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Mass-lumped Mixed Finite Elements Methods and the Abstract Gradient Scheme framework for elliptic problems

Cindy Guichard^{1,2}

¹Sorbonne Université, Université Paris Cité, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris, France

²Inria, Univ. Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, 59000 Lille, France

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Abstract

We provide a general framework for second-order elliptic problems, which includes a variety of boundary conditions. We show how one can apply mass-lumped mixed finite element to this problem, and we provide sufficient conditions for the convergence of such a method. In particular, we exhibit convergence results assuming two different type of assumptions: on one hand, we show convergence properties following the standard analysis of mixed finite elements. On the other hand, we provide conditions on one of the approximation space, which also lead to some convergence properties. We then formulate Abstract Gradient Discretization method (AGDM) based on these mass-lumped mixed finite elements, enabling us to apply this type of discretization to a variety of nonlinear problems. Finally, we illustrate our results by two examples. The first one is a second-order elliptic problem with homogeneous Neumann boundary conditions, discretized by Raviart-Thomas finite elements. We show on this example that mass-lumping leads to classical finite volume schemes. The second one is inspired by the elliptic part of a model of shallow water flows with dispersive terms. We apply on this example generalized operators, and we prove the convergence of the method used in the literature.

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1 Introduction

Following the pioneering works by Babuška [6], Mixed Finite Element (MFE) methods have been successfully analysed in the seminal Brezzi's paper [12], and extensively used for the approximation of important problems such as elasticity and electromagnetism [9] and also the Stokes problem [10], for which a natural physical framework is the saddle point problem when formulated with velocity and pressure unknowns. This work focuses on second-order elliptic problems, for which the use of MFE has also been enhanced by the introduction of the Raviart-Thomas finite element [28].

Let us give the following example of a second-order elliptic problem. We consider a function $p : \Omega \rightarrow \mathbb{R}$ solution to the problem below:

$$\begin{aligned}
 -\operatorname{div}(\Lambda \nabla p) &= r - \operatorname{div} \mathbf{F} \text{ in } \Omega, \\
 p &= 0 \text{ on } \Gamma_d, \\
 (\Lambda \nabla p - \mathbf{F}) \cdot \mathbf{n} &= g \text{ on } \Gamma_n,
 \end{aligned} \tag{1}$$

where we assume the following:

- Ω is an open bounded connected subset of \mathbb{R}^d ($d \in \mathbb{N}^*$) with Lipschitz boundary Γ such that $\Gamma = \Gamma_d \cup \Gamma_n$ with $|\Gamma_d| > 0$, (2a)
- $\mathbf{n}(\mathbf{x})$ denotes the outer normal to Γ for a.e. $\mathbf{x} \in \Gamma$, (2b)
- Λ is a measurable function from Ω to the set of $d \times d$ symmetric matrices (2c)
and there exist $\underline{\lambda}, \bar{\lambda} > 0$ such that, for a.e. $\mathbf{x} \in \Omega, \forall \zeta \in \mathbb{R}^d, \underline{\lambda}|\zeta|^2 \leq \zeta \cdot \Lambda(\mathbf{x})\zeta \leq \bar{\lambda}|\zeta|^2$,
- $r \in L^2(\Omega)$, $\mathbf{F} \in L^2(\Omega)^d$ and $g \in L^2(\Gamma)$. (2d)

Remark 1.1 (About the right-hand side). *The source term of Problem (1) recovers general cases. Indeed $f \in H^1(\Omega)'$ can be decomposed as $f = -\operatorname{div} \mathbf{F}$: it suffices to consider, owing to the Riesz Theorem, the element $w \in H^1(\Omega)$ solution of $\langle w, \cdot \rangle_{H^1(\Omega)} = \langle f, \cdot \rangle_{H^1(\Omega)', H^1(\Omega)}$, and to let $\mathbf{F} = \nabla w$.*

On the other hand, the Gradient Discretization Method (GDM) has been studied since several years as illustrated by [16]. This method gives a framework in order to simultaneously analyse the mathematical properties of a variety of numerical methods (since these methods satisfy ad-hoc properties) for the approximation of some elliptic or parabolic problems. As examples of such methods, we can cite conforming and nonconforming finite elements for instance. An extension of the GDM is the Abstract Gradient Discretisation Method (AGDM) which has been introduced in [17] with the goal to unify the treatment

of boundary conditions. The fact that MFE methods are encompassed in the Gradient Discretisation framework has been studied in [23] and in [16, Chapter 10] where this property has been shown in the case of an isotropic Poisson problem with homogeneous Dirichlet boundary conditions and right-hand side in $L^2(\Omega)$ and RTk mixed finite elements. A first originality of this work is to extend this result and to deal with a source term in $H^1(\Omega)'$ and not only in $L^2(\Omega)$, see Remark 1.1.

The GDM and AGDM use the weak form of the problem studied. This is why, in order to rewrite Problem (1) in a weak sense, we introduce the space $H_{0,\Gamma_d}^1(\Omega) = \{q \in H^1(\Omega), \gamma_{\Gamma_d} q = 0\}$ where γ_{Γ} denotes the trace mapping on the boundary Γ . We then define the space $H^G = \{(p, \gamma_{\Gamma} p), p \in H_{0,\Gamma_d}^1(\Omega)\}$. We then look for $(p, \widehat{p}) \in H^G$ such that

$$\begin{aligned} & \int_{\Omega} \Lambda(\mathbf{x}) \nabla p(\mathbf{x}) \cdot \nabla q(\mathbf{x}) \, d\mathbf{x} \\ &= \int_{\Omega} (r(\mathbf{x}) q(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \cdot \nabla q(\mathbf{x})) \, d\mathbf{x} + \int_{\Gamma_n} g(\mathbf{x}) \widehat{q}(\mathbf{x}) \, ds(\mathbf{x}), \quad \forall (q, \widehat{q}) \in H^G. \end{aligned} \quad (3)$$

Notice that the notations used here have been chosen in order to be consistent with (and even to serve as an example) the continuous abstract formalism used in Section 2. The proof that Problem (3) has one and only one solution is obtained by using the Lax-Milgram theorem. For some applications, a formulation in which the fluxes are directly computed and approximated can be useful, a such example is fluid flows in porous media as, for instance, in [2, 4]. In this work, we write the mixed formulation taking inspiration by [7]. We consider a pressure unknown which simultaneously reconstruct the value inside the domain and at the boundary, and we introduce the spaces L , \mathbf{L} and \mathbf{H}^D respectively defined by

$$L = L^2(\Omega) \times L^2(\Gamma_n), \quad \mathbf{L} = L^2(\Omega)^d \text{ and } \mathbf{H}^D = \{\mathbf{u} \in \mathbf{L}, D\mathbf{u} := (\operatorname{div}(\mathbf{u}), -\mathbf{u} \cdot \mathbf{n}) \in L\}. \quad (4)$$

Let us now define $G : H^G \rightarrow \mathbf{L}$ by $G(p, \widehat{p}) = \nabla p$. We have the following relation:

$$\forall (p, \widehat{p}) \in H^G, \quad \forall \mathbf{u} \in \mathbf{H}^D, \quad \langle (p, \widehat{p}), D\mathbf{u} \rangle_L + \langle G(p, \widehat{p}), \mathbf{u} \rangle_{\mathbf{L}} = 0,$$

where $\langle \cdot, \cdot \rangle_L$ and $\langle \cdot, \cdot \rangle_{\mathbf{L}}$ are the natural scalar products on L and \mathbf{L} . This enables us to consider the following mixed formulation: find $\bar{p} := (p, \widehat{p}) \in L$ and $\bar{\mathbf{u}} \in \mathbf{H}^D$ such that

$$\begin{aligned} \langle \Lambda^{-1} \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_{\mathbf{L}} - \langle \bar{p}, D\bar{\mathbf{v}} \rangle_L &= \langle \Lambda^{-1} \mathbf{F}, \bar{\mathbf{v}} \rangle_{\mathbf{L}}, & \forall \bar{\mathbf{v}} \in \mathbf{H}^D, \\ \langle \bar{q}, D\bar{\mathbf{u}} \rangle_L &= \langle f, \bar{q} \rangle_L, & \forall \bar{q} := (q, \widehat{q}) \in L, \end{aligned} \quad (5)$$

with $f = (r, g) \in L$. The fact that Problem (5) is equivalent to Problem (3) is stated in Theorem 2.2.

The application of mixed methods consist in selecting finite dimensional spaces $\mathbf{H}_h^D \subset \mathbf{H}^D$ and $L_h \subset L$ which satisfy Ladyzhenskaya–Babuška–Brezzi (LBB) conditions [10, Section 12.5], [12], [19] in order that the problem find $p_h \in L_h$ and $\mathbf{u}_h \in \mathbf{H}_h^D$, such that

$$\begin{aligned} \langle \Lambda^{-1} \mathbf{u}_h, \mathbf{v}_h \rangle_{\mathbf{L}} - \langle p_h, D\mathbf{v}_h \rangle_L &= \langle \Lambda^{-1} \mathbf{F}, \mathbf{v}_h \rangle_{\mathbf{L}}, & \forall \mathbf{v}_h \in \mathbf{H}_h^D, \\ \langle q_h, D\mathbf{u}_h \rangle_L &= \langle f, q_h \rangle_L, & \forall q_h \in L_h, \end{aligned} \quad (6)$$

has one and only one solution which converges to the solution $(\bar{p}, \bar{\mathbf{u}}) \in L \times \mathbf{H}^D$ of Problem (5). The numerical resolution of (6) leads to large linear systems, which can require a high time computing. Numerical strategies have been proposed to achieve lower computational costs. In [26], a review of some mass-lumping methods is proposed, interpreted as quadrature numerical methods. In [1], a mass-lumping method is used in order to compute the approximation by mixed finite element methods for the elliptic part of a dispersive model for shallow water flows. Thus, an originality of this work is to introduce and analyze the convergence of mixed finite element methods coupled with a mass-lumping operator. This approach of mass-lumping is different from [3, 5, 8] where the mass-lumping is algebraic (i.e. on the resulting linear system). With our strategy, we obtain convergence results under less regularity

assumptions (see Hypotheses (19a), (19b) and (19c) done in Theorems 3.5 and 3.7). On our introductory example, such a method consists in looking for $p_h \in L_h$ and $\mathbf{u}_h \in \mathbf{H}_h^D$, such that,

$$\begin{aligned} \langle \Lambda^{-1} \Pi_h \mathbf{u}_h, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} - \langle \bar{p}_h, D\mathbf{v}_h \rangle_L &= \langle \Lambda^{-1} \mathbf{F}, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}}, & \forall \mathbf{v}_h \in \mathbf{H}_h^D, \\ \langle q_h, D\mathbf{u}_h \rangle_{L+} &= \langle f, q_h \rangle_L, & \forall q_h \in L_h. \end{aligned} \quad (7)$$

Where in (7), we denote by $\Pi_h : \mathbf{H}_h^D \rightarrow \mathbf{L}$ an operator, which is the identity operator in the case of (6), but which can be chosen in some cases as a piecewise constant operator with disjoint supports. Then the matrix issued from the bilinear form $\langle \Lambda^{-1} \Pi_h \mathbf{u}_h, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}}$ can be diagonal (see the examples of Section 5).

In Section 3 we analyse the convergence properties of our mixed approach for an extended abstract problem. Indeed, another originality of our work is to consider a more restricted convergence sense to the method than the one which is expected in the framework of mixed finite element methods [12, 13, 24]. The authors prove that, for $\Pi_h = \text{Id}_{\mathbf{L}}$ (no mass-lumping), the convergence $p_h \rightarrow \bar{p}$ in L and $\mathbf{u}_h \rightarrow \bar{\mathbf{u}}$ in \mathbf{H}^D can only occur under full LBB conditions. In this work we show the convergence under two types of hypotheses:

- An equivalent LBB conditions adapted to our problem is established in section 3.2 leading to the convergence Theorem 3.5.
- A more regular approximation of the pressure is assumed in section 3.3 leading to the convergence Theorem 3.7.

In both cases, we obtain the convergence of $p_h \rightarrow \bar{p}$ in L and $\Pi_h \mathbf{u}_h \rightarrow \bar{\mathbf{u}}$ in \mathbf{L} . These convergence properties are then equivalent to the convergence properties obtained by a standard Finite Element Method (FEM) applied to Problem (3).

In Section 4, we show the equivalence between the mixed scheme (7) and an abstract gradient scheme [17, (2.6)] in the case of Problems (3) and (5). This enables us to define a discrete setting which can be used in more complex problems, including nonlinear terms for example, in the spirit of [16] and extending the results of [23].

The last Section 5 is devoted to two examples, one for each case of convergence. The first one is inspired from [7], i.e. a mixed problem with homogeneous Neumann boundary condition and RT0 approximation. We propose an original proof of the *inf-sup* condition based on results for finite volume schemes. We prove that the assumptions of Theorem 3.5 are fulfilled to obtain the convergence. The second example is inspired from [1], where mass-lumped mixed finite element methods are used. This case has been studied in [18, Chapter 2] in the AGDM framework but using another strategy to obtain the convergence. Here we prove that the assumptions of Theorem 3.7 are fulfilled to obtain the convergence.

Appendices contain technical results. We can mention Appendix B which establishes a link between our condition and an LBB condition of the literature.

2 Continuous abstract framework for an elliptic problem

2.1 Spaces and operators

Let us first introduce the notations and assumptions used for writing the studied elliptic problem. The notations are close to the ones used in [17] in the case of Hilbert spaces and compatible with the ones

used in the Introduction:

- \mathbf{L} and L are two separable Hilbert spaces. (8a)

- The subspace $H^G \subset L$ is dense in L . (8b)

- The linear operator $G : H^G \rightarrow \mathbf{L}$ is such that its graph $\{(\bar{p}, G\bar{p}), \bar{p} \in H^G\}$ is closed in $L \times \mathbf{L}$. (8c)

As a consequence H^G endowed with the scalar product $\langle \bar{p}, \bar{q} \rangle_{H^G} = \langle \bar{p}, \bar{q} \rangle_L + \langle G\bar{p}, G\bar{q} \rangle_{\mathbf{L}}$ is a Hilbert space.

- V is a closed subspace of L and V^\perp is its orthogonal in L . (8d)

For any $q \in L$, we write $q = q^V + q^{\perp V}$, where q^V is the orthogonal projection of q on V and $q^{\perp V}$ is the orthogonal projection of q on V^\perp .

The hypothesis that L and \mathbf{L} are separable provide the sequential weak compactness properties which are used in the proof of convergence theorems 3.5 and 3.7. As in [17] and in the introduction, we define the space \mathbf{H}^D by

$$\mathbf{H}^D = \{\bar{v} \in \mathbf{L}, \exists \bar{w} \in L, \forall \bar{q} \in H^G, \langle \bar{v}, G\bar{q} \rangle_{\mathbf{L}} + \langle \bar{w}, \bar{q} \rangle_L = 0\}. \quad (9)$$

Since, for any $\bar{v} \in \mathbf{H}^D$, there exists at most one $\bar{w} \in L$ satisfying the preceding relation (thanks to (8b)), we denote it by $D\bar{v} := \bar{w}$, where $D : \mathbf{H}^D \rightarrow L$ is a linear operator. Then \mathbf{H}^D is a Hilbert space endowed with the scalar product $\langle \bar{u}, \bar{v} \rangle_{\mathbf{H}^D} = \langle \bar{u}, \bar{v} \rangle_{\mathbf{L}} + \langle D\bar{u}, D\bar{v} \rangle_L$, and \mathbf{H}^D is dense in \mathbf{L} . Note that, as noticed in [17], we have as well

$$H^G = \{\bar{p} \in L, \exists \bar{u} \in \mathbf{L}, \forall \bar{v} \in \mathbf{H}^D, \langle D\bar{v}, \bar{p} \rangle_L + \langle \bar{v}, \bar{u} \rangle_{\mathbf{L}} = 0\},$$

and then $\bar{u} = G\bar{p}$. This allows us to generalize the Green formula,

$$\forall \bar{q} \in H^G, \forall \bar{v} \in \mathbf{H}^D, \langle \bar{v}, G\bar{q} \rangle_{\mathbf{L}} + \langle D\bar{v}, \bar{q} \rangle_L = 0. \quad (10)$$

2.2 Standard and mixed form of an elliptic problem

Under Assumptions (8), we consider the following data for an elliptic problem.

- The mapping $a : L \rightarrow L$ is linear and continuous with norm bounded by $\bar{\alpha}$ and V -coercive in (11a)

the sense that $\forall \bar{q} \in L, \underline{\alpha} \|\bar{q}^V\|_L^2 \leq \langle a(\bar{q}), \bar{q} \rangle_L$.

- The mapping $\mathbf{A} : \mathbf{L} \rightarrow \mathbf{L}$ which is linear, continuous with norm bounded by $\bar{\alpha}$, and coercive (11b)

in the sense there exists $\underline{\alpha} > 0$ such that for all $\bar{v} \in \mathbf{L}, \underline{\alpha} \|\bar{v}\|_{\mathbf{L}}^2 \leq \langle \mathbf{A}(\bar{v}), \bar{v} \rangle_{\mathbf{L}}$.

- The functions $f \in L$ and $\mathbf{F} \in \mathbf{L}$. (11c)

We consider the following weak problem:

$$\text{Find } \bar{p} \in H^G \text{ such that, } \langle \mathbf{A}(G\bar{p}), G\bar{q} \rangle_{\mathbf{L}} + \langle a(\bar{p}), \bar{q} \rangle_L = \langle f, \bar{q} \rangle_L + \langle \mathbf{F}, G\bar{q} \rangle_{\mathbf{L}}, \forall \bar{q} \in H^G. \quad (12)$$

Thanks to the above definitions of \mathbf{H}^D and D , see (9)-(10), Problem (12) is then equivalent to:

$$\begin{aligned} \text{Find } \bar{p} \in H^G \text{ such that } \mathbf{A}(G\bar{p}) - \mathbf{F} \in \mathbf{H}^D \text{ and} \\ -D(\mathbf{A}(G\bar{p}) - \mathbf{F}) + a(\bar{p}) = f \text{ in } L. \end{aligned} \quad (13)$$

Let $\bar{p} \in H^G$ the solution of (13), we denote $\bar{u} = -(\mathbf{A}(G\bar{p}) - \mathbf{F}) \in \mathbf{H}^D$. We can write a mixed version of (13):

$$\begin{aligned} \text{Find } \bar{p} \in H^G \text{ and } \bar{u} \in \mathbf{H}^D \text{ such that} \\ \mathbf{A}^{-1}\bar{u} + G\bar{p} = \mathbf{A}^{-1}\mathbf{F} \text{ in } \mathbf{L}, \\ D\bar{u} + a(\bar{p}) = f \text{ in } L. \end{aligned} \quad (14)$$

Once again, thanks to (9)-(10), we can rewrite the problem in the following weak mixed formulation:

$$\begin{aligned} & \text{Find } \bar{p} \in L \text{ and } \bar{\mathbf{u}} \in \mathbf{H}^D \text{ such that,} \\ & \langle \mathbf{A}^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_L - \langle \bar{p}, D\bar{\mathbf{v}} \rangle_L = \langle \mathbf{A}^{-1}\mathbf{F}, \bar{\mathbf{v}} \rangle_L, \quad \forall \bar{\mathbf{v}} \in \mathbf{H}^D, \\ & \langle D\bar{\mathbf{u}}, \bar{q} \rangle_L + \langle a(\bar{p}), \bar{q} \rangle_L = \langle f, \bar{q} \rangle_L, \quad \forall \bar{q} \in L. \end{aligned} \quad (15)$$

The introduction of the space V in assumptions (8) and of the linear mapping a in (11) enable us to include in this framework a variety of boundary conditions, as shown in [17]. In the case of Problem (5) in the introduction, which involves Dirichlet boundary conditions on a part of the boundary, one can let $V = \{0\}$ and $a \equiv 0$. On the contrary, in the example handled in Section 5.1, the nonhomogeneous Neumann boundary condition is handled owing to a suitable choice for V and a .

In the literature the existence and the uniqueness of the solution of a mixed problem is mainly based on a so-called *inf-sup condition*. The aim of the following result is to prove the equivalence between (12) and (15), and to formulate the *inf-sup condition* which is adapted to our problem. We need a preliminary result on the properties of the linear operator \mathbf{A}^{-1} , as detailed in the following lemma.

Lemma 2.1 (Invertibility of \mathbf{A}).

Under Assumptions (8) and (11b) on the mapping $\mathbf{A} : \mathbf{L} \rightarrow \mathbf{L}$ (linearity, continuity, coercivity), then \mathbf{A} is invertible and \mathbf{A}^{-1} is also linear, continuous with norm bounded by $\underline{\alpha}^{-1}$, and coercive in the sense that for all $\bar{\mathbf{v}} \in \mathbf{L}$, $(\underline{\alpha}/\bar{\alpha}^2)\|\bar{\mathbf{v}}\|_{\mathbf{L}}^2 \leq \langle \mathbf{A}^{-1}(\bar{\mathbf{v}}), \bar{\mathbf{v}} \rangle_L$.

Proof. First of all, we have to prove the bijectivity of \mathbf{A} . Let $a_{\mathbf{A}}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \langle \mathbf{A}(\bar{\mathbf{u}}), \bar{\mathbf{v}} \rangle_L$ which is a bilinear form on $\mathbf{L} \times \mathbf{L}$, continue and coercive (due to the ones of \mathbf{A}). And, for an arbitrary $\bar{\mathbf{w}} \in \mathbf{L}$, let $l_{\bar{\mathbf{w}}}(\bar{\mathbf{v}}) = \langle \bar{\mathbf{w}}, \bar{\mathbf{v}} \rangle_L$ which is linear and continue. Thanks to the Lax-Milgram theorem there exists one and only one $\bar{\mathbf{u}} \in \mathbf{L}$ such that $a_{\mathbf{A}}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = l_{\bar{\mathbf{w}}}(\bar{\mathbf{v}})$ for any $\bar{\mathbf{v}} \in \mathbf{L}$ which is equivalent to write $\mathbf{A}(\bar{\mathbf{u}}) = \bar{\mathbf{w}}$. Let $\bar{\mathbf{v}} \in \ker(\mathbf{A})$, thus we can write, $\underline{\alpha}\|\bar{\mathbf{v}}\|_{\mathbf{L}}^2 \leq \langle \mathbf{A}(\bar{\mathbf{v}}), \bar{\mathbf{v}} \rangle_L = 0$. Thus $\|\bar{\mathbf{v}}\|_{\mathbf{L}} = 0 \Rightarrow \bar{\mathbf{v}} = \mathbf{0}_L$. Now we study the continuity and the coercivity of \mathbf{A}^{-1} .

