



**HAL**  
open science

# NSD Edge-Colourings with Strong Assignment Constraints

Julien Bensmail, Leandro Montero

► **To cite this version:**

Julien Bensmail, Leandro Montero. NSD Edge-Colourings with Strong Assignment Constraints. 2025.  
hal-04947954

**HAL Id: hal-04947954**

**<https://hal.science/hal-04947954v1>**

Preprint submitted on 14 Feb 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# NSD Edge-Colourings with Strong Assignment Constraints

Julien Bensmail<sup>a</sup>, Leandro Montero<sup>b</sup>

<sup>a</sup>Université Côte d'Azur, CNRS, Inria, I3S, France

<sup>b</sup>IMT Atlantique, LS2N - CNRS, La Chantrerie, Nantes, France

---

## Abstract

In this work, we introduce and study a combination of two types of graph edge-colourings, namely strong edge-colourings (in which every two edges at distance at most 2 must be assigned distinct colours) and neighbour-sum-distinguishing edge-colourings (in which adjacent edges must be assigned distinct colours, and every two adjacent vertices must be incident to distinct sums of colours). In particular, we investigate how the smallest number of colours in such combined edge-colourings behaves. For several classes of graphs, we prove that such edge-colourings are very close from strong edge-colourings, in the sense that no additional colours are required. For others classes of graphs, we prove that introducing more colours is sometimes necessary. We conjecture that, in general, designing this new type of edge-colourings should always be possible provided we are allowed to introduce and assign a constant number of additional colours. We prove this conjecture for a few classes of graphs, including trees and graphs of bounded maximum degree.

*Keywords:* strong edge-colouring; NSD edge-colouring; graph colouring.

---

## 1. Introduction

In this work, we essentially introduce and study a combination of **strong edge-colourings** and **NSD edge-colourings**. So that our investigations are clear, we thus begin with recalling what all these colouring notions are about. For the sake of keeping the current introduction short, below we voluntarily focus on some of the most important aspects only; for more details on the literature, we refer the interested reader to the most recent works on these topics and notions. Be aware that, later in this work, results of less importance will be recalled, as they connect to the results we establish.

### *Strong edge-colourings*

Let  $G$  be a graph. Recall that a  $k$ -edge-colouring  $\phi$  of  $G$  is an assignment  $\phi : E(G) \rightarrow \{1, \dots, k\}$  of colours to the edges of  $G$ . In case  $\phi$  has the property that no two adjacent edges (*i.e.*, sharing a vertex) of  $G$  are assigned the same colour, then  $\phi$  is said *proper*. The smallest  $k \geq 1$  such that  $G$  admits proper  $k$ -edge-colourings is called the *chromatic index* of  $G$ , and is denoted by  $\chi'(G)$ . Obviously,  $\chi'(G)$  is always at least  $\Delta(G)$ , the maximum degree of a vertex of  $G$ , and, by a celebrated and influential result of Vizing [19], it is known that  $\chi'(G)$  cannot exceed  $\Delta(G) + 1$ . Thus, the chromatic index of any graph  $G$  is always one of two possible values, namely  $\Delta(G)$  and  $\Delta(G) + 1$ .

Strong edge-colourings are a strengthening of proper edge-colourings to pairs of edges that are within a larger range. Namely, an edge-colouring  $\phi$  of a graph  $G$  is *strong* if

every two edges at distance at most  $2^1$  are assigned distinct colours. The smallest  $k \geq 1$  in a strong  $k$ -edge-colouring of  $G$  is denoted by  $\chi'_s(G)$ , and is called the *strong chromatic index* of  $G$ . These notions were first introduced by Fouquet and Jolivet in [11]. Obviously,  $\chi'(G) \leq \chi'_s(G)$  holds for every graph  $G$ , and the most natural question is how distant from  $\chi'(G)$  can  $\chi'_s(G)$  be, in general. Since bounds on  $\chi'$  are more naturally expressed in terms of the maximum degree parameter, so are those investigated for  $\chi'_s$ ; this leads us to perhaps the most influential open question on the strong chromatic index of graphs:

**Conjecture 1.1** (Erdős, Nešetřil [8]). *For every graph  $G$  with  $\Delta(G) = \Delta$ , we have*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \text{if } \Delta \text{ is odd.} \end{cases}$$

Conjecture 1.1 is still widely open in general, as it was mostly proved to hold for  $\Delta = 3$  only, see [3, 16]. For  $\Delta = 4$ , the conjectured bound in 20, and the best bound established to date is 21, see [17]. This apart, several refinements of Conjecture 1.1 for restricted classes of graphs have also been quite investigated, for planar graphs and bipartite graphs in particular. Let us mention as well that, even under various restrictions on  $G$ , determining  $\chi'(G)$  and  $\chi'_s(G)$  for a given graph  $G$  is an NP-complete problem, see *e.g.* [14, 15].

