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# An extremal problem for the Bergman kernel of orthogonal polynomials

S. Charpentier, N. Levenberg and F. Wielonsky

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## Abstract

Let  $\Gamma \subset \mathbb{C}$  be a curve of class  $C(2, \alpha)$ . For  $z_0$  in the unbounded component of  $\mathbb{C} \setminus \Gamma$ , and for  $n = 1, 2, \dots$ , let  $\nu_n$  be a probability measure with  $\text{supp}(\nu_n) \subset \Gamma$  which minimizes the Bergman function  $B_n(\nu, z) := \sum_{k=0}^n |q_k^\nu(z)|^2$  at  $z_0$  among all probability measures  $\nu$  on  $\Gamma$  (here,  $\{q_0^\nu, \dots, q_n^\nu\}$  are an orthonormal basis in  $L^2(\nu)$  for the holomorphic polynomials of degree at most  $n$ ). We show that  $\{\nu_n\}_n$  tends weak-\* to  $\widehat{\delta}_{z_0}$ , the balayage of the point mass at  $z_0$  onto  $\Gamma$ , by relating this to an optimization problem for probability measures on the unit circle. Our proof makes use of estimates for Faber polynomials associated to  $\Gamma$ .

Keywords: orthogonal polynomials, Bergman kernel, Faber polynomials, Szegő function

MSC Classification Numbers: 30C40, 30E10

## 1 Introduction

Let  $K$  be a compact subset of the complex plane  $\mathbb{C}$  and let  $\mathcal{M}(K)$  denote the probability measures on  $K$ . Given a positive integer  $n$ , if the support of  $\nu \in \mathcal{M}(K)$  contains at least  $n + 1$  points, we can form the associated Bergman function

$$B_n(\nu, z) := \sum_{k=0}^n |q_k^\nu(z)|^2,$$

where  $\{q_0^\nu, \dots, q_n^\nu\}$  form an orthonormal basis in  $L^2(\nu)$  for  $\mathcal{P}_n$ , the holomorphic polynomials of degree at most  $n$ . One can set  $B_n(\nu, z) = +\infty$  when  $\nu$  has less than  $n + 1$  points in its support, but since we are interested in asymptotics ( $n \rightarrow \infty$ ) of Bergman functions, we may assume  $K$  contains infinitely many points. We fix  $z_0 \in \Omega$ , the unbounded component of  $\mathbb{C} \setminus K$ , and for each  $n$  we consider a probability measure  $\nu_n$  with  $\text{supp}(\nu_n) \subset K$  which minimizes the Bergman function at  $z_0$  among all such  $\nu \in \mathcal{M}(K)$ :

$$B_n(\nu_n, z_0) = \min_{\nu \in \mathcal{M}(K)} B_n(\nu, z_0).$$

The existence of at least one minimizing measure follows from the weak-\* compactness of  $\mathcal{M}(K)$  and lower semicontinuity of the map  $\nu \mapsto B_n(\nu, z_0)$ , see Lemma 2.1.

Equivalently,  $\nu_n$  solves the max-min problem

$$\max_{\nu \in \mathcal{M}(K)} \lambda_n(\nu, z_0), \quad \lambda_n(\nu, z_0) = \min_{p \in \mathcal{P}_n, p(z_0)=1} \int_K |p|^2 d\nu \leq 1, \quad (1.1)$$

where  $\lambda_n(\nu, z_0)$  is the Christoffel function of  $\nu$  at  $z_0$ . We recall that (cf., [12, Theorem 1.4])

$$\lambda_n(\nu, z) = B_n(\nu, z)^{-1}, \quad n \geq 0, \quad (1.2)$$

where we note that, with our previous convention on  $B_n(\nu, z)$ , the equality still holds when  $\nu$  has less than  $n + 1$  points.

Such an extremal measure  $\nu_n$  is called an **optimal prediction measure** (OPM) for  $K$  and  $z_0$  of order  $n$ . In general, it is not unique. For motivation to study this problem, we refer to [2] where they give a nice application to the field of optimal design for polynomial regression. Although  $B_n(\nu, z)$  is well defined only if all orthogonal polynomials up to degree  $n$ , exist,  $\lambda_n(\nu, z)$  is always defined, equal to 0 when the support of  $\nu$  consists of fewer than  $n + 1$  points. In fact,  $\lambda_n(\nu, z)$  is defined for all  $z \in \mathbb{C}$ . For an extremal measure  $\nu_n$ , all the orthogonal polynomials  $q_k^{\nu_n}$ ,  $k = 0, \dots, n$ , do exist. Note also that, for each  $n$ , the Bergman function  $B_n(\nu, z)$  only depends on a finite number of moments of the measure  $\nu$ , namely

$$m_{j,k} = \int_K z^j \bar{z}^k d\nu, \quad j, k = 0, \dots, n. \quad (1.3)$$

It is known that  $B_n(\nu_n, z_0)$  is related to the polynomial of extremal growth at  $z_0$ , see [2]. Indeed, one has

$$B_n(\nu_n, z_0) = \sup_{p \in \mathcal{P}_n} \frac{|p(z_0)|^2}{\|p\|_K^2} \leq e^{2ng_\Omega(z_0)}, \quad (1.4)$$

where the upper bound, with  $g_\Omega$  the Green function of  $\Omega$  and the point  $\infty$ , follows from the fact that

$$g_\Omega(z_0) = \sup \left\{ \frac{1}{\deg(p)} \log |p(z_0)| : p \in \cup_n \mathcal{P}_n, \|p\|_K \leq 1 \right\}. \quad (1.5)$$

Here  $\deg(p)$  denotes the degree of  $p$  and  $\|p\|_K := \sup_{z \in K} |p(z)|$ . Note that polynomials of extremal growth are also studied in the recent paper [3] where they are called dual residual polynomials.

For a general probability measure  $\nu$  on  $K$  and  $z \in \mathbb{C}$ , we have that

$$1 \geq \lambda_n(\nu, z) \geq \lambda_{n+1}(\nu, z) \geq 0$$

so that the limit

$$\lambda_\infty(\nu, z) := \lim_{n \rightarrow \infty} \lambda_n(\nu, z) \quad (1.6)$$

exists and  $0 \leq \lambda_\infty(\nu, z) \leq 1$ . It has been verified by explicit computations in [2] that if:

- i)  $K = [-1, 1]$  and  $z_0$  is real or purely imaginary,
- ii)  $K = \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$  and  $|z_0| > 1$ ,

certain sequences of optimal prediction measures  $\nu_n$  tend weak-\* to a limit, namely

$$\nu_n \rightarrow \widehat{\delta}_{z_0}, \quad n \rightarrow \infty, \quad (1.7)$$

where  $\widehat{\delta}_{z_0}$  denotes the balayage measure of  $\delta_{z_0}$ , the point mass at  $z_0$ , onto  $K$ . The authors of [2] have conjectured that this convergence holds true more generally. It is the aim of the present paper to show that the conjecture holds true for a more general class of compact sets  $K$  and points  $z_0$  outside of  $K$ . Namely, our main result is the following theorem.

