

$$\lim_{s \rightarrow \infty} \frac{sL'(s)}{L(s)} = 0, \tag{4.1}$$

then

$$G(X) \sim \frac{e^{-X}}{L(e^X)} = \frac{1}{f(X)}, \quad X \rightarrow \infty, \tag{4.2}$$

(ii) If L satisfies (3.6), then

$$G^{-1}(Y) = -\log\left(YL\left(\frac{1}{Y}\right)\right) + o(1) \quad \text{as } Y \rightarrow 0^+ \quad (4.3)$$

and in particular

$$G^{-1}(Y) = (1 + o(1))|\log Y|, \quad Y \rightarrow 0^+. \quad (4.4)$$

Proof of lemma 4.1. (i) For X large enough, by the change of variable $\eta = e^\tau$, we have

$$G(X) = \int_X^\infty \frac{d\tau}{f(\tau)} = \int_X^\infty \frac{d\tau}{e^\tau L(e^\tau)} = \int_{e^X}^\infty \frac{d\eta}{\eta^2 L(\eta)} =: Q(e^X). \quad (4.5)$$

By [8, Lemma 4.2(i)] with $p = 2$, we have $Q(s) \sim \frac{1}{sL(s)}$ as $s \rightarrow \infty$, hence (4.2).

(ii) Using (4.2) and setting $X = X(Y) := G^{-1}(Y)$, we write

$$\frac{1}{Y} = e^X L(e^X)(1 + \varepsilon(e^X)), \quad \text{with } \lim_{s \rightarrow \infty} \varepsilon(s) = 0. \quad (4.6)$$

Let $\lambda(s) := (1 + \varepsilon(s))L(s)$. Assuming $\frac{1}{2} < \alpha < 1$ without loss of generality, it follows from (2.3) that $\log(L(s)) = o(\log^{1-\alpha}s)$, hence $\log(\lambda(s)) = o(\log^\alpha s)$ as $s \rightarrow \infty$. Using [8, Lemma 7.2] (which holds under assumption (3.6)), we then deduce that $L(s\lambda(s)) \sim L(s)$ as $s \rightarrow \infty$. Going back to (4.6), we obtain $L(1/Y) = L(\lambda(e^X)e^X) \sim L(e^X)$ as $X \rightarrow \infty$, and then $e^{-X} = YL(1/Y)(1 + \varepsilon_1(Y))$ with $\lim_{Y \rightarrow 0} \varepsilon_1(Y) = 0$, hence (4.3). \square

Proof of Proposition 3.1. For clarity we split the proof into several steps.

Step 1. *Upper estimate of $|\nabla u|$.*

Lemma 4.2. *Under the assumptions of Proposition 3.1, we have*

$$\|\nabla u(t)\|_\infty \leq M_1(T-t)^{-1/2}, \quad t_0 < t < T, \quad (4.7)$$

with $t_0 = T - \delta/2$ and $M_1 = M_1(\Omega, f, M, T, \delta) > 0$.

Proof of Lemma 4.2. We first claim that, under the current assumptions,

$$\|f(u(\cdot, t))\|_\infty \leq \frac{C(f, M, T)}{T-t}, \quad T - \delta \leq t < T. \quad (4.8)$$

By (4.2), there exists $z_0 > 0$ such that

$$0 < f(X) \leq \frac{2}{G(X)}, \quad X \geq z_0. \quad (4.9)$$

Moreover, there exists $z_1 \geq z_0$ such that $L(e^{s+M}) \leq 2L(e^s)$ for all $s \geq z_1$, hence

$$G(X+M) = \int_{X+M}^\infty \frac{ds}{e^s L(e^s)} = \int_X^\infty \frac{ds}{e^{s+M} L(e^{s+M})} \geq \frac{1}{2}e^{-M}G(X), \quad X \geq z_1. \quad (4.10)$$

Set $\Sigma := \{(x, t) \in \Omega \times (T - \delta, T); u(x, t) \geq z_1 + M\}$ and let $(x, t) \in \Sigma$. By (3.3) and the decreasing monotonicity of G , we have $G(u(x, t)) \geq G(\psi(t) + M)$. Moreover, $\psi(t) \geq \|u(t)\|_\infty - M \geq z_1$ hence $G(\psi(t) + M) \geq \frac{1}{2}e^{-M}G(\psi(t))$ by (4.10). Consequently, using (4.9), we obtain

$$0 < f(u(x, t)) \leq \frac{2}{G(u(x, t))} \leq \frac{2}{G(\psi(t) + M)} \leq \frac{4e^M}{G(\psi(t))} = \frac{4e^M}{T-t}, \quad (x, t) \in \Sigma. \quad (4.11)$$

On the other hand, for $(x, t) \in (\Omega \times (T - \delta, T)) \setminus \Sigma$, we have

$$|f(u(x, t))| \leq C(f, M) \leq C(f, M) \frac{T}{T-t}.$$

This and (4.11) prove claim (4.8).

Now denote by $(e^{-tA})_{t \geq 0}$ the heat semigroup in Ω (with Dirichlet condition for $\Omega \neq \mathbb{R}^n$) and set $t_1 = T - \delta$, $t_0 = T - \delta/2$. Using the variation of constants formula, the gradient estimate for the heat semigroup (see, e.g., [30, Proposition 48.7*]), (3.3) and (4.8), we compute for $t_0 < t < T$,

$$\begin{aligned} \|\nabla u(t)\|_\infty &\leq \|\nabla e^{-(t-t_1)A}u(t_1)\|_\infty + \int_{t_1}^t \|\nabla e^{-(t-\tau)A}f(u(\tau))\|_\infty d\tau \\ &\leq C(\Omega, T)(t-t_1)^{-1/2}\|u(t_1)\|_\infty + C(\Omega, T) \int_{t_1}^t (t-\tau)^{-1/2}\|f(u(\tau))\|_\infty d\tau \\ &\leq C(\Omega, T)(G^{-1}(\delta) + M)(T-t)^{-1/2} + C(\Omega, f, M, T) \int_{t_1}^t (t-\tau)^{-1/2}(T-\tau)^{-1} d\tau. \end{aligned}$$

Setting $t^* = t - (T - t)$, we have $t - \tau \geq \frac{1}{2}(T - \tau)$ for $\tau \leq t^*$, hence

$$\begin{aligned} \int_{t_1}^t (t-\tau)^{-1/2}(T-\tau)^{-1} d\tau &\leq C \int_{t_1}^{t^*} (T-\tau)^{-\frac{3}{2}} d\tau + (T-t)^{-1} \int_{t^*}^t (t-\tau)^{-1/2} d\tau \\ &\leq C[(T-\tau)^{-\frac{1}{2}}]_{t_1}^{t^*} + 2(T-t)^{-1}(t-t^*)^{\frac{1}{2}} \leq C(T-t)^{-1/2}, \end{aligned}$$

and (4.7) follows. \square

Step 2. Rescaled equation. We set

$$y := \frac{x}{\sqrt{T-t}}, \quad s := -\log(T-t). \quad (4.12)$$

By (4.4), we may take $s_0 > 0$ large enough so that $\psi_1(s) := \psi(T - e^{-s})$ exists and is $> \frac{1}{8}s$ for $s \geq s_0$. We define the rescaled function

$$w(y, s) := u(ye^{-s/2}, T - e^{-s}) - \psi_1(s). \quad (4.13)$$

Note that it is equivalent to $u(x, t) = \psi(t) + w(y, s)$. By a simple computation using $\psi_1' = e^{-s}f(\psi_1)$, we have

$$w_s = -\frac{y}{2} \cdot \nabla w + e^{-s}u_t - e^{-s}f(\psi_1), \quad (4.14)$$

and

$$\nabla w = e^{-s/2}\nabla u, \quad \Delta w = e^{-s}\Delta u,$$

hence

$$w_s - \Delta w + \frac{1}{2}y \cdot \nabla w = e^{-s}(u_t - \Delta u - f(\psi_1)).$$

Then w is a global (in time) solution of

$$w_s - \Delta w + \frac{1}{2}y \cdot \nabla w = h(s)\left(e^w \frac{L(e^{\psi_1+w})}{L(e^{\psi_1})} - 1\right), \quad (y, s) \in \mathcal{W}, \quad (4.15)$$

where

$$\begin{aligned} \mathcal{W} &:= \{(y, s) : s_0 < s < \infty, y \in D(s)\} \quad \text{with } D(s) := e^{s/2}\Omega \\ &\text{and } h(s) := e^{-s}f(\psi_1) = e^{-s+\psi_1}L(e^{\psi_1}) \end{aligned} \quad (4.16)$$

(note that $h \equiv 1$ for $L \equiv 1$). Observe that $\lim_{\infty} D(s) = \mathbb{R}^n$ and that $\mathcal{W} = \mathbb{R}^n \times (s_0, \infty)$ for $\Omega = \mathbb{R}^n$. In term of the variables y and s and rescaled function w , (3.3) and Lemma 4.2 imply

$$w \leq M, \quad (4.17)$$

$$|\nabla w| \leq M_1, \quad (4.18)$$

where $M_1 = M_1(\Omega, M, f, T)$. By the proof of [30, Proposition 23.1] and the radial decreasing property, we have $u(0, t) = \|u(t)\|_\infty \geq \psi(t)$ hence $w(0, s) \geq 0$ for $s > s_0$, and (4.17)-(4.18) imply

$$-M_1|y| \leq w(y, s) \leq M. \quad (4.19)$$

We rewrite equation (4.15) as

$$w_s - \Delta w + \frac{1}{2}y \cdot \nabla w = e^w - 1 + H(s, y) \quad (4.20)$$

which is equivalent to

$$\rho w_s - \nabla \cdot (\rho \nabla w) = (e^w - 1)\rho + \rho H(s, y) \quad (4.21)$$

where the nonautonomous term is given by

$$H(s, y) := (h(s) - 1)(e^w - 1) + h(s)e^w \left(\frac{L(e^{\psi_1+w})}{L(e^{\psi_1})} - 1 \right). \quad (4.22)$$

Step 3. Control of nonautonomous terms. It is clear that $H \equiv 0$ when $L \equiv 1$. Now, under assumption (3.6), we shall suitably control the convergence of $h(s)$ to 1 and control the nonautonomous term H , in a way that will ensure the existence of a Liapunov functional.