#### *NSD edge-colourings*

Let  $G$  be a graph, and  $\phi$  be a proper  $k$ -edge-colouring of  $G$ . For every vertex  $v$  of  $G$ , one can consider the edges incident to  $v$ , and compute the *sum*, denoted by  $\sigma(v)$ , of the colours assigned to these edges. In case  $\phi$  has the property that  $\sigma(u) \neq \sigma(v)$  holds for every two adjacent vertices  $u$  and  $v$  of  $G$ , we say  $\phi$  is *neighbour-sum-distinguishing* (NSD for short). Last, we denote by  $\chi'_{\text{NSD}}(G)$  the *NSD chromatic index* of  $G$ , being the smallest  $k \geq 1$  (if any) such that  $G$  admits NSD  $k$ -edge-colourings.

NSD edge-colourings were introduced more recently, by Flandrin, Marczyk, Przybyło, Sacle, and Woźniak in [10]. An important point to raise right away, is that  $\chi'_{\text{NSD}}(G)$  is well defined for a connected graph  $G$  if and only if  $G$  is not  $K_2$ , the complete graph of order 2. For this reason, in this context, we focus on *nice graphs*, being those graphs with no connected components isomorphic to  $K_2$ . As earlier, NSD edge-colourings are restricted proper edge-colourings, so for every nice graph  $G$  we have  $\chi'(G) \leq \chi'_{\text{NSD}}(G)$ ; so, once again, bounds on  $\chi'_{\text{NSD}}$  are better expressed as functions of the maximum degree, and one can wonder how larger than  $\chi'$  can  $\chi'_{\text{NSD}}$  be, in general. The main conjecture here, is that we should always have  $\chi'_{\text{NSD}}(G) \leq \Delta(G) + 2$  for a nice graph  $G$ , unless  $G = C_5$ , the cycle of length 5, for which  $\chi'_{\text{NSD}}(C_5) = \Delta(C_5) + 3$ . The general conjecture is thus:

**Conjecture 1.2** (Flandrin, Marczyk, Przybyło, Sacle, Woźniak [10]). *If  $G$  is a nice graph with  $\Delta(G) = \Delta$ , then  $\chi'_{\text{NSD}}(G) \leq \Delta + 3$ .*

The best bound on  $\chi'_{\text{NSD}}$  we are aware of to date, is that  $\chi'_{\text{NSD}}(G) \leq 2\Delta + 2$  holds for any nice graph  $G$  with maximum degree  $\Delta$  (see [20]). This apart, most works on the topic focused on particular classes of graphs, such as graphs of bounded maximum degree and planar graphs. Algorithmic aspects have also been considered in [5]; namely, determining  $\chi'_{\text{NSD}}(G)$  for a given graph  $G$  is NP-complete in general.

---

<sup>1</sup>Two edges  $e$  and  $f$  of a graph are at distance at most 2 if either  $e$  and  $f$  share a vertex (distance 1), or a vertex incident to  $e$  is adjacent to one incident to  $f$  (distance 2).

### *Combining strong and NSD edge-colourings*

In this work, we introduce and study a new type of edge-colourings, being essentially a combination of both strong edge-colourings and NSD edge-colourings. Namely, given a graph  $G$  and a  $k$ -edge-colouring  $\phi$  of  $G$ , we say  $\phi$  is *neighbour-sum-distinguishing* (and *strong*) (NSDS for short) if  $\phi$  is both strong and NSD. That is, 1) no two edges at distance at most 2 are assigned the same colour by  $\phi$ , and 2) no two adjacent vertices are incident to the same sum of colours. Now, we define the *NSDS chromatic index*  $\chi'_{\text{NSDS}}(G)$  of  $G$  as the smallest  $k \geq 1$  (if any) such that  $G$  admits NSDS  $k$ -edge-colourings.

Our main intent, through studying NSDS edge-colourings, is to determine how the strong and NSD requirements may interact in general. Obviously, we have  $\max\{\chi'_s(G), \chi'_{\text{NSD}}(G)\} \leq \chi'_{\text{NSDS}}(G)$  for all nice graphs  $G$ . Generally speaking, the strong chromatic index and the NSD chromatic index behave in rather different ways, w.r.t. the maximum degree  $\Delta$ . Namely, comparing Conjectures 1.1 and 1.2, the former parameter is rather a quadratic function of  $\Delta$ , while the latter one is more of a linear one. Consequently, a first guess one may have, is that strong edge-colourings require so many colours that, perhaps, among all optimal strong edge-colourings (*i.e.*, assigning the least number of colours possible) there might be one that is also NSD. While we first prove, later on in this work, that this is indeed true for several classes of graphs, we also prove that there exist arbitrarily large connected graphs  $G$  with  $\chi'_s(G) < \chi'_{\text{NSDS}}(G)$ . In our opinion, this fact legitimates the study of NSDS edge-colourings as such.