**Theorem 1.1.** *Assume  $K$  is a compact subset bounded by a curve  $\Gamma \in C(2, \alpha)$ ,  $0 < \alpha < 1$  (i.e.,  $\Gamma$  can be parameterized by a function of class  $C(2, \alpha)$ ). For  $z_0 \in \Omega$ , any sequence of optimal prediction measures  $\{\nu_n\}_n$  tends weak- $*$  to  $\widehat{\delta}_{z_0}$ , the balayage of  $\delta_{z_0}$  onto  $\Gamma$ .*

Here,  $C(k, \alpha)$  denotes the class of  $k$ -times continuously differentiable functions with  $k$ -th derivative satisfying a Lipschitz condition of order  $\alpha$ .

After some general preliminaries in the next section, in section 3 we complement the study in [2] of the case of  $K = \overline{\mathbb{D}}$ , the closed unit disk. We show in Theorem 3.2 that for  $z \in \mathbb{D}$ , the balayage  $\widehat{\delta}_z$  to  $\mathbb{T} := \partial\mathbb{D}$  is the unique maximizer of  $\lambda_\infty(\mu, z)$  from (1.6) among  $\mu \in \mathcal{M}(\mathbb{T})$ . We then study the more general case of  $K$  bounded by a  $C(1, \alpha)$  curve  $\Gamma$  in Section 4, and then by a  $C(2, \alpha)$  curve in Section 5. To derive Theorem 1.1 in this setting, for  $z \in \Omega$  we make a connection between  $\widetilde{\lambda}_\infty(\nu, z)$ , a modification of  $\lambda_\infty$  for measures  $\nu$  supported on  $\Gamma$ , with  $\lambda_\infty(\Phi_*\nu, 1/\overline{\Phi(z)})$  where  $\Phi_*\nu$  is the push-forward of  $\nu$  on  $\mathbb{T}$ ,  $\Phi$  being a conformal map from the exterior of  $\Gamma$  to the exterior of  $\mathbb{T}$ . After some preliminary results, an outline of the proof is given in Section 4, followed by the details. We conclude with an interesting observation on the distinction between the cases of  $K$  being a curve versus  $K$  being an arc.

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## 2 General preliminaries

We begin with an elementary result.

**Lemma 2.1.** *Let  $K$  be a subset of  $\mathbb{C}$ ,  $n$  a given positive integer, and  $z \in \mathbb{C}$ . Then the map  $\nu \in \mathcal{M}(K) \mapsto B_n(\nu, z) \in (0, \infty]$  is weak- $*$  lower semicontinuous.*

*Proof.* By (1.2), it is equivalent to check that the map  $\nu \mapsto \lambda_n(\nu, z)$  is upper semicontinuous, which is true since, for each  $p \in \mathcal{P}_n$ ,  $p(z) = 1$ , the map

$$\nu \mapsto \int_K |p|^2 d\nu$$

is weak- $*$  continuous, and  $\lambda_n(\nu, z)$  is obtained as a minimum of such maps.  $\square$

We continue with some observations related to [2].

1. The max-min in (1.1) coincides with the min-max for general compact  $K$ , namely

$$\max_{\nu \in \mathcal{M}(K)} \min_{p \in \mathcal{P}_n, p(z_0)=1} \int_K |p|^2 d\nu = \min_{p \in \mathcal{P}_n, p(z_0)=1} \max_{\nu \in \mathcal{M}(K)} \int_K |p|^2 d\nu. \quad (2.1)$$

This follows from the classical minimax theorem, see the proof of [2, Proposition 2.1].

2. Let  $K \subset \mathbb{C}$  be compact and contain infinitely many points and fix  $z_0 \notin K$ . For  $n \in \mathbb{N}$ , let

$$M_n = M_n(z_0, K) := \sup\{|p(z_0)| : p \in \mathcal{P}_n, \|p\|_K \leq 1\}. \quad (2.2)$$

There exists a unique  $p_n \in \mathcal{P}_n$  with  $\|p_n\|_K = 1$  and  $p_n(z_0) = M_n$ ; in [2] this is called the *polynomial of extremal growth relative to  $K$  at  $z_0$* . Indeed, note that

$$M_n = \sup\left\{\frac{|p(z_0)|}{\|p\|_K} : p \in \mathcal{P}_n\right\} = \left[\inf\left\{\frac{\|p\|_K}{|p(z_0)|} : p \in \mathcal{P}_n\right\}\right]^{-1}$$

and

$$\begin{aligned} \inf\left\{\frac{\|p\|_K}{|p(z_0)|} : p \in \mathcal{P}_n\right\} &= \inf\{\|p\|_K : p \in \mathcal{P}_n, |p(z_0)| = 1\} \\ &= \inf\{\|1 - Q\|_K : Q \in \mathcal{P}_n, Q(z_0) = 0\}. \end{aligned}$$

Let  $V_n := \{Q \in \mathcal{P}_n, Q(z_0) = 0\}$ . This is an  $n$ -dimensional complex vector space, and clearly each nonzero  $Q \in V_n$  has at most  $n - 1$  zeros in  $K$  (since  $Q(z_0) = 0$ ). By the classical Haar uniqueness theorem in Chebyshev approximation (cf., [1], [6, Theorem 19]), every continuous, complex-valued function on  $K$  admits a unique best sup-norm approximant from  $V_n$ . Applying this to the constant function 1 there exists a unique  $Q_n \in V_n$  with  $M_n = [\|1 - Q_n\|_K]^{-1}$ , and thus  $p_n = 1 - Q_n$ .

3. From 2. and Remark 2.3 in [2], it follows that the support of an OPM  $\nu_n$  of order  $n$  for  $K$  and  $z_0$  as in 2. is contained in

$$S_n(K) := \{z \in K : |p_n(z)| = \|p_n\|_K\}.$$

The set  $\{z \in \mathbb{C} : |p_n(z)| = \|p_n\|_K\}$  is a real algebraic curve in  $\mathbb{R}^2$  of degree at most  $2n$ . In particular, for  $z_0 \in \Omega$ , the unbounded component of  $\mathbb{C} \setminus K$ , if  $p_n$  is non-constant, any OPM  $\nu_n$  for  $K$  is supported on  $\partial\Omega$ . A necessary and sufficient condition that  $p_n$  be non-constant is that  $z_0$  lie outside of

$$\widehat{K}_n := \{z \in \mathbb{C} : |q_n(z)| \leq \|q_n\|_K \text{ for all } q_n \in \mathcal{P}_n\},$$

the  $n$ -th order polynomial hull of  $K$ . Since these sets  $\widehat{K}_n$  decrease to

$$\widehat{K} := \{z \in \mathbb{C} : |q(z)| \leq \|q\|_K \text{ for all } q \in \bigcup_n \mathcal{P}_n\},$$

the polynomial hull of  $K$ , and  $\Omega = \mathbb{C} \setminus \widehat{K}$ , by appealing to either the Hilbert lemniscate theorem (cf., [8, Theorem 5.5.8]) or simply Runge's theorem, for any  $z_0 \in \Omega$ , there exists  $n_0$  sufficiently large so that  $p_n$  is non-constant for  $n \geq n_0$ .