Lemma 4.3. *Under the assumptions of Proposition 3.1, there exist $C_0 > 0$ and $s_1 \geq s_0$ such that*

$$|h(s) - 1| \leq Cs^{-\alpha}, \quad s > s_1 \quad (4.23)$$

and

$$\|H(s, \cdot)\|_\infty \leq C_0 s^{-\alpha} \log s, \quad s > s_1. \quad (4.24)$$

Proof. Integrating by parts, we have

$$G(X) = \int_X^\infty \frac{dz}{f(z)} = \int_X^\infty \frac{e^{-z} dz}{L(e^z)} = \frac{e^{-X}}{L(e^X)} - \int_X^\infty \frac{L'(e^z)}{L^2(e^z)} dz,$$

hence

$$|e^X L(e^X) G(X) - 1| \leq C e^X L(e^X) \int_X^\infty \frac{e^{-z}}{\log^\alpha(e^z) L(z)} dz \leq C \frac{e^X L(e^X) G(X)}{X^\alpha}, \quad X \geq 2. \quad (4.25)$$

In particular $\lim_{X \rightarrow \infty} e^X L(e^X) G(X) = 1$ and then we get

$$|e^X L(e^X) G(X) - 1| \leq CX^{-\alpha}, \quad X \geq 2.$$

Since $G(\psi_1(s)) = G(\psi(T - e^{-s})) = e^{-s}$, we have $h(s) = G(\psi_1(s))e^{\psi_1(s)}L(e^{\psi_1(s)})$. Since also $\psi_1(s) = G^{-1}(e^{-s}) \sim s$ owing to (4.4), property (4.23) follows.

Let us next prove (4.24). We write $H(s, y) \equiv H_1(s, y) + H_2(s, y)$ according to (4.22). Using (4.17) and (4.23), we have

$$|H_1(s, y)| := |(h(s) - 1)(e^w - 1)| \leq \frac{C}{s^\alpha},$$

which shows that H_1 satisfies the desired property for all $s > s_0$. Now we shall show that H_2 also satisfies (4.24). We proceed similarly as in [7, Lemma 3.1]. Taking $s_1 > s_0$ sufficiently large, we have

$$e^{\psi_1(s)} \geq e^{s/4}, \quad s \geq s_1. \quad (4.26)$$

Let $Q = \mathcal{W} \cap (\mathbb{R}^n \times [s_1, \infty))$ and $E = \{(y, s) \in Q; e^{w(y, s)} \leq s^{-\alpha}\}$. By (4.23) and the slow variation property of L , we have

$$|H_2(s, y)| \leq C_0 e^w \leq C_0 s^{-\alpha}, \quad (y, s) \in E. \quad (4.27)$$

Next consider the case when $(y, s) \in Q \setminus E$. Then, taking s_1 large enough, we have $e^{w+\psi_1} \geq s^{-\alpha} e^{s/4} \geq e^{s/8} \geq 2$. By assumption (3.6),

$$\Sigma(X) := \sup_{z \geq X} \left| \frac{zL'(z)}{L(z)} \right| \leq C(\log X)^{-\alpha}, \quad X \geq 2.$$

Therefore, $\Sigma(e^{s/8}) \leq C(\log(e^{s/8}))^{-\alpha} \leq Cs^{-\alpha}$, as well as $M \geq e^w \geq s^{-\alpha}$. Using (4.26), it follows that

$$\begin{aligned} \left| \log\left(\frac{L(e^{w+\psi_1})}{L(e^{\psi_1})}\right) \right| &= \left| \int_{e^{w+\psi_1}}^{e^{\psi_1}} \frac{L'(z)}{L(z)} dz \right| \leq \Sigma\left(\min\{e^{w+\psi_1}, e^{\psi_1}\}\right) \left| [\log z]_{e^{w+\psi_1}}^{e^{\psi_1}} \right| \\ &\leq \Sigma\left(e^{s/8}\right) |\log e^w| \leq Cs^{-\alpha} \log s. \end{aligned}$$

Hence,

$$|H_2(s, y)| \leq 2M \left| \frac{L(e^{w+\psi_1})}{L(e^{\psi_1})} - 1 \right| \leq Cs^{-\alpha} \log s.$$

Combining with (4.27), we deduce that

$$\|H_2(s, \cdot)\|_\infty \leq Cs^{-\alpha} + Cs^{-\alpha} \log s \leq Cs^{-\alpha} \log s, \quad s \geq s_1.$$

□

Step 4. Weighted energy functional and its properties. The Liapunov functional for equation (4.21) will be given by the weighted energy defined as follows:

$$\mathcal{E}[w](s) := E[w](s) + C_1 s^{-\gamma} \quad (4.28)$$

where $\gamma = \alpha - \frac{1}{2}$,

$$E[w] := \int_{D(s)} \left(\frac{1}{2} |\nabla w|^2 + w - e^w \right) \rho dy,$$

and $C_1 > 0$ is a constant.

Lemma 4.4. *For all $s > s_0$, there exists $M_0 > 0$ (independent of s, y) such that*

$$\frac{d}{ds} \mathcal{E}[w](s) \leq -\frac{1}{2} \int_{D(s)} w_s^2 \rho \leq 0 \quad (4.29)$$

and

$$\mathcal{E}[w](s) \geq -M_0. \quad (4.30)$$

Proof. Fix any $s_1 > s_0$. By parabolic regularity (see, e.g., [7, Lemma 3.2.] for details), we have

$$D^2 w, (1 + |y|)^{-1} w_s, (1 + |y|)^{-1} \nabla w_s \in BC(\overline{\mathcal{W}}_{s_1}), \quad (4.31)$$

where $\mathcal{W}_{s_1} = \mathcal{W} \cap (\mathbb{R}^n \times (s_0, s_1))$ and BC denotes the set of bounded continuous functions. Using the exponential decay of ρ , this guarantees the convergence and the differentiability of the various integrals and justifies the integrations by parts in the rest of the proof.

First, we compute the variation of the first part $E[w](s)$ of the energy. We have

$$\begin{aligned} \frac{d}{ds} \int_{D(s)} w \rho dy &= \int_{D(s)} \rho w_s dy + \frac{1}{2} \int_{\partial D(s)} w \rho (y \cdot \nu) d\sigma, \\ \frac{d}{ds} \int_{D(s)} e^w \rho dy &= \int_{D(s)} \rho w_s e^w dy + \frac{1}{2} \int_{\partial D(s)} e^w \rho (y \cdot \nu) d\sigma, \end{aligned} \quad (4.32)$$

where $d\sigma$ denotes the surface measure on $\partial D(s)$ and ν the exterior unit normal on $\partial D(s)$.

Now, we control the variation of the term involving ∇w by using integration by parts as follows

$$\begin{aligned} \frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho &= 2 \int_{D(s)} \nabla w_s \cdot (\rho \nabla w) + \frac{1}{2} \int_{\partial D(s)} |\nabla w|^2 \rho (y \cdot \nu) d\sigma \\ &= -2 \int_{D(s)} w_s \nabla \cdot (\rho \nabla w) + 2 \int_{D(s)} \rho w_s (\nabla w \cdot \nu) d\sigma + \frac{1}{2} \int_{\partial D(s)} \rho |\nabla w|^2 (y \cdot \nu) d\sigma, \end{aligned}$$

and then use $w_s = -\frac{y}{2} \cdot \nabla w - h(s)$ (owing to (4.14)) and $\nabla w = (\nabla w \cdot \nu)\nu$ on $\partial D(s)$, hence $w_s(\nabla w \cdot \nu) = -\frac{1}{2}|\nabla w|^2(y \cdot \nu) - h(s)\nabla w \cdot \nu$, which yields

$$\frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho = -2 \int_{D(s)} w_s \nabla \cdot (\rho \nabla w) - \frac{1}{2} \int_{\partial D(s)} \rho |\nabla w|^2 (y \cdot \nu) d\sigma - 2h(s) \int_{\partial D(s)} \rho \nabla w \cdot \nu d\sigma. \quad (4.33)$$