This work is organised as follows. To get a first look into NSDS edge-colourings, we start by considering common classes of graphs in Section 2, for which we prove that the NSDS chromatic index is always equal to the strong chromatic index. In Section 3, we prove that this phenomenon is not systematic, that is, there exist graphs  $G$  for which  $\chi'_s(G) < \chi'_{\text{NSDS}}(G)$  holds. Consequently, we give a conjecture on the precise relationship between the strong chromatic index and the NSDS chromatic index in Section 4, which conjecture we support by providing some general approaching bounds in the same section, and then better bounds for specific classes of graphs in Section 5. We end up in Section 6 with concluding words, including perspectives for further work on the topic.

## **2. Early results and classes of graphs**

In this section, we give first observations and results on the NSDS chromatic index. To begin with, we clarify which graphs we are dealing with, *i.e.*, which graphs  $G$  have  $\chi'_{\text{NSDS}}(G)$  defined. Obviously, all graphs admit strong edge-colourings. Regarding NSD edge-colourings however, recall that a connected graph admits such edge-colourings if and only if it is not isomorphic to  $K_2$ . Thus,  $K_2$  is an example of connected graph for which the NSDS chromatic index is not finite. It turns out this is the only exception, and that  $\chi'_{\text{NSDS}}(G)$  is defined if and only if  $G$  is nice, *i.e.*, has no  $K_2$  as a connected component.

**Observation 2.1.** *If  $G$  is a graph, then  $\chi'_{\text{NSDS}}(G)$  is well defined if and only if  $G$  is nice.*

This can be proved *e.g.* through greedy arguments. Such arguments, however, would be very reminiscent of upcoming ones we will use later to prove Theorems 4.2 and 4.3. Thus, in case the reader requires more formal arguments at this point, we refer them to the proofs of these two later results. To be more precise, the proof of Theorem 4.2 is inspired by the proof that  $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$  holds for any graph  $G$  with maximum degree  $\Delta$ , which essentially consists in colouring edges greedily.

As for very early bounds on the NSDS chromatic index, as mentioned before, we can directly apply its definition to relate it with the maximum degree, the chromatic index, and the strong chromatic index; namely:

**Observation 2.2.** *If  $G$  is a nice graph with  $\Delta(G) = \Delta$ , then*

$$\Delta \leq \chi'(G) \leq \chi'_s(G) \leq \chi'_{\text{NSDS}}(G).$$

In particular, in any graph  $G$ , for every edge  $uv$ , all edges incident to either  $u$  or  $v$  must receive distinct colours by a strong edge-colouring. As a result,  $\max_{uv \in E(G)} d(u) + d(v) - 1$  is an obvious lower bound on the strong chromatic index of  $G$ .

**Observation 2.3.** *If  $G$  is a nice graph, then*

$$\max_{uv \in E(G)} d(u) + d(v) - 1 \leq \chi'_s(G) \leq \chi'_{\text{NSDS}}(G).$$

In the rest of this section, we determine the exact value of the NSDS chromatic index of some classes of graphs of interest. We begin with cycles; recall that, for any  $n \geq 3$ , we denote by  $C_n$  the cycle of length  $n$ . This case is more interesting than it might seem, as, here, all of the strong, NSD, and NSDS chromatic indices coincide.

**Theorem 2.4.** *For any cycle  $C_n$  with  $n \geq 3$ , we have:*

$$\chi'_{\text{NSDS}}(C_n) = \chi'_s(C_n) = \chi'_{\text{NSD}}(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod{3}; \\ 5 & \text{if } n = 5; \\ 4 & \text{otherwise.} \end{cases}$$

*Proof.* Let us denote by  $e_0, \dots, e_{n-1}$  the consecutive edges of  $C_n$  (where, throughout, operations over the subscripts are modulo  $n$ ), and consider any NSDS edge-colouring  $\phi$  of  $C_n$ . For every  $i \in \{0, \dots, n-1\}$ , we note that  $\phi(e_i)$  must be different from all of  $\phi(e_{i-2}), \phi(e_{i-1}), \phi(e_{i+1}), \phi(e_{i+2})$  (for  $\phi$  to be strong), and to  $\phi(e_{i-2})$  and  $\phi(e_{i+2})$  (for  $\phi$  to be NSD; namely, if  $e_i = uv$ , so that  $u$  is not in conflict with its second neighbour, and similarly for  $v$ , respectively). Thus, we can equivalently see  $\phi$  as a proper vertex-colouring of  $G$ , the graph obtained from  $C_n$  by adding a vertex  $v_i$  for every edge  $e_i$ , and adding an edge  $v_i v_j$  whenever  $e_i$  and  $e_j$  are at distance at most 2 in  $C_n$ . Note that  $G$  is nothing but  $C_n^2$ , the square of  $C_n$ . By all these arguments, we deduce that  $\chi'_{\text{NSDS}}(C_n) = \chi(C_n^2)$ , from which we derive the claimed equalities right away.  $\square$

We now establish some connection between NSDS edge-colourings and other types of distinguishing labellings, from which we get to deduce the exact value of the NSDS chromatic index for a lot more classes of nice graphs. This is based on the fact that, for classes of nice graphs such as complete graphs and complete bipartite graphs, we are actually dealing with  $2K_2$ -free graphs (*i.e.*, in which any two edges are at distance at most 2), for which the strong chromatic index is exactly the number of edges. In particular, for such graphs, since, by a strong edge-colouring, every colour is assigned to exactly one edge only, and the number of assigned colours is rather large, one may think there should always be a way to permute colours so that neighbours are also distinguished by sums.