- Let $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbf{L}$ such that $\mathbf{A}(\bar{\mathbf{u}}) = \bar{\mathbf{v}}$. The coercivity of \mathbf{A} allows to write $\underline{\alpha}\|\bar{\mathbf{u}}\|_{\mathbf{L}}^2 \leq \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle_L$ which implies, thanks to the Cauchy-Schwarz inequality, $\underline{\alpha}\|\mathbf{A}^{-1}(\bar{\mathbf{v}})\|_{\mathbf{L}}^2 \leq \langle \bar{\mathbf{v}}, \mathbf{A}^{-1}(\bar{\mathbf{v}}) \rangle_L \leq \|\bar{\mathbf{v}}\|_{\mathbf{L}}\|\mathbf{A}^{-1}(\bar{\mathbf{v}})\|_{\mathbf{L}}$ and thus the continuity of \mathbf{A}^{-1} .
- Let $\bar{\mathbf{v}} \in \mathbf{L}$. Owing to the coercivity of \mathbf{A} we can write $\langle \mathbf{A}^{-1}(\bar{\mathbf{v}}), \bar{\mathbf{v}} \rangle_L = \langle \mathbf{A}^{-1}(\bar{\mathbf{v}}), \mathbf{A}(\mathbf{A}^{-1}(\bar{\mathbf{v}})) \rangle_L = \langle \mathbf{A}(\mathbf{A}^{-1}(\bar{\mathbf{v}})), \mathbf{A}^{-1}(\bar{\mathbf{v}}) \rangle_L \geq \underline{\alpha}\|\mathbf{A}^{-1}(\bar{\mathbf{v}})\|_{\mathbf{L}}^2$. Moreover thanks to the continuity of \mathbf{A} we can write $\|\mathbf{A}(\mathbf{A}^{-1}(\bar{\mathbf{v}}))\|_{\mathbf{L}} \leq \bar{\alpha}\|\mathbf{A}^{-1}(\bar{\mathbf{v}})\|_{\mathbf{L}}$. Thus we recover the expected result.

□

Theorem 2.2 (Existence and uniqueness for the abstract weak problems).

Under Assumptions (8) and (11), the following assertions are equivalent:

- (i) $L = \text{Im}(D) + V$,
- (ii) Problem (12) has one and only solution, denoted by \bar{p} , and there exists $C > 0$ independent of f and \mathbf{F} , such that $\|\bar{p}\|_{H^G} \leq C(\|f\|_L + \|\mathbf{F}\|_L)$. Then $\bar{\mathbf{u}} = -(\mathbf{A}(G\bar{p}) - \mathbf{F}) \in \mathbf{H}^D$ and $(\bar{p}, \bar{\mathbf{u}})$ is solution to (15).
- (iii) The following *inf-sup condition* holds:

$$\exists \beta > 0 \text{ s.t. } \forall \bar{q} \in L, \quad \sup_{\bar{\mathbf{v}} \in \mathbf{H}^D, \|\bar{\mathbf{v}}\|_{\mathbf{H}^D} = 1} \langle \bar{q}^{\perp V}, D\bar{\mathbf{v}} \rangle_L \geq \beta \|\bar{q}^{\perp V}\|_L. \quad (16)$$

- (iv) Problem (15) has one and only one solution $(\bar{p}, \bar{\mathbf{u}})$ and \bar{p} is solution to (12).

Proof. (i) \Rightarrow (ii) Applying [17, Theorem 3.2] in the case of a Hilbert space L , we get that the assumption $L = \text{Im}(\mathbf{D}) + V$ implies that the norms $\|\bar{q}\|_{H^G} = (\|\bar{q}\|_L^2 + \|\mathbf{G}\bar{q}\|_L^2)^{\frac{1}{2}}$ and $\|\bar{q}\|_{H^G, V} = (\|\bar{q}^V\|_L^2 + \|\mathbf{G}\bar{q}\|_L^2)^{\frac{1}{2}}$ are equivalent for any $\bar{q} \in H^G$.

We introduce the bilinear form $a_{LM} : H^G \times H^G \rightarrow \mathbb{R}$ such that $a_{LM}(\bar{p}, \bar{q}) = \langle \mathbf{A}(\mathbf{G}\bar{p}), \mathbf{G}\bar{q} \rangle_L + \langle a(\bar{p}), \bar{q} \rangle_L$ and the linear one $l_{LM} : H^G \rightarrow \mathbb{R}$ such that $l_{LM}(\bar{q}) = \langle f, \bar{q} \rangle_L + \langle \mathbf{F}, \mathbf{G}\bar{q} \rangle_L$. Thanks to the assumptions on the operators \mathbf{A} and a , we obtain that $a_{LM}(\bar{p}, \bar{p}) \geq \underline{\alpha} \|\bar{p}\|_{H^G, V}^2$ and, due to the equivalence of the norms, the coercivity of a_{LM} . Using Cauchy-Schwarz inequality and the continuity of the operators \mathbf{A} and a , we have,

$$|a_{LM}(\bar{p}, \bar{q})| \leq \bar{\alpha} (\|\mathbf{G}\bar{p}\|_L \|\mathbf{G}\bar{q}\|_L + \|\bar{p}\|_L \|\bar{q}\|_L) \leq \bar{\alpha} \|\bar{p}\|_{H^G} \|\bar{q}\|_{H^G},$$

which gives the continuity of a_{LM} . Once again the Cauchy-Schwarz inequality leads to the continuity of l_{LM} : $|l_{LM}(\bar{q})| \leq (\|f\|_L^2 + \|\mathbf{F}\|_L^2)^{\frac{1}{2}} \|\bar{q}\|_{H^G}$. Finally, the Lax-Milgram theorem gives us the existence and the uniqueness of the solution $\bar{p} \in H^G$ to (12). The estimation given is obtain by taking $\bar{q} = \bar{p}$ and by using same arguments as above.

Writing that $\langle \mathbf{A}(\mathbf{G}\bar{p}) - \mathbf{F}, \mathbf{G}\bar{q} \rangle_L = \langle f - a(\bar{p}), \bar{q} \rangle_L$, we get by definition of \mathbf{D} that $\bar{\mathbf{u}} = -(\mathbf{A}(\mathbf{G}\bar{p}) - \mathbf{F}) \in \mathbf{H}^D$ and $\mathbf{D}\bar{\mathbf{u}} = f - a(\bar{p})$. We then have

$$\begin{aligned} \mathbf{A}^{-1}\bar{\mathbf{u}} + \mathbf{G}\bar{p} &= \mathbf{A}^{-1}\mathbf{F} \text{ in } L, \\ \mathbf{D}\bar{\mathbf{u}} + a(\bar{p}) &= f \text{ in } L. \end{aligned}$$

This yields, for $\bar{\mathbf{v}} \in \mathbf{H}^D$ and $\bar{q} \in L$,

$$\begin{aligned} \langle \mathbf{A}^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_L + \langle \mathbf{G}\bar{p}, \bar{\mathbf{v}} \rangle_L &= \langle \mathbf{A}^{-1}\mathbf{F}, \bar{\mathbf{v}} \rangle_L, \\ \langle \mathbf{D}\bar{\mathbf{u}}, \bar{q} \rangle_L + \langle a(\bar{p}), \bar{q} \rangle_L &= \langle f, \bar{q} \rangle_L. \end{aligned}$$

We then conclude that $(\bar{p}, \bar{\mathbf{u}})$ is solution to (15) since $\langle \mathbf{G}\bar{p}, \bar{\mathbf{v}} \rangle_L = -\langle \bar{p}, \mathbf{D}\bar{\mathbf{v}} \rangle_L$ by (10).

(ii) \Rightarrow (iii) Let $\bar{q} \in L$. We apply the fact that (12) has one and only one solution in the case where $\mathbf{A} = \text{Id}$, $a = \mathcal{P}_V$ (the orthogonal projection on V), $\mathbf{F} = 0$ and $f = \bar{q}^{\perp V}$. Note that, for any element $\bar{q} \in H^G$, the notation \bar{q}^V is a shortcut for $\mathcal{P}_V(\bar{q})$. Thus let $\bar{p} \in H^G$ such that we have,

$$\forall \tilde{q} \in H^G, \langle \mathbf{G}\bar{p}, \mathbf{G}\tilde{q} \rangle_L + \langle \bar{p}^V, \tilde{q} \rangle_L = \langle \bar{q}^{\perp V}, \tilde{q} \rangle_L,$$

which yields, by (10), $-\mathbf{D}(\mathbf{G}\bar{p}) = \bar{q}^{\perp V} - \bar{p}^V$. From (ii) we have $\|\bar{p}\|_{H^G} \leq C \|\bar{q}^{\perp V}\|_L$, which shows that $\|\mathbf{D}(\mathbf{G}\bar{p})\|_L \leq (C+1) \|\bar{q}^{\perp V}\|_L$, and $\|\mathbf{G}\bar{p}\|_L \leq C \|\bar{q}^{\perp V}\|_L$. Hence $\|\mathbf{G}\bar{p}\|_{\mathbf{H}^D} \leq (2C+1) \|\bar{q}^{\perp V}\|_L$. Letting $\tilde{q} = \bar{q}^{\perp V}$, we deduce $-\langle \mathbf{D}(\mathbf{G}\bar{p}), \bar{q}^{\perp V} \rangle_L = \|\bar{q}^{\perp V}\|_L^2$. If $\|\bar{q}^{\perp V}\|_L \neq 0$, we get that $\mathbf{u} = -\frac{\mathbf{G}\bar{p}}{\|\mathbf{G}\bar{p}\|_{\mathbf{H}^D}}$ is such that $\|\mathbf{u}\|_{\mathbf{H}^D} \leq 1$ and

$$\langle \mathbf{D}\mathbf{u}, \bar{q}^{\perp V} \rangle_L \geq \frac{1}{2C+1} \|\bar{q}^{\perp V}\|_L.$$

Hence (16) holds with $\beta = \frac{1}{2C+1}$.

(iii) \Rightarrow (iv) In order to apply the Theorem A.1 in appendix, we take $\widetilde{\mathbf{H}}^D := \mathbf{H}^D$, $\widetilde{L} := L$, still $\mathcal{H} = \mathbf{H}^D \times L$ and we define $\mathbb{A}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) := \langle \mathbf{A}^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_L$ and $\mathbb{a}(\bar{p}, \bar{q}) := \langle a(\bar{p}), \bar{q} \rangle_L$. Then Hypotheses (63) are satisfied, due to Assumptions (11a) and (11b), and Hypothesis (64) is given by (iii). With these definitions, we obtain, for any $(\bar{p}, \bar{\mathbf{u}}), (\bar{q}, \bar{\mathbf{v}}) \in \mathcal{H} \times \mathcal{H}$, the following bilinear form

$$E((\bar{p}, \bar{\mathbf{u}}), (\bar{q}, \bar{\mathbf{v}})) := \langle \mathbf{A}^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle_L - \langle \bar{p}, \mathbf{D}\bar{\mathbf{v}} \rangle_L + \langle \bar{q}, \mathbf{D}\bar{\mathbf{u}} \rangle_L + \langle a(\bar{p}), \bar{q} \rangle_L.$$

We notice that any solution $(\bar{p}, \bar{\mathbf{u}})$ to (15) is such that, for any $(\bar{q}, \bar{\mathbf{v}}) \in \mathcal{H}$,

$$E((\bar{p}, \bar{\mathbf{u}}), (\bar{q}, \bar{\mathbf{v}})) = \langle \mathbf{A}^{-1}\mathbf{F}, \bar{\mathbf{v}} \rangle_L + \langle f, \bar{q} \rangle_L. \quad (17)$$

We therefore apply Theorem A.1, noticing that in this case $\mathcal{P}_{\widetilde{L}} = \text{Id}_L$. Then thanks to [12, Corollary 0.1], we can say that there exists one and only one $(\bar{p}, \bar{\mathbf{u}}) \in \mathcal{H}$ such that (17) holds and therefore (15)

holds. From the first equation of (15) and using (10), we get that $\bar{p} \in H^G$ and that $\bar{\mathbf{u}} = -(\mathbf{A}(G\bar{p}) - \mathbf{F})$. The second equation shows that $\bar{\mathbf{u}} \in \mathbf{H}^D$ and considering $\bar{q} \in H^G$ in this equation yields (12).

(iv) \Rightarrow (i) We consider, for a given $\bar{q} \in L$, Problem (12) with $\mathbf{A} = \text{Id}$, $a = \mathcal{P}_V$, $\mathbf{F} = 0$, $f = \bar{q}$. Then the solution $(\bar{p}, \bar{\mathbf{u}})$ to (15) in this case is such that $\bar{q} = D\bar{\mathbf{u}} + \bar{p}^V$ and thus $L = \text{Im}(D) + V$. \square

Remark 2.3 (Link with the literature). *We notice that the mixed problem (15) contains the additional term $\langle a(\bar{p}), \bar{q} \rangle_L$ compared to the usual literature on mixed problems, see for instance [12, Problem (1.1)] or [19, Problem (80)]. It is due to this additional term that the inf-sup condition (16) is only written in V^\perp instead of L . If $V = \{0\}$ and $a \equiv 0$, then $V^\perp = L$ and this inf-sup condition is the standard one (see [12, Condition (2.2)] or [19, Condition (85)]). We also notice that the bilinear form $\langle \mathbf{A}^{-1}\bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_L \geq \frac{\alpha}{\alpha^2} \|\bar{\mathbf{u}}\|_{\mathbf{H}^D}^2$ for any function $\bar{\mathbf{u}}$ such that $D\bar{\mathbf{u}} = 0$, which means that Conditions [12, (2.4) – (2.5)] or [19, (88) – (89)] are satisfied. So in this particular case $V = \{0\}$, we find the three standard conditions which ensure the existence and uniqueness of the mixed problem, see [12, Proposition 2.1] or [19, Lemma 5.2].*

3 Mass-lumped mixed finite element approximation

This section is devoted to the study of a mixed Finite Element Method (FEM), possibly mass-lumped, for the approximation of Problem (15).

3.1 Scheme, estimate and general properties

Considering two finite dimensional spaces $L_h \subset L$ and $\mathbf{H}_h^D \subset \mathbf{H}^D$, we introduce the following mass-lumped mixed finite element scheme:

$$\begin{aligned} &\text{Find } p_h \in L_h \text{ and } \mathbf{u}_h \in \mathbf{H}_h^D \text{ such that,} \\ &\langle \mathbf{A}^{-1}\Pi_h \mathbf{u}_h, \Pi_h \mathbf{v}_h \rangle_L - \langle p_h, D\mathbf{v}_h \rangle_L = \langle \mathbf{A}^{-1}\mathbf{F}, \Pi_h \mathbf{v}_h \rangle_L, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D, \\ &\langle D\mathbf{u}_h, q_h \rangle_L + \langle a(p_h), q_h \rangle_L = \langle f, q_h \rangle_L, \quad \forall q_h \in L_h, \end{aligned} \quad (18)$$

where $\Pi_h : \mathbf{H}_h^D \rightarrow \mathbf{L}$ is a linear mapping which plays an important role in the numerical method. In standard mixed finite elements, we let $\Pi_h = \text{Id}_L$. Let us observe that the mass matrix resulting from the bilinear form $\langle \mathbf{A}^{-1}\Pi_h \mathbf{u}_h, \Pi_h \mathbf{v}_h \rangle_L$ can lead to diagonal matrices, if Π_h is a piecewise constant reconstruction with disjoint supports.

For a sequence of strictly positive reals $(h_n)_{n \in \mathbb{N}}$ such that $h_n \rightarrow 0$ as $n \rightarrow +\infty$, and, for any $n \in \mathbb{N}$, we consider finite dimensional spaces $L_{h_n} \subset L$, $\mathbf{H}_{h_n}^D \subset \mathbf{H}^D$ and a linear mapping $\Pi_{h_n} : \mathbf{H}_{h_n}^D \rightarrow \mathbf{L}$, such that the following properties hold.

- The operators Π_{h_n} preserve the \mathbf{L} -norm in the following sense: there exists $\rho > 0$, independent of n , with

$$\forall n \in \mathbb{N}, \forall \mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D, \quad \frac{1}{\rho} \|\mathbf{v}_{h_n}\|_{\mathbf{L}} \leq \|\Pi_{h_n} \mathbf{v}_{h_n}\|_{\mathbf{L}} \leq \rho \|\mathbf{v}_{h_n}\|_{\mathbf{L}}. \quad (19a)$$

- The following condition is a conformity condition

$$\forall (\mathbf{v}_{h_n})_{n \in \mathbb{N}} \text{ with } \mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D, \left((\|\mathbf{v}_{h_n}\|_{\mathbf{L}})_{n \in \mathbb{N}} \text{ bounded} \right) \Rightarrow \left(\forall \bar{\mathbf{v}} \in \mathbf{L}, \lim_{n \rightarrow \infty} \langle \mathbf{v}_{h_n} - \Pi_{h_n} \mathbf{v}_{h_n}, \bar{\mathbf{v}} \rangle_{\mathbf{L}} = 0 \right), \quad (19b)$$

which ensures that \mathbf{v}_{h_n} and $\Pi_{h_n} \mathbf{v}_{h_n}$ have the same weak limit (if the limit exists).

- The two conditions below are interpolation conditions

$$\forall \bar{\mathbf{v}} \in \mathbf{H}^D, \lim_{n \rightarrow \infty} \inf_{\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D} (\|\bar{\mathbf{v}} - \Pi_{h_n} \mathbf{v}_{h_n}\|_{\mathbf{L}}^2 + \|D\bar{\mathbf{v}} - D\mathbf{v}_{h_n}\|_{\mathbf{L}}^2) = 0, \quad (19c)$$

and

$$\forall \bar{q} \in L, \lim_{n \rightarrow \infty} \inf_{q_{h_n} \in L_{h_n}} \|\bar{q} - q_{h_n}\|_L = 0. \quad (19d)$$

- For any $q_{h_n} \in L_{h_n}$, we write $q_{h_n} = q_{h_n}^{\perp V} + q_{h_n}^V$, with $q_{h_n}^{\perp V} \in V^\perp$ and $q_{h_n}^V \in V$. At the discrete level the *inf-sup condition* which is assumed is that there exists $\beta > 0$, independent of n , such that

$$\forall n \in \mathbb{N}, \forall q_{h_n} \in L_{h_n}, \sup_{\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D, \|\mathbf{v}_{h_n}\|_{\mathbf{H}^D} = 1} \langle q_{h_n}^{\perp V}, D\mathbf{v}_{h_n} \rangle_L \geq \beta \|q_{h_n}^{\perp V}\|_L. \quad (19e)$$

The following lemma shows that the continuous *inf-sup* inequality is a consequence of the above assumptions.

Lemma 3.1. *Under Assumptions (8) and (19d), the discrete inf-sup condition (19e) implies the continuous inf-sup one (16).*

Proof. Let $\bar{q} \in L$ with $\bar{q}^{\perp V} \neq 0$, and, thanks to (19d), let $(q_{h_n})_{n \in \mathbb{N}}$ such that $\|\bar{q} - q_{h_n}\|_L \rightarrow 0$ as $n \rightarrow +\infty$. For any $n \in \mathbb{N}$, let $\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D$ with $\|\mathbf{v}_{h_n}\|_{\mathbf{H}^D} = 1$ such that, with (19e),

$$\langle q_{h_n}^{\perp V}, D\mathbf{v}_{h_n} \rangle_L \geq \beta \|q_{h_n}^{\perp V}\|_L. \quad (20)$$

Since the sequence $(\mathbf{v}_{h_n})_{n \in \mathbb{N}}$ is bounded, one can extract a subsequence, again denoted $(\mathbf{v}_{h_n})_{n \in \mathbb{N}}$, and $\bar{\mathbf{v}} \in \mathbf{H}^D$, such that $\mathbf{v}_{h_n} \rightharpoonup \bar{\mathbf{v}}$ for the weak topology of \mathbf{H}^D (which is separable using (8a)). We have that

$$\|\bar{\mathbf{v}}\|_{\mathbf{H}^D} \leq \liminf_{n \rightarrow +\infty} \|\mathbf{v}_{h_n}\|_{\mathbf{H}^D} = 1,$$

and we also have, by letting $n \rightarrow +\infty$ in (20) for the subsequence,

$$\langle \bar{q}^{\perp V}, D\bar{\mathbf{v}} \rangle_L \geq \beta \|\bar{q}^{\perp V}\|_L,$$

which shows that $\bar{\mathbf{v}} \neq 0$. This yields

$$\langle \bar{q}^{\perp V}, D \frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|_{\mathbf{H}^D}} \rangle_L \geq \frac{\beta}{\|\bar{\mathbf{v}}\|_{\mathbf{H}^D}} \|\bar{q}^{\perp V}\|_L \geq \beta \|\bar{q}^{\perp V}\|_L,$$

which concludes the proof of (16). \square

Theorem 3.2 (Existence and uniqueness for the mixed abstract finite element scheme).

Under Assumptions (8) and (11), assuming that the discrete inf-sup condition (19e) and that the condition (19a) hold, then for all $n \in \mathbb{N}$, Problem (18) has one and only solution $p_{h_n} \in L_{h_n}$ and $\mathbf{u}_{h_n} \in \mathbf{H}_{h_n}^D$ such that there exists $C > 0$, only depending on $\underline{\alpha}, \bar{\alpha}, \rho$ and β with

$$\|\mathbf{u}_{h_n}\|_L^2 + \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L^2 + \|p_{h_n}\|_L^2 \leq C(\|\mathbf{F}\|_L^2 + \|f\|_L^2), \quad (21)$$

where we denote by $\mathcal{P}_{L_{h_n}}$ the orthogonal projection on the closed space L_{h_n} .

Proof. As done in the proof of (iii) \Rightarrow (iv) of Theorem 2.2, we apply Theorem A.1 in appendix. We let $\widetilde{\mathbf{H}}^D := \mathbf{H}_{h_n}^D$, $\widetilde{L} := L_{h_n}$, we let again $\mathcal{H} = \mathbf{H}^D \times L$ and we define $\mathbb{A}(\mathbf{u}_{h_n}, \mathbf{v}_{h_n}) := \langle \mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}, \Pi_{h_n} \mathbf{v}_{h_n} \rangle_L$ and $\mathbf{a}(p_{h_n}, q_{h_n}) := \langle a(p_{h_n}), q_{h_n} \rangle_L$. We again deduce (63) from Assumptions (11a) and (11b), using Hypothesis (19a), and we obtain (64) from (19e).