Also, we have $\partial D(s) = e^{s/2} \partial \Omega$ hence $\int_{\partial D(s)} d\sigma \leq C(\Omega) e^{s/2}$ and $|y| = R e^{s/2}$ on $\partial D(s)$. Consequently, using (4.18), we get (here and in the rest of the proof, C denotes a generic positive constant independent of s, y)

$$\begin{aligned} \int_{\partial D(s)} \rho |\nabla w|^2 |y \cdot \nu| d\sigma &\leq C M_1^2 e^{s/2} \exp[-(R e^{s/2})^2/4] \leq C \exp(-c e^s), \\ \int_{\partial D(s)} \rho |\nabla w \cdot \nu| d\sigma &\leq C M_1 e^{s/2} \exp[-(R e^{s/2})^2/4] \leq C \exp(-c e^s), \end{aligned} \quad (4.34)$$

and using $w = -\psi_1$ on $\partial D(s)$ we have

$$\left| \int_{\partial D(s)} w \rho (y \cdot \nu) d\sigma - \int_{\partial D(s)} e^w \rho (y \cdot \nu) d\sigma \right| \leq C \psi_1(s) e^{s/2} \exp[-(R e^{s/2})^2/4] \leq C \exp(-c e^s). \quad (4.35)$$

Combining (4.21) and (4.32)-(4.35), we obtain

$$\begin{aligned} \frac{d}{ds} E[w] &\leq - \int_{D(s)} w_s^2 \rho + \int_{D(s)} w_s H(s, y) \rho + C \exp(-c e^s) \\ &\leq -\frac{1}{2} \int_{D(s)} w_s^2 \rho + \frac{1}{2} \int_{D(s)} H^2(s, y) \rho + C \exp(-c e^s) \end{aligned} \quad (4.36)$$

and (4.24) and $\alpha > \frac{1}{2}$ guarantee that, for some $C_1 > 0$,

$$\frac{1}{2} \int_{D(s)} H^2(s, y) \rho dy + C \exp(-c e^s) \leq \gamma C_1 s^{-\alpha - \frac{1}{2}} \quad \text{for all } s \geq s_0. \quad (4.37)$$

It then follows from (4.28) and (4.36) that

$$\frac{d}{ds} \mathcal{E}[w](s) = \frac{d}{ds} E[w](s) - \gamma C_1 s^{-\alpha - \frac{1}{2}} \leq -\frac{1}{2} \int_{D(s)} w_s^2 \rho < 0.$$

Combining estimate (4.19) with exponential decay of ρ and boundedness of $|\nabla w|$ we obtain (4.30). \square

Step 5. Convergence and conclusion. By (4.29)-(4.30), we have

$$\ell := \lim_{s \rightarrow \infty} \mathcal{E}[w](s) \in [-M_0, \infty). \quad (4.38)$$

Pick any sequence $s_j \rightarrow \infty$ and set $z_j := w(y, s + s_j)$. Since $-c|y| \leq z_j(y, s) \leq \tilde{M}$ and $|\nabla z_j(y, s)| \leq M_1$ (due to (4.19) and (4.18)), it follows from (4.20), (4.24) and parabolic estimates that the sequence $(z_j)_j$ is precompact in $C_{loc}^{2,1}(\mathbb{R}^n \times [0, 1])$. Consequently, there exists a subsequence (still denoted s_j) and a solution ϕ of

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w = e^w - 1 \quad \text{in } \mathbb{R}^n \times [0, 1] \quad (4.39)$$

such that $w(\cdot, \cdot + s_j) \rightarrow \phi$ in $C_{loc}^{2,1}(\mathbb{R}^n \times [0, 1])$. Moreover $|\nabla \phi|$ is bounded in $\mathbb{R}^n \times [0, 1]$ and $-c|y| \leq \phi(y, s) \leq \tilde{M}$. Let $K > 0$. Using (4.29), (4.38) and the fact that $B_K \subset D(s)$ for all sufficiently large s , we deduce that

$$\int_0^1 \int_{B_K} (\partial_s z_j)^2 \rho dy ds \leq \int_{s_j}^{s_j+1} \int_{D(s)} (\partial_s w)^2 \rho dy ds \leq 2\mathcal{E}[w](s_j) - 2\mathcal{E}[w](s_j+1) \rightarrow 0, \quad j \rightarrow \infty.$$

By Fatou's lemma, and since $K > 0$ is arbitrary, we deduce that $\partial_s \phi \equiv 0$ in B_K . Therefore, by (4.39), we have $\Delta \phi - \frac{1}{2} y \cdot \nabla \phi = 1 - e^\phi$ in \mathbb{R}^n . Moreover, since u is radially symmetric, so is ϕ . In terms of the variables (x, t) and u , in view of (4.12) and (4.13), we have $w(y, s_j) = u(y\sqrt{T-t_j}, t_j) - \psi(t_j)$, which concludes the proof. \square

Proposition 3.2 follows from Proposition 3.1 and the following two lemmas. The first one shows that there are no nontrivial radially decreasing backward self-similar profiles that correspond to a time increasing solution. This lemma is essentially due to [3] (under slightly different form; see [3, Lemma 4.2(a)]). We give a proof in Appendix (significantly simpler than that in [3]; moreover we remove the boundedness assumption on ϕ').

Lemma 4.5. *Let $c(r) := \frac{n-1}{r} - \frac{r}{2}$ and $F(s) := e^s - 1$. Let ϕ be a solution of*

$$\begin{cases} \phi'' + c(r)\phi' + F(\phi) = 0, & 0 < r < \infty \\ \phi(0) \geq 0, \phi'(0) = 0 \end{cases} \quad (4.40)$$

such that $\phi' \leq 0$. If $g(r) := 1 + \frac{1}{2}r\phi'(r) \geq 0$ for all $r \geq 0$ then $\phi \equiv 0$.

The second lemma shows that the assumption $u_t \geq 0$ implies the type I blowup property (3.3). The proof is based on the argument in [14] (see also [3]), combined with the properties of f and the asymptotics of the ODE solution obtained in Lemma 4.1.

Lemma 4.6. *Under the assumptions of Theorem 2.1, estimate (3.3) is satisfied.*

Proof. Under assumptions (2.1)-(2.3), we have in particular

$$\lim_{s \rightarrow \infty} \frac{sL'(s)}{L(s)} = \lim_{s \rightarrow \infty} \frac{s^2L''(s)}{L(s)} = 0 \quad (4.41)$$

and there exists $A > 0$ such that

$$f \in C^2([A, \infty)) \quad \text{and} \quad f, f', f'' > 0 \quad \text{on} \quad [A, \infty) \quad (4.42)$$

and $G : [A, \infty) \rightarrow (0, G(A)]$ is decreasing and C^2 . Now, since $\lim_{t \rightarrow T} \|u(t)\|_\infty = \infty$, there exists $(x_0, t_0) \in B_R \times (0, T)$ such that $u(x_0, t_0) \geq A$ and, setting $R' = |x_0|$ and using $u_r \leq 0$ and $u_t \geq 0$, we have

$$u(x, t) \geq A, \quad (x, t) \in Q := \bar{B}_{R'} \times [t_0, T). \quad (4.43)$$

Set $J = u_t - \delta f(u)$, with $\delta > 0$ to be determined. By direct calculation, we have

$$J_t - \Delta J - f'(u)J = f''(u)|\nabla u|^2 \geq 0 \quad \text{in } Q.$$

On the other hand, as a consequence of Lemma 5.1(ii) below (which is proved independently), 0 is the only blow-up point of u . Therefore, $f(u) \leq C_0$ and $u_t \geq C > 0$ on $\partial_p Q$, which implies that $J > 0$ in $\partial_p Q$ for some sufficiently small $\delta > 0$. By the maximum principle, we deduce that $J > 0$ in Q . An integration yields the following upper bound

$$u(0, t) \leq G^{-1}(\delta(T - t)) \quad \text{in } [t_0, T).$$

Using the radial decreasing property of u and the expansion (4.3) of G^{-1} in Lemma 4.1, we obtain (3.3). \square

Proof of Proposition 3.2. The assumptions of Proposition 3.1 are verified (including (3.3) owing to Lemma 4.6). It thus suffices to prove that $\phi \equiv 0$ is the only possible limit in (3.7) under the assumptions of Theorem 2.1. Using the notation of Step 5 of the proof of Proposition 3.1, we have

$$\lim_{j \rightarrow \infty} w_s(y, s + s_j) = 0, \quad (y, s) \in \mathbb{R}^n \times [0, 1]$$

and it follows from (4.14) that

$$w_s(y, s + s_j) + \frac{y}{2} \cdot \nabla w(y, s + s_j) + h(s + s_j) \geq 0, \quad (y, s) \in \mathbb{R}^n \times [0, 1]. \quad (4.44)$$

Since $\lim_{s \rightarrow \infty} h(s) = 1$ by (4.23), we deduce that $\phi \in \mathcal{S}$ and satisfies

$$1 + \frac{r}{2}\phi'(r) \geq 0, \quad r \geq 0. \quad (4.45)$$

Lemma 4.5 then guarantees that $\phi \equiv 0$, which concludes the proof. \square

5 Proof of upper part of Theorem 2.1

The upper estimate of u in Theorem 2.1 will be a consequence of the following result from [8] (see [8, Theorem 3.1 and formula (4.22)]).