In other words, for  $2K_2$ -free graphs on  $m$  edges, finding a NSDS  $m$ -edge-colouring is equivalent to assigning each of labels  $1, \dots, m$  to exactly one edge, so that every two adjacent vertices have distinct sums. Graphs admitting such labellings have actually been studied in literature under the term *locally antimagic graphs*, introduced in [4, 6].

**Observation 2.5.** *If  $G$  is a  $2K_2$ -free locally antimagic graph, then*

$$\chi'_{\text{NSDS}}(G) = \chi'_s(G) = |E(G)|.$$

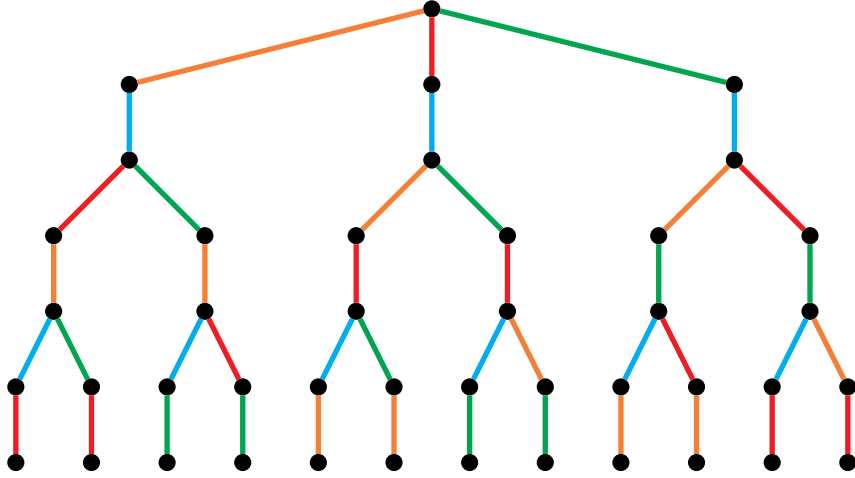


Figure 1: A graph  $G$  with  $\chi'_{\text{NSDS}}(G) \geq \chi'_s(G) + 1$ . The four edge colours (blue, red, green, orange) depict the unique strong 4-edge-colouring of  $G$ , up to permuting colours.

The main conjecture from [4, 6] is that all nice graphs should be locally antimagic. This conjecture was proved independently by Haslegrave [12] and Szabo Lyngsie and Zhong [18]. Combining this and Observation 2.5, we thus obtain that  $\chi'_{\text{NSDS}}(G) = \chi'_s(G) = |E(G)|$  holds for  $G$  being a nice complete graph, or a nice complete bipartite graph, in which case  $G$  is obviously  $2K_2$ -free. Another consequence is that  $\chi'_{\text{NSDS}}(G) = \chi'_s(G) = |E(G)|$  holds for  $G = C_5^\Delta$  ( $\Delta \geq 3$ ), where  $C_5^\Delta$ , called a *blown-up*  $C_5$ , is, in brief, a graph of maximum degree  $\Delta$  believed to be the only connected graph for which the bound in Conjecture 1.1 is attained [8]. More generally speaking, the fact that all nice graphs are locally antimagic implies that  $\chi'_{\text{NSDS}}(G) \leq |E(G)|$  holds for every nice graph  $G$ .

From comments above regarding the “worst” graphs w.r.t the strong chromatic index, one may think that, perhaps, the equality  $\chi'_{\text{NSDS}}(G) = \chi'_s(G)$  holds for every nice graph  $G$ . In the next section, we prove this presumption is wrong in general.

### 3. The strong and NSDS chromatic indices can differ

We here prove that there exist nice graphs  $G$  with  $\chi'_s(G) < \chi'_{\text{NSDS}}(G)$ . That is, we exhibit arbitrarily large connected graphs  $G$  with  $\chi'_{\text{NSDS}}(G) \geq \chi'_s(G) + 1$ . A remarkable fact is that we prove such phenomena do not occur solely in graphs with low degree.