With these definitions, we obtain, for any $(p_{h_n}, \mathbf{u}_{h_n}), (q_{h_n}, \mathbf{v}_{h_n}) \in \mathcal{H} \times \mathcal{H}$, the following bilinear form

$$E((p_{h_n}, \mathbf{u}_{h_n}), (q_{h_n}, \mathbf{v}_{h_n})) := \langle \mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}, \Pi_{h_n} \mathbf{v}_{h_n} \rangle_L - \langle p_{h_n}, D\mathbf{v}_{h_n} \rangle_L + \langle q_{h_n}, D\mathbf{u}_{h_n} \rangle_L + \langle a(p_{h_n}), q_{h_n} \rangle_L.$$

We notice that any solution $(p_{h_n}, \mathbf{u}_{h_n})$ to (18) is such that, for any $(q_{h_n}, \mathbf{v}_{h_n}) \in \mathcal{H}$,

$$E((p_{h_n}, \mathbf{u}_{h_n}), (q_{h_n}, \mathbf{v}_{h_n})) = \langle \mathbf{A}^{-1} \mathbf{F}, \Pi_{h_n} \mathbf{v}_{h_n} \rangle_L + \langle f, q_{h_n} \rangle_L. \quad (22)$$

Then, applying Theorem A.1, we get the existence of $\theta > 0$, only depending on $\rho, \underline{\alpha}, \bar{\alpha}$ and β , such that

$$E((\mathbf{u}_{h_n}, p_{h_n}), (\mathbf{v}_{h_n}, q_{h_n})) \geq \theta \left(\|\mathbf{u}_{h_n}\|_L^2 + \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L^2 + \|p_{h_n}\|_L^2 \right)^{1/2} \left(\|\mathbf{v}_{h_n}\|_{\mathbf{H}^D}^2 + \|q_{h_n}\|_L^2 \right)^{1/2}.$$

Using the preceding equation, a Cauchy-Schwarz inequality on (22), the continuity of \mathbf{A}^{-1} and (19a), we can write

$$\|\mathbf{u}_{h_n}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{L_{h_n}} \mathbf{D}\mathbf{u}_{h_n}\|_{\mathbf{L}}^2 + \|p_{h_n}\|_{\mathbf{L}}^2 \leq \frac{1}{\theta \|(q_{h_n}, \mathbf{v}_{h_n})\|_{\mathcal{H}}^{1/2}} \left(\frac{\rho}{\underline{\alpha}} \|\mathbf{F}\|_{\mathbf{L}} \|\mathbf{v}_{h_n}\|_{\mathbf{L}} + \|f\|_{\mathbf{L}} \|q_{h_n}\|_{\mathbf{L}} \right).$$

Using $\|\mathbf{v}_{h_n}\|_{\mathbf{L}} \leq \|\mathbf{v}_{h_n}\|_{\mathbf{H}^{\mathbf{D}}}$ and another Cauchy-Schwarz inequality, we obtain (21). Noticing that (18) provides a square linear system, we get from (21) that the only solution of this linear system with a null right-hand side is null. This shows the existence and uniqueness of the solution. \square

Remark 3.3. *The proof of Theorem 3.2 cannot be done by applying [12, Theorem 2.1] (as also noticed in Remark A.2), since Hypothesis (H2) of [12, Theorem 2.1] is not satisfied, and the bilinear form E , in the proof above, includes the additional term in a . As a consequence, the conclusion of Theorem 3.2 is also weaker (note the presence of a projection in the left-hand side of (21)).*

We then have the following convergence lemma.

Lemma 3.4 (Weak convergence of the solution of the mixed abstract finite element scheme is strong). *Under Assumptions (8) and (11), let $(L_{h_n})_{n \in \mathbb{N}}$ and $(\mathbf{H}_{h_n}^{\mathbf{D}})_{n \in \mathbb{N}}$ be families of finite dimensional spaces, let $\beta > 0$ and $\rho > 0$ such that, for any $n \in \mathbb{N}$: $L_{h_n} \subset L$ and $\mathbf{H}_{h_n}^{\mathbf{D}} \subset \mathbf{H}^{\mathbf{D}}$, the condition on linear mapping (19a), the conformity condition (19b), the consistency condition (19d) and the discrete inf-sup condition (19e) hold. Let us assume that the solution $(p_{h_n}, \mathbf{u}_{h_n}) \in L_{h_n} \times \mathbf{H}_{h_n}^{\mathbf{D}}$ of Problem (18) converges to the solution $(\bar{p}, \bar{\mathbf{u}}) \in L \times \mathbf{H}^{\mathbf{D}}$ of Problem (15) (which is unique owing to Lemma 3.1) in the following sense: as $n \rightarrow +\infty$, $p_{h_n} \rightharpoonup \bar{p}$ in L , $\Pi_{h_n} \mathbf{u}_{h_n} \rightharpoonup \bar{\mathbf{u}}$ in \mathbf{L} (for the weak topologies). Then, as $n \rightarrow +\infty$, we have the convergences $p_{h_n} \rightarrow \bar{p}$ in L , $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in \mathbf{L} (for the strong topologies).*

Proof. Firstly, letting $\bar{\mathbf{v}} = \bar{\mathbf{u}}$ and $\bar{q} = \bar{p}$ and adding the two equations in (15), we get that

$$\langle \mathbf{A}^{-1} \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_{\mathbf{L}} + \langle a(\bar{p}), \bar{p} \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1} \mathbf{F}, \bar{\mathbf{u}} \rangle_{\mathbf{L}} + \langle f, \bar{p} \rangle_{\mathbf{L}}. \quad (23)$$

And letting $\mathbf{v}_{h_n} = \mathbf{u}_{h_n}$ and $q_{h_n} = p_{h_n}$ and adding the two equations in (18), we get that

$$\langle \mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}, \Pi_{h_n} \mathbf{u}_{h_n} \rangle_{\mathbf{L}} + \langle a(p_{h_n}), p_{h_n} \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1} \mathbf{F}, \Pi_{h_n} \mathbf{u}_{h_n} \rangle_{\mathbf{L}} + \langle f, p_{h_n} \rangle_{\mathbf{L}}. \quad (24)$$

Moreover, thanks to (11a), $\langle a(p_{h_n} - \bar{p}), p_{h_n} - \bar{p} \rangle_{\mathbf{L}} \geq 0$, thus we have $\liminf_{n \rightarrow +\infty} \langle a(p_{h_n}), p_{h_n} \rangle_{\mathbf{L}} \geq \langle a(\bar{p}), \bar{p} \rangle_{\mathbf{L}}$. This yields, letting $n \rightarrow +\infty$ in (24) and using (23),

$$\limsup_{n \rightarrow +\infty} \langle \mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}, \Pi_{h_n} \mathbf{u}_{h_n} \rangle_{\mathbf{L}} \leq \langle \mathbf{A}^{-1} \mathbf{F}, \bar{\mathbf{u}} \rangle_{\mathbf{L}} + \langle f, \bar{p} \rangle_{\mathbf{L}} - \langle a(\bar{p}), \bar{p} \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1} \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_{\mathbf{L}}.$$

We then deduce that

$$0 \leq \limsup_{n \rightarrow +\infty} \langle \mathbf{A}^{-1} (\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}), (\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}) \rangle_{\mathbf{L}} \leq 0,$$

which provides, using the coercivity of \mathbf{A}^{-1} , the convergence of $\Pi_{h_n} \mathbf{u}_{h_n}$ to $\bar{\mathbf{u}}$ and the convergence of $\langle \mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}, \Pi_{h_n} \mathbf{u}_{h_n} \rangle_{\mathbf{L}}$ to $\langle \mathbf{A}^{-1} \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle_{\mathbf{L}}$. Then, using (23) and (24), we obtain that $\langle a(p_{h_n}), p_{h_n} \rangle_{\mathbf{L}}$ converges to $\langle a(\bar{p}), \bar{p} \rangle_{\mathbf{L}}$. We have $p_{h_n} = p_{h_n}^{\perp V} + p_{h_n}^V$ and $\bar{p} = \bar{p}^{\perp V} + \bar{p}^V$. Hence, thanks to (11a), we can write

$$\underline{\alpha} \|p_{h_n}^V - \bar{p}^V\|_{\mathbf{L}}^2 \leq \langle a(p_{h_n} - \bar{p}), p_{h_n} - \bar{p} \rangle_{\mathbf{L}},$$

thus we get that $p_{h_n}^V \rightarrow \bar{p}^V$ in L . Now we introduce $\bar{p}_{h_n} \in L_{h_n}$ an interpolation of \bar{p} , such that, using (19d), $\|\bar{p} - \bar{p}_{h_n}\|_{\mathbf{L}} \rightarrow 0$ as $n \rightarrow \infty$. Notice that we also have $\|(\bar{p} - \bar{p}_{h_n})^V\|_{\mathbf{L}} \rightarrow 0$ and therefore $\|(p_{h_n} - \bar{p}_{h_n})^V\|_{\mathbf{L}} \rightarrow 0$ as $n \rightarrow \infty$. Using the first equations of (18) and (15) and by introducing $\langle \bar{p}_{h_n}, \mathbf{D}\mathbf{v}_{h_n} \rangle_{\mathbf{L}}$ on each side, we can write

$$\begin{aligned} & \langle \mathbf{A}^{-1} (\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} \rangle_{\mathbf{L}} - \langle p_{h_n} - \bar{p}_{h_n}, \mathbf{D}\mathbf{v}_{h_n} \rangle_{\mathbf{L}} \\ &= \langle \mathbf{A}^{-1} (\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} \rangle_{\mathbf{L}} - \langle (p_{h_n} - \bar{p}_{h_n})^V, \mathbf{D}\mathbf{v}_{h_n} \rangle_{\mathbf{L}} - \langle (p_{h_n} - \bar{p}_{h_n})^{\perp V}, \mathbf{D}\mathbf{v}_{h_n} \rangle_{\mathbf{L}} \\ &= \langle \mathbf{A}^{-1} (\mathbf{F} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} - \mathbf{v}_{h_n} \rangle_{\mathbf{L}} - \langle \bar{p} - \bar{p}_{h_n}, \mathbf{D}\mathbf{v}_{h_n} \rangle_{\mathbf{L}}, \quad \forall \mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^{\mathbf{D}}. \end{aligned}$$

Owing to (19e), we choose \mathbf{v}_{h_n} with $\|\mathbf{v}_{h_n}\|_{\mathbf{H}^D} = 1$ and $\langle (p_{h_n} - \bar{p}_{h_n})^{\perp V}, D\mathbf{v}_{h_n} \rangle_L \geq \beta \|(p_{h_n} - \bar{p}_{h_n})^{\perp V}\|_L$ (the extremum is reached in finite dimension). We thus get, from the two above relations,

$$\begin{aligned} \beta \|(p_{h_n} - \bar{p}_{h_n})^{\perp V}\|_L &\leq \langle \mathbf{A}^{-1}(\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} \rangle_L - \langle \mathbf{A}^{-1}(\mathbf{F} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} - \mathbf{v}_{h_n} \rangle_L \\ &\quad - \langle (p_{h_n} - \bar{p}_{h_n})^V, D\mathbf{v}_{h_n} \rangle_L + \langle \bar{p} - \bar{p}_{h_n}, D\mathbf{v}_{h_n} \rangle_L. \end{aligned}$$

Using the Cauchy-Schwarz inequality, the continuity of \mathbf{A}^{-1} , (19a) and the definition of $\|\mathbf{v}_{h_n}\|_{\mathbf{H}^D}$ (implying that $\|\mathbf{v}_{h_n}\|_L$ and $\|D\mathbf{v}_{h_n}\|_L$ are bounded by 1), we can write,

$$\beta \|(p_{h_n} - \bar{p}_{h_n})^{\perp V}\|_L \leq \frac{\rho}{\alpha} \|\Pi_{h_n} \mathbf{u}_{h_n} - \bar{\mathbf{u}}\|_L + \|\bar{p} - \bar{p}_{h_n}\|_L + \|(p_{h_n} - \bar{p}_{h_n})^V\|_L - \langle \mathbf{A}^{-1}(\mathbf{F} - \bar{\mathbf{u}}), \Pi_{h_n} \mathbf{v}_{h_n} - \mathbf{v}_{h_n} \rangle_L.$$

Using the previous convergence results and (19b) for the last term on the right, we obtain that $p_{h_n}^{\perp V}$ converges to $\bar{p}^{\perp V}$ in L and conclude the proof of the lemma. \square

3.2 Convergence under LBB conditions

We recall that our framework includes two major differences with the standard mixed finite element theory (on one side, the presence of V and a which leads to a modification in the inf-sup condition, on the other side, the presence of the mass-lumping operator Π_{h_n}). In this section, we obtain convergence results assuming the following hypothesis: there exists $C_B > 0$ such that

$$\forall n \in \mathbb{N}, \forall \mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D, \|D\mathbf{v}_{h_n}\|_L \leq C_B (\|\mathbf{v}_{h_n}\|_L + \|\mathcal{P}_{L_{h_n}} D\mathbf{v}_{h_n}\|_L). \quad (25)$$

We prove in Theorem B.1 in the appendix that this assumption is equivalent to [12, Hypothesis H2] or [19, Assumptions (97) – (98)] in the case $V = \{0\}$ and $\Pi_{h_n} = \text{Id}$. Hence completing, together with the *inf-sup* condition (19e), the so-called Ladyzhenskaya–Babuška–Brezzi (LBB) conditions. For this reason, we call Condition (25) an LBB condition. This assumption is satisfied by the example of Raviart–Thomas finite elements with full Neumann boundary conditions in Section 5.1. We have the following convergence theorem.

Theorem 3.5 (Convergence under LBB conditions).

Under Assumptions (8) and (11), let $(L_{h_n})_{n \in \mathbb{N}}$ and $(\mathbf{H}_{h_n}^D)_{n \in \mathbb{N}}$ be families of finite dimensional spaces such that, for any $n \in \mathbb{N} : L_{h_n} \subset L$ and $\mathbf{H}_{h_n}^D \subset \mathbf{H}^D$, conditions (19) hold as well as (25). Then the solution $(p_{h_n}, \mathbf{u}_{h_n})$ of (18) converges to the solution $(\bar{p}, \bar{\mathbf{u}}) \in L \times \mathbf{H}^D$ of Problem (15) (which is unique owing to Lemma 3.1) in the following sense: as $n \rightarrow +\infty$, $p_{h_n} \rightarrow \bar{p}$ in L , $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in L and $D\mathbf{u}_{h_n}$ weakly converges to $D\bar{\mathbf{u}}$ in L .

Proof. Let us first observe that, owing to (21) and applying (25) (which implies that $D\mathbf{u}_{h_n}$ remains bounded if \mathbf{u}_{h_n} and $\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}$ are bounded), and thanks to (8a), we can find $(\bar{p}, \bar{\mathbf{u}}, \bar{\delta}) \in L \times L \times L$ and construct a subsequence of $(h_n)_{n \in \mathbb{N}}$, again denoted $(h_n)_{n \in \mathbb{N}}$, such that as $n \rightarrow \infty$, $p_{h_n} \rightarrow \bar{p}$ in L , $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in L and $D\mathbf{u}_{h_n} \rightarrow \bar{\delta}$ in L (all convergences being weak). From (19b), we get that $\mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$. Passing to the limit, for any $\varphi \in H^G$, in (given by (10)) $\langle \mathbf{u}_{h_n}, G\varphi \rangle_L + \langle D\mathbf{u}_{h_n}, \varphi \rangle_L = 0$, we get that

$$\langle \bar{\mathbf{u}}, G\varphi \rangle_L + \langle \bar{\delta}, \varphi \rangle_L = 0,$$

which proves, using again (10), that $\bar{\mathbf{u}} \in \mathbf{H}^D$ and $\bar{\delta} = D\bar{\mathbf{u}}$.

Then, considering a sequence $(q_{h_n})_{n \in \mathbb{N}}$ converging to \bar{q} in L thanks to (19d), we can let $n \rightarrow \infty$ in the second equation of (18), and we get that $(\bar{\mathbf{u}}, \bar{p})$ is solution to the second equation of (15). For any $\bar{\mathbf{v}} \in \mathbf{H}^D$, we deduce from (19c) the existence of $\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D$ such that $\Pi_{h_n} \mathbf{v}_{h_n} \rightarrow \bar{\mathbf{v}}$ in L . Letting $\bar{\mathbf{v}} = \mathbf{v}_{h_n}$ in the first equation of (18), we can let $n \rightarrow \infty$. By weak/strong convergence, we deduce that $(\bar{p}, \bar{\mathbf{u}})$ is solution of to the first equation of (15). Hence, by uniqueness of the limit, this convergence holds for the whole sequence. We conclude the strong convergence property using Lemma 3.4. \square

Remark 3.6 (Convergence results in the standard LBB framework). *The convergence results proved on $D\mathbf{u}_{h_n}$ in Theorem 3.5 are weaker than those which are proved in [12, Theorem 2.1] and [19, Theorem 5.3], in which it is proved that the convergence of $D\mathbf{u}_{h_n}$ to $D\bar{\mathbf{u}}$ is strong and that error estimates are available. In order to retrieve such results, more hypotheses than (19b) have to be done on Π_{h_n} , which can be the object of further works.*

3.3 Convergence under regular approximation of the pressure

In this section, we assume the *inf-sup* condition (19e) but we no longer assume Condition (25). Hence we replace the LBB framework by a stronger assumption than (19d) on the interpolation properties of the spaces L_{h_n} :

$$\forall n \in \mathbb{N}, L_{h_n} \subset H^G \text{ and } \forall \varphi \in H^G, \lim_{n \rightarrow \infty} \inf_{q_{h_n} \in L_{h_n}} \|\varphi - q_{h_n}\|_{H^G} = 0. \quad (26)$$

This framework corresponds to that which is assumed in [1] as detailed in the example provided in Section 5.2. We have the following convergence theorem.

Theorem 3.7 (Convergence of the solution of the mixed abstract finite element scheme).

Under Assumptions (8) and (11), let $(L_{h_n})_{n \in \mathbb{N}}$ and $(\mathbf{H}_{h_n}^D)_{n \in \mathbb{N}}$ be families of finite dimensional spaces such that, for any $n \in \mathbb{N}$: $L_{h_n} \subset H^G$ and $\mathbf{H}_{h_n}^D \subset \mathbf{H}^D$, conditions (19) hold as well as the additional regularity condition on the pressure space (26). Then the solution $(p_{h_n}, \mathbf{u}_{h_n})$ of (18) converges to the solution $(\bar{p}, \bar{\mathbf{u}}) \in L \times \mathbf{H}^D$ of Problem (15) (which is unique owing to Lemma 3.1) in the following sense: as $n \rightarrow +\infty$, $p_{h_n} \rightarrow \bar{p}$ in L and $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in \mathbf{L} .

Proof. Let us first observe that, owing to (21), we can find $(\bar{p}, \bar{\mathbf{u}}) \in L \times \mathbf{L}$ and construct a subsequence of $(h_n)_{n \in \mathbb{N}}$, again denoted $(h_n)_{n \in \mathbb{N}}$, such that as $n \rightarrow \infty$, $p_{h_n} \rightarrow \bar{p}$ in L and $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in \mathbf{L} (both convergences being weak). From (19b), we get that $\mathbf{u}_{h_n} \rightharpoonup \bar{\mathbf{u}}$. Let $\varphi \in H^G$ and let $(\varphi_{h_n})_{n \in \mathbb{N}}$ converging to φ in H^G thanks to (26). Using (10), we write the second equation of (18) as

$$-\langle \mathbf{u}_{h_n}, G\varphi_{h_n} \rangle_{\mathbf{L}} + \langle a(p_{h_n}), \varphi_{h_n} \rangle_{\mathbf{L}} = \langle f, \varphi_{h_n} \rangle_{\mathbf{L}}.$$

Using weak/strong convergence for the terms of the left-hand side, we get, letting $n \rightarrow \infty$,

$$-\langle \bar{\mathbf{u}}, G\varphi \rangle_{\mathbf{L}} + \langle a(\bar{p}), \varphi \rangle_{\mathbf{L}} = \langle f, \varphi \rangle_{\mathbf{L}}.$$

Since the above relation holds for any $\varphi \in H^G$, we obtain that $\bar{\mathbf{u}} \in \mathbf{H}^D$ and that $D\bar{\mathbf{u}} = f - a(\bar{p})$. Hence $(\bar{p}, \bar{\mathbf{u}})$ is solution to the second equation of (18). Considering $\bar{\mathbf{v}} \in \mathbf{H}^D$, we deduce from (19c) the existence of $\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^D$ such that $\Pi_{h_n} \mathbf{v}_{h_n} \rightarrow \bar{\mathbf{v}}$ in \mathbf{H}^D . Letting $\bar{\mathbf{v}} = \mathbf{v}_{h_n}$ in the first equation of (18), we can let $n \rightarrow \infty$ and by weak/strong convergence, we deduce that $(\bar{p}, \bar{\mathbf{u}})$ is solution of to the first equation of (18). Hence, by uniqueness of the limit, this convergence holds letting $n \rightarrow +\infty$. We conclude the strong convergence property using Lemma 3.4. \square

Remark 3.8 (Convergence results provided by Theorem 3.7). *No kind of convergence of $D\mathbf{u}_{h_n}$ to $D\bar{\mathbf{u}}$ is provided by Theorem 3.7. Indeed, Assumption (26) is done in order to replace, since $\varphi \in H^G$, $\langle D\bar{\mathbf{u}}, \varphi \rangle_{\mathbf{L}}$ by $-\langle \bar{\mathbf{u}}, G\varphi \rangle_{\mathbf{L}}$ in the second equation of (18).*

4 An AGDM from the mass-lumped mixed FEM

In this section we firstly define an Abstract Gradient Discretization Method (AGDM) for the approximation of Problem (12). To do this, as in the references [16, 17], we will define discretization items as well as the associated properties in order to obtain error estimates. The important objective of this section will be to show that the mixed Finite Element Methods (FEM), possibly mass-lumped, described by (18) enter into this AGDM framework.

4.1 AGDM for an elliptic problem

The following definitions give an open framework which includes a variety of methods (see [16, 17]).

Definition 4.1 (Abstract Gradient Discretisation). *Under Assumptions (8), an abstract gradient discretisation is defined by $\mathcal{D} = (X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ where:*

1. *The set of discrete unknowns $X_{\mathcal{D}}$ is a finite dimensional vector space on \mathbb{R} .*
2. *The “function” reconstruction $P_{\mathcal{D}} : X_{\mathcal{D}} \rightarrow L$ is a linear mapping.*
3. *The “gradient” reconstruction $G_{\mathcal{D}} : X_{\mathcal{D}} \rightarrow \mathbf{L}$ is, also, a linear mapping.*
4. *The mappings $P_{\mathcal{D}}$ and $G_{\mathcal{D}}$ are such that the following quantity is a norm on $X_{\mathcal{D}}$:*

$$\|q\|_{\mathcal{D}} := (\|(P_{\mathcal{D}}q)^V\|_L^2 + \|G_{\mathcal{D}}q\|_{\mathbf{L}}^2)^{1/2}$$

As in [17], given \mathcal{D} an AGD in the sense of Definition (4.1), the Abstract Gradient Scheme (AGS) for the approximation of Problem (12) is then obtained by replacing the continuous space and operators by the discrete ones:

$$\text{Find } p \in X_{\mathcal{D}} \text{ such that, } \langle \mathbf{A}(G_{\mathcal{D}}p), G_{\mathcal{D}}q \rangle_{\mathbf{L}} + \langle a(P_{\mathcal{D}}p), P_{\mathcal{D}}q \rangle_L = \langle f, P_{\mathcal{D}}q \rangle_L + \langle \mathbf{F}, G_{\mathcal{D}}q \rangle_{\mathbf{L}}, \quad \forall q \in X_{\mathcal{D}}. \quad (27)$$

Below we recall some key notions to the present abstract setting, see [17, Section 3.2].

Definition 4.2 (Coercivity). *If \mathcal{D} is a gradient discretisation in the sense of Definition 4.1, let $C_{\mathcal{D}}$ be the norm of $P_{\mathcal{D}}$:*

$$C_{\mathcal{D}} = \max_{q \in X_{\mathcal{D}} \setminus \{0\}} \frac{\|P_{\mathcal{D}}q\|_L}{\|q\|_{\mathcal{D}}}. \quad (28)$$

A sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of gradient discretisations is coercive if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_n} \leq C_P$ for all $n \in \mathbb{N}$.