Thus if, e.g.,  $K$  is an ellipse of the form

$$K = \{(x, y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 = 1\}$$

with  $a \neq b$  and  $z_0$  lies outside  $K$ , by Bezout's theorem  $S_n(K)$  contains at most  $4n$  points. Since an OPM  $\nu_n$  exists, the support of  $\nu_n$  contains at least  $n + 1$  points. On the other hand, we recall in the next section that for the unit circle  $\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  and a point  $z_0$  with  $|z_0| > 1$ , there exist OPM's  $\nu_n$  which are absolutely continuous with respect to arclength measure and hence with support  $\mathbb{T}$ . It follows from 3. that OPM's for  $\mathbb{D}$  and  $\mathbb{T}$  coincide. More generally, if  $K$  is a compact subset bounded by a  $C(2, \alpha)$  curve  $\Gamma = \partial\Omega$  (as in section 4) and  $z_0 \in \Omega$ , OPM's  $\nu_n$  for  $K$  and  $\Gamma$  coincide, at least for  $n$  sufficiently large, which we will always assume.

### 3 The unit disk $\mathbb{D}$

We begin by recalling that the harmonic measure  $\omega_{\mathbb{D}}(z, t)$  for the disk  $\mathbb{D}$  and a point  $z = |z|e^{i\phi} \in \mathbb{D}$  is given by

$$d\omega_{\mathbb{D}}(z, t) = \frac{1 - |z|^2}{|e^{it} - z|^2} \frac{dt}{2\pi} = \left[ \sum_{k=-\infty}^{\infty} |z|^{|k|} e^{ik(\phi-t)} \right] \frac{dt}{2\pi} = \operatorname{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) \frac{dt}{2\pi} =: P_z(e^{it}) \frac{dt}{2\pi}, \quad (3.1)$$

see e.g. [8, Chapter 4.3]. In particular,  $d\omega_{\mathbb{D}}(0, t) = dt/2\pi$ . It may also be defined as the balayage  $\widehat{\delta}_z$  of the Dirac mass  $\delta_z$  onto the unit circle  $\mathbb{T}$ , see [9, Appendix A.3] or, by conformal invariance, the balayage  $\widehat{\delta}_{1/\bar{z}}$  of  $\delta_{1/\bar{z}}$  onto  $\mathbb{T}$ .

**Definition 3.1.** A positive and finite measure  $\mu$  on the unit circle  $\mathbb{T}$  satisfies the Szegő condition if its density  $f = d\mu/d\theta$  satisfies

$$\int_{\mathbb{T}} \log f d\theta > -\infty.$$

Then, the Szegő function is defined by

$$D(\mu, z) = \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log f(t) dt \right), \quad |z| < 1. \quad (3.2)$$

Note that, with  $\mu_a$  the absolutely continuous part of  $\mu$ , and  $\lambda > 0$ , one has

$$D(\mu, z) = D(\mu_a, z), \quad D(\lambda\mu, z) = \sqrt{\lambda} D(\mu, z). \quad (3.3)$$

It is known, see [10, Theorem 2.4.1], that for any measure  $\mu$  satisfying the Szegő condition,

$$\lambda_{\infty}(\mu, z) = (1 - |z|^2) |D(\mu, z)|^2, \quad |z| < 1. \quad (3.4)$$

We also recall that for any measure  $\mu$  on  $\mathbb{T}$  the Christoffel function satisfies

$$|z|^{2n} \lambda_n(\mu, z) = \lambda_n(\mu, 1/\bar{z}), \quad z \neq 0, \quad (3.5)$$

see e.g. [10, Lemma 2.2.8]. These relations (3.4) and (3.5) will be crucial in the sequel, as will the unicity in the next result.

**Theorem 3.2.** *Let  $z \in \mathbb{D}$ . The unique probability measure  $\mu$  on  $\mathbb{T}$  that maximizes  $\lambda_{\infty}(\mu, z)$ , is the balayage measure  $\widehat{\delta}_z$ .*

*Proof.* Let  $\mu$  be a probability measure on  $\mathbb{T}$ . By [10, Theorem 2.7.15], we have  $\lambda_{\infty}(\mu, z) = 0$  for any  $z \in \mathbb{D}$  precisely when  $\mu$  does not belong to the Szegő class. Thus, we may assume that  $\mu$  belongs to the Szegő class. By (3.4), we are led to maximize  $|D(\mu, z)|$ . Let  $\mu = \mu_a + \mu_s$ ,  $\mu_a = gdt$ ,  $g \in L^1(\mathbb{T})$ , be the Radon-Nikodym decomposition of the measure  $\mu$ . From (3.3), we see that the larger the mass of  $\mu_a$ , the larger the modulus of  $D(\mu, z)$ . Thus,  $\mu_s$  should vanish, that is,  $\mu$  has to be absolutely continuous.

We write

$$d\mu = f(e^{it}) d\omega_{\mathbb{D}}(0, t) = \frac{f(e^{it})}{P_z(e^{it})} d\omega_{\mathbb{D}}(z, t).$$

Then since

$$\int_{-\pi}^{\pi} \log f(e^{it}) d\omega_{\mathbb{D}}(0, t) = \int_{-\pi}^{\pi} \log \frac{f(e^{it})}{P_z(e^{it})} d\omega_{\mathbb{D}}(z, t) + \int_{-\pi}^{\pi} \log P_z(e^{it}) d\omega_{\mathbb{D}}(z, t),$$

it suffices to maximize the entropy functional

$$\int_{-\pi}^{\pi} \log \frac{f(e^{it})}{P_z(e^{it})} d\omega_{\mathbb{D}}(z, t) = \int_{-\pi}^{\pi} \log \left( \frac{d\mu}{d\omega_{\mathbb{D}}(z, t)} \right) d\omega_{\mathbb{D}}(z, t)$$

over absolutely continuous probability measures  $\mu$ . Jensen's inequality yields that the maximum is attained, uniquely, when  $d\mu/d\omega_{\mathbb{D}}(z, t) = 1$ ; i.e.,  $d\mu = d\omega_{\mathbb{D}}(z, t)$ .  $\square$

It is proved in [2] that, for a given degree  $n$  and  $z_0 = |z_0|e^{i\phi}$  with  $|z_0| > 1$ , the harmonic measure (3.1) for  $1/\bar{z}_0$ ,

$$d\mu_P(\theta) := d\omega_{\mathbb{D}}(1/\bar{z}_0, \theta) = \left[ \sum_{k=-\infty}^{\infty} |z_0|^{-|k|} e^{ik(\phi-\theta)} \right] \frac{d\theta}{2\pi},$$

is an OPM of order  $n$  for  $\bar{\mathbb{D}}$  and  $z_0$ , as well as any measure  $\mu$  whose moments

$$m_k = m_k(\mu) := \int_{\mathbb{T}} z^k d\mu, \quad k = 0, \pm 1, \dots, \pm n$$

coincide with those of the harmonic measure:

$$m_k(\mu_P) := \int_{\mathbb{T}} z^k d\mu_P = \int_{\mathbb{T}} z^k d\hat{\delta}_{1/\bar{z}_0} = \begin{cases} (\bar{z}_0)^{-k}, & k \geq 0, \\ z_0^k, & k < 0, \end{cases} \quad |k| = 0, 1, \dots, n.$$

Moreover, from (1.4) and (2.2), since  $M_n(z_0, \mathbb{T}) = |z_0|^n$  we have, for  $n \geq 0$ ,

$$B_n(\mu_P, z_0) = |z_0|^{2n}, \quad \lambda_n(\mu_P, z_0) = |z_0|^{-2n}. \quad (3.6)$$

## 4 Preliminaries when $K$ is bounded by a $C(1, \alpha)$ curve

Let  $K$  be a connected, simply connected, compact subset of  $\mathbb{C}$ , with nonempty interior. Let  $\Omega$  be the unbounded component of  $\mathbb{C} \setminus K$  and  $\Omega_{\infty} := \Omega \cup \{\infty\}$ . Let  $\Phi$  be the conformal map from  $\Omega$  to  $\mathbb{C} \setminus \mathbb{D}$ , with  $\Phi(\infty) = \infty$  and  $\Phi'(\infty) > 0$ . In this section, we assume that  $\Gamma = \partial\Omega$  is a  $C(1, \alpha)$  curve. For  $r \geq 1$ , we define the level curves of  $\Phi$ ,

$$\Gamma_r := \{z \in \Omega : |\Phi(z)| = r\}.$$

We will need several results.