**Theorem 3.1.** *For every  $\Delta \geq 3$ , there exist arbitrarily large trees  $G$  with  $\Delta(G) = \Delta$  and  $\chi'_{\text{NSDS}}(G) \geq \chi'_s(G) + 1$ .*

*Proof.* Let  $\Delta \geq 3$  be fixed, and set  $s = 2\Delta - 2$ . Consider  $G$ , the tree obtained as follows:

- start from an initial vertex  $r$ , and make it adjacent to  $\Delta$  leaves;
- then repeat the following process at least  $s$  times:
  1. denoting by  $L$  the (current) set of leaves, for every  $l \in L$  make  $l$  adjacent to  $\Delta - 2$  new leaves;
  2. denoting by  $L'$  the resulting set of  $|L|(\Delta - 2)$  leaves, for every  $l' \in L'$  make  $l'$  adjacent to  $\Delta - 1$  new leaves.

In the resulting graph  $G$ , note that any edge not incident to a leaf is incident to a vertex with degree  $\Delta - 1$  and to one with degree  $\Delta$ . Also, for every non-leaf vertex  $v$  of  $G$ , we have either  $\text{dist}(r, v) \equiv 0 \pmod{2}$  and  $d(v) = \Delta$ , or  $\text{dist}(r, v) \equiv 1 \pmod{2}$  and  $d(v) = \Delta - 1$ .

Since  $G$  has vertices with degree  $\Delta$  and  $\Delta - 1$ , we have  $\chi'_s(G) = \Delta + (\Delta - 1) - 1 = 2\Delta - 2 = s$ , according to [9]. Now, due to the diameter of  $G$ , we have the following properties (in which, for any vertex  $u$  and any proper edge-colouring, we denote by  $C(u)$  the set of (pairwise distinct) colours assigned to the edges incident to  $u$ ):

**Claim 3.2.** *Let  $k \in \{1, \dots, s\}$  be any integer, and  $X \cup Y$  be any partition of  $\{1, \dots, s\} \setminus \{k\}$  with  $|X| = |Y| + 1$ . Then, by any strong  $s$ -edge-colouring  $\phi$  of  $G$ , there is an edge  $uv$  with:*

- $d(u) = \Delta$  and  $d(v) = \Delta - 1$ ;
- $\phi(uv) = k$ ; and
- $C(u) \setminus \{k\} = X$  and  $C(v) \setminus \{k\} = Y$ .

*Proof of the claim.* Note that, for every vertex  $v$  adjacent to  $r$  (thus with  $d(v) = \Delta - 1$ ), by  $\phi$  we have  $C(r) \cup C(v) = \{1, \dots, s\}$ . From this, we deduce that if  $r'$  is another vertex with degree  $\Delta$  adjacent to  $v$ , then  $C(r') = C(r) \setminus \{\phi(rv)\} \cup \{\phi(vr')\}$ , where  $\phi(vr') \notin C(r)$ . In other words,  $C(r)$  and  $C(r')$  have exactly  $\Delta - 1$  elements in common. As a consequence, since all vertices at distance exactly 2 from  $r$  are of degree  $\Delta$ , then for any  $x \in C(r)$  and  $y \notin C(r)$ , there is a vertex  $r'$  with degree  $\Delta$  at distance exactly 2 from  $r$  such that  $C(r') = C(r) \setminus \{x\} \cup \{y\}$ .

Now, starting from  $u = r$ , until  $C(u) = X \cup \{k\}$  we can repeatedly find (at distance 2) a new vertex  $u'$  with degree  $\Delta$  such that  $C(u')$  has one more element in common with  $X \cup \{k\}$  than  $C(u)$  does, and set  $u'$  as the next  $u$ . Since, in  $G$ , there are vertices of degree  $\Delta$  at distance  $i$  from  $r$  for all even distances in  $\{2, \dots, 2s\}$ , by repeating this process we must find, within  $s$  steps, a vertex  $u$  of degree  $\Delta$  such that  $C(u) = X \cup \{k\}$ . Now, assuming  $v$  is the unique neighbour of  $u$  such that  $\phi(uv) = k$ , we have  $C(u) \cup C(v) = \{1, \dots, s\}$ , and  $C(u) \cap C(v) = \{k\}$ . Thus,  $C(u) \setminus \{k\} = X$  and  $C(v) \setminus \{k\} = \{1, \dots, s\} \setminus X \cup \{k\} = Y \cup \{k\}$ . The claim thus holds.  $\diamond$

We also need the following tool:

**Claim 3.3.** *For every even integer  $2n \geq 4$ , there is a  $k \in \{1, \dots, 2n\}$  such that  $\{1, \dots, 2n\} \setminus \{k\}$  admits a partition  $X \cup Y$  with  $|X| = n$ ,  $|Y| = n - 1$ , and  $\sum_{i \in X} i = \sum_{i \in Y} i$ .*