Definition 4.3 (Consistency). *If \mathcal{D} is a gradient discretisation in the sense of Definition 4.1, let $S_{\mathcal{D}} : H^G \rightarrow [0, +\infty)$ be given by*

$$\forall \varphi \in H^G, \quad S_{\mathcal{D}}(\varphi) = \min_{q \in X_{\mathcal{D}}} (\|P_{\mathcal{D}}q - \varphi\|_L + \|G_{\mathcal{D}}q - G\varphi\|_{\mathbf{L}}). \quad (29)$$

A sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of gradient discretisations is consistent if

$$\forall \varphi \in H^G, \quad \lim_{n \rightarrow \infty} S_{\mathcal{D}_n}(\varphi) = 0. \quad (30)$$

Definition 4.4 (Limit-conformity). *If \mathcal{D} is a gradient discretisation in the sense of Definition 4.1, let $W_{\mathcal{D}} : \mathbf{H}^D \rightarrow [0, +\infty)$ be given by*

$$\forall \varphi \in \mathbf{H}^D, \quad W_{\mathcal{D}}(\varphi) = \sup_{q \in X_{\mathcal{D}} \setminus \{0\}} \frac{|\langle \varphi, G_{\mathcal{D}}q \rangle_{\mathbf{L}} + \langle D\varphi, P_{\mathcal{D}}q \rangle_L|}{\|q\|_{\mathcal{D}}}. \quad (31)$$

A sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of gradient discretisations is limit-conforming if

$$\forall \varphi \in \mathbf{H}^D, \quad \lim_{n \rightarrow \infty} W_{\mathcal{D}_n}(\varphi) = 0. \quad (32)$$

We can comment that the *coercivity* is a kind of discrete Poincaré inequality, the *limit-conformity* mimics at the discrete level the duality relation (10) whereas the *consistency* is closed to an error interpolation in the Finite Element formalism. Let us underline that the *limit-conformity* property implies the *coercivity* property, as stated by [17, Lemma 3.10]. It is not mandatory in our studied case, but notice that, in the literature of AGDM, additional properties can be requested in the case of some nonlinear problems called *compactness* property (see [17, Definition 3.9]) and a *piecewise constant* property on $P_{\mathcal{D}}$ could be useful. These properties allow us to exhibit the following error estimate theorem [17, Theorem 5.2].

Theorem 4.5 (Error estimate in the AGDM literature.). *Under Assumptions (8) and (11), let $\bar{p} \in H^G$ be the solution of Problem (12) (see Theorem 2.2) and let \mathcal{D} be an AGD in the sense of Definition 4.1. Then there exists one and only one $p \in X_{\mathcal{D}}$ solution to the AGS (27). This solution satisfies:*

$$\begin{aligned} \|G\bar{p} - G_{\mathcal{D}}p\|_{\mathbf{L}} &\leq \frac{1}{\underline{\alpha}} \left(W_{\mathcal{D}}(\mathbf{A}(G\bar{p}) - \mathbf{F}) + (\bar{\alpha}(1 + C_{\mathcal{D}}) + \underline{\alpha})S_{\mathcal{D}}(\bar{p}) \right), \\ \|\bar{p} - P_{\mathcal{D}}p\|_{\mathbf{L}} &\leq \frac{1}{\underline{\alpha}} \left(C_{\mathcal{D}}W_{\mathcal{D}}(\mathbf{A}(G\bar{p}) - \mathbf{F}) + (\bar{\alpha}C_{\mathcal{D}}(1 + C_{\mathcal{D}}) + \underline{\alpha})S_{\mathcal{D}}(\bar{p}) \right), \end{aligned} \quad (33)$$

where $C_{\mathcal{D}}$ defined by (28) is the norm of the linear reconstruction operator $P_{\mathcal{D}}$, $S_{\mathcal{D}}$ defined by (29) measures the consistency and $W_{\mathcal{D}}$ defined by (31) measures the conformity of the approximation spaces and operators.

4.2 Construction of the Abstract Gradient Discretisation

Our aim is to construct an AGDM still denoted $\mathcal{D} = (X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ in the sense of Definition 4.1 using the mass-lumped Mixed Finite Element framework. In this section, we assume that Assumptions (8) and (11b) hold.

Definition 4.6 (Abstract mass-lumped Mixed Finite Element Gradient Discretisation (AMFE-GD)). *Under Assumptions (8), let \mathbf{A} be a linear operator such that (11b) holds. Let $L_h \subset L$ and $\mathbf{H}_h^D \subset \mathbf{H}^D$ be finite dimensional subspaces. Let $\Pi_h : \mathbf{H}_h^D \rightarrow \mathbf{L}$ be a linear mapping such that there exists $\rho > 0$, with*

$$\forall \mathbf{v}_h \in \mathbf{H}_h^D, \quad \frac{1}{\rho} \|\mathbf{v}_h\|_{\mathbf{L}} \leq \|\Pi_h \mathbf{v}_h\|_{\mathbf{L}} \leq \rho \|\mathbf{v}_h\|_{\mathbf{L}}. \quad (34)$$

Let $(\varphi_i)_{i \in I}$ be a basis of L_h and $(\psi_j)_{j \in J}$ be a basis of \mathbf{H}_h^D . We define $\mathcal{D} = (X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ by:

1. $X_{\mathcal{D}} = \{p = (p_i)_{i \in I} : p_i \in \mathbb{R}, \forall i \in I\}$.
2. $\forall p \in X_{\mathcal{D}}, P_{\mathcal{D}}p = \sum_{i \in I} p_i \varphi_i$.
3. $G_{\mathcal{D}} : X_{\mathcal{D}} \rightarrow \mathbf{L}$ is defined, for any $p \in X_{\mathcal{D}}$, by

$$G_{\mathcal{D}}p := \mathbf{A}^{-1} \Pi_h \mathbf{G}_h, \text{ where,} \quad (35a)$$

$$\mathbf{G}_h \in \mathbf{H}_h^D \text{ is s.t. } \forall \mathbf{v}_h \in \mathbf{H}_h^D, \langle \mathbf{A}^{-1} \Pi_h \mathbf{G}_h, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}}p, D\mathbf{v}_h \rangle_{\mathbf{L}} = 0. \quad (35b)$$

Let us notice that (35) provides a definition for the discrete abstract gradient $G_{\mathcal{D}}$. Indeed, the bilinear form $\mathbb{A} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbb{R}$ defined by $\mathbb{A}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{A}^{-1} \Pi_h \mathbf{u}, \Pi_h \mathbf{v} \rangle_{\mathbf{L}}$ is continuous and coercive on \mathbf{L} owing to Hypotheses (11b) and (34). Since $\mathbf{H}_h^D \subset \mathbf{L}$ is finite dimensional, it is a closed subspace of \mathbf{L} . For a given $p \in X_{\mathcal{D}}$, the mapping $\mathbf{v} \mapsto -\langle P_{\mathcal{D}}p, D\mathbf{v} \rangle_{\mathbf{L}}$ is a linear form which is continuous since the space \mathbf{H}_h^D is finite dimensional. Hence, by Lax-Milgram Theorem, there exists one and only one \mathbf{G}_h solution to (35b), which justifies Definition (35).

In terms of implementation, we look for $\mathbf{G}_h \in \mathbf{H}_h^D := \text{span}(\psi_j)_{j \in J}$, so we have to find scalars $(\omega_j)_{j \in J}$ such that $\mathbf{G}_h = \sum_{j \in J} \omega_j \psi_j$. Moreover, (35) yields to

$$\langle G_{\mathcal{D}}p, \Pi_h \psi_k \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}}p, D\psi_k \rangle_{\mathbf{L}} = 0, \quad \forall k \in J.$$

Thus we obtain

$$\sum_{j \in J} \omega_j \langle \mathbf{A}^{-1} \Pi_h \psi_j, \Pi_h \psi_k \rangle_{\mathbf{L}} = -\langle P_{\mathcal{D}}p, D\psi_k \rangle_{\mathbf{L}}, \quad \forall k \in J. \quad (36)$$

These equations (36) lead to a linear system with unknowns the scalars $(\omega_j)_{j \in J}$. Thus (36) gives a way to compute the family $(\omega_j)_{j \in J}$ which ensures the computation of \mathbf{G}_h and at last of $G_{\mathcal{D}}$. The size of the

linear system is thus linked to the dimension of \mathbf{H}_h^D , while the linear system resulting from the abstract gradient scheme (27) is sized to the dimension of L_h . Finally, this is consistent with the size of the linear systems resulting from the mixed methods.

Note that, in some cases but not all, it is possible to define Π_h (generally piecewise constant with disjoint supports, in the case where \mathbf{A} is a local operator) such that the linear system (36) is diagonal, which corresponds to the term ‘‘mass lumping’’ as shown by the examples given in Section 5.

The following theorem shows the equivalence between the scheme (27) using the AMFE-GD method as defined by Definition 4.6, and the mass-lumped mixed finite element method presented in Section 3.

Theorem 4.7 (Equivalence between the mass-lumped Mixed Finite Element scheme and the Abstract Gradient Scheme).

Let Assumptions (8) and (11) hold. Let $L_h \subset L$ and $\mathbf{H}_h^D \subset \mathbf{H}^D$ be finite dimensional subspaces. Let $\Pi_h : \mathbf{H}_h^D \rightarrow \mathbf{L}$ be a linear mapping such that there exists $\rho > 0$ with (34). Let $(\varphi_i)_{i \in I}$ be a basis of L_h and let $(X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ be the AGD of Definition 4.6.

In the case $\mathbf{F} = 0$, the following assertions are equivalent:

- (i) $(p_h, \mathbf{u}_h) \in L_h \times \mathbf{H}_h^D$ is a solution of the mixed finite element scheme (18).
- (ii) $p_h = P_{\mathcal{D}}p$ and $\mathbf{u}_h = -\mathbf{G}_h$ with
 - * $p := (p_i)_{i \in I}$ is a solution of the AGS (27),
 - * $G_{\mathcal{D}}p = \mathbf{A}^{-1}\Pi_h\mathbf{G}_h$,

For general $\mathbf{F} \in \mathbf{L}$, and assuming that

$$\mathbf{A} \text{ is self-adjoint,} \quad (37)$$

the preceding equivalence remains true with replacing $\mathbf{u}_h = -\mathbf{G}_h$ by $\mathbf{u}_h = \mathbf{F}_h - \mathbf{G}_h$ with $\mathbf{F}_h \in \mathbf{H}_h^D$ such that

$$\langle \Pi_h \mathbf{F}_h, \mathbf{A}^{-1}\Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} = \langle \mathbf{F}, \mathbf{A}^{-1}\Pi_h \mathbf{v}_h \rangle_{\mathbf{L}}, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D. \quad (38)$$

Proof.

Let us first remark that the existence of $\mathbf{F}_h \in \mathbf{H}_h^D$ such that

$$\langle \Pi_h \mathbf{F}_h, \mathbf{A}^{-1}\Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1}\Pi_h \mathbf{F}_h, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} = \langle \mathbf{F}, \mathbf{A}^{-1}\Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1}\mathbf{F}, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}}, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D, \quad (39)$$

holds true if $\mathbf{F} = 0$ (letting $\mathbf{F}_h = 0$) for general \mathbf{A} satisfying Hypotheses (11b). It also holds true for general $\mathbf{F} \in \mathbf{L}$ when \mathbf{A} is satisfying both Hypotheses (11b) and (37), which imply that \mathbf{A}^{-1} is self-adjoint as well. Indeed, the coercivity of \mathbf{A}^{-1} (see Lemma 2.1) and the properties of Π_h ensure the existence and the uniqueness of \mathbf{F}_h solution to (38), which, together with (37), provides (39).

The proof of the theorem is therefore complete by showing the equivalence of (i) and (ii) with $\mathbf{u}_h = \mathbf{F}_h - \mathbf{G}_h$, assuming (39).

[(i) \Rightarrow (ii)]. Let $(p_h, \mathbf{u}_h) \in L_h \times \mathbf{H}_h^D$ be a solution of scheme (18). Let $q \in X_{\mathcal{D}}$ and $q_h := P_{\mathcal{D}}q \in L_h$. We select this q_h as test function in the second equation of (18). Thus we have $\langle \mathbf{D}\mathbf{u}_h, P_{\mathcal{D}}q \rangle_{\mathbf{L}} + \langle a(p_h), P_{\mathcal{D}}q \rangle_{\mathbf{L}} = \langle f, P_{\mathcal{D}}q \rangle_{\mathbf{L}}$. Owing to (35), there exists $\mathbf{G}_h \in \mathbf{H}_h^D$ such that $\langle P_{\mathcal{D}}q, \mathbf{D}\mathbf{u}_h \rangle_{\mathbf{L}} = -\langle G_{\mathcal{D}}q, \Pi_h \mathbf{u}_h \rangle_{\mathbf{L}}$ and $G_{\mathcal{D}}q = \mathbf{A}^{-1}\Pi_h \mathbf{G}_h$. So using (39), we can write $\langle \Pi_h \mathbf{F}_h, G_{\mathcal{D}}q \rangle_{\mathbf{L}} = \langle \mathbf{F}, G_{\mathcal{D}}q \rangle_{\mathbf{L}}$. This leads to

$$\langle \Pi_h (\mathbf{F}_h - \mathbf{u}_h), G_{\mathcal{D}}q \rangle_{\mathbf{L}} + \langle a(p_h), P_{\mathcal{D}}q \rangle_{\mathbf{L}} = \langle f, P_{\mathcal{D}}q \rangle_{\mathbf{L}} + \langle \mathbf{F}, G_{\mathcal{D}}q \rangle_{\mathbf{L}}. \quad (40)$$

Let $p := (p_i)_{i \in I} \in X_{\mathcal{D}}$ and let $p_h = P_{\mathcal{D}}p$. Again owing to (35), we can write $\langle G_{\mathcal{D}}p, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}}p, \mathbf{D}\mathbf{v}_h \rangle_{\mathbf{L}} = 0, \forall \mathbf{v}_h \in \mathbf{H}_h^D$. By combining with the first equation of (18), we obtain

$$\langle \mathbf{A}^{-1}\Pi_h \mathbf{u}_h, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} + \langle G_{\mathcal{D}}p, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}} = \langle \mathbf{A}^{-1}\mathbf{F}, \Pi_h \mathbf{v}_h \rangle_{\mathbf{L}}, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D,$$

Thanks to (39), we can write

$$\langle G_{\mathcal{D}p}, \Pi_h \mathbf{v}_h \rangle_L = \langle \mathbf{A}^{-1} \Pi_h (\mathbf{F}_h - \mathbf{u}_h), \Pi_h \mathbf{v}_h \rangle_L, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D,$$

which allows us to check from (35) that $G_{\mathcal{D}p} = \mathbf{A}^{-1} \Pi_h \mathbf{G}_h$ with $\mathbf{G}_h = \mathbf{F}_h - \mathbf{u}_h$. We retrieve the AGS (27) thanks to (40).

[(i) \Leftarrow (ii)]. Let $(X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ the AGD of Definition 4.6 and $p \in X_{\mathcal{D}}$ the solution of the associated AGS (27). Using (35a), we write $G_{\mathcal{D}p} = \mathbf{A}^{-1} \Pi_h \mathbf{G}_h$ with $\mathbf{G}_h \in \mathbf{H}_h^D$ satisfying (35b). We define $\mathbf{u}_h = \mathbf{F}_h - \mathbf{G}_h$, thus $\mathbf{u}_h \in \mathbf{H}_h^D$ and $G_{\mathcal{D}p} = \mathbf{A}^{-1} \Pi_h (\mathbf{F}_h - \mathbf{u}_h)$. We deduce from (35b) that

$$\langle \mathbf{A}^{-1} \Pi_h (\mathbf{F}_h - \mathbf{u}_h), \Pi_h \mathbf{v}_h \rangle_L + \langle P_{\mathcal{D}p}, D\mathbf{v}_h \rangle_L = 0, \quad \forall \mathbf{v}_h \in \mathbf{H}_h^D. \quad (41)$$

We also define $p_h = P_{\mathcal{D}p} \in L_h$ and by injecting (39) in (41), we recover the first equation (18) of the mixed abstract element scheme. Now, we inject $G_{\mathcal{D}p} = \mathbf{A}^{-1} \Pi_h (\mathbf{F}_h - \mathbf{u}_h)$ in the AGS (27), thus we obtain

$$\langle \Pi_h (\mathbf{F}_h - \mathbf{u}_h), G_{\mathcal{D}q} \rangle_L + \langle a(p_h), P_{\mathcal{D}q} \rangle_L = \langle f, P_{\mathcal{D}q} \rangle_L + \langle \mathbf{F}, G_{\mathcal{D}q} \rangle_L, \quad \forall q \in X_{\mathcal{D}}.$$

Thanks to (35a), there exists $\widehat{\mathbf{G}}_h \in \mathbf{H}_h^D$ such that $G_{\mathcal{D}q} = \mathbf{A}^{-1} \Pi_h \widehat{\mathbf{G}}_h$ which involves, also by using (39), that, $-\langle \Pi_h \mathbf{u}_h, G_{\mathcal{D}q} \rangle_L + \langle a(p_h), P_{\mathcal{D}q} \rangle_L = \langle f, P_{\mathcal{D}q} \rangle_L, \forall q \in X_{\mathcal{D}}$. Thanks to (35b), and since $\mathbf{u}_h \in \mathbf{H}_h^D$, we obtain,

$$\langle D\mathbf{u}_h, P_{\mathcal{D}q} \rangle_L + \langle a(p_h), P_{\mathcal{D}q} \rangle_L = \langle f, P_{\mathcal{D}q} \rangle_L, \quad \forall q \in X_{\mathcal{D}}.$$

Finally, by taking $q_h := P_{\mathcal{D}q}$, we recover the second equation of (18). □

As mentioned in the Introduction 1, in the previous works [16] or [23], a GDM was already built from discrete spaces involved in a mixed finite element method. But the proof of the equivalence between the gradient scheme and the mixed finite element scheme is only done in the case where $\mathbf{F} = 0$, for standard divergence and gradient operators, and RTk mixed finite elements. In this paper, these results are proved to hold true in the abstract gradient discretization framework and with more general assumptions.

The following result is inspired from [16, Theorem 10.3]. The convergence properties of the mixed finite element method are then used for proving that AMFE-GD fulfills the required properties for leading as well to convergence properties.

Theorem 4.8 (Properties of AMFE-GD).

Under Assumptions (8) and (11b), let $(L_{h_n})_{n \in \mathbb{N}}$ and $(\mathbf{H}_{h_n}^D)_{n \in \mathbb{N}}$ be families of finite dimensional spaces and $(\Pi_{h_n})_{n \in \mathbb{N}}$ be a family of linear operators such that, for any $n \in \mathbb{N} : L_{h_n} \subset L$ and $\mathbf{H}_{h_n}^D \subset \mathbf{H}^D$, conditions (19) hold. We assume one of the two following assumptions:

1. *either (25) holds (full LBB hypotheses),*
2. *or (26) holds (regular interpolation of the pressure: $L_{h_n} \subset H^G$).*

For any $n \in \mathbb{N}$, let $\mathcal{D}_n = (X_{\mathcal{D}_n}, P_{\mathcal{D}_n}, G_{\mathcal{D}_n})$ be the AMFE-GD defined by Definition 4.6 from the pair of spaces $(\mathbf{H}_{h_n}^D, L_{h_n})$. Then $(\mathcal{D}_n)_{n \in \mathbb{N}}$ is coercive, limit-conforming and consistent in the sense of Definitions 4.2, 4.4 and 4.3.

Proof.

Although the *limit-conformity* property (checked on all functions of \mathbf{H}^D) implies the *coercivity* property, since we use the *coercivity* property to prove the *limit-conformity* (using approximation spaces), we have to check separately these properties.

Coercivity. Let $n \in \mathbb{N}$ and $q \in X_{\mathcal{D}_n}$. We denote $q_{h_n} = P_{\mathcal{D}_n} q$ with $q_{h_n} \in L_{h_n}$, $q_{h_n} = q_{h_n}^{\perp V} + q_{h_n}^V$. Then $\|q\|_{\mathcal{D}_n} = (\|(P_{\mathcal{D}_n} q)^V\|_L^2 + \|G_{\mathcal{D}_n} q\|_L^2)^{1/2} = (\|q_{h_n}^V\|_L^2 + \|G_{\mathcal{D}_n} q\|_L^2)^{1/2}$. Using (19e) and the fact that $\mathbf{H}_{h_n}^D$ is finite dimensional, we can find $\boldsymbol{\psi}_{h_n} \in \mathbf{H}_{h_n}^D$ such that $\|\boldsymbol{\psi}_{h_n}\|_{\mathbf{H}^D} = 1$ and

$$\langle q_{h_n}^{\perp V}, D\boldsymbol{\psi}_{h_n} \rangle_L \geq \beta \|q_{h_n}^{\perp V}\|_L.$$

By (35), we have $\langle G_{\mathcal{D}_n} q, \Pi_{h_n} \psi_{h_n} \rangle_{\mathbf{L}} = -\langle q_{h_n}, D\psi_{h_n} \rangle_{\mathbf{L}}$ which can be rewritten as $\langle G_{\mathcal{D}_n} q, \Pi_{h_n} \psi_{h_n} \rangle_{\mathbf{L}} = -\langle q_{h_n}^{\perp V}, D\psi_{h_n} \rangle_{\mathbf{L}} - \langle q_{h_n}^V, D\psi_{h_n} \rangle_{\mathbf{L}}$. Using the Cauchy-Schwarz inequality yields

$$\beta \|q_{h_n}^{\perp V}\| \leq \|\Pi_{h_n} \psi_{h_n}\|_{\mathbf{L}} \|G_{\mathcal{D}_n} q\|_{\mathbf{L}} + \|D\psi_{h_n}\|_{\mathbf{L}} \|q_{h_n}^V\|_{\mathbf{L}} \leq \|\psi_{h_n}\|_{\mathbf{H}^D} (\rho \|G_{\mathcal{D}_n} q\|_{\mathbf{L}} + \|q_{h_n}^V\|_{\mathbf{L}}) = \rho \|G_{\mathcal{D}_n} q\|_{\mathbf{L}} + \|q_{h_n}^V\|_{\mathbf{L}}.$$

Hence we have

$$\begin{aligned} \|P_{\mathcal{D}_n} q\|_{\mathbf{L}}^2 &= \|q_{h_n}\|_{\mathbf{L}}^2 = \|q_{h_n}^V\|_{\mathbf{L}}^2 + \|q_{h_n}^{\perp V}\|_{\mathbf{L}}^2 \leq \left(1 + \frac{2}{\beta^2}\right) \|q_{h_n}^V\|_{\mathbf{L}}^2 + \frac{2\rho^2}{\beta^2} \|G_{\mathcal{D}_n} q\|_{\mathbf{L}}^2 \\ &\leq \underbrace{\max\left(1 + \frac{2}{\beta^2}, \frac{2\rho^2}{\beta^2}\right)}_{:= C_P^2} \|q\|_{\mathcal{D}_n}^2. \end{aligned} \quad (42)$$

Limit-conformity. Take a sequence $(q_n)_{n \in \mathbb{N}}$ such that $q_n \in X_{\mathcal{D}_n}$ for all $n \in \mathbb{N}$, and $(\|q_n\|_{\mathcal{D}_n})_{n \in \mathbb{N}}$ is bounded. Let $\psi \in \mathbf{L}$. Using (19c), we can choose, for any $n \in \mathbb{N}$, $\psi_{h_n} \in \mathbf{H}_{h_n}^D$ such that $\|\psi - \Pi_{h_n} \psi_{h_n}\|_{\mathbf{L}} \rightarrow 0$ and $\|D\psi - D\psi_{h_n}\|_{\mathbf{L}} \rightarrow 0$ as $n \rightarrow \infty$. Thanks to (35), we have, $\langle G_{\mathcal{D}_n} q_n, \Pi_{h_n} \psi_{h_n} \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D\psi_{h_n} \rangle_{\mathbf{L}} = 0$. We can write

$$\langle G_{\mathcal{D}_n} q_n, \psi \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D\psi \rangle_{\mathbf{L}} = \langle G_{\mathcal{D}_n} q_n, (\psi - \Pi_{h_n} \psi_{h_n}) \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D(\psi - \psi_{h_n}) \rangle_{\mathbf{L}}.$$

Applying the Cauchy-Schwarz inequality leads to

$$|\langle G_{\mathcal{D}_n} q_n, \psi \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D\psi \rangle_{\mathbf{L}}| \leq \|G_{\mathcal{D}_n} q_n\|_{\mathbf{L}} \|\psi - \Pi_{h_n} \psi_{h_n}\|_{\mathbf{L}} + \|P_{\mathcal{D}_n} q_n\|_{\mathbf{L}} \|D(\psi - \psi_{h_n})\|_{\mathbf{L}}.$$

Thanks to (42), we have

$$|\langle G_{\mathcal{D}_n} q_n, \psi \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D\psi \rangle_{\mathbf{L}}| \leq (\|\psi - \Pi_{h_n} \psi_{h_n}\|_{\mathbf{L}} + C_P \|D(\psi - \psi_{h_n})\|_{\mathbf{L}}) \|q\|_{\mathcal{D}_n}.$$

Thus $|\langle G_{\mathcal{D}_n} q_n, \psi \rangle_{\mathbf{L}} + \langle P_{\mathcal{D}_n} q_n, D\psi \rangle_{\mathbf{L}}| \rightarrow 0$ as $n \rightarrow \infty$ which is equivalent to the *limit-conformity* property, as shown in [16, Lemma 2.17].