We recall a result about sequences of conformal maps, see [13], Theorem 4 of Section 2.3.

**Theorem 4.1.** *Let  $J \subset \mathbb{C}$  be a Jordan curve and let  $D$  be the bounded component of  $\mathbb{C} \setminus J$ . Let  $\{D_n\}_{n=1}^{\infty}$  be a sequence of bounded, simply connected domains such that  $\overline{D}_{n+1} \subset D_n$  for each  $n$  and*

$$\bigcap_{n=1}^{\infty} D_n = \overline{D}.$$

*Given  $z_0 \in D$ , let  $F, F_n, n \geq 1$ , be the conformal mappings of  $D, D_n, n \geq 1$ , onto  $\mathbb{D}$  which take  $z_0$  to the origin and such that  $F'(z_0) > 0$  and  $F'_n(z_0) > 0$  for each  $n$ . Then we have*

$$\lim_{n \rightarrow \infty} F_n(z) = F(z) \quad \text{uniformly, for all } z \in \overline{D}.$$

We will also make use of the Faber polynomials  $F_n, n \geq 0$ , of the interior of  $K$ , see [11]. They are defined by the following identity, see [11, p.62]:

$$F_n(z) = \Phi^n(z) + \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\Phi^n(t)}{t-z} dt, \quad |\Phi(z)| > r \geq 1,$$

where we recall that  $\Gamma_r := \{z \in \Omega : |\Phi(z)| = r\}$ .

**Proposition 4.2** ([11, p.61]). *Let  $\Gamma$  be a  $C(1, \alpha)$  curve, and let  $F_n, n \geq 0$ , be the associated Faber polynomials. Let  $F$  be a closed subset of the interior of  $K$ . Then, there is a constant  $c(F)$  such that*

$$|F_n(t)| \leq \frac{c(F)}{n^\alpha}, \quad t \in F.$$

**Remark 4.3.** The above result on the decrease of Faber polynomials in  $K$  also holds for piecewise analytic curves  $\Gamma$ , see [5, Theorem 1].

**Proposition 4.4** ([11, Theorem 2 p.68]). *When  $\Gamma$  is a  $C(p+1, \alpha)$  curve,  $p \in \mathbb{N}$ , the following estimate holds:*

$$F_n(z) = \Phi^n(z) + \mathcal{O}\left(\frac{\log n}{n^{p+\alpha}}\right), \quad n \rightarrow \infty, \tag{4.1}$$

*uniformly for  $z \in \overline{\Omega}$ .*

We denote by  $A(\overline{\Omega}_\infty)$  the set of functions analytic in a neighborhood of  $\overline{\Omega}_\infty$ .

**Proposition 4.5.** *Given a function  $g \in A(\overline{\Omega}_\infty)$ ,  $Q_n$  any polynomial of degree at most  $n$ , and  $P_n$  the unique polynomial of degree at most  $n$  such that*

$$Q_n(z)g(z) - P_n(z) = \mathcal{O}(z^{-1}), \quad z \rightarrow \infty, \tag{4.2}$$

*one has*

$$Q_n(z)g(z) - P_n(z) = -\frac{1}{2\pi i} \int_{\Gamma_g} Q_n(t) \frac{g(t)}{t-z} dt \tag{4.3}$$

*for  $z$  outside of  $\Gamma_g$ , a simple, positively oriented, curve lying in  $K$  and in the domain of analyticity of  $g$ .*

*Proof.* Because of (4.2), the identity (4.3) is a simple consequence of Cauchy's formula applied to the difference  $Q_n(z)g(z) - P_n(z)$  outside of  $\Gamma_g$ .  $\square$



Let  $\mu$  be a probability measure on  $\Gamma$ . We set, for  $z \in \Omega$ ,

$$\widetilde{B}_n(\mu, z) := \frac{B_n(\mu, z)}{|\Phi(z)|^{2n}} \quad \text{and} \quad \widetilde{\lambda}_n(\mu, z) := |\Phi(z)|^{2n} \lambda_n(\mu, z), \quad n \geq 0, \quad (4.4)$$

and

$$\widetilde{B}_\infty(\mu, z) := \limsup_{n \rightarrow \infty} \widetilde{B}_n(\mu, z) \leq \infty \quad \text{and} \quad \widetilde{\lambda}_\infty(\mu, z) := \liminf_{n \rightarrow \infty} \widetilde{\lambda}_n(\mu, z) \geq 0. \quad (4.5)$$

In fact, in Lemma 4.6 below, we show that the limits exist in (4.5). Note that since  $B_n(\mu, z)$  is weak-\* continuous for our class of measures so is  $\widetilde{B}_n(\mu, z)$ .

The idea behind our proof that the weak-\* limit of any subsequence  $\{\nu_n\}_{n \in Y}$ ,  $Y \subset \mathbb{N}$  of OPM's for  $\Gamma$  and  $z$  is  $\widehat{\delta}_z$ , the balayage of the point mass at  $z$  to  $\Gamma$ , is as follows. Using Propositions 4.2, 4.4 and 4.5, we first show in Lemma 4.6 and Corollary 4.7 that for any probability measure  $\mu$  on  $\Gamma$ ,  $\widetilde{\lambda}_\infty(\mu, z)$  (and hence  $\widetilde{B}_\infty(\mu, z)$ ) is related to  $\lambda_\infty(\Phi_*\mu, 1/\overline{\Phi(z)})$  where  $\Phi_*\mu \in \mathcal{M}(\mathbb{T})$ . The crux of the matter is to then show that if  $\alpha$  is a weak-\* limit of a subsequence  $\{\nu_n\}_{n \in Y_1}$ ,  $Y_1 \subset Y$  then (a perturbation of) a ‘‘diagonal subsequence’’  $\{\widetilde{B}_n(\nu_n, z)\}_{n \in Y_1}$ , converges to (a perturbation of)  $\widetilde{B}_\infty(\alpha, z)$  (Lemma 5.4). As in the proof of Lemma 4.6, we use Faber polynomials in Lemma 5.1 as a tool to prove a sort of monotonicity of  $\{\widetilde{B}_n(\mu, z)\}$  in  $n$  for general  $\mu$  which is needed to apply Dini's theorem to conclude the proof of Lemma 5.4. After the proof of our main result, we make a remark to indicate a relationship with an underlying general potential-theoretic question.