*Proof of the claim.* The proof is by induction on  $n \geq 2$ . For  $n = 2$ , we can consider e.g.  $k = 4$ ,  $X = \{1, 2\}$ , and  $Y = \{3\}$ ; while, for  $n = 3$ , we can consider e.g.  $k = 5$ ,  $X = \{1, 3, 4\}$ , and  $Y = \{2, 6\}$ . Now, assuming the claim holds for all values up to some even  $n - 2$ , we prove it for  $n$ . By the induction hypothesis, for  $n' = n - 2$ , the claim holds for some  $k'$ ,  $X'$ , and  $Y'$ . The crucial point, now, is that  $\{1, \dots, 2n\} \setminus \{1, \dots, 2n'\} = \{2n - 3, 2n - 2, 2n - 1, 2n\}$ ; thus, a solution for  $n$  is e.g.  $k = k'$ ,  $X = X' \cup \{2n - 3, 2n\}$ , and  $Y = Y' \cup \{2n - 1, 2n - 2\}$ .  $\diamond$

We are now ready for the proof (see Figure 1 for an illustration). By Claim 3.3, whatever  $\Delta \geq 3$  is, there is a  $k \in \{1, \dots, s\}$  such that  $\{1, \dots, s\} \setminus \{k\}$  partitions into  $X \cup Y$ , where  $|X| = \Delta - 1$ ,  $|Y| = \Delta - 2$ , and  $\sum_{i \in X} i = \sum_{i \in Y} i$ . Now, by Claim 3.2 and by construction of  $G$ , by any strong  $s$ -edge-colouring  $\phi$  there must be an edge  $uv$  where  $d(u) = \Delta$ ,  $d(v) = \Delta - 1$ ,  $\phi(uv) = k$ , and  $C(u) \setminus \{k\} = X$  and  $C(v) \setminus \{k\} = Y$ . Thus,  $\sigma(u) = \sigma(v)$  and  $\phi$  cannot be NSDS. Therefore,  $\chi'_{\text{NSDS}}(G) > s = \chi'_s(G)$ .  $\square$

As will be proved in a next section (through Theorem 5.1), for nice trees  $G$  we actually have  $\chi'_{\text{NSDS}}(G) \leq \chi'_s(G) + 1$ . Thus, for the trees  $G$  we consider in the proof of Theorem 3.1

we actually have  $\chi'_{\text{NSDS}}(G) = \chi'_s(G) + 1$ . In particular, from our construction we do not obtain that there exist graphs  $G$  with  $\chi'_{\text{NSDS}}(G) \geq \chi'_s(G) + 2$ .

#### 4. A conjecture, and some bounds on the NSDS chromatic index

As established in the previous section, it is not true that we always have equality between the strong and NSDS chromatic indices. Still, from our first investigations, we believe that strong edge-colourings require so many colours that, in general, it should be possible to change/permute colours locally to also achieve the NSD requirement. Note as well that the graphs we designed in the proof of Theorem 3.1 are very particular, in the sense that they admit a unique optimal strong edge-colouring (up to permuting colours), and that the vertex degrees are mostly the same (making more probable sum conflicts by an edge-colouring). From this, we believe the following conjecture is reasonable:

**Conjecture 4.1.** *There is some positive constant  $c \geq 1$ , such that, for all nice graphs  $G$ , we have  $\chi'_{\text{NSDS}}(G) \leq \chi'_s(G) + c$ .*

Towards Conjecture 4.1, in what follows we establish two general upper bounds on  $\chi'_{\text{NSDS}}$ . The first one is a quadratic function of the maximum degree  $\Delta$ , and is essentially obtained through greedy arguments. The second one is a function of the strong chromatic index and of the chromatic number<sup>2</sup>. The first bound is not too distant from the naive upper bound on the strong chromatic index, as it only differs from it by an additive term  $2\Delta$ . The second bound is particularly interesting in contexts of graphs for which the strong chromatic index and chromatic number are low (see later for some examples).

We start off with our bound based on greedy arguments:

**Theorem 4.2.** *If  $G$  is a nice graph with  $\Delta(G) = \Delta$ , then*

$$\chi'_{\text{NSDS}}(G) \leq 2\Delta^2 - 1.$$

*Proof.* The proof is by induction on  $|V(G)|$ . As a base case, it is easy to check that the claim holds whenever  $|V(G)| = 3$ ; thus, we proceed with proving the general case.

Set  $k = 2\Delta^2 - 1$ . We may assume that  $G$  is connected. Since  $G$  is nice, it has a vertex  $u$  with degree  $d$  at least 2. We denote by  $v_1, \dots, v_d$  the  $d$  neighbours of  $u$ , and set  $G' = G - u$ . Note that, isolated edges of  $G'$  apart, we have that  $G'$  is nice and admits a NSDS  $k$ -edge-colouring  $\phi'$  by the induction hypothesis. Our goal is to extend  $\phi'$  to a NSDS  $k$ -edge-colouring of  $G$  by properly assigning colours to the edges incident  $u$  and to the isolated edges of  $G'$ . Note that each  $e$  of these isolated edges is of the form either  $v_i v_j$  for some distinct  $i, j \in \{1, \dots, d\}$ , or  $v_i w$  where  $w \notin \{u, v_1, \dots, v_d\}$  is a degree-1 vertex. We first extend  $\phi'$  to any such  $e$  (if any) by assigning any colour (in  $\{1, \dots, k\}$ ) to  $e$ . For now, note that, setting  $e = xy$ , we have  $\sigma(x) = \sigma(y)$ , which will be solved upon colouring the  $uv_i$ 's. Note also that, at this point, we still have that the partial edge-colouring of  $G$  is strong (this is because any two isolated edges of  $G'$  are at distance at least 3 in  $G$ ).