Consistency. Following [16, Lemma 2.16], we prove (29) on a dense subset of H^G . We introduce $H_S := \{\varphi \in H^G, \mathbf{A}G\varphi \in \mathbf{H}^D\}$. In Lemma C.1, we prove that H_S is dense in H^G .

Let $\varphi \in H_S$. We notice that $\bar{p} = \varphi$ and $\bar{\mathbf{u}} = -\mathbf{A}G\varphi$ is then solution to (15), letting $\mathbf{F} = 0$, $f = -\mathbf{D}\mathbf{A}G\varphi + \mathcal{P}_V(\varphi)$ and $a = \mathcal{P}_V$ (these choices being the simplest ones for applying the above convergence results). The *inf-sup* condition (19e) and the interpolation property (19d) allow us to apply Lemma 3.1, which shows that the continuous *inf-sup* condition (16) holds, and therefore we can apply Theorem 2.2. Hence this solution $(\bar{p}, \bar{\mathbf{u}})$ is the unique one. On the other hand, we consider $(p_{h_n}, \mathbf{u}_{h_n}) \in L_{h_n} \times \mathbf{H}_{h_n}^D$ the solution of the associated discrete problem given by (18). From Theorem 4.7 (which can be applied in this case for general \mathbf{A} since $\mathbf{F} = 0$), we know that $p_n \in X_{\mathcal{D}_n}$, defined by $p_{h_n} = P_{\mathcal{D}_n} p_n$, is such that $G_{\mathcal{D}_n} p_n = -\mathbf{A}^{-1} \Pi_{h_n} \mathbf{u}_{h_n}$. If (25) holds, we apply Theorem 3.5, otherwise, if (26) holds, we apply Theorem 3.7. In both cases, we obtain that $p_{h_n} \rightarrow \bar{p}$ in L and $\Pi_{h_n} \mathbf{u}_{h_n} \rightarrow \bar{\mathbf{u}}$ in \mathbf{L} . This proves that $P_{\mathcal{D}_n} p_n \rightarrow \bar{p}$ in L and $G_{\mathcal{D}_n} p_n \rightarrow G\varphi$ in \mathbf{L} and concludes the proof of the *consistency*. \square

5 Examples of application

In this section we consider two examples with $\Omega \subset \mathbb{R}^2$. For each of them, we firstly define a continuous problem which enters into our abstract framework. We then define a numerical scheme based on mass-lumped mixed finite elements. The first example illustrates a case of convergence with full LBB conditions, as in Section 3.2. The second one shows a case where the space $L_h \subset H^G$, enabling the application of the result of Section 3.3.

5.1 RT0 finite elements on rectangles

In [16, Chapter 10], it is shown that the scheme using RTk mixed finite element applied to a second order elliptic equation with homogeneous Dirichlet boundary conditions is a gradient scheme, and therefore convergence results with minimal regularity apply. We focus in this section on two different features. Firstly, we introduce the mass-lumping operator Π_h , which is shown to yield a finite volume scheme on rectangular grid for the Laplace problem. Secondly, we consider nonhomogeneous Neumann boundary conditions.

Let us observe that mass-lumping with RT0 mixed finite element can be obtained by the algebraic condensation of the mass matrix $(\langle \mathbf{A}^{-1} \boldsymbol{\psi}_k, \boldsymbol{\psi}_j \rangle_{\mathbf{L}})_{j,k}$, which leads to replacing the matrix by a diagonal matrix. This technique is used in [3, 5]: an error estimate provides a convergence result for which the authors assume that $\bar{p} \in C^{3,1}(\bar{\Omega})$ and $\bar{\mathbf{u}} \in (C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega))^d$, see [3, 5] for the definition of these spaces, and they assume that the *inf-sup* condition holds through the existence of suitable Fortin operator (see [19, Theorem 5.4]) adapted to the boundary conditions which are done. The technique of algebraic condensation of the mass matrix is also studied in [8] in the case of triangular meshes and homogeneous Dirichlet boundary conditions. The authors obtain the convergence result [8, Proposition 5], in which they assume $\bar{p} \in H^1(\Omega)$ and $\bar{\mathbf{u}} \in H^1(\Omega)^d$ with $D\bar{\mathbf{u}} \in H^1(\Omega)$.

These additional regularity properties are not necessary in our framework designed for handling more general elliptic problems. The mass matrix $(\langle \mathbf{A}^{-1} \Pi_h \boldsymbol{\psi}_k, \Pi_h \boldsymbol{\psi}_j \rangle_{\mathbf{L}})_{j,k}$ in (18), obtained through the use of the mass-lumping operator Π_h , identified as an operator acting on the space \mathbf{L} , is diagonal in the examples provided in this section. We can then prove the convergence of the scheme (as well as the properties needed for the convergence of the AGS (27)), only assuming the regularity $\bar{p} \in H^1(\Omega)$ and $\bar{\mathbf{u}} \in H_{\text{div}}(\Omega)$. Note that this regularity is the one which is obtained, only assuming the regularity $f \in L^2(\Omega)$ and $\mathbf{F} \in L^2(\Omega)^d$ for the right-hand-side, and is the minimal regularity for Problem (12).

The proof of the *inf-sup* condition in the case of nonhomogeneous Neumann boundary conditions on the whole boundary is studied in [7], where the authors use a regularity result by Grisvard [25]. We propose below an original method, which can apply to a large variety of problems. This method is based on a finite volume scheme, which leads to fluxes which can be used for reconstructing an element of $H_{\text{div}}(\Omega)$ enabling us to prove the *inf-sup* condition.

5.1.1 Continuous problem

We consider the problem (1) where we modify the boundary condition. Hence the objective is to approximate a function $p : \Omega \rightarrow \mathbb{R}$ solution to the following nonhomogeneous Neumann problem:

$$\begin{aligned} -\operatorname{div}(\Lambda \nabla p) &= r - \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ (\Lambda \nabla p - \mathbf{F}) \cdot \mathbf{n} &= g \text{ on } \Gamma, \end{aligned} \quad (43)$$

where we still assume (2) and add the hypothesis that $\int_{\Omega} r(\mathbf{x}) d\mathbf{x} + \int_{\Gamma} g(\mathbf{x}) ds(\mathbf{x}) = 0$. This last condition is called the *compatibility condition* in [20] for the purely diffusive Neumann problem, since, if there exists a solution to (43), integrating the first equation of (43) and using the second one leads to this condition.

In order to provide a weak form for Problem (43), we define the space $H^G = \{(p, \gamma_{\Gamma} p), p \in H^1(\Omega)\}$, where $\gamma_{\Gamma} : H^1(\Omega) \rightarrow L^2(\Gamma)$ denotes the trace mapping on the boundary. We then look for $(p, \hat{p}) \in H^G$ such that

$$\begin{aligned} &\int_{\Omega} \Lambda(\mathbf{x}) \nabla p(\mathbf{x}) \cdot \nabla q(\mathbf{x}) d\mathbf{x} \\ &= \int_{\Omega} (r(\mathbf{x}) q(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \cdot \nabla q(\mathbf{x})) d\mathbf{x} + \int_{\Gamma} g(\mathbf{x}) \hat{q}(\mathbf{x}) ds(\mathbf{x}), \quad \forall (q, \hat{q}) \in H^G. \end{aligned} \quad (44)$$

In fact, if (44) has a solution (p, \hat{p}) , then $(p + C, \hat{p} + C)$, where $C \in \mathbb{R}$, is also a solution. Hence we fix C by modifying the initial bilinear form, in a way which is similar to [16, 3.1.3] or [17]. We consider the

following formulation: find $(p, \widehat{p}) \in H^G$ such that

$$\begin{aligned} & \int_{\Omega} \Lambda(\mathbf{x}) \nabla p(\mathbf{x}) \cdot \nabla q(\mathbf{x}) \, d\mathbf{x} + \left(\int_{\Omega} p(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma} \widehat{p}(\mathbf{x}) \, ds(\mathbf{x}) \right) \left(\int_{\Omega} q(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma} \widehat{q}(\mathbf{x}) \, ds(\mathbf{x}) \right) \\ & = \int_{\Omega} (r(\mathbf{x}) q(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \cdot \nabla q(\mathbf{x})) \, d\mathbf{x} + \int_{\Gamma} g(\mathbf{x}) \widehat{q}(\mathbf{x}) \, ds(\mathbf{x}), \quad \forall (q, \widehat{q}) \in H^G. \end{aligned} \quad (45)$$

Then the proof that Problem (45) has one and only one solution is obtained by using the Lax-Milgram theorem, thanks to the following mean value Poincaré inequality: there exists $C_P > 0$ only depending on Ω such that

$$\|p\|_{L^2(\Omega)} \leq C_P \left(\|\nabla p\|_{L^2(\Omega)^d} + \left| \int_{\Omega} p(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma} \widehat{p}(\mathbf{x}) \, ds(\mathbf{x}) \right| \right), \quad \forall (p, \widehat{p}) \in H^G,$$

which can easily be proved with reasoning by contradiction (see for instance [11, Chapter 8, page 246]). Letting $(q, \widehat{q}) = (\chi_{\Omega}, \chi_{\Gamma})$, where $\chi_{\Omega}(\mathbf{x}) = 1$ for a.e. $\mathbf{x} \in \Omega$ and $\chi_{\Gamma}(\mathbf{x}) = 1$ for a.e. $\mathbf{x} \in \Gamma$, we obtain, owing to the *compatibility condition*, that (p, \widehat{p}) satisfies

$$\int_{\Omega} p(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma} \widehat{p}(\mathbf{x}) \, ds(\mathbf{x}) = 0,$$

and is therefore also a solution to (44). Note that imposing nonhomogeneous Neumann boundary conditions in mixed formulations can also be done by using penalty methods (see for example [7, 15]).

In order to retrieve the formalism used in Section 2, we define the spaces $L = L^2(\Omega) \times L^2(\Gamma)$, $\mathbf{L} = L^2(\Omega)^d$, the notation $f = (r, g) \in L$ and the operator $G : H^G \rightarrow \mathbf{L}$ by $G\bar{q} = \nabla q$ for all $\bar{q} := (q, \widehat{q}) \in H^G$. Then the Problem (45) can be rewrite as Problem (12) using $\mathbf{A} = \Lambda$ and the mapping

$$a(\bar{q}) = \langle \bar{q}, (\chi_{\Omega}, \chi_{\Gamma}) \rangle_L (\chi_{\Omega}, \chi_{\Gamma}) \text{ for all } \bar{q} \in L.$$

A main point to check that Assumptions (8) and (11) are satisfied in this case is to well define the space V in order to obtain (11a). We define $V = \{ \kappa(\chi_{\Omega}, \chi_{\Gamma}), \kappa \in \mathbb{R} \}$ and thus $a(\bar{q}) = (|\Omega| + |\Gamma|) \bar{q}^V$ for any $\bar{q} \in L$ which implies that $\langle a(\bar{q}), \bar{q} \rangle_L = (\chi_{\Omega}, \chi_{\Gamma}) \|\bar{q}^V\|_L^2$ and thus (11a).

As in Section 2, in order to exhibit the mixed form of the problem, we define

$$\mathbf{H}^D = \{ \mathbf{u} \in \mathbf{L}, D\mathbf{u} := (\operatorname{div}(\mathbf{u}), -\mathbf{u} \cdot \mathbf{n}) \in L \}, \quad (46)$$

and thus we have the following relation, equivalent to (10):

$$\forall (p, \widehat{p}) \in H^G, \quad \forall \mathbf{u} \in \mathbf{H}^D, \quad \langle (p, \widehat{p}), D\mathbf{u} \rangle_L + \langle G(p, \widehat{p}), \mathbf{u} \rangle_L = 0.$$

Then Hypotheses (8) and (11) are satisfied by the spaces and operators introduced in this section. Since the study of Problem (45) shows that item (ii) of Theorem 2.2 holds, item (iv) implies that the mixed weak formulation, given by (15), has one and only one solution.

5.1.2 Definitions of the discrete spaces \mathbf{H}_h^D, L_h and operator Π_h

We define in this section the discrete spaces \mathbf{H}_h^D, L_h and operator Π_h involved in the mixed discrete Problem (18) and in Definition 4.6 for an AGDM, using the Raviart-Thomas mixed finite elements on a rectangular grid. We therefore assume that $\Omega \subset \mathbb{R}^2$ is a connected polygonal domain, whose the boundary is the finite union of segments of the form $[a, b] \times \{c\}$ or $\{c\} \times [a, b]$, as shown in Figure 1.

Let \mathcal{M} be a rectangular mesh of Ω , such that there exist two sequences $x_1 < \dots < x_N$ and $y_1 < \dots < y_M$ such that, for any element K of \mathcal{M} , there exists $i \in \{1, \dots, N-1\}$ and $j \in \{1, \dots, M-1\}$ with $K_{i+\frac{1}{2}, j+\frac{1}{2}} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$. We denote by \mathcal{M}_K the set of all $L \in \mathcal{M}$ such that L is sharing an edge with K .

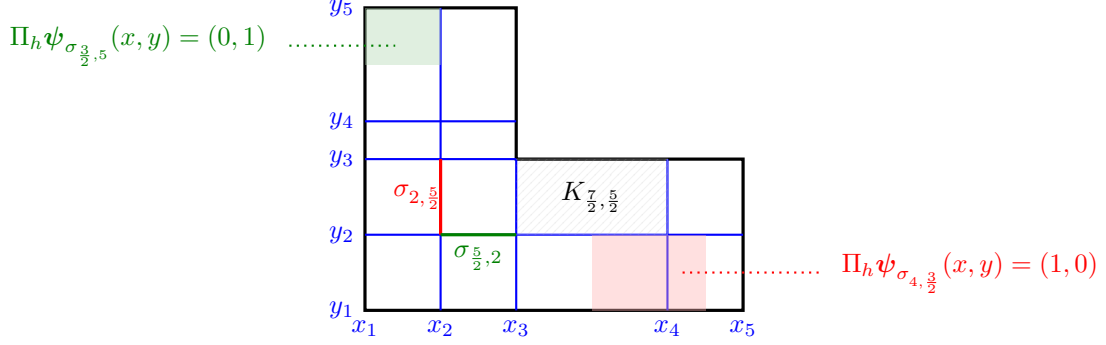


Figure 1: An example of polygonal domain with edges parallel to the axes which illustrate some notations. The support of $\Pi_h \psi_{\sigma_{4, \frac{3}{2}}}$ is colored in red and that of $\Pi_h \psi_{\sigma_{\frac{3}{2}, 5}}$ in green.

We then denote by $\mathcal{E}^{(y)}$ the set of all $\sigma_{i+\frac{1}{2}, j} = (x_i, x_{i+1}) \times \{y_j\} \subset \partial K$ and by $\mathcal{E}^{(x)}$ the set of all $\sigma_{i, j+\frac{1}{2}} := \{x_i\} \times (y_j, y_{j+1}) \subset \partial K$ for any $K \in \mathcal{M}$. We then denote by $\mathcal{E} = \mathcal{E}^{(x)} \cup \mathcal{E}^{(y)}$, by \mathcal{E}_{int} the set of interior edges and by \mathcal{E}_{ext} the set of exterior edges. We denote by $\mathcal{E}_{K, \text{ext}}$ the set of all $\sigma \in \mathcal{E}_{\text{ext}}$ such that $\sigma \subset \partial K$.

The RT0 method consists in defining a basis of functions $(\psi_\sigma)_{\sigma \in \mathcal{E}}$ for the velocity approximation space \mathbf{H}_h^D , which are piecewise affine. Let us give the expression of these functions ψ_σ in the cells of the mesh, together with the definition of $\Pi_h \psi_\sigma$.

1. For any $\sigma = \sigma_{i+\frac{1}{2}, j}$:

- if $K = K_{i+\frac{1}{2}, j+\frac{1}{2}} \in \mathcal{M}$, then $\psi_\sigma(x, y) = (0, \frac{y_{j+1}-y}{y_{j+1}-y_j})$ for a.e. $(x, y) \in K$ and $\Pi_h \psi_\sigma(x, y) = (0, 1)$ for a.e. $(x, y) \in (x_i, x_{i+1}) \times (y_j, \frac{1}{2}(y_j + y_{j+1}))$ and $(0, 0)$ for a.e. $(x, y) \in (x_i, x_{i+1}) \times (\frac{1}{2}(y_j + y_{j+1}), y_{j+1})$.
- if $M = K_{i+\frac{1}{2}, j-\frac{1}{2}} \in \mathcal{M}$, then $\psi_\sigma(x, y) = (0, \frac{y-y_{j-1}}{y_j-y_{j-1}})$ for a.e. $(x, y) \in M$ and $\Pi_h \psi_{\sigma_{i+\frac{1}{2}, j}}(x, y) = (0, 1)$ for a.e. $(x, y) \in (x_i, x_{i+1}) \times (\frac{1}{2}(y_{j-1} + y_j), y_j)$ and $(0, 0)$ for a.e. $(x, y) \in (x_i, x_{i+1}) \times (y_{j-1}, \frac{1}{2}(y_{j-1} + y_j))$.
- $\psi_\sigma(x, y) = (0, 0)$ and $\Pi_h \psi_\sigma(x, y) = (0, 0)$ otherwise.

2. For any $\sigma = \sigma_{i, j+\frac{1}{2}}$:

- if $K = K_{i+\frac{1}{2}, j+\frac{1}{2}} \in \mathcal{M}$, then $\psi_\sigma(x, y) = (\frac{x_{i+1}-x}{x_{i+1}-x_i}, 0)$ for a.e. $(x, y) \in K$ and $\Pi_h \psi_\sigma(x, y) = (1, 0)$ for a.e. $(x, y) \in (x_i, \frac{1}{2}(x_i + x_{i+1})) \times (y_j, y_{j+1})$ and $(0, 0)$ for a.e. $(x, y) \in (\frac{1}{2}(x_i + x_{i+1}), x_{i+1}) \times (y_j, y_{j+1})$;
- if $M = K_{i-\frac{1}{2}, j+\frac{1}{2}} \in \mathcal{M}$, then $\psi_\sigma(x, y) = (\frac{x-x_{i-1}}{x_i-x_{i-1}}, 0)$ for a.e. $(x, y) \in M$ and $\Pi_h \psi_\sigma(x, y) = (1, 0)$ for a.e. $(x, y) \in (\frac{1}{2}(x_{i-1} + x_i), x_i) \times (y_j, y_{j+1})$ and $(0, 0)$ for a.e. $(x, y) \in (x_{i-1}, \frac{1}{2}(x_{i-1} + x_i)) \times (y_j, y_{j+1})$;
- $\psi_\sigma(x, y) = (0, 0)$ and $\Pi_h \psi_\sigma(x, y) = (0, 0)$ otherwise.

Let us now define L_h for the pressure approximation. For $K \in \mathcal{M}$, we define the basis functions $\varphi_K \in L^2(\Omega)$ by $\varphi_K(\mathbf{x}) = 1$ for a.e. $\mathbf{x} \in K$, and 0 a.e. in $\Omega \setminus \{K\}$. For $\sigma \in \mathcal{E}_{\text{ext}}$, we define the basis functions $\varphi_\sigma \in L^2(\Gamma)$ by $\varphi_\sigma(\mathbf{x}) = 1$ for a.e. $\mathbf{x} \in \sigma$, and 0 a.e. in $\Gamma \setminus \{\sigma\}$. Then $L_h =: \text{span}\{(\varphi_K, \varphi_\sigma), K \in \mathcal{M}, \sigma \in \mathcal{E}_{\text{ext}}\}$.

5.1.3 Verification of assumptions of Theorem 3.5

Lemma 5.1. *Conditions (19) hold when defining \mathbf{H}_h^D, L_h and Π_h as in Section 5.1.2, as well as the LBB condition (25).*

Proof. For the sake of brevity, we only focus on the main ideas enabling the proof of the conditions involving Π_h and the LBB conditions under full Neumann boundary conditions.