Theorem 5.1. *Let $\Omega = \mathbb{R}^n$ or $\Omega = B_R$, and $u_0 \geq 0$ be radially symmetric, nonincreasing in $|x|$ and $T < \infty$. Assume that*

$$f \in C^1([0, \infty)), \quad f(0) \geq 0, \quad \lim_{s \rightarrow \infty} f(s) = \infty,$$

$$f \text{ is } C^2 \text{ and } \frac{f}{\log f} \text{ is convex at } \infty, \quad \int^{\infty} \frac{\log f(s)}{f(s)} ds < \infty.$$

If $\Omega = \mathbb{R}^n$ assume also that u_0 is nonconstant. Then there exist $A_0, \rho > 0$ such that

$$u(x, t) \leq H^{-1} \left(H(u(0, t)) + \frac{|x|^2}{4} \right) \quad \text{in } B_\rho \times (T - \rho, T), \quad (5.1)$$

where

$$H(X) = \int_X^\infty \frac{A_0 + \log f(s)}{f(s)} ds.$$

Moreover, there exist constants $A, M > 0$ such that

$$-u_r \geq \frac{rf(u)}{2(A + \log f(u))}, \quad \text{in } Q := \{(x, t) \in B_{R/2} \times (T/2, T); u(x, t) \geq M\}. \quad (5.2)$$

We first apply Theorem 5.1, establishing the required properties of f . This is the contents of the following lemma.

Lemma 5.1. (i) *Let $f(s) = e^s L(e^s)$ satisfy (2.1) and*

$$\frac{sL'(s)}{L(s)} = o(1), \quad \frac{s^2 L''(s)}{L(s)} = o(1) \quad \text{as } s \rightarrow \infty. \quad (5.3)$$

Then, we have

$$\lim_{s \rightarrow \infty} f(s) = \infty \quad (5.4)$$

and

$$\frac{f}{\log f} \text{ is convex at } \infty \text{ and } \int^{\infty} \frac{\log f}{f} < \infty. \quad (5.5)$$

(ii) *Under the assumptions of Theorem 2.1, properties (5.1) and (5.2) are satisfied.*

Proof. (i) Since for any $\mu > 0$ there exists s_0 such that $s^\mu L(s) \geq \frac{1}{2}$ for all $s \geq s_0$, we have $\lim_{s \rightarrow \infty} f = \infty$ and there exists $M > 0$ such that $f > e^4$ and $0 < 1/\log f < \frac{1}{4}$ in $[M, \infty)$ and $f \in C^2(M, \infty)$. Set $F = \frac{f}{\log f}$, we shall show that

$$F'' \geq 0 \quad \text{in } [M, \infty). \quad (5.6)$$

We have $f'/f = 1 + \theta(e^s)$, where $\theta(X) = \frac{XL'(X)}{L(X)}$, then

$$F'(s) = \frac{f'}{\log f} - \frac{f'}{(\log f)^2} = \Psi F, \quad \text{where } \Psi(s) = (1 + \theta(e^s)) \left(1 - \frac{1}{\log f(s)}\right)$$

and $\frac{F''}{F} = \Psi' + \Psi^2$. Since

$$\Psi'(s) = e^s \theta'(e^s) \left(1 - \frac{1}{\log f(s)}\right) + (1 + \theta(e^s))^2 \frac{1}{\log f(s)}$$

and (5.3) implies $\lim_{X \rightarrow \infty} \theta(X) = \lim_{X \rightarrow \infty} X\theta'(X) = 0$, we deduce that $\lim_{s \rightarrow \infty} \Psi(s) = 1$ and $\lim_{s \rightarrow \infty} \Psi'(s) = 0$, so that $\lim_{s \rightarrow \infty} \frac{F''}{F} = 1$, hence (5.6).

For the second assertion of (5.5), using that for any $\mu > 0$ there exists $X_0 > 0$ such that $s^\mu L(s)$ is increasing and $\log f(s) \leq 2s$ on $[X_0, \infty)$, we have (with $\mu = \frac{1}{4}$)

$$\int^{\infty} \frac{\log f}{f} = \int^{\infty} \left(\frac{\log f(s)}{e^{s/4}} \right) \left(\frac{1}{e^{s/2}} \right) \left(\frac{1}{e^{s/4} L(e^s)} \right) ds \lesssim \int^{\infty} \frac{1}{e^{s/2}} ds < \infty.$$

Hence (5.5).

(ii) This is a consequence of assertion (i) and of Theorem 5.1. \square

Building on property (4.2), we next relate the asymptotics of H and G and their inverses, making use of the slow variation assumption $sL'(s) = o(L(s))$ as $s \rightarrow \infty$.

Lemma 5.2. *Let $f(s) = e^s L(e^s)$ and assume that L is C^1 and positive for large s and satisfies (4.1).*

(i) *As $X \rightarrow \infty$, we have*

$$H(X) \sim \frac{Xe^{-X}}{L(e^X)} = \frac{X}{f(X)}, \quad (5.7)$$

$$H(X) \sim G(X)|\log(G(X))|. \quad (5.8)$$

(ii) *If a function ε satisfies $\lim_{X \rightarrow \infty} \varepsilon(X) = 0$ (resp., $\lim_{Y \rightarrow 0^+} \varepsilon(Y) = 0$) then*

$$\lim_{X \rightarrow \infty} \frac{G(X + \varepsilon(X))}{G(X)} = 1 \quad \left(\text{resp., } \lim_{Y \rightarrow 0^+} \left\{ G^{-1}[(1 + \varepsilon(Y))Y] - G^{-1}(Y) \right\} = 0 \right) \quad (5.9)$$

and the similar properties hold for H, H^{-1} .

(iii) *We have*

$$\lim_{Y \rightarrow 0^+} \left(H^{-1}(Y) - G^{-1}\left(\frac{Y}{|\log Y|}\right) \right) = 0. \quad (5.10)$$

Proof of Lemma 5.2. We may fix $s_0 > 0$ such that $L(e^s) > 0$ for all $s \geq s_0$.

(i) Let $X > s_0$. Since $\log L(\eta) = o(\log \eta)$ as $\eta \rightarrow \infty$ by (4.1), we have

$$H(X) \sim \int_X^\infty \frac{\log f(s)}{f(s)} ds = \int_X^\infty \frac{s + \log L(e^s)}{e^s L(e^s)} ds \sim \int_X^\infty \frac{s}{e^s L(e^s)} ds,$$

and (5.7) follows from (4.2) applied with $\tilde{L}(s) = \frac{L(s)}{\log s}$. Using (4.2) again, which implies $|\log G(X)| \sim X$ as $X \rightarrow \infty$, we deduce (5.8) from (5.7).

(ii) Let $\epsilon \in (0, 1/2)$. Since L has slow variation at ∞ , we easily deduce from (4.2) the existence of $\eta, X_0 > 0$ such that

$$G(X + \eta) \geq (1 - \epsilon)G(X) \quad \text{and} \quad G(X - \eta) \leq (1 + \epsilon)G(X) \quad \text{for all } X \geq X_0.$$

Since G is decreasing at ∞ , this implies property (5.9) for G .

For $Y \in (0, Y_1]$ with $Y_1 > 0$ sufficiently small, using (4.5) we have

$$G^{-1}(Y) = \log Q^{-1}(Y).$$

This and [8, formula (4.32)] for $p = 2$ imply property (5.9) for G^{-1} . Since H is the function G corresponding to $\tilde{L}(\tau) = \frac{L(\tau)}{A_0 + \log \tau + \log L(\tau)}$ and \tilde{L} satisfies (4.1), the similar properties hold for H, H^{-1} .

(iii) For all $Y > 0$ sufficiently small we may set $X = G^{-1}\left(\frac{Y}{|\log Y|}\right)$ ($\rightarrow \infty$ as $Y \rightarrow 0^+$), so that $\frac{Y}{|\log Y|} = G(X)$. As $Y \rightarrow 0^+$, we have $\log G(X) \sim \log Y$ so that, owing to (5.8),

$$Y = G(X)|\log Y| \sim G(X)|\log G(X)| \sim H(X).$$

Applying property (5.9) for H^{-1} , we get (5.10). □

Combining the results in Lemmas 5.1(ii) and 5.2, and also making use of property (3.8) in Proposition 3.2 at $x = 0$, we are now in a position to complete the proof.