**Lemma 4.6.** *Let  $z \in \Omega$  and let  $\mu$  be a measure on  $\Gamma$ . We have*

$$\widetilde{\lambda}_\infty(\mu, z) = \inf \left\{ \int_\Gamma |f|^2 d\mu, f \in A(\overline{\Omega}_\infty), f(z) = 1 \right\}. \quad (4.6)$$

Moreover,  $\widetilde{\lambda}_\infty(\mu, z) = \lim_{n \rightarrow \infty} \widetilde{\lambda}_n(\mu, z)$ , i.e., the limit exists.

*Proof.* We first show

$$\widetilde{\lambda}_\infty(\mu, z) \geq \inf \left\{ \int_\Gamma |f|^2 d\mu, f \in A(\overline{\Omega}_\infty), f(z) = 1 \right\}.$$

Let  $\Psi$  be a conformal map from  $U = \text{Int}(K)$  to  $\mathbb{D}$ . We consider the level curves  $\widetilde{\Gamma}_k := \{|\Psi| = 1 - 1/k\}$ ,  $k = 2, 3, \dots$ , and let  $\Omega_k$  be the domain outside of  $\widetilde{\Gamma}_k$ . Then  $\Omega_k \supset \Omega_{k+1} \supset \Omega$ , and  $\Omega = \text{Int}(\cap_k \Omega_k)$ . Letting  $\Phi_k$  denote the conformal map from  $\Omega_k$  to  $\mathbb{C} \setminus \mathbb{D}$  with  $\Phi_k(\infty) = \infty$  and  $\Phi'_k(\infty) > 0$ , it follows from Theorem 4.1 that  $\Phi_k$  converges locally uniformly in  $\Omega$  to  $\Phi$ .

Fix  $k \in \mathbb{N}$ . We have, for each  $n$ ,

$$\begin{aligned} \widetilde{\lambda}_n(\mu, z) &= |\Phi(z)|^{2n} \inf \left\{ \int_\Gamma |p|^2 d\mu, p \in \mathcal{P}_n, p(z) = 1 \right\} \\ &\geq |\Phi(z)|^{2n} \inf \left\{ \int_\Gamma \frac{|p|^2}{(|\Phi_k|)^{2n}} d\mu, p \in \mathcal{P}_n, p(z) = 1 \right\} \\ &\geq |\Phi(z)|^{2n} \inf \left\{ \int_\Gamma |f|^2 d\mu, f \in A(\overline{\Omega}_\infty), f(z) = \Phi_k(z)^{-n} \right\}. \\ &= \left| \frac{\Phi(z)}{\Phi_k(z)} \right|^{2n} \inf \left\{ \int_\Gamma |f|^2 d\mu, f \in A(\overline{\Omega}_\infty), f(z) = 1 \right\}. \end{aligned}$$

In the first inequality, we have used that  $|\Phi_k| > 1$  on  $\Gamma$ , and in the second inequality, we have used that  $p/\Phi_k^n$  is analytic in a neighborhood of  $\overline{\Omega_\infty}$ .

Letting  $k$  tend to infinity, since  $z \in \Omega$  we have  $\Phi_k(z) \rightarrow \Phi(z)$  as  $k \rightarrow \infty$ , and thus

$$\tilde{\lambda}_n(\mu, z) \geq \inf \left\{ \int_{\Gamma} |f|^2 d\mu, f \in A(\overline{\Omega_\infty}), f(z) = 1 \right\},$$

which implies the desired inequality:

$$\liminf_{n \rightarrow \infty} \tilde{\lambda}_n(\mu, z) = \tilde{\lambda}_\infty(\mu, z) \geq \inf \left\{ \int_{\Gamma} |f|^2 d\mu, f \in A(\overline{\Omega_\infty}), f(z) = 1 \right\}.$$

To show that

$$\inf \left\{ \int_{\Gamma} |f|^2 d\mu, f \in A(\overline{\Omega_\infty}), f(z) = 1 \right\} \geq \limsup_{n \rightarrow \infty} \tilde{\lambda}_n(\mu, z),$$

given  $\epsilon > 0$ , take  $g \in A(\overline{\Omega_\infty})$  with  $g(z) = 1$  and

$$\int_{\Gamma} |g|^2 d\mu \leq (1 + \epsilon)^2 \inf \left\{ \int_{\Gamma} |f|^2 d\mu, f \in A(\overline{\Omega_\infty}), f(z) = 1 \right\}.$$

We show that for  $n \geq n_0(\epsilon)$ , we can find  $p_n \in \mathcal{P}_n$  such that

$$\int_{\Gamma} |p_n|^2 d\mu \leq (1 + \epsilon)^2 \int_{\Gamma} |g|^2 d\mu \quad \text{and} \quad p_n(z) = \Phi^n(z). \quad (4.7)$$

Applying Proposition 4.5 with the function  $g$  and the polynomial  $Q_n = F_n$ , the  $n$ -th Faber polynomial for  $K$ , and making use of Proposition 4.2, we conclude that, for some constant  $c$  independent of  $n$ ,

$$|F_n(t)g(t) - P_n(t)| \leq \frac{c}{n^\alpha}, \quad t \in \Gamma \cup \{z\}.$$

By Proposition 4.4 applied with  $p = 0$ , we have  $F_n \rightarrow \Phi^n$  on  $\Gamma$  uniformly, and  $F_n(z) \rightarrow \Phi^n(z)$ . Since  $|\Phi| = 1$  on  $\Gamma$  and  $g(z) = 1$  we get

$$|P_n| \rightarrow |g|, \quad \text{uniformly on } \Gamma, \quad \text{and} \quad P_n(z) \rightarrow \Phi^n(z).$$

Thus, we get, for  $n \geq n_0(\epsilon)$ , that  $p_n := (\Phi^n(z)/P_n(z))P_n$  satisfy (4.7).  $\square$

For a measure  $\mu$  on  $\Gamma$  we have  $\Phi_*\mu$  is a measure on the circle  $\mathbb{T}$ . From (1.6),

$$\lambda_\infty(\Phi_*\mu, 1/\overline{\Phi(z)}) = \lim_{n \rightarrow \infty} \lambda_n(\Phi_*\mu, 1/\overline{\Phi(z)}).$$

**Corollary 4.7.** *For any measure  $\mu$  on  $\Gamma$ , it holds that*

$$\tilde{\lambda}_\infty(\mu, z) = \lambda_\infty(\Phi_*\mu, 1/\overline{\Phi(z)}), \quad z \in \Omega. \quad (4.8)$$

*Proof.* One has, in view of (4.6) and (3.5),

$$\begin{aligned} \tilde{\lambda}_\infty(\mu, z) &= \inf \left\{ \int_{\Gamma} |f|^2 d\mu, f \in A_\epsilon(\Gamma), f(z) = 1 \right\} \\ &= \inf \left\{ \int_{\Gamma} |f|^2 d\Phi_*\mu, f \in A_\epsilon(\mathbb{T}), f(\Phi(z)) = 1 \right\} = \lim_{n \rightarrow \infty} |\Phi(z)|^{2n} \lambda_n(\Phi_*\mu, \Phi(z)) \\ &= \lim_{n \rightarrow \infty} \lambda_n(\Phi_*\mu, 1/\overline{\Phi(z)}) = \lambda_\infty(\Phi_*\mu, 1/\overline{\Phi(z)}). \end{aligned}$$

$\square$

## 5 Case of $K$ bounded by a curve $\Gamma \in C(2, \alpha)$

With the same notation as section 4 we now assume that  $\Gamma = \partial\Omega$  is a  $C(2, \alpha)$  curve. We start with proving a weak monotonicity of the sequence  $\{\widetilde{B}_n(\mu, z)\}_n$  for  $\mu$  on  $\Gamma$ .