- The proof of consistency condition (19c) can be found in [19] for the $H_{\text{div}}(\Omega)$ case. The main differences here are the convergence of Π_h and of the normal trace term. This can be done using a dense subspace of \mathbf{H}_h^D and an interpolation operator.
- The consistency condition (19d) is proved using a dense subspace of L and a cell and face constant interpolation.
- Inequality (25) is obtained with $C_B = 1$ as a consequence of $D\mathbf{H}_h^D \subset L_h$.
- In order to prove (19a), we decompose any $\mathbf{v}_h \in \mathbf{H}_h^D$ as $\mathbf{v}_h = \mathbf{v}_h^{(1)} e_1 + \mathbf{v}_h^{(2)} e_2$ with $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. We focus on $\mathbf{v}_h^{(1)}$ since the other term is similar. Following the previous notations about the discretisation of Ω , for any $i \in \{1, \dots, N\}$, we introduce $\varphi_i : [x_1, x_N] \rightarrow [0, 1]$ be the continuous piecewise affine function such that $\varphi_i(x_i) = 1$ and $\varphi_i(x_j) = 0$ for $j \neq i$. And let $\widehat{\varphi}_i : [x_1, x_N] \rightarrow [0, 1]$ be the discontinuous piecewise constant function such that $\widehat{\varphi}_i(x) = 1$ on $(x_{i-\frac{1}{2}}, x_i)$ if $i > 1$ and on $(x_i, x_{i+\frac{1}{2}})$ if $i < N$ and 0 elsewhere and where $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$. For a given $j \in \{1, \dots, M-1\}$ and $y \in [y_j, y_{j+1}]$, there exist scalars $(\alpha_{i,j})_i$ such that $\mathbf{v}_h^{(1)}(x, y) = \sum_{i=1}^N \alpha_{i,j} \varphi_i(x)$ and $\Pi_h \mathbf{v}_h^{(1)}(x, y) = \sum_{i=1}^N \alpha_{i,j} \widehat{\varphi}_i(x)$. We have to compare $I_\varphi := \int_{x_1}^{x_N} (\sum_{i=1}^N \alpha_{i,j} \varphi_i(s))^2 ds$ and $I_{\widehat{\varphi}} = \int_{x_1}^{x_N} (\sum_{i=1}^N \alpha_{i,j} \widehat{\varphi}_i(s))^2 ds$. We notice that for $i = 1, \dots, N-1$,

$$\int_{x_i}^{x_{i+1}} \left(\sum_{k=1}^N \alpha_{k,j} \varphi_k(s) \right)^2 ds = \frac{(x_{i+1} - x_i)}{3} (\alpha_{i,j}^2 + \alpha_{i,j} \alpha_{i+1,j} + \alpha_{i+1,j}^2),$$

and

$$\int_{x_i}^{x_{i+1}} \left(\sum_{k=1}^N \alpha_{k,j} \widehat{\varphi}_k \right)^2 ds = \frac{(x_{i+1} - x_i)}{2} (\alpha_{i,j}^2 + \alpha_{i+1,j}^2).$$

Since

$$\frac{1}{2} (\alpha_{i,j}^2 + \alpha_{i+1,j}^2) \leq \alpha_{i,j}^2 + \alpha_{i,j} \alpha_{i+1,j} + \alpha_{i+1,j}^2 \leq \frac{3}{2} (\alpha_{i,j}^2 + \alpha_{i+1,j}^2),$$

we obtain $\frac{1}{3} I_{\widehat{\varphi}} \leq I_\varphi \leq I_{\widehat{\varphi}}$. By integrate over y , we get $\|\mathbf{v}_h^{(1)}\|_{\mathbf{L}}^2 \leq \|\Pi_h \mathbf{v}_h^{(1)}\|_{\mathbf{L}}^2 \leq 3 \|\mathbf{v}_h^{(1)}\|_{\mathbf{L}}^2$. Thus we obtain (19a) with $\rho = \sqrt{3}$.

- In order to prove (19b), we consider a function $\bar{\mathbf{v}}$ belonging to a dense subspace of \mathbf{L} such that, for a given $j \in \{1, \dots, M-1\}$ and $y \in [y_j, y_{j+1}]$, we have $\psi_y(s) := \bar{\mathbf{v}}^{(1)}(s, y) \in C^1([x_1, x_N])$. We reuse, for any $\mathbf{v}_h \in \mathbf{H}_h^D$, the above notation $\mathbf{v}_h^{(1)}(x, y) = \sum_{i=1}^N \alpha_{i,j} \varphi_i(x)$ and we study the quantity

$$\begin{aligned} I_y &= \int_{x_1}^{x_N} \left(\sum_{i=1}^N \alpha_{i,j} (\varphi_i(s) - \widehat{\varphi}_i(s)) \right) \psi_y(s) ds \\ &= \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \left(\alpha_{i,j} (\varphi_i(s) - \widehat{\varphi}_i(s)) + \alpha_{i+1,j} (\varphi_{i+1}(s) - \widehat{\varphi}_{i+1}(s)) \right) \psi_y(s) ds. \end{aligned}$$

Since $\int_{x_i}^{x_{i+1}} (\varphi_i(s) - \widehat{\varphi}_i(s)) ds = 0$ for any $i \in \{1, \dots, N-1\}$, we can write,

$$\begin{aligned} \left| \int_{x_i}^{x_{i+1}} \alpha_{i,j} (\varphi_i(s) - \widehat{\varphi}_i(s)) \psi_y(s) ds \right| &= \left| \int_{x_i}^{x_{i+1}} \alpha_{i,j} (\varphi_i(s) - \widehat{\varphi}_i(s)) (\psi_y(s) - \psi_y(x_i)) ds \right| \\ &\leq \int_{x_i}^{x_{i+1}} |\alpha_{i,j}| |\varphi_i(s) - \widehat{\varphi}_i(s)| \underbrace{|\psi_y(s) - \psi_y(x_i)|}_{\leq (x_{i+1} - x_i) \max(|\psi'_y|)} ds. \end{aligned}$$

Since $|\varphi_i(s) - \widehat{\varphi}_i(s)| \leq 1$, we have

$$\left| \int_{x_i}^{x_{i+1}} \alpha_{i,j} (\varphi_i(s) - \widehat{\varphi}_i(s)) \psi_y(s) ds \right| \leq (x_{i+1} - x_i)^2 \max(|\psi'_y|) |\alpha_{i,j}| \leq h(x_{i+1} - x_i) \max(|\psi'_y|) |\alpha_{i,j}|.$$

Using the same computation for $\int_{x_i}^{x_{i+1}} \alpha_{i+1,j} (\varphi_{i+1}(s) - \widehat{\varphi}_{i+1}(s)) \psi_y(s) ds$, we thus obtain

$$|I_y| \leq h \max(|\psi'_y|) \sum_{i=1}^{N-1} (x_{i+1} - x_i) (|\alpha_{i,j}| + |\alpha_{i+1,j}|).$$

We can notice, using the support of each $\widehat{\varphi}_i$, that

$$\int_{x_1}^{x_N} |\Pi_h \mathbf{v}_h^{(1)}(s, y)| ds = \frac{1}{2} \sum_{i=1}^{N-1} (x_{i+1} - x_i) (|\alpha_{i,j}| + |\alpha_{i+1,j}|),$$

which yields $|I_y| \leq 2h \max(|\psi'_y|) \int_{x_1}^{x_N} |\Pi_h \mathbf{v}_h^{(1)}(s, y)| ds$ and therefore, since $\psi'_y(s) = \partial_x \bar{\mathbf{v}}(s, y)$,

$$\left| \int_{x_1}^{x_N} (\mathbf{v}_h^{(1)}(s, y) - \Pi_h \mathbf{v}_h^{(1)}(s, y)) \bar{\mathbf{v}}^{(1)}(s, y) ds \right| \leq 2h \|\partial_x \bar{\mathbf{v}}\|_\infty \int_{x_1}^{x_N} |\Pi_h \mathbf{v}_h^{(1)}(s, y)| ds.$$

Integrating with respect to y , we obtain

$$\left| \int_{\Omega} (\mathbf{v}_h^{(1)}(\mathbf{x}) - \Pi_h \mathbf{v}_h^{(1)}(\mathbf{x})) \bar{\mathbf{v}}^{(1)}(\mathbf{x}) d\mathbf{x} \right| \leq 2h \|\partial_x \bar{\mathbf{v}}\|_\infty \int_{\Omega} |\Pi_h \mathbf{v}_h^{(1)}(\mathbf{x})| d\mathbf{x}.$$

Then, using the Cauchy-Schwarz inequality, we obtain

$$\left| \int_{\Omega} (\mathbf{v}_h^{(1)}(\mathbf{x}) - \Pi_h \mathbf{v}_h^{(1)}(\mathbf{x})) \bar{\mathbf{v}}^{(1)}(\mathbf{x}) d\mathbf{x} \right| \leq 2h \|\partial_x \bar{\mathbf{v}}\|_\infty |\Omega|^{\frac{1}{2}} \|\Pi_h \mathbf{v}_h^{(1)}\|_{L^2(\Omega)}.$$

Considering a sequence h_n such that $\|\mathbf{v}_{h_n}^{(1)}\|_{L^2(\Omega)}$ remains bounded, we get (19b) using (19a) and $\lim_{n \rightarrow +\infty} h_n = 0$.

- In order to check the *inf-sup* condition (19e), we firstly remark that $L_h \cap V^\perp = \{\bar{q}_h^\perp, \forall \bar{q}_h \in L_h\}$ due to the definitions of V and L_h . We consider a given $\bar{q}_h \in L_h \cap V^\perp = \{(q_h, \widehat{q}_h) \in L_h : \int_{\Omega} q_h(\mathbf{x}) d\mathbf{x} + \int_{\Gamma} \widehat{q}_h(\mathbf{x}) ds(\mathbf{x}) = 0\}$. By definitions of L_h and V^\perp , we deduce that \bar{q}_h is cell and face constant and we have

$$\sum_{K \in \mathcal{M}} |K| q_K + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \widehat{q}_\sigma = 0,$$

where q_K and \widehat{q}_σ denote the constant values of \bar{q}_h in K and in σ . We then consider the element $\bar{\omega}_h = (\omega_h, \widehat{\omega}_h) \in L_h$, which is such that

$$\sum_{K \in \mathcal{M}} |K| \omega_K = 0, \tag{47}$$

(hence we don't assume that $\bar{\omega}_h \in V^\perp$) and which satisfies the following finite volume scheme [21, Chapter III,(10.7) – (10.8)], for all $K \in \mathcal{M}$:

$$- \sum_{M \in \mathcal{M}_K} \tau_{KM}(\omega_M - \omega_K) - \sum_{\sigma \in \mathcal{E}_{K,\text{ext}}} \tau_{K\sigma}(\hat{\omega}_\sigma - \omega_K) = |K|q_K, \quad (48)$$

and

$$-\tau_{K\sigma}(\omega_K - \hat{\omega}_\sigma) = |\sigma| \hat{q}_\sigma, \forall \sigma \in \mathcal{E}_{K,\text{ext}}, \quad (49)$$

where, for any neighbour M of K , $\tau_{KM} = \frac{|\partial K \cap \partial M|}{d_{KM}}$, with d_{KM} being the distance between the centers of gravity of K and M , and where for any $\sigma \in \mathcal{E}_{K,\text{ext}}$, $\tau_{K\sigma} = \frac{|\sigma|}{d_{K\sigma}}$, with $d_{K\sigma}$ being the orthogonal distance between the center of gravity of K and σ . From [21, Chapter III, Lemma 10.1], we get the existence and uniqueness of $(\omega_K)_{K \in \mathcal{M}}$ solution of (47)-(48)-(49), and therefore that of $(\hat{\omega}_\sigma)_{\sigma \in \mathcal{E}_{\text{ext}}}$ thanks to (49). We define

$$\|\bar{\omega}_h\|_h^2 := \sum_{\sigma \in \mathcal{E}} \frac{|\sigma|}{d_\sigma} (\delta_\sigma \bar{\omega}_h)^2, \quad (50)$$

with,

- if $\sigma \in \mathcal{E}_{\text{int}}$ is shared by the cells K and M , $d_\sigma = d_{KM}$ and $\delta_\sigma \bar{\omega}_h := \omega_M - \omega_K$ with M, K oriented in the sens that if $\sigma = \sigma_{i,j+\frac{1}{2}}$ then $K = K_{i-\frac{1}{2},j+\frac{1}{2}}$, if $\sigma = \sigma_{i+\frac{1}{2},j}$ then $K = K_{i+\frac{1}{2},j-\frac{1}{2}}$ and always $M = K_{i+\frac{1}{2},j+\frac{1}{2}}$,
- if $\sigma \in \mathcal{E}_{\text{ext}}$ belongs to the cell $K = K_{i+\frac{1}{2},j+\frac{1}{2}}$, $d_\sigma = d_{K,\sigma}$ and $\delta_\sigma \bar{\omega}_h := \hat{\omega}_\sigma - \omega_K$ if $\sigma = \sigma_{i+1,j+\frac{1}{2}}$ or $\sigma = \sigma_{i+\frac{1}{2},j+1}$, $\delta_\sigma \bar{\omega}_h := \omega_K - \hat{\omega}_\sigma$ if $\sigma = \sigma_{i,j+\frac{1}{2}}$ or $\sigma = \sigma_{i+\frac{1}{2},j}$.

We get, by multiplying (48) by ω_K and (49) by $\hat{\omega}_\sigma$ (since $\tau_{KM} = \tau_{MK}$), using the Cauchy-Schwarz inequality,

$$\|\bar{\omega}_h\|_h^2 = \sum_{K \in \mathcal{M}} |K|q_K \omega_K + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \hat{q}_\sigma \hat{\omega}_\sigma \leq \|q_h\|_{L^2(\Omega)} \|\omega_h\|_{L^2(\Omega)} + \|\hat{q}_h\|_{L^2(\Gamma)} \|\hat{\omega}_h\|_{L^2(\Gamma)}. \quad (51)$$

We then use (47) and apply [21, Chapter III, Lemma 10.2] (*Mean Poincaré inequality*), which proves that there exists $C_1 > 0$ only depending on Ω such that

$$\|\omega_h\|_{L^2(\Omega)} \leq C_1 \|\bar{\omega}_h\|_h. \quad (52)$$

From [21, Chapter III, Lemma 10.5] (*Trace inequality*), defining $K_\sigma \in \mathcal{M}$ for any $\sigma \in \mathcal{E}_{\text{ext}}$ such that $\sigma \in \mathcal{E}_K$, we get that there exists $C_2 > 0$ only depending on Ω , such that

$$\left(\sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \omega_{K_\sigma}^2 \right)^{1/2} \leq C_2 (\|\bar{\omega}_h\|_h + \|\omega_h\|_{L^2(\Omega)}). \quad (53)$$

We then notice that, using the Young inequality and (52)-(53),

$$\begin{aligned} \|\hat{\omega}_h\|_{L^2(\Gamma)}^2 &= \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \hat{\omega}_\sigma^2 \leq 2 \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \omega_{K_\sigma}^2 + 2 \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| (\hat{\omega}_\sigma - \omega_{K_\sigma})^2 \\ &\leq 2 \sum_{\sigma \in \mathcal{E}_{\text{ext}}} |\sigma| \omega_{K_\sigma}^2 + 2 \text{diam}(\Omega) \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \tau_{K_\sigma} (\hat{\omega}_\sigma - \omega_{K_\sigma})^2 \\ &\leq \left(2C_2^2 (1 + C_1)^2 + 2 \text{diam}(\Omega) \right) \|\bar{\omega}_h\|_h^2. \end{aligned}$$

Using the previous inequality and the Cauchy-Schwarz inequality on (51) leads to the existence of C_3 only depending on Ω such that

$$\|\bar{\omega}_h\|_h \leq C_3 (\|q_h\|_{L^2(\Omega)} + \|\hat{q}_h\|_{L^2(\Gamma)}). \quad (54)$$

We then define $\mathbf{v}_h \in \mathbf{H}_h^D$ by

$$\mathbf{v}_h(\mathbf{x}) = - \sum_{\sigma \in \mathcal{E}_K} Q_\sigma \boldsymbol{\psi}_\sigma(\mathbf{x}), \text{ with } Q_\sigma = \frac{\delta_\sigma \bar{\omega}_h}{d_\sigma}, \text{ for all } \mathbf{x} \in K \text{ and } K \in \mathcal{M}. \quad (55)$$

We then notice that, for $K = K_{i+\frac{1}{2}, j+\frac{1}{2}}$ and $\mathbf{x} \in K$, $\operatorname{div} \boldsymbol{\psi}_\sigma(\mathbf{x}) = \frac{1}{x_{i+1}-x_i} = \frac{|\sigma|}{|K|}$ if $\sigma = \sigma_{i+1, j+\frac{1}{2}}$, $\operatorname{div} \boldsymbol{\psi}_\sigma(\mathbf{x}) = -\frac{|\sigma|}{|K|}$ if $\sigma = \sigma_{i, j+\frac{1}{2}}$, $\operatorname{div} \boldsymbol{\psi}_\sigma(\mathbf{x}) = \frac{1}{y_{j+1}-y_j} = \frac{|\sigma|}{|K|}$ if $\sigma = \sigma_{i+\frac{1}{2}, j+1}$, $\operatorname{div} \boldsymbol{\psi}_\sigma(\mathbf{x}) = -\frac{|\sigma|}{|K|}$ if $\sigma = \sigma_{i+\frac{1}{2}, j}$. Therefore, dividing (48) by $|K| = (x_{i+1} - x_i)(y_{j+1} - y_j)$ and using that $\tau_{K, \sigma} = \frac{|\sigma|}{d_\sigma}$, we get that

$$- \sum_{\sigma \in \mathcal{E}_K} \frac{\delta_\sigma \bar{\omega}_h}{d_\sigma} \operatorname{div} \boldsymbol{\psi}_\sigma(\mathbf{x}) = q_K,$$

and, for $\sigma \in \mathcal{E}_{ext}$ which belongs to the cell K and for $\mathbf{x} \in \sigma$, dividing (49) by $|\sigma|$,

$$- \frac{\delta_\sigma \bar{\omega}_h}{d_\sigma} \boldsymbol{\psi}_\sigma(\mathbf{x}) \cdot \mathbf{n} = \hat{q}_\sigma.$$

Therefore (55) implies that

$$D\mathbf{v}_h = \bar{q}_h = (q_h, \hat{q}_h). \quad (56)$$

Remarking that $\|\Pi_h \mathbf{v}_h\|_{L^2(K)}^2 = \sum_{\sigma \in \mathcal{E}_K} Q_\sigma^2 |\sigma| d_{K, \sigma}$, we obtain

$$\|\Pi_h \mathbf{v}_h\|_{L^2(\Omega)}^2 = \sum_{\sigma \in \mathcal{E}} Q_\sigma^2 |\sigma| d_\sigma = \|\bar{\omega}_h\|_h^2. \quad (57)$$

Using (56)-(19a) and then (54)-(57), we can write

$$\|\mathbf{v}_h\|_{\mathbf{H}^D}^2 \leq \rho^2 \|\Pi_h \mathbf{v}_h\|_{L^2(\Omega)}^2 + \|\bar{q}_h\|_{L_h}^2 \leq C_4 \|\bar{q}_h\|_{L_h}^2, \quad (58)$$

where $C_4 > 0$ is only depending on Ω . We introduce $\tilde{\mathbf{v}}_h = \frac{\mathbf{v}_h}{\|\mathbf{v}_h\|_{\mathbf{H}^D}}$. Gathering the previous results leads to

$$\langle \bar{q}_h^{\perp V}, D\tilde{\mathbf{v}}_h \rangle_L = \frac{\|\bar{q}_h\|_{L_h}^2}{\|\mathbf{v}_h\|_{\mathbf{H}^D}} \geq \frac{\|\bar{q}_h\|_{L_h}}{\sqrt{C_4}},$$

which concludes the proof of (19e). □

Let us observe that using Scheme (7) is then identical to the finite volume scheme used for proving the inf-sup property.

5.2 Mass lumped mixed finite element for dispersive shallow water flow

We show in this section that one of the schemes introduced in [1] for the approximation of the elliptic part of dispersive shallow water flows can be analysed through the results given in this work.

5.2.1 Continuous problem

In [1], the authors study a model of shallow water flow, in which a hyperbolic model for the transport of momentum and mass is coupled with an elliptic model, dedicated to the modelisation of dispersive effects. We focus in this section on the discretisation of this elliptic model which can be written through the following functions: the water height $H : \Omega \rightarrow \mathbb{R}$, the bathymetry $z_b : \Omega \rightarrow \mathbb{R}$ and the function $\zeta = H + \frac{\alpha^2}{2} z_b$, where $\alpha > 0$ is a parameter tuning the physics of the dispersive effects (various values are assigned to this parameter in the literature: in the ‘‘Green-Naghdi model’’, one lets $\alpha = \sqrt{3}$ whereas

$\alpha = 2$ in [1], see also [14] for a discussion about the value of α). Specific first order differential operators, denoted by ∇_{sw}^α and div_{sw}^α , called *shallow-water* gradient and divergence in [1, Section 2.4.1], are then defined in the following way: for $\varphi : \Omega \rightarrow \mathbb{R}$,

$$\nabla_{sw}^\alpha \varphi = (H \nabla \varphi + \varphi \nabla \zeta, -\alpha \varphi)^T, \quad (59a)$$

and for $\mathbf{u} = (\mathbf{v}, w)^T$ with $\mathbf{v} : \Omega \rightarrow \mathbb{R}^2$ and $w : \Omega \rightarrow \mathbb{R}$, we set

$$\text{div}_{sw}^\alpha \mathbf{u} = \text{div}(H\mathbf{v}) - \mathbf{v} \cdot \nabla \zeta + \alpha w. \quad (59b)$$

Then the problem reads: find $p : \Omega \rightarrow \mathbb{R}$ (called in [1] the non-hydrostatic pressure) and $\mathbf{u} = (\mathbf{v}, w)^T : \Omega \rightarrow \mathbb{R}^3$ (which is the velocity field) such that,

$$H\mathbf{u} + \nabla_{sw}^\alpha p = \mathbf{g} \text{ in } \Omega, \quad (60a)$$

$$\text{div}_{sw}^\alpha \mathbf{u} = f \text{ in } \Omega. \quad (60b)$$

The functions f and \mathbf{g} used in the right-hand-side of Problem (60) represent the coupling effects at each time step with the hyperbolic part of the model in [1]. We consider a simpler situation than the mixed Dirichlet and Neumann boundary conditions assumed in [1], by prescribing homogeneous Dirichlet boundary conditions on the whole boundary:

$$p = 0 \text{ on } \Gamma. \quad (61)$$

In order to enter into the hypotheses (8) of our paper, we define $L = L^2(\Omega)$, $\mathbf{L} = L^2(\Omega)^3$, $H^G = H_0^1(\Omega)$, $\mathbf{H}^D = \{\boldsymbol{\omega} \in L^2(\Omega)^3 \mid \text{D}(\boldsymbol{\omega}) \in L^2(\Omega)\}$, $\text{G} = \nabla_{sw}^\alpha$ and $\text{D} = \text{div}_{sw}^\alpha$, $V = \{0\}$ (and therefore $V^\perp = L$). We enter into hypotheses (11), defining for $\mathbf{u} = (\mathbf{v}, w)^T$ the operator \mathbf{A} by $\mathbf{A}\mathbf{u} = H^{-1}\mathbf{u}$, letting $a \equiv 0$, defining the right-hand side by $\mathbf{F} = H^{-1}\mathbf{g}$ and assuming:

$$\bullet H \in C^1(\overline{\Omega}) \text{ and there exist } \underline{\alpha}, \overline{\alpha} > 0 \text{ such that, for } \mathbf{x} \in \Omega, \underline{\alpha} \leq H(\mathbf{x}) \leq \overline{\alpha}, \quad (62a)$$

$$\bullet \zeta \in C^2(\overline{\Omega}), \quad (62b)$$

$$\bullet \mathbf{F} \in L^2(\Omega)^3, f \in L^2(\Omega). \quad (62c)$$

We notice that the operators $\text{G} = \nabla_{sw}^\alpha$ and $\text{D} = \text{div}_{sw}^\alpha$ fulfill the abstract Green formula (10). Under these notations, Problem (12) provides a weak sense to Problem (60)-(61), and the existence and the uniqueness of the solution of the mixed problem is a consequence of Theorem 2.2, since the Lax-Milgram theorem implies that item (ii) holds.