Proof of upper part of Theorem 2.1. Since $m(t) := u(0, t) = \|u(t)\|_\infty \rightarrow \infty$, we have $G(m(t)) \rightarrow 0^+$ as $t \rightarrow T$. Applying property (5.8) to inequality (5.1), we get

$$u(x, t) \leq H^{-1}\left(H(m) + \frac{|x|^2}{4}\right) = H^{-1}\left((1 + o(1))\left(G(m)|\log(G(m))| + \frac{|x|^2}{4}\right)\right)$$

as $(x, t) \rightarrow (0, T)$. Next applying Lemma 5.2 (ii) and (iii), we obtain

$$u(x, t) \leq G^{-1} \left(\frac{G(m)|\log(G(m))| + \frac{|x|^2}{4}}{\left| \log \left(G(m)|\log(G(m))| + \frac{|x|^2}{4} \right) \right|} \right) + o(1). \quad (5.11)$$

Also, for (x, t) close enough to $(0, T)$, we have $G(m) \leq G(m)|\log(G(m))| + \frac{|x|^2}{4} < 1$, so that

$$\begin{aligned} \left| \log \left(G(m)|\log(G(m))| + \frac{|x|^2}{4} \right) \right| &\leq \min \left\{ |\log(G(m))|, \left| \log \left(\frac{|x|^2}{4} \right) \right| \right\} \\ &= (1 + o(1)) \min \left\{ |\log(G(m))|, |\log |x|^2| \right\} \end{aligned}$$

as $(x, t) \rightarrow (0, T)$, hence

$$\frac{G(m)|\log(G(m))| + \frac{|x|^2}{4}}{\left| \log \left(G(m)|\log(G(m))| + \frac{|x|^2}{4} \right) \right|} \geq (1 + o(1)) \left(G(m) + \frac{|x|^2}{4|\log |x|^2|} \right).$$

Since G^{-1} is decreasing near 0^+ , we thus deduce from (5.11) that

$$u(x, t) \leq G^{-1} \left((1 + o(1)) \left(G(m) + \frac{|x|^2}{4|\log |x|^2|} \right) \right) + o(1), \quad (x, t) \rightarrow (0, T). \quad (5.12)$$

On the other hand, by Proposition 3.2 for $x = 0$, we have

$$\lim_{t \rightarrow T} (m(t) - G^{-1}(T - t)) = 0,$$

then by Lemma 5.2(ii) we obtain

$$G(m) = (1 + o(1))(T - t). \quad (5.13)$$

Combining (5.12), (5.13) and applying Lemma 5.2(ii) once again, we obtain the upper part of (2.8). \square

6 Proof of Theorem 2.2

Recalling definition (2.11), we respectively denote by $(v, w) := \int_{\mathbb{R}^n} v w \rho dy$ and $\|v\| = (v, v)^{\frac{1}{2}}$ the inner product and the norm of the Hilbert space L_ρ^2 (restated in terms of $z = -\varphi$, instead of φ in [8, section 6.3]). The following proposition follows from the proof in [8, section 6]. It was actually assumed there that φ is bounded but, owing to the fast decay of the weight ρ , the proof remains valid without change whenever φ is polynomially bounded at space infinity.

Proposition 6.1. *Let z be a radially symmetric nonincreasing solution of*

$$\partial_s z + \mathcal{L}z - z = F(z, s) + \tau(s)z + B(y, s) \quad \text{in } \mathbb{R}^n \times (s_1, \infty),$$

where

$$\tau(s) = o\left(\frac{1}{s}\right), \quad s \rightarrow \infty,$$

$$|B| + |\nabla B| \leq C e^{-s}, \quad |F(\xi, s)| \leq C \xi^2, \quad |\partial_\xi F(\xi, s)| \leq C |\xi|, \quad s \geq s_1,$$

and

$$V(s) := \lim_{\xi \rightarrow 0} \frac{F(\xi, s)}{\xi^2} = k + K(s), \quad \text{with } k > 0 \text{ and } \lim_{s \rightarrow \infty} K(s) = 0.$$

We also assume that there exists $M_0 > 0$ such that

$$|z(y, s)| \leq M_0(1 + |y|), \quad |\nabla z| \leq M_0 \quad \text{and} \quad \lim_{s \rightarrow \infty} z(y, s) = 0 \text{ uniformly for } y \text{ bounded},$$

and that

$$\|z(\cdot, s)\| \geq c s^{-1}, \quad s \geq s_1.$$

Then

$$z(y, s) = \frac{2n - |y|^2}{8ks} + o\left(\frac{1}{s}\right) \quad \text{as } s \rightarrow \infty, \quad (6.1)$$

with convergence in $H_\rho^1(\mathbb{R}^n)$ and uniformly for y bounded.

We shall derive Theorem 2.2 as consequence of this Proposition. To this end, we verify the required assumptions in several subsections for clarity.

6.1 Setting the problem

Since we want to apply Proposition 6.1, we first need to extend the solution to \mathbb{R}^n by means of a suitable cut-off in order to handle the case $\Omega = B_R$ (actually, the cut-off procedure can be applied even in the case $\Omega = \mathbb{R}^n$, so as to take advantage of the support and regularity properties of the solution).

Namely, for δ given by (7.3) below, we introduce a cut-off function $\phi \in C^\infty(\mathbb{R}^n)$, radially symmetric and nonincreasing in $|x|$, such that

$$\phi(x) = \begin{cases} 1, & 0 \leq |x| \leq \delta/2 \\ 0, & |x| \geq \delta \end{cases} \quad (6.2)$$

and we then set $\tilde{u}(x, t) = \phi(x)u(x, t)$. We see that \tilde{u} is solution of

$$\tilde{u}_t - \Delta \tilde{u} = f(\tilde{u}) + A(x, t), \quad x \in \mathbb{R}^n, \quad 0 < t < T$$

where

$$A(x, t) = \phi f(u) - f(\phi u) - u \Delta \phi - 2 \nabla u \cdot \nabla \phi.$$

Note that

$$\text{supp}(A) \subset \{(x, t) \in \mathbb{R}^n \times (0, T); \delta/2 \leq |x| \leq \delta\} \quad \text{and} \quad |A| + |\nabla A| \leq C, \quad (6.3)$$

since 0 is the only blow-up point of u . Also, by (4.42), (7.3) and parabolic regularity we see that

$$\nabla \tilde{u} \in C^{2,1}(\mathbb{R}^n \times (T - \delta, T)). \quad (6.4)$$

Proposition 6.1 will be applied to the function z obtained by rescaling \tilde{u} by the similarity variables (y, s) (cf. (4.12)) and ODE normalization around $(0, T)$:

$$\tilde{u}(x, t) = z(y, s) + \psi(t). \quad (6.5)$$

Let $s_0 := -\log T$. In the rest of the proof, we will denote by s_1 a (sufficiently large) time $> \max(s_0, 1)$, which may vary from line to line. By direct calculation, similar to (4.15), we see that z is a global solution of

$$z_s + \mathcal{L}z = Z(z, s) + B(y, s), \quad y \in \mathbb{R}^n, \quad s \in (s_0, \infty), \quad (6.6)$$

where

$$\mathcal{L}z = -\Delta z + \frac{1}{2}y \cdot \nabla z$$

is the Hermite operator, and

$$Z(\xi, s) := h(s) \left(e^\xi \frac{L(e^{\psi_1 + \xi})}{L(e^{\psi_1})} - 1 \right), \quad (6.7)$$

$$h(s) = e^{-s + \psi_1(s)} L(e^{\psi_1(s)}), \quad \psi_1(s) = \psi(T - e^{-s}), \quad B(y, s) := \frac{A(ye^{-s/2}, T - e^{-s})}{e^s}. \quad (6.8)$$

We then rewrite equation (6.6) as

$$z_s + \mathcal{L}z - z = W(z, s) + B(y, s) \equiv \tau(s)z + F(z, s) + B(y, s), \quad y \in \mathbb{R}^n, \quad s \geq s_1, \quad (6.9)$$

with

$$W(\xi, s) := Z(\xi, s) - \xi, \quad \tau(s) := \partial_\xi W(0, s), \quad F(\xi, s) = W(\xi, s) - \tau(s)\xi. \quad (6.10)$$

We note that, in the case corresponding to $f(u) = e^u$, we have $\psi_1(s) = s$ and then

$$h(s) \equiv 1 \quad \text{and} \quad Z(\xi, s) = e^\xi - 1.$$

Combining with (6.10), $W(\xi, s)$ in this case was bounded by a quadratic function. The function W here (for $f(s) = e^s L(e^s)$) has a nonzero linear part, but its decay as $s \rightarrow \infty$ will turn out to be sufficiently fast so as to permit the required properties. Under the assumptions of Theorem 2.1, by (4.18)-(4.19) and Proposition 3.2, there exist $c, M_0 > 1$ such that

$$-c|y| \leq z(y, s) \leq M_0, \quad |\nabla z| \leq M_0 \quad \text{and} \quad \lim_{s \rightarrow \infty} z(y, s) = 0 \quad \text{uniformly for } y \text{ bounded}, \quad (6.11)$$

and, moreover, by (6.3),

$$\text{supp}(B) \subset \{(y, s) \in \mathbb{R}^n \times (s_0, \infty) : \frac{\delta}{2}e^{s/2} \leq |y| \leq \delta e^{s/2}\} \quad \text{and} \quad |B| + |\nabla B| \leq Ce^{-s}. \quad (6.12)$$

We also record the following easy bounds for ψ :

$$\frac{c_1}{T-t} \leq f(\psi(t)) \leq \frac{c_2}{T-t}, \quad t \rightarrow T, \quad (6.13)$$

for some constants $c_1, c_2 > 0$. Estimate (6.13) follows from the fact that $T-t = G(\psi(t)) \sim \frac{1}{f(\psi(t))}$ as $t \rightarrow T$ (cf. (4.2) in Lemma 5.2).