**Lemma 5.1.** *Let  $z \in \Omega$  be fixed. Let  $\mu$  be any measure supported on  $\Gamma$  such that the orthogonal polynomials are well-defined up to degree  $N$ . Let  $n < N$ . Then there exist positive numbers  $c_n \geq 1$  such that*

$$\widetilde{B}_{N-n}(\mu, z) \leq c_n \widetilde{B}_N(\mu, z), \quad c_n = 1 + \mathcal{O}\left(\frac{\log n}{n^{1+\alpha}}\right), \quad \text{as } n, N \rightarrow \infty, \quad (5.1)$$

where the  $c_n$ 's are independent of the measure  $\mu$ .

*Proof.* Let  $n < N$ . We will prove that

$$|\Phi(z)|^{2n} \lambda_N(\mu, z) \leq c_n \lambda_{N-n}(\mu, z)$$

for appropriate  $c_n$ , which is equivalent to the inequality in (5.1).

To estimate  $|\Phi(z)|^{2n} \lambda_N(\mu, z)$ ,

$$\begin{aligned} |\Phi(z)|^{2n} \lambda_N(\mu, z) &= |\Phi(z)|^{2n} \min_{p \in \mathcal{P}_N, p(z) \neq 0} \frac{\int_{\Gamma} |p|^2 d\mu}{|p(z)|^2} \\ &\leq |\Phi(z)|^{2n} \min_{p \in \mathcal{P}_{N-n}, p(z) \neq 0} \frac{\int_{\Gamma} |F_n p|^2 d\mu}{|F_n(z) p(z)|^2} \\ &= \frac{|\Phi(z)|^{2n}}{|F_n(z)|^2} \cdot \min_{p \in \mathcal{P}_{N-n}, p(z) \neq 0} \frac{\int_{\Gamma} |F_n p|^2 d\mu}{|p(z)|^2}. \end{aligned}$$

To estimate  $\int_{\Gamma} |F_n p|^2 d\mu$ , we get

$$\int_{\Gamma} |F_n p|^2 d\mu \leq \|F_n\|_{\Gamma}^2 \int_{\Gamma} |p|^2 d\mu$$

so that

$$|\Phi(z)|^{2n} \lambda_N(\mu, z) \leq \frac{|\Phi(z)|^{2n}}{|F_n(z)|^2} \|F_n\|_{\Gamma}^2 \cdot \min_{p \in \mathcal{P}_{N-n}, p(z) \neq 0} \frac{\int_{\Gamma} |p|^2 d\mu}{|p(z)|^2}.$$

The last minimum equals  $\lambda_{N-n}(\mu, z)$  and we need estimate

$$\frac{|\Phi(z)|^{2n}}{|F_n(z)|^2} \cdot \|F_n\|_{\Gamma}^2$$

from above. Using the estimate (4.1) with  $p = 1$  for each piece, we obtain

$$|\Phi(z)|^{2n} \lambda_N(\mu, z) \leq \left(1 + \mathcal{O}\left(\frac{\log n}{n^{1+\alpha}}\right)\right) \cdot \lambda_{N-n}(\mu, z),$$

from which the existence of the  $c_n$ 's follows. The proof shows that they are independent of the measure  $\mu$ .  $\square$

**Remark 5.2.** In the particular case of  $\Gamma = \mathbb{T}$ , the unit circle, and  $\mu = d\theta/2\pi$ , the family  $\{z^n\}_{n \in \mathbb{N}}$  is an orthonormal basis, and

$$B_n(\mu, z) = \frac{|z|^{2n+2} - 1}{|z|^2 - 1}.$$

The inequality  $\widetilde{B}_{n-1}(\mu, z) \leq \widetilde{B}_n(\mu, z)$  is true since it is equivalent to  $|z|^{2n+2} - 1 \leq |z|^{2n+2}$ . For the harmonic measure  $\mu_P$  in (3.1), from (3.6) we have  $B_n(\mu_P, z_0) = |z_0|^{2n}$  so that  $\widetilde{B}_n(\mu_P, z_0) = 1$  for all  $n$ .

In Proposition 5.3 and Lemma 5.4, the point  $z$  is fixed and for any measure  $\mu$ , we will simply write  $\widetilde{B}_n(\mu)$ ,  $\widetilde{B}_\infty(\mu)$  instead of  $\widetilde{B}_n(\mu, z)$ ,  $\widetilde{B}_\infty(\mu, z)$ , and similarly for other expressions depending on  $z$ .

**Proposition 5.3.** *Fix  $z \in \Omega$ . Assume that a subsequence  $\{\nu_{\varphi(n)}\}_n$  of a sequence  $\{\nu_n\}_n$  of OPM's tends weak-\* to a limit measure  $\alpha$ . Then  $\alpha$  satisfies the following:*

1) *For all integers  $k$ , we have*

$$\widetilde{B}_k(\alpha) \leq 1 \leq \widetilde{\lambda}_k(\alpha). \quad (5.2)$$

2)  *$\alpha$  has an infinite number of points in its support.*

*Proof.* To show 1), for a given  $k$ , we have

$$\widetilde{B}_k(\alpha) \leq \liminf_n \widetilde{B}_k(\nu_{\varphi(n)}) \leq \liminf_n c_{\varphi(n)-k} \widetilde{B}_{\varphi(n)}(\nu_{\varphi(n)}) = \liminf_n \widetilde{B}_{\varphi(n)}(\nu_{\varphi(n)}) \leq 1.$$

Here the first inequality follows from lower semicontinuity of  $B_k$  (and hence  $\widetilde{B}_k$ ), recall Lemma 2.1. The second inequality and the equality use Lemma 5.1, while the final inequality uses (1.4) and the fact that  $|\Phi(z)| = e^{g\Omega(z)}$ ,  $z \in \Omega$ . The second inequality in (5.2) is equivalent to the first one.

We prove 2) by contradiction. Assume that  $\alpha$  has, say,  $k$  points in its support. Then,  $B_k(\alpha) = \infty$ , hence  $\widetilde{B}_k(\alpha) = \infty$ , which contradicts the first inequality in (5.2), and proves 2). Note, in particular, that orthogonal polynomials of all degrees are well-defined for the measure  $\alpha$ .  $\square$

From (5.2), all numbers  $\widetilde{B}_n(\alpha)$ ,  $n \geq 0$ , are less than or equal to 1, and thus

$$\widetilde{B}_\infty(\alpha) = \lim_n \widetilde{B}_n(\alpha) \leq 1$$

(recall from Lemma 4.6 the limit exists).