5.2.2 Definitions of the discrete spaces \mathbf{H}_h^D, L_h and operator Π_h

Following the $\mathbb{P}^1/\mathbb{P}^1$ approximation method described in [1, Section 4.1], we consider the following numerical approximation. The domain Ω is meshed by a conforming triangular mesh of Ω in the sense of [16, Definition 7.4] or [20, Definition 1.50]. This mesh is denoted by $\mathfrak{T}_h = (\mathcal{M}_h, \mathcal{F}_h, \mathcal{C}_h, \mathcal{V}_h)$, where \mathcal{M}_h is the set of triangular cells, \mathcal{F}_h of edges, \mathcal{C}_h of the cell barycenters and \mathcal{V}_h of the vertices of the triangles and h denotes the *mesh size*, i.e. the maximal diameter of the *cells*. We introduce the family $(\phi_i)_{i=1}^N$ of the \mathbb{P}_1 finite element shape functions associated to the vertices, denoted by $\mathbf{x}_i \in \mathcal{V}_h \subset \overline{\Omega}$ for $i = 1, \dots, N$. The function ϕ_i is affine in each $K \in \mathcal{M}_h$, satisfies $\phi_i(\mathbf{x}_i) = 1$ and $\phi_i(\mathbf{x}_j) = 0$ for any $j \neq i$. For each $i = 1, \dots, N$, we denote by $\mathcal{M}_i \subset \mathcal{M}_h$ the set of the triangles having \mathbf{x}_i as a vertex. For each $K \in \mathcal{M}_h$, we denote by $\mathcal{F}_K \subset \mathcal{F}_h$ and $\mathcal{V}_K \subset \{1, \dots, N\}$ the set of its edges and the index of its vertices.

We then define the \mathbb{P}^1 finite element space $Q_h = \text{span}\{\phi_i, i = 1, \dots, N\}$, and the spaces $L_h = Q_h \cap H_0^1(\Omega)$ (note that we impose the boundary condition on the elements of L_h) and $\mathbf{H}_h^D = Q_h^3$.

If the components of $\mathbf{v}_h = (\sum_{i \in I} \alpha_i^{(k)} \phi_i)_{k=1,2,3} \in \mathbf{H}_h^D$ are ordered as $(\alpha_i^{(1)})_{i=1}^N, (\alpha_i^{(2)})_{i=1}^N, (\alpha_i^{(3)})_{i=1}^N$, the mass matrix $(\langle \mathbf{A}^{-1} \boldsymbol{\psi}_k, \boldsymbol{\psi}_j \rangle_{\mathbf{L}})_{j,k}$ in (18) without mass lumping operator is given by $\begin{pmatrix} M_h & 0 & 0 \\ 0 & M_h & 0 \\ 0 & 0 & M_h \end{pmatrix}$ with $M_h = (\int_{\Omega} H(\mathbf{x}) \phi_j(\mathbf{x}) \phi_i(\mathbf{x}) d\mathbf{x})_{i,j=1}^N$. An algebraic condensation of this matrix is described in [1, (4.13)]. Let us retrieve this algebraic condensation by defining a mass lumping operator $\Pi_h : \mathbf{H}_h^D \rightarrow \mathbf{L}$. We first define the dual cells $(\omega_i)_{i=1}^N$ by their interior edges, joining the middle of the edges and the barycenter of the triangles (see Figure 2). Let $(\widehat{\phi}_i)_{i=1}^N$ be the characteristic functions of the cells $(\omega_i)_{i=1}^N$, which means that $\widehat{\phi}_i(\mathbf{x}) = 1$ if $\mathbf{x} \in \omega_i$, and $\widehat{\phi}_i(\mathbf{x}) = 0$ if $\mathbf{x} \in \Omega \setminus \omega_i$. Then, for any function $\mathbf{v}_h = (\sum_{i \in I} \alpha_i^{(k)} \phi_i)_{k=1,2,3} \in \mathbf{H}_h^D$, we define

$$\Pi_h \mathbf{v}_h = \left(\sum_{i \in I} \alpha_i^{(k)} \widehat{\phi}_i \right)_{k=1,2,3}.$$

Then the matrix $(\langle \mathbf{A}^{-1} \Pi_h \boldsymbol{\psi}_k, \Pi_h \boldsymbol{\psi}_j \rangle_{\mathbf{L}})_{j,k}$ in (18) has the form $\begin{pmatrix} \widehat{M}_h & 0 & 0 \\ 0 & \widehat{M}_h & 0 \\ 0 & 0 & \widehat{M}_h \end{pmatrix}$, where the matrix \widehat{M}_h is diagonal and, for $i = 1, \dots, N$,

$$\widehat{M}_{hii} = \int_{\omega_i} H(\mathbf{x}) d\mathbf{x} = H_i |\omega_i| = H_i \sum_{K \in \mathcal{M}_i} \frac{|K|}{3},$$

where we denote by $H_i = \frac{1}{|\omega_i|} \int_{\omega_i} H(\mathbf{x}) d\mathbf{x}$ the average value of H in ω_i . We then retrieve [1, (4.13)].

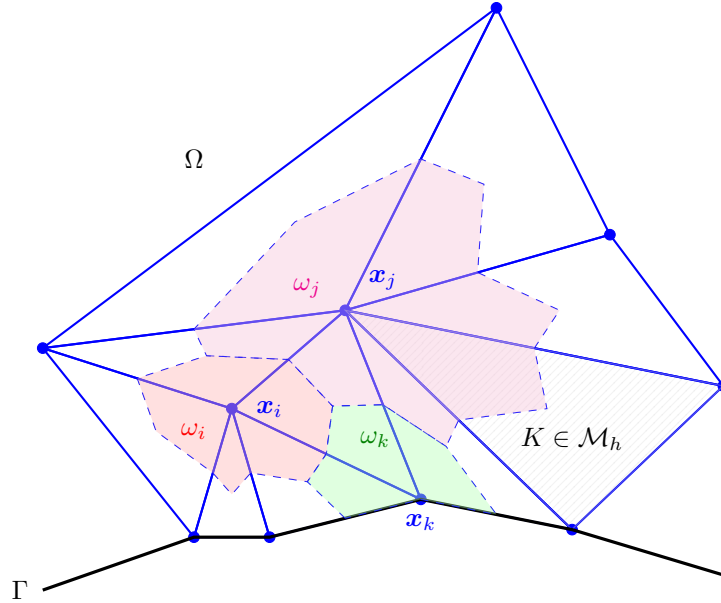


Figure 2: Some cells of the triangular mesh (solid blue lines), three cells of the barycentric dual mesh (dashed lines). The boundary Γ of Ω is the black line.

5.2.3 Verification of assumptions of Theorem 3.7

Lemma 5.2. *Conditions (19) hold when defining \mathbf{H}_h^D, L_h and Π_h as in Section 5.2.2 as well as the consistency condition (26).*

Proof. As in the proof of Lemma 5.1, we only give the main ideas of the proofs for the sake of brevity. We first observe that (26) is a standard property of \mathbb{P}^1 finite elements (see [20, Corollary 1.110]).

- The proof of consistency condition (19c) can be done using a dense subspace of \mathbf{H}_h^D . The control of the term with D is obtained using the definition of div_{sw}^α , standard property of \mathbb{P}^1 finite elements and the regularity assumptions on H and ζ . For the term involving Π_h , we use that for $\mathbf{v}_h \in \mathbf{H}_h^D$ each component $k \in \{1, 2, 3\}$ satisfies (see for instance [16, (8.18)])

$$\|\mathbf{v}_h^{(k)} - \Pi_h \mathbf{v}_h^{(k)}\|_{L^2(\Omega)} \leq h \|\nabla \mathbf{v}_h^{(k)}\|_{L^2(\Omega)^2}.$$

- In order to check the *inf-sup* condition (19e), which reads, in this setting, find $\beta > 0$ independent of n , such that $\forall n \in \mathbb{N}, \forall q_h \in L_h$,

$$\sup_{\mathbf{v}_h \in \mathbf{H}_h^D, \|\mathbf{v}_h\|_{\mathbf{H}^D} = 1} \int_{\Omega} \left(\text{div}(H \mathbf{v}_h^{(1),(2)})(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \cdot \nabla \zeta(\mathbf{x}) + \alpha \mathbf{v}_h^{(3)}(\mathbf{x}) \right) q_h(\mathbf{x}) \, d\mathbf{x} \geq \beta \|q_h\|_{L^2(\Omega)},$$

we notice that selecting $\mathbf{v}_h = (0, 0, q_h / \|q_h\|_{L^2(\Omega)})^T$ directly provides the relation with $\beta = \alpha$.

- We prove (19a) on each coordinates of any $\mathbf{v}_h = (\mathbf{v}_h^{(1)}, \mathbf{v}_h^{(2)}, \mathbf{v}_h^{(3)})^T \in \mathbf{H}_h^D$. Let $k \in \{1, 2, 3\}$, as above, we write $\mathbf{v}_h^{(k)}(\mathbf{x}) = \sum_{i=1}^N \alpha_i^{(k)} \phi_i(\mathbf{x})$ and $\Pi_h \mathbf{v}_h^{(k)}(\mathbf{x}) = \sum_{i=1}^N \alpha_i^{(k)} \widehat{\phi}_i(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. Thus we can write,

$$\|\mathbf{v}_h^{(k)}\|_{L^2(\Omega)}^2 = \sum_{K \in \mathcal{M}_h} \int_K \left(\sum_{i \in \mathcal{V}_K} \alpha_i^{(k)} \phi_i(\mathbf{x}) \right)^2 d\mathbf{x}, \quad \|\Pi_h \mathbf{v}_h^{(k)}\|_{L^2(\Omega)}^2 = \sum_{K \in \mathcal{M}_h} \int_K \left(\sum_{i \in \mathcal{V}_K} \alpha_i^{(k)} \widehat{\phi}_i(\mathbf{x}) \right)^2 d\mathbf{x},$$

where $\mathcal{V}_K = \{i \in I \mid \mathbf{x}_i \in K\}$. But, due to the mass lumping, we can write that, for any $K \in \mathcal{M}$,

$$\int_K \left(\sum_{i \in \mathcal{V}_K} \alpha_i^{(k)} \widehat{\phi}_i(\mathbf{x}) \right)^2 d\mathbf{x} = \int_K \left(\alpha_{i_1}^{(k)} \widehat{\phi}_{i_1}(\mathbf{x}) + \alpha_{i_2}^{(k)} \widehat{\phi}_{i_2}(\mathbf{x}) + \alpha_{i_3}^{(k)} \widehat{\phi}_{i_3}(\mathbf{x}) \right)^2 d\mathbf{x} = \frac{|K|}{3} \boldsymbol{\alpha}_K^T \boldsymbol{\alpha}_K,$$

with $\boldsymbol{\alpha}_K := (\alpha_{i_1}^{(k)}, \alpha_{i_2}^{(k)}, \alpha_{i_3}^{(k)})^T$ and where $\mathcal{V}_K := \{i_1, i_2, i_3\}$ is the set of indices of the vertices of the triangular cell K . On the other hand we have,

$$\int_K \left(\sum_{i \in \mathcal{V}_K} \alpha_i^{(k)} \phi_i(\mathbf{x}) \right)^2 d\mathbf{x} = \boldsymbol{\alpha}_K^T M_K \boldsymbol{\alpha}_K,$$

where $M_K = \frac{|K|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. The matrix M_K is symmetric positive definite and satisfies,

$$\lambda_{\min} \boldsymbol{\alpha}_K^T \boldsymbol{\alpha}_K \leq \boldsymbol{\alpha}_K^T M_K \boldsymbol{\alpha}_K \leq \lambda_{\max} \boldsymbol{\alpha}_K^T \boldsymbol{\alpha}_K,$$

where λ_{\min} (respectively λ_{\max}) is the minimal (respectively maximal) eigen value of M_K . We have $\lambda_{\min} = \frac{|K|}{12}$ and $\lambda_{\max} = \frac{|K|}{3}$. Combining the previous results, we obtain,

$$\|\mathbf{v}_h^{(k)}\|_{L^2(\Omega)}^2 \leq \|\Pi_h \mathbf{v}_h^{(k)}\|_{L^2(\Omega)}^2 \leq 4 \|\mathbf{v}_h^{(k)}\|_{L^2(\Omega)}^2.$$

This proves (19a) with $\rho = 2$.

- The proof of (19b) follows the same ideas as that of Lemma 5.1. Let $\mathbf{v}_h \in \mathbf{H}_h^D$ such that $\mathbf{v}_h^{(k)}(\mathbf{x}) = \sum_{i \in I} \alpha_i^{(k)} \phi_i(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ and $k \in \{1, 2, 3\}$. We study, for any function $\bar{\mathbf{v}} \in C^1(\bar{\Omega})^3$, the quantity,

$$I_k = \int_{\Omega} \left(\mathbf{v}_h^{(k)}(\mathbf{x}) - \Pi_h \mathbf{v}_h^{(k)}(\mathbf{x}) \right) \bar{\mathbf{v}}(\mathbf{x})^{(k)} d\mathbf{x} = \sum_{i=1}^N \alpha_i^{(k)} \int_{\Omega} \left(\phi_i(\mathbf{x}) - \widehat{\phi}_i(\mathbf{x}) \right) \bar{\mathbf{v}}(\mathbf{x})^{(k)} d\mathbf{x}.$$

Since, for all $i \in I$, $\int_{\Omega} \phi_i(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \widehat{\phi}_i(\mathbf{x}) d\mathbf{x} = |\omega_i|$, we can write,

$$\begin{aligned} \left| \int_{\Omega} (\phi_i(\mathbf{x}) - \widehat{\phi}_i(\mathbf{x})) \overline{\mathbf{v}}(\mathbf{x})^{(k)} d\mathbf{x} \right| &= \left| \int_{\Omega} (\phi_i(\mathbf{x}) - \widehat{\phi}_i(\mathbf{x})) (\overline{\mathbf{v}}(\mathbf{x})^{(k)} - \overline{\mathbf{v}}(\mathbf{x}_i)^{(k)}) d\mathbf{x} \right| \\ &\leq \int_{\Omega} |\phi_i(\mathbf{x}) - \widehat{\phi}_i(\mathbf{x})| \underbrace{|\overline{\mathbf{v}}(\mathbf{x})^{(k)} - \overline{\mathbf{v}}(\mathbf{x}_i)^{(k)}|}_{\leq h \|\nabla \overline{\mathbf{v}}^{(k)}\|_{L^\infty(\Omega)}} d\mathbf{x}. \end{aligned}$$

Thus we have,

$$|I_k| \leq 2h \|\nabla \overline{\mathbf{v}}^{(k)}\|_{L^\infty(\Omega)} \left(\sum_{i=1}^N |\alpha_i^{(k)}| |\omega_i| \right).$$

Then, using the Cauchy-Schwarz inequality, we obtain,

$$\begin{aligned} |I_k| &\leq 2h \|\nabla \overline{\mathbf{v}}^{(k)}\|_{L^\infty(\Omega)} \underbrace{\left(\sum_{i=1}^N |\omega_i| \right)^{\frac{1}{2}}}_{\leq |\Omega|^{\frac{1}{2}}} \underbrace{\left(\sum_{i=1}^N |\omega_i| (\alpha_i^{(k)})^2 \right)^{\frac{1}{2}}}_{= \|\Pi_h \mathbf{v}_h^{(k)}\|_{L^2(\Omega)}}. \end{aligned}$$

The preceding inequality immediately implies (19b). □

Remark 5.3. In [18, Chapter 2], the convergence has already been established, using a specific method of proof specific. Whereas our proof based on Theorem 3.7 can be reused easily for other numerical approaches.

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A Properties of mixed bilinear forms

In this section, we present and prove a theorem on a general bilinear form. This result is used in the proofs of Theorem 2.2 (continuous setting) and 3.2 (discrete setting). Thanks to the definitions of \mathbf{H}^D and \mathbf{D} given by (9)-(10), and under Assumptions (8), we assume the following,

$\mathbb{A} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbb{R}$, and $\mathbf{a} : L \times L \rightarrow \mathbb{R}$ are bilinear forms such that there exist $0 < \underline{\alpha} \leq \overline{\alpha}$ with

$$\underline{\alpha} \|\overline{\mathbf{u}}\|_{\mathbf{L}}^2 \leq \mathbb{A}(\overline{\mathbf{u}}, \overline{\mathbf{u}}) \text{ and } |\mathbb{A}(\overline{\mathbf{u}}, \overline{\mathbf{v}})| \leq \overline{\alpha} \|\overline{\mathbf{u}}\|_{\mathbf{L}} \|\overline{\mathbf{v}}\|_{\mathbf{L}}, \text{ for all } \overline{\mathbf{u}}, \overline{\mathbf{v}} \in \mathbf{H}^D, \quad (63a)$$

$$\underline{\alpha} \|\overline{\mathbf{p}}^V\|_L^2 \leq \mathbf{a}(\overline{\mathbf{p}}, \overline{\mathbf{p}}) \text{ and } |\mathbf{a}(\overline{\mathbf{p}}, \overline{\mathbf{q}})| \leq \overline{\alpha} \|\overline{\mathbf{p}}\|_L \|\overline{\mathbf{q}}\|_L, \text{ for all } \overline{\mathbf{p}}, \overline{\mathbf{q}} \in L. \quad (63b)$$

Let $\beta > 0$ and let $\widetilde{L} \subseteq L$ and $\widetilde{\mathbf{H}}^D \subseteq \mathbf{H}^D$ be closed subspaces such that

$$\forall \widetilde{\mathbf{q}} \in \widetilde{L}, \sup_{\overline{\mathbf{v}} \in \widetilde{\mathbf{H}}^D, \|\overline{\mathbf{v}}\|_{\mathbf{H}^D} = 1} \langle \widetilde{\mathbf{q}}^{\perp V}, \mathbf{D}\overline{\mathbf{v}} \rangle_L \geq \beta \|\widetilde{\mathbf{q}}^{\perp V}\|_L. \quad (64)$$

Theorem A.1. Under Assumptions (8), (63) and (64), we denote by $\mathcal{H} = L \times \mathbf{H}^D$ the Hilbert space endowed with the norm $\|(\overline{\mathbf{p}}, \overline{\mathbf{u}})\|_{\mathcal{H}}^2 = \|\overline{\mathbf{p}}\|_L^2 + \|\overline{\mathbf{u}}\|_{\mathbf{L}}^2 + \|\mathbf{D}\overline{\mathbf{u}}\|_L^2$. Let $E : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be the bilinear form defined by

$$E((\overline{\mathbf{p}}, \overline{\mathbf{u}}), (\overline{\mathbf{q}}, \overline{\mathbf{v}})) = \mathbb{A}(\overline{\mathbf{u}}, \overline{\mathbf{v}}) - \langle \overline{\mathbf{p}}, \mathbf{D}\overline{\mathbf{v}} \rangle_L + \langle \overline{\mathbf{q}}, \mathbf{D}\overline{\mathbf{u}} \rangle_L + \mathbf{a}(\overline{\mathbf{p}}, \overline{\mathbf{q}}). \quad (65)$$

Then there exists $\theta > 0$, only depending on $\underline{\alpha}$, $\bar{\alpha}$ and β such that

$$\forall (\tilde{q}, \tilde{v}) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}}, \quad \sup_{(\tilde{p}, \tilde{u}) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}}, \|\tilde{p}, \tilde{u}\|_{\mathcal{H}} \leq 1} E((\tilde{p}, \tilde{u}), (\tilde{q}, \tilde{v})) \geq \theta \left(\|\tilde{v}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}}^2 + \|\tilde{q}\|_{\mathbf{L}}^2 \right)^{1/2}, \quad (66)$$

and

$$\forall (\tilde{p}, \tilde{u}) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}}, \quad \sup_{(\tilde{q}, \tilde{v}) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}}, \|\tilde{q}, \tilde{v}\|_{\mathcal{H}} \leq 1} E((\tilde{p}, \tilde{u}), (\tilde{q}, \tilde{v})) \geq \theta \left(\|\tilde{u}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{L}} D \tilde{u}\|_{\mathbf{L}}^2 + \|\tilde{p}\|_{\mathbf{L}}^2 \right)^{1/2}, \quad (67)$$

where $\mathcal{P}_{\tilde{L}}$ denotes the orthogonal projection on the closed space \tilde{L} .

Remark A.2. In the use of Theorem A.1 in the proof of 2.2, we let $\tilde{L} = L$, $\widetilde{\mathbf{H}}^{\mathbf{D}} = \mathbf{H}^{\mathbf{D}}$ and $\mathcal{P}_{\tilde{L}} = \text{Id}_L$ and then Theorem (A.1) allows to obtain the same existence and uniqueness result as [12, Corollary 0.1]. On the contrary, in the use of Theorem A.1 in the proof of Theorem 3.2, since the right-hand side of inequalities (66) and (67) include the terms $\|\mathcal{P}_{\tilde{L}} D \cdot\|_{\mathbf{L}}^2$ instead of $\|D \cdot\|_{\mathbf{L}}^2$, we get a weaker estimate, leading to weaker convergence properties.

Proof. We remark that $E(\cdot, \cdot)$ is bilinear continuous on $\mathcal{H} \times \mathcal{H}$. Let us prove (66) for a given $(\tilde{q}, \tilde{v}) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}} \setminus \{(0, 0)\}$. In the following proof, for $i \in \{1, 2, 3, 4\}$, we define quantities $(\tilde{p}_i, \tilde{u}_i) \in \tilde{L} \times \widetilde{\mathbf{H}}^{\mathbf{D}}$ such that $\|(\tilde{p}_i, \tilde{u}_i)\|_{\mathcal{H}} \leq C_i \|(\tilde{q}, \tilde{v})\|_{\mathcal{H}}$, with C_i only depending on $\bar{\alpha}, \underline{\alpha}, \beta$.