We claim that there must be some  $\alpha \in \{1, \dots, k\}$  so that, through assigning colour  $\alpha$  to any  $uv_i$ , we preserve the strongness of the resulting partial colouring, and, besides isolated edges, no two adjacent vertices of  $G'$  have the same sum. Regarding the first requirement, note that  $\alpha$  must avoid any colour assigned to the at most  $\Delta - 1 + (\Delta - 1)^2 = \Delta^2 - \Delta$  edges at distance at most 2 from  $v_i$  in  $G'$ , and any colour assigned to the at most  $(\Delta - 1)^2 = \Delta^2 - 2\Delta + 1$

---

<sup>2</sup>Recall that, for a graph  $G$ , the *chromatic number*  $\chi(G)$  is the smallest  $k \geq 1$  such that  $V(G)$  can be partitioned into  $k$  stable sets (forming a *proper  $k$ -vertex-colouring* of  $G$ ).



edges incident, in  $G'$ , to the other  $v_i$ 's. Regarding the second requirement,  $\alpha$  must also guarantee that the resulting sum  $\sigma(v_i)$  is different from the sums of the at most  $\Delta - 1$  neighbours of  $v_i$  in  $G'$ . Thus, all in all, there are at most

$$(\Delta^2 - \Delta) + (\Delta^2 - 2\Delta + 1) + (\Delta - 1) = 2\Delta^2 - 2\Delta$$

colours that cannot be assigned to  $uv_i$ .

Now consider every  $i = 1, \dots, d-1$  in order. For every  $i$  considered this way, we assign a colour to  $uv_i$  that fulfils all of the constraints above, and also that is different from the colours assigned to  $uv_1, \dots, uv_{i-1}$ . For a technical reason to be explained in what follows, for  $i = d-1$  we also choose the colour of  $uv_{d-1}$  so that the resulting partial sum of  $u$  is different from the current partial sum of  $v_d$ . Note that we can extend  $\phi'$  to  $uv_1, \dots, uv_{d-1}$  so that all these conditions are fulfilled, provided

$$k > (2\Delta^2 - 2\Delta) + (\Delta - 2) + 1 = 2\Delta^2 - \Delta - 1.$$

It remains to colour  $uv_d$ . Note that the fact that we currently have  $\sigma(u) \neq \sigma(v_d)$  guarantees we cannot get a conflict between  $u$  and  $v_d$ , whatever colour we assign to  $uv_d$ . Note however that, upon colouring  $uv_d$ , the sum of  $u$  gets fully determined, and it has to be different from all of  $\sigma(v_1), \dots, \sigma(v_{d-1})$ , which forbids at most  $\Delta - 1$  colours to be assigned to  $uv_d$ . Also, as earlier, the colour assigned to  $uv_d$  must be different from all colours assigned to  $uv_1, \dots, uv_{d-1}$ . Thus, in total, we can find an open colour for  $uv_d$  provided

$$k > (2\Delta^2 - 2\Delta) + 2(\Delta - 1) = 2\Delta^2 - 2.$$

Thus, we can always extend  $\phi'$  to  $G$ , and the claimed bound holds. In particular, let us mention that, through the process above, isolated edges of  $G'$  get treated naturally.  $\square$

We now prove our bound involving the strong chromatic index and the chromatic number. An important tool we will use here and later on, is Alon's Combinatorial Nullstellensatz [1], which reads as follows:

**Combinatorial Nullstellensatz.** *Let  $\mathbb{F}$  be an arbitrary field, and let  $f = f(x_1, \dots, x_n)$  be a polynomial in  $\mathbb{F}[x_1, \dots, x_n]$ . Suppose the total degree of  $f$  is  $\sum_{i=1}^n t_i$ , where each  $t_i$  is a nonnegative integer, and suppose the coefficient of  $\prod_{i=1}^n x_i^{t_i}$  is nonzero. If  $S_1, \dots, S_n$  are subsets of  $\mathbb{F}$  with  $|S_i| > t_i$ , then there are  $s_1 \in S_1, \dots, s_n \in S_n$  so that  $f(s_1, \dots, s_n) \neq 0$ .*

**Theorem 4.3.** *If  $G$  is a nice graph with  $\chi(G) = k$  and  $\chi'_s(G) = s$ , then  $\chi'_{\text{NSDS}}(G) \leq 2ks$ .*