**Lemma 5.4.** *Let  $\{\nu_n\}_n$  be a sequence of OPM's on  $K$ , with  $\nu_n$  of order  $n$ . For any subsequence  $\{\nu_{\varphi_1(n)}\}_n$  of  $\{\nu_n\}_n$  with a weak-\* limit  $\alpha$ , there is a subsequence  $\{\nu_{\varphi_2(n)}\}_n$  of  $\{\nu_{\varphi_1(n)}\}_n$  such that*

$$\lim_n \widetilde{B}_{\varphi_2(n)}(\nu_{\varphi_2(n)}) = \widetilde{B}_\infty(\alpha). \quad (5.3)$$

*Proof.* Note that, by Proposition 5.3, the weak-\* convergence  $\nu_{\varphi_1(n)} \rightarrow \alpha$  implies that orthogonal polynomials for the limit measure  $\alpha$  exist for any degree  $n \geq 0$  and

$$\widetilde{B}_\infty(\alpha) = \lim_n \widetilde{B}_{\varphi_1(n)}(\alpha).$$

If the sequence  $\{\nu_{\varphi_1(n)}\}$  contains an element which appears infinitely many times, then  $\alpha$  is equal to this element; hence we may assume that each element in the sequence  $\nu_{\varphi_1(n)}$  appears at most a finite number of times. For a technical reason in the sequel of the proof (in the definition of the functions  $F_n$  below), we replace the sequence  $\nu_{\varphi_1(n)}$  with the subsequence, still denoted  $\nu_{\varphi_1(n)}$ , where we keep only the last occurrence of each repeated element. Hence, with that change, each element in the sequence  $\{\nu_{\varphi_1(n)}\}$  appears exactly once.

We choose the subsequence  $\{\nu_{\varphi_2(n)}\}_n$  of  $\{\nu_{\varphi_1(n)}\}_n$  in such a way that

$$\forall n \geq 1, \quad n \leq \varphi_2(n) - \varphi_2(n-1) < \varphi_2(n+1) - \varphi_2(n). \quad (5.4)$$

For a measure  $\mu$ , we set

$$\widetilde{C}_n(\mu) := \left( \prod_{k=0}^n c_{\varphi_2(k) - \varphi_2(k-1)} \right) \widetilde{B}_{\varphi_2(n)}(\mu), \quad n \in \mathbb{N}, \quad (5.5)$$

where the  $c_k$  are the constants in (5.1) (recall that they are independent of  $\mu$ ). The sequence  $\widetilde{C}_n(\mu)$  is increasing with  $n$ . Indeed,

$$\widetilde{C}_n(\mu) / \widetilde{C}_{n-1}(\mu) = c_{\varphi_2(n) - \varphi_2(n-1)} \widetilde{B}_{\varphi_2(n)}(\mu) / \widetilde{B}_{\varphi_2(n-1)}(\mu) \geq 1,$$

where the inequality comes from (5.1).

For the measure  $\alpha$  we also define

$$\widetilde{C}_\infty(\alpha) := L \widetilde{B}_\infty(\alpha), \quad L := \prod_{k=0}^{\infty} c_{\varphi_2(k) - \varphi_2(k-1)} \geq 1,$$

The infinite product in the definition of  $L$  converges because of (5.4) and the asymptotic behavior of the  $c_k$  as  $k$  tends to infinity, see (5.1). Also, by the choice of the subsequence  $\{\nu_{\varphi_1(n)}\}_n$ , we have

$$\widetilde{C}_\infty(\alpha) = \lim_{n \rightarrow \infty} \widetilde{C}_n(\alpha). \quad (5.6)$$

The set of measures  $S = \{\nu_{\varphi_2(0)}, \nu_{\varphi_2(1)}, \dots, \alpha\}$  is compact. Consider the array of values taken by the functions  $F_0, F_1, \dots, F_n, \dots, F_\infty$  on  $S$ :

$F_\infty$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_\infty(\alpha)$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$F_n$	$\widetilde{C}_\infty(\alpha)$	$\dots$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_n(\nu_{\varphi_2(n)})$	$\rightarrow$	$\widetilde{C}_n(\alpha)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_1$	$\widetilde{C}_\infty(\alpha)$	$\widetilde{C}_1(\nu_{\varphi_2(1)})$	$\dots$		$\rightarrow$	$\widetilde{C}_1(\alpha)$
$F_0$	$\widetilde{C}_0(\nu_{\varphi_2(0)})$	$\widetilde{C}_0(\nu_{\varphi_2(1)})$	$\dots$		$\rightarrow$	$\widetilde{C}_0(\alpha)$
	$\nu_{\varphi_2(0)}$	$\nu_{\varphi_2(1)}$	$\dots$	$\nu_{\varphi_2(n)}$	$\rightarrow$	$\alpha$

where all values above the ascending main diagonal  $\widetilde{C}_0(\nu_{\varphi_2(0)}), \widetilde{C}_1(\nu_{\varphi_2(1)}), \dots, \widetilde{C}_n(\nu_{\varphi_2(n)}), \dots$  are equal to  $\widetilde{C}_\infty(\alpha)$ . Note our choice of the subsequence  $\{\nu_{\varphi_1(n)}\}$  insures each  $F_n$  is well-defined. The following properties are satisfied:

a) The function  $F_\infty$  is constant, hence continuous on  $S$ .

b) For each  $n$ ,  $F_n$  is continuous at  $\alpha$  because  $\widetilde{C}_n(\nu_{\varphi_2(k)}) \rightarrow \widetilde{C}_n(\alpha)$  as  $n \leq k \rightarrow \infty$ . To see this, using (5.5) we have

$$\widetilde{C}_n(\nu_{\varphi_2(k)}) = \left( \prod_{p=0}^n c_{\varphi_2(p)-\varphi_2(p-1)} \right) \widetilde{B}_{\varphi_2(n)}(\nu_{\varphi_2(k)})$$

and  $\widetilde{B}_{\varphi_2(n)}(\nu_{\varphi_2(k)}) \rightarrow \widetilde{B}_{\varphi_2(n)}(\alpha)$  as  $k \rightarrow \infty$  since  $\nu_{\varphi_2(k)} \rightarrow \alpha$  weak- $*$ .

c) At each  $\nu_{\varphi_2(n)}$ , the sequence of functions  $F_0, F_1, \dots, F_n, \dots$  increases to  $\widetilde{C}_\infty(\alpha)$ . Indeed, by (5.1), we have

$$\forall k \leq n-1, \quad \widetilde{C}_k(\nu_{\varphi_2(n)}) \leq \widetilde{C}_{k+1}(\nu_{\varphi_2(n)}), \quad \text{and} \quad \widetilde{C}_n(\nu_{\varphi_2(n)}) \leq \widetilde{C}_n(\alpha) \leq \widetilde{C}_\infty(\alpha),$$

where the next-to-last inequality uses that  $\nu_{\varphi_2(n)}$  is an optimal prediction measure.

d) At  $\alpha$ , the sequence of functions  $F_0, F_1, \dots, F_n, \dots$  also increases to  $\widetilde{C}_\infty(\alpha)$ . This is a consequence of (5.1) and (5.6).