- Let $(\tilde{p}_1, \tilde{u}_1) = (\tilde{q}, \tilde{v})$. Then

$$E((\tilde{p}_1, \tilde{u}_1), (\tilde{q}, \tilde{v})) = \mathbb{A}(\tilde{v}, \tilde{v}) + \mathbf{a}(\tilde{q}, \tilde{q}) \geq \underline{\alpha} \|\tilde{v}\|_{\mathbf{L}}^2 + \underline{\alpha} \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}^2. \quad (68)$$

- Let $(\tilde{p}_2, \tilde{u}_2) = (0, \hat{\mathbf{u}}_2 \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}})$ where $\hat{\mathbf{u}}_2 \in \widetilde{\mathbf{H}}^{\mathbf{D}}$ is such that $\|\hat{\mathbf{u}}_2\|_{\mathbf{H}^{\mathbf{D}}} = 1$ and $\langle \tilde{q}, D \hat{\mathbf{u}}_2 \rangle_{\mathbf{L}} \geq \frac{\beta}{2} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}$ thanks to the (64). This leads to

$$E((\tilde{p}_2, \tilde{u}_2), (\tilde{q}, \tilde{v})) = \mathbb{A}(\mathbf{u}_2, \tilde{v}) + \langle D \mathbf{u}_2, \tilde{q}^{\perp \mathbf{V}} \rangle_{\mathbf{L}} + \langle D \mathbf{u}_2, \tilde{q}^{\mathbf{V}} \rangle_{\mathbf{L}}.$$

By construction of \tilde{u}_2 , using (63a) and the Cauchy-Schwarz inequality, we can write

$$E((\tilde{p}_2, \tilde{u}_2), (\tilde{q}, \tilde{v})) \geq \frac{\beta}{2} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 - \bar{\alpha} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}} \|\tilde{v}\|_{\mathbf{L}} - \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}} \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}.$$

Thanks to the Young inequality, we can write

$$\bar{\alpha} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}} \|\tilde{v}\|_{\mathbf{L}} \leq \frac{\beta}{8} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 + \frac{2\bar{\alpha}^2}{\beta} \|\tilde{v}\|_{\mathbf{L}}^2,$$

and

$$\|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}} \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}} \leq \frac{\beta}{8} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 + \frac{2}{\beta} \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}^2.$$

This leads to

$$E((\tilde{p}_2, \tilde{u}_2), (\tilde{q}, \tilde{v})) \geq \frac{\beta}{4} \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 - \frac{2\bar{\alpha}^2}{\beta} \|\tilde{v}\|_{\mathbf{L}}^2 - \frac{2}{\beta} \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}^2. \quad (69)$$

- Let $(\tilde{p}_3, \tilde{u}_3) = (-\mathcal{P}_{\tilde{L}} D \tilde{v}, 0)$. Then $E((\tilde{p}_3, \tilde{u}_3), (\tilde{q}, \tilde{v})) = \|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}}^2 + \mathbf{a}(\mathcal{P}_{\tilde{L}}(D \tilde{v}), \tilde{q})$. Using (63b) and the Cauchy-Schwarz inequality, we have

$$E((\tilde{p}_3, \tilde{u}_3), (\tilde{q}, \tilde{v})) \geq \|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}}^2 - \bar{\alpha} \|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}} \|\tilde{q}\|_{\mathbf{L}}.$$

The Young inequality again allows to write $E((\tilde{p}_3, \tilde{u}_3), (\tilde{q}, \tilde{v})) \geq \frac{\|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}}^2}{2} - \frac{\bar{\alpha}^2}{2} \|\tilde{q}\|_{\mathbf{L}}^2$. Using $\|\tilde{q}\|_{\mathbf{L}}^2 = \|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 + \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}^2$, we deduce

$$E((\tilde{p}_3, \tilde{u}_3), (\tilde{q}, \tilde{v})) \geq \frac{\|\mathcal{P}_{\tilde{L}} D \tilde{v}\|_{\mathbf{L}}^2}{2} - \frac{\bar{\alpha}^2}{2} (\|\tilde{q}^{\perp \mathbf{V}}\|_{\mathbf{L}}^2 + \|\tilde{q}^{\mathbf{V}}\|_{\mathbf{L}}^2). \quad (70)$$

- For $\theta_2, \theta_3 > 0$, we define $(\tilde{p}_4, \tilde{\mathbf{u}}_4) = (\tilde{p}_1, \tilde{\mathbf{u}}_1) + \theta_2(\tilde{p}_2, \tilde{\mathbf{u}}_2) + \theta_3(\tilde{p}_3, \tilde{\mathbf{u}}_3)$. By using (68)+ $\theta_2 \times$ (69)+ $\theta_3 \times$ (70), this leads to

$$\begin{aligned} E((\tilde{p}_4, \tilde{\mathbf{u}}_4), (\tilde{q}, \tilde{\mathbf{v}})) &\geq (\underline{\alpha} - \theta_2 \frac{2\bar{\alpha}^2}{\beta}) \|\tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \frac{\theta_3}{2} \|\mathcal{P}_{\tilde{\mathbf{L}}} \mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}}^2 \\ &\quad + (\underline{\alpha} - \theta_2 \frac{2}{\beta} - \theta_3 \frac{\bar{\alpha}^2}{2}) \|\tilde{q}^V\|_{\mathbf{L}}^2 + \left(\frac{\beta}{4} \theta_2 - \theta_3 \frac{\bar{\alpha}^2}{2} \right) \|\tilde{q}^{\perp V}\|_{\mathbf{L}}^2. \end{aligned}$$

By defining $\theta_2 = \min\left(\frac{\beta \underline{\alpha}}{4\bar{\alpha}^2}, \frac{\beta \underline{\alpha}}{8}\right)$, $\theta_3 = \min\left(\frac{\beta \theta_2}{4\bar{\alpha}^2}, \frac{\underline{\alpha}}{2\bar{\alpha}^2}\right)$ and $\theta_4 = \min\left(\frac{\underline{\alpha}}{2}, \frac{\theta_3}{2}, \frac{\beta}{8} \theta_2\right)$, we can write,

$$E((\tilde{p}_4, \tilde{\mathbf{u}}_4), (\tilde{q}, \tilde{\mathbf{v}})) \geq \theta_4 (\|\tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{\mathbf{L}}} \mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\tilde{q}^V\|_{\mathbf{L}}^2 + \|\tilde{q}^{\perp V}\|_{\mathbf{L}}^2) = \theta_4 (\|\tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{\mathbf{L}}} \mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\tilde{q}\|_{\mathbf{L}}^2).$$

Since, by construction, we have $\|(\tilde{p}_4, \tilde{\mathbf{u}}_4)\|_{\mathcal{H}} \leq \|(\tilde{q}, \tilde{\mathbf{v}})\|_{\mathcal{H}} + \theta_2 \|\tilde{q}\|_{\mathbf{L}} + \theta_3 \|\mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}} \leq \tilde{\theta} \|(\tilde{q}, \tilde{\mathbf{v}})\|_{\mathcal{H}}$ with $\tilde{\theta} = 1 + \theta_2 + \theta_3$, we can define $(\tilde{p}, \tilde{\mathbf{u}}) = \frac{(\tilde{p}_4, \tilde{\mathbf{u}}_4)}{\tilde{\theta} \|(\tilde{q}, \tilde{\mathbf{v}})\|_{\mathcal{H}}}$. We then have $\|(\tilde{p}, \tilde{\mathbf{u}})\|_{\mathcal{H}} \leq 1$ and

$$E((\tilde{p}, \tilde{\mathbf{u}}), (\tilde{q}, \tilde{\mathbf{v}})) \geq \frac{\theta_4}{\tilde{\theta}} \frac{\|\tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{\mathbf{L}}} \mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\tilde{q}\|_{\mathbf{L}}^2}{\|(\tilde{q}, \tilde{\mathbf{v}})\|_{\mathcal{H}}} \geq \frac{\theta_4}{\tilde{\theta}} \left(\|\tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\mathcal{P}_{\tilde{\mathbf{L}}} \mathbf{D} \tilde{\mathbf{v}}\|_{\mathbf{L}}^2 + \|\tilde{q}\|_{\mathbf{L}}^2 \right)^{1/2},$$

which shows (66).

We then deduce (67) from (66) by exchanging the arguments and using $-\mathbf{D}$ instead of \mathbf{D} . \square

B Equivalence between (25) and an LBB condition

We notice that Problem (2.1) in [12] or Problem (96) in [19] correspond to Problem (18) in the particular case $V = \{0\}$, $a \equiv 0$ and $\Pi_h = \text{Id}_{\mathbf{L}}$. Therefore we explore in this section the equivalence between condition (25) and the standard conditions (2.2)-(2.4)-(2.5) in [12] or (97)-(98)-(99) in [19], which correspond to (19e)-(72)-(73) below. We also remind the definitions of $\mathbf{H}^{\mathbf{D}}$ and \mathbf{D} given by (9)-(10).

Lemma B.1 (Equivalence between (25) and LBB condition).

Under Assumptions (8) (with the additional hypothesis that $V = \{0\}$, $a \equiv 0$) and (11), let $(L_{h_n})_{n \in \mathbb{N}}$ and $(\mathbf{H}_{h_n}^{\mathbf{D}})_{n \in \mathbb{N}}$ be families of finite dimensional spaces such that, for any $n \in \mathbb{N}$: $L_{h_n} \subset L$ and $\mathbf{H}_{h_n}^{\mathbf{D}} \subset \mathbf{H}^{\mathbf{D}}$ and such that the discrete inf-sup condition (19e) holds. Let us denote

$$W_{h_n} = \{\mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^{\mathbf{D}}, \forall q_{h_n} \in L_{h_n}, \langle \mathbf{D} \mathbf{v}_{h_n}, q_{h_n} \rangle_{\mathbf{L}} = 0\}.$$

Then the following assertions are equivalent.

(i) *There exists $C_B > 0$ such that*

$$\forall n \in \mathbb{N}, \forall \mathbf{v}_{h_n} \in \mathbf{H}_{h_n}^{\mathbf{D}}, \|\mathbf{D} \mathbf{v}_{h_n}\|_{\mathbf{L}} \leq C_B (\|\mathbf{v}_{h_n}\|_{\mathbf{L}} + \|\mathcal{P}_{L_{h_n}} \mathbf{D} \mathbf{v}_{h_n}\|_{\mathbf{L}}), \quad (71)$$

which is similar to (25).

(ii) *There exist $\alpha > 0$ and $\alpha^* > 0$ such that*

$$\forall n \in \mathbb{N}, \forall \mathbf{u}_{h_n} \in W_{h_n}, \sup_{\mathbf{v}_{h_n} \in W_{h_n}, \|\mathbf{v}_{h_n}\|_{\mathbf{H}^{\mathbf{D}}}=1} \langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \mathbf{v}_{h_n} \rangle_{\mathbf{L}} \geq \alpha \|\mathbf{u}_{h_n}\|_{\mathbf{H}^{\mathbf{D}}} \quad (72)$$

and

$$\forall n \in \mathbb{N}, \forall \mathbf{v}_{h_n} \in W_{h_n}, \sup_{\mathbf{u}_{h_n} \in W_{h_n}, \|\mathbf{u}_{h_n}\|_{\mathbf{H}^{\mathbf{D}}}=1} \langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \mathbf{v}_{h_n} \rangle_{\mathbf{L}} \geq \alpha^* \|\mathbf{v}_{h_n}\|_{\mathbf{H}^{\mathbf{D}}}. \quad (73)$$

Proof. (i) \implies (ii)

Let $n \in \mathbb{N}$ and $\mathbf{u}_{h_n} \in W_{h_n}$. By definition of W_{h_n} , we have $\langle D\mathbf{u}_{h_n}, q_{h_n} \rangle_L = 0$ for an arbitrary $q_{h_n} \in L_{h_n}$. Since $\langle D\mathbf{u}_{h_n}, q_{h_n} \rangle_L = \langle \mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}, q_{h_n} \rangle_L$, we obtain $\langle \mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}, q_{h_n} \rangle_L = 0$. The particular choice $q_{h_n} = \mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}$ leads to $\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n} = 0$ and therefore (71) yields $\|D\mathbf{u}_{h_n}\|_L \leq C_B \|\mathbf{u}_{h_n}\|_L$. From Lemma 2.1, we get $\frac{\alpha}{\alpha^2} \|\mathbf{u}_{h_n}\|_L^2 \leq \langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \mathbf{u}_{h_n} \rangle_L$. Hence

$$\|\mathbf{u}_{h_n}\|_{\mathbf{H}^D}^2 = \|D\mathbf{u}_{h_n}\|_L^2 + \|\mathbf{u}_{h_n}\|_L^2 \leq (C_B^2 + 1) \|\mathbf{u}_{h_n}\|_L^2 \leq (C_B^2 + 1) \frac{\bar{\alpha}^2}{\alpha} \langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \mathbf{u}_{h_n} \rangle_L.$$

This gives, assuming $\mathbf{u}_{h_n} \neq 0$,

$$\|\mathbf{u}_{h_n}\|_{\mathbf{H}^D} \leq (C_B^2 + 1) \frac{\bar{\alpha}^2}{\alpha} \langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \frac{\mathbf{u}_{h_n}}{\|\mathbf{u}_{h_n}\|_{\mathbf{H}^D}} \rangle_L,$$

which proves (72), and

$$\|\mathbf{u}_{h_n}\|_{\mathbf{H}^D} \leq (C_B^2 + 1) \frac{\bar{\alpha}^2}{\alpha} \langle \mathbf{A}^{-1} \frac{\mathbf{u}_{h_n}}{\|\mathbf{u}_{h_n}\|_{\mathbf{H}^D}}, \mathbf{u}_{h_n} \rangle_L,$$

which proves (73).

(ii) \implies (i)

Let us first remark that, for any $\mathbf{u}_{h_n} \in W_{h_n}$, using (72), we can select $\mathbf{v}_{h_n} \in W_{h_n}$ with $\|\mathbf{v}_{h_n}\|_{\mathbf{H}^D} = 1$ (hence $\|\mathbf{v}_{h_n}\|_L \leq 1$) and $\langle \mathbf{A}^{-1} \mathbf{u}_{h_n}, \mathbf{v}_{h_n} \rangle_L \geq \alpha \|\mathbf{u}_{h_n}\|_{\mathbf{H}^D}$. We then have, again using Lemma 2.1,

$$\alpha \|\mathbf{u}_{h_n}\|_{\mathbf{H}^D} \leq \frac{1}{\alpha} \|\mathbf{u}_{h_n}\|_L \|\mathbf{v}_{h_n}\|_L \leq \frac{1}{\alpha} \|\mathbf{u}_{h_n}\|_L. \quad (74)$$

We now consider any $\mathbf{u}_{h_n} \in \mathbf{H}_{h_n}^D$, that we decompose in the orthogonal sum (in \mathbf{H}^D) by $\mathbf{u}_{h_n} = \mathbf{u}_{h_n}^W + \mathbf{u}_{h_n}^{\perp W}$ where $\mathbf{u}_{h_n}^W \in W_{h_n}$ and $\mathbf{u}_{h_n}^{\perp W} \in W_{h_n}^\perp$. The assumption $V = \{0\}$ implies $V^\perp = L$ and thus (19e) is identical to [19, (85)], thus we can apply [19, Lemma 5.1]. Together with [19, (87)], for $q_{h_n} \in L_{h_n}$ such that $\|q_{h_n}\|_{L_{h_n}} = 1$ and

$$\langle D\mathbf{u}_{h_n}^{\perp W}, q_{h_n} \rangle_L = \max \{ \langle D\mathbf{u}_{h_n}^{\perp W}, \tilde{q}_{h_n} \rangle_L, \tilde{q}_{h_n} \in L_{h_n}, \|\tilde{q}_{h_n}\|_{L_{h_n}} = 1 \},$$

this lemma allows us to write,

$$\beta \|\mathbf{u}_{h_n}^{\perp W}\|_{\mathbf{H}^D} \leq \langle D\mathbf{u}_{h_n}^{\perp W}, q_{h_n} \rangle_L = \langle \mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}^{\perp W}, q_{h_n} \rangle_L \leq \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}^{\perp W}\|_L = \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L. \quad (75)$$

Using the properties of norms, we have

$$\|D\mathbf{u}_{h_n}\|_L \leq \|\mathbf{u}_{h_n}\|_{\mathbf{H}^D} \leq \|\mathbf{u}_{h_n}^W\|_{\mathbf{H}^D} + \|\mathbf{u}_{h_n}^{\perp W}\|_{\mathbf{H}^D}.$$

Together with (74) and (75), this gives,

$$\|D\mathbf{u}_{h_n}\|_L \leq \frac{1}{\alpha \underline{\alpha}} \|\mathbf{u}_{h_n}^W\|_L + \frac{1}{\beta} \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L.$$

Since $\|\mathbf{u}_{h_n}^W\|_L \leq \|\mathbf{u}_{h_n}\|_L + \|\mathbf{u}_{h_n}^{\perp W}\|_L$, we obtain again using (75)

$$\|D\mathbf{u}_{h_n}\|_L \leq \frac{1}{\alpha \underline{\alpha}} \left(\|\mathbf{u}_{h_n}\|_L + \frac{1}{\beta} \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L \right) + \frac{1}{\beta} \|\mathcal{P}_{L_{h_n}} D\mathbf{u}_{h_n}\|_L,$$

which gives (71). □

C Density result

This section is devoted to the proof of density used in the consistency step of the proof of Theorem 4.8.

Lemma C.1 (Density of H_S in H^G). *Under Assumptions (8) and (11b), let us define $H_S := \{\varphi \in H^G, \mathbf{AG}\varphi \in \mathbf{H}^D\}$ where D and \mathbf{H}^D are defined by (9)-(10). Then H_S is dense in H^G .*

Proof. Let $\varphi \in H^G$ be given. For a given integer $n > 0$, we define $\varphi_n \in H^G$ such that

$$\frac{1}{n} \langle \mathbf{AG}\varphi_n, \mathbf{G}\psi \rangle_{\mathbf{L}} + \langle \varphi_n, \psi \rangle_L = \langle \varphi, \psi \rangle_L, \quad \forall \psi \in H^G. \quad (76)$$

The existence and uniqueness of φ_n is a consequence of the Lax-Milgram theorem, since the coercivity of the bilinear form $\frac{1}{n} \langle \mathbf{AG}\cdot, \mathbf{G}\cdot \rangle_{\mathbf{L}} + \langle \cdot, \cdot \rangle_L$ in H^G is a consequence of Hypothesis (11b). Letting $\psi = \varphi_n$ and using (11b), we get

$$\frac{1}{n} \underline{\alpha} \|\mathbf{G}\varphi_n\|_{\mathbf{L}}^2 + \|\varphi_n\|_L^2 \leq \|\varphi\|_L \|\varphi_n\|_L. \quad (77)$$

We therefore obtain that $\|\varphi_n\|_L^2 \leq \|\varphi\|_L \|\varphi_n\|_L$, yielding

$$\|\varphi_n\|_L \leq \|\varphi\|_L. \quad (78)$$

Owing to the separability hypothesis (8a), equation (78) implies the existence of a subsequence of $(\varphi_n)_{n>0}$, again denoted $(\varphi_n)_{n>0}$, which weakly converges in L to some function $\bar{\varphi}$.

We deduce from (77)-(78) that

$$\|\mathbf{G}\varphi_n\|_{\mathbf{L}} \leq \frac{\sqrt{n}}{\sqrt{\underline{\alpha}}} \|\varphi\|_L. \quad (79)$$

For a given $\psi \in H^G$, we have, using (79), that

$$\frac{1}{n} |\langle \mathbf{AG}\varphi_n, \mathbf{G}\psi \rangle_{\mathbf{L}}| \leq \frac{\bar{\alpha}}{\sqrt{n}\sqrt{\underline{\alpha}}} \|\varphi\|_L \|\mathbf{G}\psi\|_{\mathbf{L}}.$$

This provides

$$\lim_{n \rightarrow \infty} \frac{1}{n} \langle \mathbf{AG}\varphi_n, \mathbf{G}\psi \rangle_{\mathbf{L}} = 0,$$

and enables us to let $n \rightarrow \infty$ in (76). We thus obtain $\langle \bar{\varphi}, \psi \rangle_L = \langle \varphi, \psi \rangle_L$. The density of H^G in L implies that $\bar{\varphi} = \varphi$. By uniqueness of the limit, we deduce the weak convergence of the whole sequence to φ . Moreover, since (78) implies

$$\|\varphi_n - \varphi\|_L^2 = \|\varphi_n\|_L^2 - 2\langle \varphi_n, \varphi \rangle_L + \|\varphi\|_L^2 \leq 2(\|\varphi\|_L^2 - \langle \varphi_n, \varphi \rangle_L),$$

we get, by letting $n \rightarrow \infty$ in the preceding inequality, that

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_L^2 = 0,$$

hence showing the convergence of φ_n to φ in L and not only its weak convergence.

From (76), we deduce by definition (10) that $D\mathbf{AG}\varphi_n = n(\varphi_n - \varphi) \in H^G \subset L$, which proves that $\varphi_n \in H_S$ and that $D\mathbf{AG}\varphi_n \in H^G$. We can therefore let $\psi = -D\mathbf{AG}\varphi_n$ in (76). We thus obtain by applying three times (10) that

$$\frac{1}{n} \|D\mathbf{AG}\varphi_n\|_L^2 + \langle \mathbf{G}\varphi_n, \mathbf{AG}\varphi_n \rangle_{\mathbf{L}} = \langle \mathbf{G}\varphi, \mathbf{AG}\varphi_n \rangle_{\mathbf{L}}.$$

Hence we obtain that

$$\langle \mathbf{G}\varphi_n, \mathbf{AG}\varphi_n \rangle_{\mathbf{L}} \leq \langle \mathbf{G}\varphi, \mathbf{AG}\varphi_n \rangle_{\mathbf{L}}, \quad (80)$$

leading, owing to (11b), to

$$\underline{\alpha} \|\mathbf{G}\varphi_n\|_{\mathbf{L}}^2 \leq \bar{\alpha} \|\mathbf{G}\varphi\|_{\mathbf{L}} \|\mathbf{G}\varphi_n\|_{\mathbf{L}}.$$

This provides,

$$\|G\varphi_n\|_{\mathbf{L}} \leq \frac{\bar{\alpha}}{\alpha} \|G\varphi\|_{\mathbf{L}}.$$

Applying again the separability hypothesis (8a), this implies the existence of a subsequence of $(\varphi_n)_{n>0}$, again denoted $(\varphi_n)_{n>0}$, such that $(G\varphi_n)_{n>0}$ weakly converges in \mathbf{L} to some function \bar{G} . Letting $n \rightarrow \infty$ in the relation

$$\langle G\varphi_n, \bar{\mathbf{u}} \rangle_{\mathbf{L}} + \langle \varphi_n, D\bar{\mathbf{u}} \rangle_{\mathbf{L}} = 0, \quad \forall \bar{\mathbf{u}} \in \mathbf{H}^{\mathbf{D}},$$

we obtain

$$\langle \bar{G}, \bar{\mathbf{u}} \rangle_{\mathbf{L}} + \langle \varphi, D\bar{\mathbf{u}} \rangle_{\mathbf{L}} = 0, \quad \forall \bar{\mathbf{u}} \in \mathbf{H}^{\mathbf{D}},$$

which shows that $\bar{G} = G\varphi$. Here again, the uniqueness of the limit implies the weak convergence of the whole sequence. Since (11b) and (80) imply

$$\begin{aligned} \underline{\alpha} \|G(\varphi_n - \varphi)\|_{\mathbf{L}}^2 &\leq \langle \mathbf{A}G(\varphi_n - \varphi), G(\varphi_n - \varphi) \rangle_{\mathbf{L}} \\ &= \langle \mathbf{A}G\varphi_n, G\varphi_n \rangle_{\mathbf{L}} - 2\langle \mathbf{A}G\varphi_n, G\varphi \rangle_{\mathbf{L}} + \langle \mathbf{A}G\varphi, G\varphi \rangle_{\mathbf{L}} \\ &\leq \langle \mathbf{A}G\varphi, G\varphi \rangle_{\mathbf{L}} - \langle \mathbf{A}G\varphi_n, G\varphi \rangle_{\mathbf{L}}, \end{aligned}$$

we get, by letting $n \rightarrow \infty$ in the preceding inequality, that

$$\lim_{n \rightarrow \infty} \|G(\varphi_n - \varphi)\|_{\mathbf{L}}^2 = 0.$$

We thus obtain the convergence of $G\varphi_n$ to $G\varphi$ in \mathbf{L} and not only its weak convergence. This concludes the proof of the density of H_S in H^G . \square

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