6.2 Properties of the equation for z

Lemma 6.1. *We have*

$$\tau(s) = o\left(\frac{1}{s}\right), \quad s \rightarrow \infty, \quad (6.14)$$

$$|F(z, s)| \leq Cz^2, \quad |\partial_\xi F(z, s)| \leq C|z|, \quad y \in \mathbb{R}^n, \quad s \geq s_1 \quad (6.15)$$

and

$$\lim_{\xi \rightarrow 0} \frac{F(\xi, s)}{\xi^2} = \frac{1}{2}h(s) \left(1 + 3e^{\psi_1} \frac{L'(e^{\psi_1})}{L(e^{\psi_1})} + e^{2\psi_1} \frac{L''(e^{\psi_1})}{L(e^{\psi_1})}\right), \quad s \geq s_1. \quad (6.16)$$

Proof of Lemma 6.1. We compute

$$\partial_\xi W(\xi, s) = \partial_\xi Z(\xi, s) - 1 = h(s) \left(e^\xi \frac{L(e^{\psi_1+\xi})}{L(e^{\psi_1})} + e^{\psi_1-2\xi} \frac{L'(e^{\psi_1-\xi})}{L(e^{\psi_1})} \right) - 1.$$

Recall $h(s) = e^{-s+\psi_1(s)}L(e^{\psi_1(s)})$ with $\psi_1(s) := \psi(T-e^{-s}) = G^{-1}(e^{-s})$. As a special case of [8, Lemma 6.1], under assumption (2.3), recall $Q(X) = \int_X^\infty \frac{d\tau}{\tau^2 L(\tau)}$, we have the expansion

$$Q(X)(XL(X) + X^2L'(X)) = 1 + o\left(\frac{1}{\log X}\right), \quad \text{as } X \rightarrow \infty. \quad (6.17)$$

Moreover, we have, from (4.5), $Q(e^{\psi(s)}) = G(\psi_1(s)) = e^{-s}$ for $s > s_0$ large. Using (6.17) and (4.26) we then obtain (6.14).

To check (6.15), since L is C^2 for X large and $W(0, s) = 0$, we may use Taylor's formula with integral remainder to write, for $s \geq s_1$ and $\xi \leq M_0$,

$$F(\xi, s) = W(\xi, s) - \partial_\xi W(0, s)\xi = \xi^2 \int_0^1 \partial_\xi^2 W(t\xi, s)(1-t)dt, \quad (6.18)$$

$$\partial_\xi F(\xi, s) = \partial_\xi W(\xi, s) - \partial_\xi W(0, s) = \xi \int_0^1 \partial_\xi^2 W(t\xi, s)dt, \quad (6.19)$$

where

$$\partial_\xi^2 W(\xi, s) = h(s) \left(e^\xi \frac{L(e^{\psi_1+\xi})}{L(e^{\psi_1})} + 3e^{\psi_1+2\xi} \frac{L'(e^{\psi_1+\xi})}{L(e^{\psi_1})} + e^{2\psi_1+3\xi} \frac{L''(e^{\psi_1+\xi})}{L(e^{\psi_1})} \right). \quad (6.20)$$

By (4.23) and (4.41), $\partial_\xi^2 W$ satisfies the bound

$$\begin{aligned} |\partial_\xi^2 W(\xi, s)| &\leq C \left\{ \frac{L(e^{\psi_1+\xi})}{L(e^{\psi_1})} + e^{\psi_1+\xi} \frac{|L'(e^{\psi_1-\xi})|}{L(e^{\psi_1})} + e^{2(\psi_1+\xi)} \frac{|L''(e^{\psi_1+\xi})|}{L(e^{\psi_1})} \right\} \\ &\leq C \frac{L(e^{\psi_1+\xi})}{L(e^{\psi_1})} \left\{ 1 + e^{\psi_1+\xi} \frac{|L'(e^{\psi_1+\xi})|}{L(e^{\psi_1+\xi})} + e^{2(\psi_1+\xi)} \frac{|L''(e^{\psi_1+\xi})|}{L(e^{\psi_1+\xi})} \right\} \leq C, \quad -1 \leq \xi \leq M_0, \end{aligned}$$

This, combined with (6.18)-(6.19), yields (6.15) at points (y, s) such that $z(y, s) \geq -1$.

To check (6.15) at points such that $z(y, s) < -1$, in view of (6.10) and (6.14), it suffices to verify that

$$\sup_{s > s_1, y \in \mathbb{R}^n} (|Z(z(y, s), s)| + |\partial_\xi Z(z(y, s), s)|) < \infty. \quad (6.21)$$

To this end, we first note that (2.1)-(2.3) guarantees the existence of X_0 such that $1/2 \leq \frac{f'(X)}{f(X)} \leq 2$ and $f(X)$ is increasing on $[X_0, \infty)$, and we write

$$Z(\xi, s) = h(s) \left(\frac{f(\psi_1 + \xi)}{f(\psi_1)} - 1 \right), \quad \partial_\xi Z(\xi, s) = h(s) \left(\frac{f'(\psi_1 + \xi)}{f(\psi_1)} \right), \quad (6.22)$$

where we denote $\psi_1 = \psi_1(s)$ for simplicity. Let $\xi \in (-\infty, M_0]$. If $\psi_1 + \xi \geq X_0$, then $0 \leq \frac{f(\psi_1 + \xi)}{f(\psi_1)} \leq \frac{f(M_0 + \psi_1)}{f(\psi_1)} \leq C$ and $0 \leq \frac{f'(\psi_1 + \xi)}{f(\psi_1)} \leq \frac{2f(\psi_1 + \xi)}{f(\psi_1)} \leq \frac{2f(\psi_1 + M_0)}{f(\psi_1)} \leq C$, owing to the fact that L has slow variation at infinity. Whereas, if $\psi_1 + \xi \leq X_0$, then $|\frac{f(\psi_1 + \xi)}{f(\psi_1)}| \leq \frac{C}{f(\psi_1)} \leq C$ and $|\frac{f'(\psi_1 + \xi)}{f(\psi_1)}| \leq C \frac{1}{f(\psi_1)} \leq C$. In view of (4.23) and (6.22), this proves (6.21).

Finally, (6.16) is a consequence of (6.18) and (6.20). \square

6.3 Nonexponential decay of z and completion of proof of Theorem 2.2

The following lemma provides a lower bound on the decay of z in L_ρ^2 , which is one of the key assumptions in Proposition 6.1. As in [8, 33], this lower bound is obtained as consequence of a lower bound on $|u_r|$ (cf. (5.2)).

Lemma 6.2. *Under the assumptions of Theorem 2.2, there exists $c > 0$ such that*

$$\|z(\cdot, s)\| \geq cs^{-1}, \quad s \geq s_1. \quad (6.23)$$

Proof. Fix $K > 0$ and $\varepsilon > 0$. By (6.11) there exists $t_0(K, \varepsilon) \in (T/2, T)$ such that

$$u(r, t) \geq \psi(t) - \varepsilon \geq M, \quad 0 \leq r \leq K\sqrt{T-t}, \quad t_0 < t < T,$$

where M is as in (5.2). Then, from (5.2), $u = u(r, t)$ with $r = |x|$, satisfies (see also [8, formula (4.22)])

$$-u_r(r, t) \geq \frac{rf(u)}{2(A + \log f(u))}, \quad 0 \leq r \leq K\sqrt{T-t}, \quad t_0 < t < T. \quad (6.24)$$

We write $z = z(\eta, s)$ with $\eta = |y|$ and $\varphi = -z$, hence $\|z(s)\| = \|\varphi(s)\|$. Observing that the RHS of (6.24) is an increasing function of u for large u and using the properties of L , we deduce that, for all $s_2 = s_2(K) \geq \max(s_0, 1)$ sufficiently large, there exists $c_1 = c_1(\varepsilon) > 0$,

$$\begin{aligned} \varphi_\eta(\eta, s) &:= -z_\eta(\eta, s) = -\sqrt{T-t} u_r(\eta\sqrt{T-t}, t) \\ &\geq \frac{\eta(T-t)f(\psi(t) - \varepsilon)}{2(A + \log f(\psi(t) - \varepsilon))} \geq \frac{c_1\eta(T-t)f(\psi(t))}{(A + \log f(\psi(t)))}. \end{aligned} \quad (6.25)$$

Inequality (6.13) guarantees that $\log(f(\psi(t))) \leq C|\log(T-t)|$ as $t \rightarrow T$. This combined with (6.13), (6.25) implies

$$\varphi_\eta(\eta, s) \geq \frac{c\eta}{s} \quad \text{for all } 0 \leq \eta \leq K \text{ and } s > s_2,$$

where the constant $c > 0$ is independent of K .

Now, choose $K = 2(1 + c^{-1})$ and take any $s \geq s_2(K)$. If $\varphi(1, s) \geq -1/s$ then it follows that

$$\varphi(\eta, s) = \varphi(1, s) + \int_1^\eta \varphi_\eta(z, s) dz \geq \frac{-1 + c(\eta - 1)}{s} \geq \frac{1}{s}, \quad K - 1 \leq \eta \leq K,$$

and hence $\|\varphi(\cdot, s)\| \geq (\int_{K-1 \leq |y| \leq K} \rho)^{1/2} s^{-1}$. Otherwise, we have $\varphi(1, s) \leq -1/s$ and, since φ is a nondecreasing function of η , we get $\varphi(\eta, s) \leq -1/s$ for $\eta \in [0, 1]$; hence $\|\varphi(\cdot, s)\| \geq (\int_{|y| \leq 1} \rho)^{1/2} s^{-1}$. We conclude that $\|\varphi(\cdot, s)\| \geq cs^{-1}$ for all $s \geq s_2$. \square

Proof of Theorem 2.2. Using (6.9), (6.11), (6.12), (6.23) and Lemma 6.1, the conclusion follows from Proposition 6.1 with $k = \frac{1}{2}$ (due to (6.16) and (4.23)). \square

7 Proof of lower estimate of Theorem 2.1 and of Corollary 2.1

We shall derive the lower estimate of Theorem 2.1 as a consequence of Theorem 2.2. To this end we modify the argument in [8, Section 5] for the case $f(u) = u^p L(u)$, the main difference here being the additive instead of multiplicative effect of the ODE term $G^{-1}(T - t)$.