Hence, from Dini's theorem, we may conclude that the convergence is uniform which implies that  $\widetilde{C}_n(\nu_{\varphi_2(n)}) \rightarrow \widetilde{C}_\infty(\alpha)$  and thus also (5.3).  $\square$

*Proof of Theorem 1.1.* Let  $\{\nu_{\varphi_1(n)}\}_n$  be a subsequence of  $\{\nu_n\}_n$  which converges weak- $*$  to a probability measure  $\alpha$ . From Lemma 5.4, there exists a subsequence  $\{\nu_{\varphi_2(n)}\}_n$  of  $\{\nu_{\varphi_1(n)}\}_n$  such that, as  $n$  tends to infinity,

$$\widetilde{B}_{\varphi_2(n)}(\nu_{\varphi_2(n)}, z_0) \rightarrow \widetilde{B}_\infty(\alpha, z_0).$$

By definition of the OPM's,

$$\forall \mu \in \mathcal{M}(\Gamma), \quad B_{\varphi_2(n)}(\nu_{\varphi_2(n)}, z_0) \leq B_{\varphi_2(n)}(\mu, z_0); \quad \text{hence} \quad \widetilde{B}_{\varphi_2(n)}(\nu_{\varphi_2(n)}, z_0) \leq \widetilde{B}_{\varphi_2(n)}(\mu, z_0).$$

Letting  $n$  tend to infinity, we get  $\widetilde{B}_\infty(\alpha, z_0) \leq \widetilde{B}_\infty(\mu, z_0)$  which shows that  $\alpha$  minimizes  $\widetilde{B}_\infty(\mu, z_0)$  over  $\mu \in \mathcal{M}(\Gamma)$ , or equivalently, maximizes

$$\widetilde{\lambda}_\infty(\mu, z_0) = \liminf_{n \rightarrow \infty} \widetilde{\lambda}_n(\mu, z_0)$$

over measures  $\mu \in \mathcal{M}(\Gamma)$ . By Corollary 4.7, this is equivalent to the fact that  $\Phi_*\alpha$  maximizes  $\lambda_\infty(\nu, 1/\overline{\Phi(z)})$  over measures  $\nu \in \mathcal{M}(\mathbb{T})$ . Finally, Theorem 3.2 shows that

$$\Phi_*\alpha = \widehat{\delta}_{1/\overline{\Phi(z_0)}} = \widehat{\delta}_{\Phi(z_0)},$$

where the balayage is onto  $\mathbb{T}$ . By conformal invariance of the balayage, we obtain that  $\alpha$  equals  $\widehat{\delta}_{z_0}$ , the balayage of  $\delta_{z_0}$  onto  $\Gamma$ .  $\square$

We end with a discussion of a related asymptotic problem. For a connected, simply connected, compact subset  $K$  of  $\mathbb{C}$  we recall from (1.4) that for  $z_0 \in \Omega$ ,

$$B_n(\nu_n, z_0) = \sup_{p \in \mathcal{P}_n} \frac{|p(z_0)|^2}{\|p\|_K^2} \leq e^{2ng_\Omega(z_0)} = |\Phi(z_0)|^{2n}.$$

In fact, from the first equality together with (1.5) it follows that

$$\lim_{n \rightarrow \infty} \frac{B_n(\nu_n, z_0)^{1/2n}}{|\Phi(z_0)|} = \lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0)^{1/2n} = 1.$$

There is the deeper question as to whether the limit of the sequence  $\{\widetilde{B}_n(\nu_n, z_0)\}_n$  – without the  $1/2n$  power – exists. Clearly

$$0 \leq \liminf_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0) \leq \limsup_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0) \leq 1.$$

1. For the case of the unit circle, since

$$B_n(\nu_n, z_0) = \sup_{p \in \mathcal{P}_n} \frac{|p(z_0)|^2}{\|p\|_K^2} = |z_0|^{2n} = |\Phi(z_0)|^{2n},$$

recall (3.1) and (3.6), we have  $\widetilde{B}_n(\nu_n, z_0) = 1$  for all  $n$ .

2. As a corollary of Lemma 5.4 and Theorem 1.1, it follows that *for a  $C(2, \alpha)$  curve  $\Gamma$ , we have*

$$\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0) = 1 \tag{5.7}$$

for all  $z_0 \in \Omega$ .

3. For the interval  $[-1, 1]$ , the existence of this limit for  $z_0 \notin [-1, 1]$  was shown by Yuditskii [14] and Peherstorfer [7]; their proofs are very technical. Writing  $\psi(z) := z + \sqrt{z^2 - 1}$  for the conformal map from  $\mathbb{C} \setminus [-1, 1]$  onto  $\mathbb{C} \setminus \mathbb{D}$ , we have  $g_\Omega(z) = \log |\psi(z)|$ . Two special cases are more easily computed. First, for  $x \in \mathbb{R} \setminus [-1, 1]$ , the polynomial  $p_n$  in (2.2) is the Chebyshev polynomial

$$T_n(z) = \frac{1}{2} ((\psi(z))^n + (\psi(z))^{-n}).$$

Thus for such  $x$ , from (2.1),

$$\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, x) = \lim_{n \rightarrow \infty} \frac{1}{2} \frac{|(\psi(x))^n + (\psi(x))^{-n}|}{|\psi(x)|^n} = \frac{1}{2}.$$

Next, for  $z = ia$ ,  $a \in \mathbb{R}$ ,  $|a| > 1$ , from [2]

$$|p_n(ia)| = \sqrt{a^2 + 1} [|a| + \sqrt{a^2 + 1}]^{n-1}.$$

Since  $|\psi(ia)| = |a + \sqrt{a^2 + 1}|$ , we have, for  $a > 0$ ,

$$\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, ia) = \lim_{n \rightarrow \infty} \frac{\sqrt{a^2 + 1} [|a| + \sqrt{a^2 + 1}]^{n-1}}{|a + \sqrt{a^2 + 1}|^n} = \frac{\sqrt{a^2 + 1}}{a + \sqrt{a^2 + 1}}.$$

The results in [14] and [7] seem to indicate that, as with these special cases, for any  $z_0 \notin [-1, 1]$ ,

$$\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0) < 1. \tag{5.8}$$

4. For a circular arc  $A_\alpha := \{z \in \mathbb{C} : |z| = 1, |\arg z| \leq \alpha\}$ ,  $0 < \alpha < \pi$ , Eichinger [4] shows that  $\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0)$  exists for any  $z_0$  with  $|z_0| \neq 1$  and he calculates this limit.

Concerning 2., in particular, for the confocal ellipses

$$E_r := \{z \in \mathbb{C} : |z - 1| + |z + 1| = r + 1/r\}$$

(5.7) holds for all points  $z_0$  outside  $E_r$  for each  $r > 1$ . As  $r \rightarrow 1$ , these ellipses converge to the interval  $[-1, 1]$ , which, according to 3., fails to have this property. We know of no general results on existence of the limit of the sequence  $\{\widetilde{B}_n(\nu_n, z_0)\}_n$ .

**Remark 5.5.** For the interval  $[-1, 1]$ , or, more generally, for a real analytic arc  $\gamma$ , there is a problem with generalizing Lemma 4.6, Corollary 4.7, and the “weak monotonicity” lemma, Lemma 5.1. Indeed, if such results were true for  $[-1, 1]$ , then the proofs of Proposition 5.3 and hence Lemma 5.4 and Theorem 1.1 would be valid as well. However, equation (5.7) then gives

$$\lim_{n \rightarrow \infty} \widetilde{B}_n(\nu_n, z_0) = 1$$

which contradicts (5.8). Thus other ideas or techniques are required to deal with arcs.

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