Let $\delta > 0$ (to be chosen below). We shall use the following property of $(S_\delta(t))_{t \geq 0}$, the Dirichlet heat semigroup on $B_\delta \subset \mathbb{R}^n$ (see [8, Lemma 5.1]).

Lemma 7.1. (i) *There exists a constant $\lambda = \lambda(n, \delta) > 0$ such that, for any $K, N > 0$,*

$$\left[S_\delta(t)(N - K|x|^2) \right]_+ \geq e^{-\lambda t} [N - K(|x|^2 + 2nt)]_+ \quad \text{in } Q := \overline{B}_{\delta/2} \times [0, \infty). \quad (7.1)$$

(ii) *For all $\phi \in L^\infty(B_\delta)$, we have*

$$|(S_\delta(t)\phi)(x)| \leq C(n)t^{-n/4} \left(\int_{B_\delta} |\phi(z)|^2 e^{-\frac{|z|^2}{4t}} dz \right)^{1/2} e^{\frac{|x|^2}{2t}}, \quad x \in B_\delta, \quad t > 0. \quad (7.2)$$

Proof of lower estimate of Theorem 2.1. Step 1. Comparison argument. Similar to (4.43), there exists $\delta \in (0, 1)$ with $\delta < \min(\sqrt{T}, G(M))$, and $\delta < R/4$ if $\Omega = B_R$, such that

$$u(x, t) \geq \tilde{A} := G^{-1}(G(A) - \delta) > A \quad \text{in } Q := \overline{B}_\delta \times [T - \delta^2, T], \quad (7.3)$$

where A is as in (4.42). Fix any $\sigma \in (0, \delta^2)$ (to be chosen later) and set

$$t_0 = T - \sigma. \quad (7.4)$$

As in [8] (see also [14]), we introduce the following comparison function:

$$U = U_\sigma(x, t) = G^{-1} \left\{ G(V) + t_0 - t \right\}, \quad \text{with } V(x, t) := A + S(t - t_0)(u(t_0) - A). \quad (7.5)$$

Since $u(t_0) \geq A$ in \overline{B}_δ , we have $0 \leq S(t - t_0)(u(t_0) - A) \leq \|u(t_0)\|_\infty - A$, so that $A \leq V \leq \|u(t_0)\|_\infty$, hence $\tau_0 := G(\|u(t_0)\|_\infty) \leq G(V) \leq G(A)$. Letting

$$E := \{ \tau > t_0; G(V) + t_0 - t > 0, \text{ on } \overline{B}_\delta \times [t_0, \tau] \}; \quad \tau_1 = \sup E, \quad Q_1 := \overline{B}_\delta \times [t_0, \tau_1),$$

we see that E is nonempty, $\tau_1 \geq t_0 + \tau_0$, U is well defined and smooth in Q_1 . By the same way as in [8, section 5 Step 2], using (4.42), we have

$$\partial_t U - \Delta U \leq f(U) \quad \text{in } Q_1.$$

Moreover, setting $T_1 = \min\{T, \tau_1\}$, on $\partial B_\delta \times [t_0, T_1)$ we have $V = A$, hence $G(V) + t_0 - t \geq G(A) - \delta$, so that $U \leq G^{-1}(G(A) - \delta) = \tilde{A}$. Since also $U(t_0) = u(t_0)$, it follows from the comparison principle that $u \geq U$ in $B_\delta \times [t_0, T_1)$ hence in particular $\tau_1 \geq T$ (since otherwise u would blow up before $t = T$). Consequently

$$u \geq U \quad \text{in } B_\delta \times [t_0, T). \quad (7.6)$$

Step 2. Propagation of lower estimate on $u(t_0)$ in the region $|x| = O(\sqrt{T - t_0})$. By Theorem 2.2, we have

$$u(y\sqrt{\sigma}, T - \sigma) - G^{-1}(\sigma) = -\frac{|y|^2 - 2n}{4|\log \sigma|} + \frac{\mathcal{R}(y, |\log \sigma|)}{|\log \sigma|}$$

with $\lim_{s \rightarrow \infty} \|\mathcal{R}(\cdot, s)\|_{H_p^1} = 0$, which rewrites in original variables as

$$u(x, T - \sigma) = G^{-1}(\sigma) + \frac{2n}{4|\log \sigma|} - \frac{|x|^2}{4\sigma|\log \sigma|} + \frac{\mathcal{R}_\sigma(x)}{|\log \sigma|}, \quad \text{where } \mathcal{R}_\sigma(x) = \mathcal{R}(x\sigma^{-1/2}, |\log \sigma|).$$

Therefore we have

$$\begin{aligned} V(x, t) &= A - S(t - t_0)A + S(t - t_0)u(t_0) \geq S(t - t_0)u(t_0) \\ &\geq S(t - t_0) \left(G^{-1}(\sigma) + \frac{2n}{4|\log \sigma|} - \frac{|x|^2}{4\sigma|\log \sigma|} \right) + \frac{1}{|\log \sigma|} S(t - t_0) \mathcal{R}_\sigma \equiv V_1 + V_2 \end{aligned}$$

hence, since $V \geq 0$,

$$V \geq (V_1)_+ - |V_2| \quad \text{in } B_\delta \times (t_0, T). \quad (7.7)$$

Using Lemma 7.1(i) and recalling (7.4), we may estimate $(V_1)_+$ from below for $t \in [t_0, T)$ as:

$$\begin{aligned} (V_1(t))_+ &\geq e^{-\lambda(t-t_0)} \left[G^{-1}(\sigma) + \frac{2n}{4|\log \sigma|} - \frac{|x|^2 + 2n(t-t_0)}{4\sigma|\log \sigma|} \right]_+ \chi_{B_{\delta/2}} \\ &= e^{-\lambda\sigma} \left[G^{-1}(\sigma) - \frac{|x|^2}{4\sigma|\log \sigma|} \right]_+ \chi_{B_{\delta/2}}. \end{aligned} \quad (7.8)$$

To control V_2 , we observe that, for $t \in [t_0, T)$,

$$\int_{B_\delta} \mathcal{R}_\sigma^2(z) e^{-\frac{|z|^2}{4(t-t_0)}} dz \leq \int_{B_\delta} \mathcal{R}^2(z\sigma^{-1/2}, |\log \sigma|) e^{-\frac{|z|^2}{4\sigma}} dz \leq \sigma^{n/2} \|\mathcal{R}(\cdot, |\log \sigma|)\|_{L_\rho^2}^2$$

and then we use Lemma 7.1(ii) to write

$$\begin{aligned} |S(t-t_0)\mathcal{R}_\sigma|(x) &\leq C(n)(t-t_0)^{-n/4} \left(\int_{B_\delta} \mathcal{R}_\sigma^2(z) e^{-\frac{|z|^2}{4(t-t_0)}} dz \right)^{1/2} e^{\frac{|x|^2}{2(t-t_0)}} \\ &\leq C(n)(t-t_0)^{-n/4} \sigma^{n/4} e^{\frac{|x|^2}{2(t-t_0)}} \|\mathcal{R}(\cdot, |\log \sigma|)\|_{L_\rho^2}, \quad x \in B_\delta, t \in [t_0, T). \end{aligned}$$

Denoting $D_\sigma = \overline{B}_{\sqrt{\sigma}} \times [T - \frac{\sigma}{2}, T)$ and using (7.4), we obtain

$$\sup_{D_\sigma} |V_2| \leq C(n) \frac{\|\mathcal{R}(\cdot, |\log \sigma|)\|_{L_\rho^2}}{|\log \sigma|} = \frac{\varepsilon(\sigma)}{|\log \sigma|}. \quad (7.9)$$

Here and in the rest of the proof ε denotes a generic function such that $\lim_{\sigma \rightarrow 0} \varepsilon(\sigma) = 0$. Assuming $\sigma \leq \delta^2/4$, it follows from (7.7)-(7.9) that

$$V \geq e^{-\lambda\sigma} \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \quad \text{in } D_\sigma. \quad (7.10)$$

Moreover, we may choose $\sigma_0 \in (0, \delta^2/4)$ such that the RHS of (7.10) is $> A$ for all $\sigma \in (0, \sigma_0]$. Therefore

$$0 < G(V) + t_0 - t \leq \mu(\sigma, t) := G \left[e^{-\lambda\sigma} \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \right] + T - t - \sigma \quad \text{in } D_\sigma,$$

where the RHS is $< G(A)$ hence, by (7.5), (7.6),

$$u \geq G^{-1}(\mu(\sigma, t)) \quad \text{in } D_\sigma. \quad (7.11)$$

Step 3. *Asymptotics of $\mu(\sigma, t)$ and conclusion.* Fix any x such that $|x|^2 \leq \sigma_0$. Choosing

$$\sigma = |x|^2, \quad (7.12)$$

it follows from (7.11) that

$$u(x, t) \geq G^{-1}(\mu(\sigma, t)) \quad \text{for } T - \frac{\sigma}{2} \leq t < T. \quad (7.13)$$

To control μ , we shall use the following lemma, which complements Lemma 5.2(ii).

Lemma 7.2. *There exist $X_1, C_1 > 0$ such that, for all $\varepsilon \in (0, 1/2)$ and $X \geq X_1$,*

$$f(X - \varepsilon) \geq (1 - \varepsilon)^2 f(X) \quad (7.14)$$

and

$$G(X - \varepsilon) \leq (1 + C_1\varepsilon)G(X). \quad (7.15)$$

Proof of Lemma 7.2. By [8, Lemma 7.2], there exists $X_1 > 0$ such that, for all $\varepsilon \in (0, 1/2)$ and $X \geq X_1$, $L(e^{X-\varepsilon})/L(e^X) \geq 1 - \varepsilon$, so that

$$f(X - \varepsilon) = e^{-\varepsilon} e^X L(e^{X-\varepsilon}) \geq f(X) e^{-\varepsilon} (1 - \varepsilon) \geq (1 - \varepsilon)^2 f(X),$$

hence (7.14). From the later, we deduce

$$G(X - \varepsilon) = \int_{X-\varepsilon}^{\infty} \frac{ds}{f(s)} = \int_X^{\infty} \frac{dt}{f(t-\varepsilon)} \leq (1 - \varepsilon)^{-2} G(X),$$

and (7.15) follows. \square

Recall that $G^{-1}(\sigma) \sim |\log \sigma|$ as $\sigma \rightarrow 0^+$ (cf. Lemma 4.1(ii)) and that G, G^{-1} are decreasing near ∞ and 0^+ respectively. Now, for any $\sigma > 0$ sufficient small, we first use (7.15) to get, for some $C > 0$,

$$\begin{aligned} G \left[e^{-\lambda\sigma} \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \right] &\leq G \left[G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} - \lambda\sigma \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \right] \\ &\leq \left(1 + C\sigma \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \right) G \left[G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right]. \end{aligned}$$

Using the mean value theorem, there exists $\theta(\sigma) \in (0, 1)$ such that

$$G \left[G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right] = \sigma - G' \left[G^{-1}(\sigma) - \theta(\sigma) \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right] \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|}.$$

By (4.2) in Lemma 4.1 and (7.14), we have

$$\begin{aligned} -G' \left[G^{-1}(\sigma) - \theta(\sigma) \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right] &= \frac{1}{f \left[G^{-1}(\sigma) - \theta(\sigma) \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right]} = (1 + \varepsilon(\sigma)) G \left[G^{-1}(\sigma) - \theta(\sigma) \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right] \\ &= (1 + \varepsilon(\sigma)) \left[1 + \frac{C_1}{|\log \sigma|} \right] \sigma \leq (1 + \varepsilon(\sigma)) \sigma. \end{aligned}$$

Consequently,

$$\begin{aligned} \mu(\sigma, t) &\leq \left(1 + C\sigma \left(G^{-1}(\sigma) - \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \right) \left(\sigma + \frac{(1 + \varepsilon(\sigma))\sigma}{4|\log \sigma|} \right) + T - t - \sigma \\ &\leq \left(1 + C\sigma G^{-1}(\sigma) \right) \left(1 + \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} \right) \sigma + T - t - \sigma \\ &= T - t + \sigma \frac{1 + \varepsilon(\sigma)}{4|\log \sigma|} + C\sigma^2 G^{-1}(\sigma) \leq (1 + \varepsilon(\sigma)) \left(T - t + \frac{\sigma}{4|\log \sigma|} \right). \end{aligned}$$

By (7.12), (7.13), using Lemma 5.2(ii), we thus deduce that, as $(x, t) \rightarrow (0, T)$,

$$u(x, t) \geq G^{-1} \left(T - t + \frac{|x|^2}{4|\log |x|^2|} \right) + o(1), \quad \text{uniformly for } |x|^2 \geq 2(T - t).$$

On the other hand, using (3.8) in Proposition 3.2 and (5.9) again, we have, as $(x, t) \rightarrow (0, T)$,

$$u(x, t) = G^{-1}(T - t) + o(1) = G^{-1} \left(T - t + \frac{|x|^2}{4|\log |x|^2|} \right) + o(1), \quad \text{uniformly for } |x|^2 \leq 2(T - t).$$

The lower part of (2.8) follows. \square

Proof of Corollary 2.1. (i) Letting $t \rightarrow T$ in the RHS of (2.8) and using the continuity of G^{-1} , we get

$$u(x, T) := \lim_{t \rightarrow T} u(x, t) = G^{-1} \left(\frac{|x|^2}{4|\log |x|^2|} \right) + o(1) \quad \text{as } x \rightarrow 0.$$

(ii) Let $K > 0$. In view of (2.8) and (5.9), setting $\xi = x/\sqrt{(T-t)|\log(T-t)|}$, it suffices to show that

$$\lim_{t \rightarrow T} \left(\sup_{0 < |\xi| \leq K} |\delta(x, t)| \right) = 0, \quad \text{where } \delta(x, t) := \frac{T-t + \frac{|x|^2}{4|\log|x|^2|}}{(T-t)(1 + \frac{|\xi|^2}{4})} - 1. \quad (7.16)$$

We first control $\delta(x, t)$ as follows

$$|\delta(x, t)| = \frac{|\xi|^2 \left| \frac{|\log(T-t)|}{|\log|x|^2|} - 1 \right|}{4 \left(1 + \frac{|\xi|^2}{4} \right)} \leq \left| \frac{|\log(T-t)|}{|\log|x|^2|} - 1 \right| |\xi|^2.$$

If $0 < |\xi|^2 |\log(T-t)| \leq 1$ then $|\log(T-t)| \leq |\log|x|^2|$ for t close to T , hence $|\delta(x, t)| \leq |\xi|^2 \leq \frac{1}{|\log(T-t)|}$. Whereas if $|\xi|^2 |\log(T-t)| \geq 1$ and $|\xi| \leq K$ then, for all $\varepsilon > 0$, we have $(1+\varepsilon)^{-1} |\log(T-t)| \leq |\log|x|^2| \leq |\log(T-t)|$ for t close to T , hence $|\delta(x, t)| \leq K^2 \varepsilon$. Property (7.16) follows, hence Corollary 2.1(ii). \square

Remark 7.1. Let us justify the claim made in Remark 3.3, that, for $n \leq 2$, Theorems 2.1-2.2 remain valid if we replace the time increasing assumption $u_t \geq 0$ by the type I blow-up condition (3.3). Indeed the assumption $u_t \geq 0$ was used only at the following places:

- to ensure (3.3) (cf. Lemma 4.6), which is used later on;
- to rule out nonzero limits $\phi \in \mathcal{S}$ when deducing Proposition 3.2 from Proposition 3.1 (see Lemma 4.5). But for $n \leq 2$, we have $\mathcal{S} = \{0\}$ by [3, Lemma 4.3(a)];
- to guarantee the largeness of u in the proof of the lower estimate of Theorem 2.1 (cf. (7.3)). But, as a consequence of Proposition 3.2, the “no-needle” property $\lim_{(x,t) \rightarrow (0,T)} u(x, t) = \infty$ can be shown by a similar argument as in [7, Proposition 2.1], which does not require $u_t \geq 0$.

8 Appendix: Proof of Lemma 4.5

We give a proof of Lemma 4.5, that we used in the proof of Proposition 3.2. We note that our proof is significantly simpler than that of [3, Lemma 4.2(a)], owing to the introduction of the auxiliary function Z below.

Proof of Lemma 4.5. Assume for contradiction that $\phi \not\equiv 0$, hence $\phi(0) > 0$ by local uniqueness for problem (4.40). By direct calculation, we have

$$g'' + c(r)g' + F(\phi)g = 0, \quad g(0) > 0, \quad g'(0) = 0. \quad (8.1)$$

Since $g \geq 0$ by assumption and $g \not\equiv 0$, local uniqueness for (8.1) yields that actually $g > 0$ on $[0, \infty)$. Set

$$Z := e^{2\phi}(ge^{-\phi})' = e^\phi(g' - \phi'g).$$

Using (4.40) and (8.1), we compute, for $r > 0$:

$$e^{-\phi}Z' = \phi'(g' - \phi'g) + (g'' - \phi''g - \phi'g') = -\phi'^2g - cg' - F(\phi)g + g(c\phi' + F(\phi)) = -\phi'^2g - c(g' - \phi'g).$$

Letting $\mu(r) = r^{n-1}e^{-r^2/4}$, we deduce

$$(\mu Z)' = \mu(Z' + cZ) = -\mu\phi'^2ge^\phi \leq 0, \quad r > 0. \quad (8.2)$$

Note that $\phi''(0) = -n^{-1}F(\phi(0)) < 0$, hence $\phi' < 0$ for $r > 0$ small. Since $Z(0) = 0$, integrating (8.2) and using also $g > 0$ and e^ϕ nonincreasing, we obtain, for some constant $C > 0$,

$$Z(r) = -\mu^{-1}(r) \int_0^r [\mu\phi'^2ge^\phi](s)ds \leq -Cr^{1-n}e^{r^2/4}e^{\phi(r)}, \quad r \geq 1,$$

hence

$$(ge^{-\phi})'(r) = [e^{-2\phi}Z](r) \leq -Cr^{1-n}e^{r^2/4}e^{-\phi(r)} \leq -Cr^{1-n}e^{-\phi(0)}e^{r^2/4}, \quad r \geq 1.$$

But this implies $(ge^{-\phi})(r) \rightarrow -\infty$ as $r \rightarrow \infty$, contradicting $g \geq 0$. \square

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