*Proof.* We can assume  $G$  is connected. Let  $\phi$  be a strong  $s$ -edge-colouring of  $G$ , and  $\psi$  be a proper  $k$ -vertex-colouring of  $G$ . For every  $i \in \{1, \dots, s\}$ , we set  $\mathcal{C}_i = \{2k(i-1) + 1, \dots, 2k(i-1) + 2k\}$ . Note that every edge-colouring of  $G$  where each edge  $e$  is assigned a colour from  $\mathcal{C}_{\phi(e)}$  is strong. Our goal is to assign colours from  $\mathcal{C}_1, \dots, \mathcal{C}_s$  to the edges of  $G$  this way, in such a way that, for most vertices, sums are distinguished modulo  $k$ , which will be achieved by making sure that  $\sigma(v) \equiv \psi(v) \pmod{k}$  holds for all vertices  $v$ . Note that the  $\mathcal{C}_i$ 's form a pool of  $2ks$  colours.

Choose  $r$  any vertex of degree  $d \geq 2$  of  $G$ , and let  $T$  be spanning tree of  $G$  rooted at  $r$  where  $d_T(r) = d_G(r)$ . To begin with, we assign any colour in  $\mathcal{C}_{\phi(e)}$  to every edge  $e$  in  $E(G) \setminus E(T)$ . It remains to colour the edges of  $T$ . We consider the vertices of  $T$  one by one, in reverse order of their distance to  $r$  in  $T$ ; that is, we first consider the deepest vertices in  $T$ , then their parents, and so on. Let  $v$  be any vertex considered this way (*i.e.*, all vertices of  $T$  that are deeper than  $v$  have been considered earlier). Then all edges

incident to  $v$ , but one  $uv$ , where  $u$  is closer (in  $T$ ) to  $r$ , have been coloured. If  $u = r$ , then we leave things as is and consider upcoming arguments below. Otherwise,  $u \neq r$ , in which case we simply assign to  $uv$  a colour in  $\mathcal{C}_{\phi(uv)}$  that results in  $\sigma(v) \equiv \psi(v) \pmod{k}$ . This is always possible since  $\mathcal{C}_{\phi(uv)}$  contains  $2k$  consecutive values. And we pursue with another  $v$ .

We then get to the point where all edges of  $T$  but those  $rv_1, \dots, rv_d$  incident to  $r$  are coloured, and we have  $\sigma(v) \equiv \psi(v) \pmod{k}$  for all  $v \in V(T) \setminus \{r, v_1, \dots, v_d\}$ . Note that, by assigning a colour in  $\mathcal{C}_{\phi(rv_i)}$  to every  $v_i$  so that  $\sigma(v_i) \equiv \psi(v_i) \pmod{k}$ , we would almost be done; the only problem is that, proceeding this way, we have no guarantee to obtain  $\sigma(r) \equiv \psi(r) \pmod{k}$  as a result, thus to avoid having  $r$  involved in conflicts. Recall however that, for every  $i \in \{1, \dots, d\}$ , set  $\mathcal{C}_{\phi(rv_i)}$  contains  $2k$  consecutive values, so there are actually two values  $\alpha_i, \beta_i \in \mathcal{C}_{\phi(rv_i)}$  which, when assigned to  $rv_i$ , result in  $\sigma(v_i) \equiv \psi(v_i) \pmod{k}$ . We claim that, because  $d \geq 2$ , there is always a way to assign a colour in  $\{\alpha_i, \beta_i\}$  to every edge  $rv_i$  so that, as a result, we obtain  $\sigma(r) \notin \{\sigma(v_1), \dots, \sigma(v_d)\}$  (which would yield our conclusion, as all neighbours of  $r$  in  $G$  lie in  $T$ ). There are multiple ways to prove this, one of the simplest ones being through algebraic arguments. Namely, for every  $i \in \{1, \dots, d\}$ , let us denote by  $n_i \geq 0$  the current value of  $\sigma(v_i)$  (*i.e.*, without taking  $rv_i$ , which is uncoloured, into account), and let  $X_i$  be a variable modelling a colour assigned to  $rv_i$ . Note that, through assigning any colours  $x_1, \dots, x_d$  to  $rv_1, \dots, rv_d$ , respectively, the only way to get a sum conflict between  $r$  and some  $v_i$  is to have  $x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_d = n_i$ . So, all these constraints can be modelled through the polynomial

$$P(X_1, \dots, X_d) = \prod_{i=1}^d \left( \left( \sum_{j=1}^d X_j \right) - X_i - n_i \right).$$

Now, if there are values  $x_1, \dots, x_d$  in  $\{\alpha_1, \beta_1\}, \dots, \{\alpha_d, \beta_d\}$ , respectively, for  $X_1, \dots, X_d$ , respectively, such that  $P(x_1, \dots, x_d) \neq 0$ , then assigning colours  $x_1, \dots, x_d$  to  $rv_1, \dots, rv_d$ , respectively, yields our conclusion. Since, in the expansion of  $P$ , the monomial  $X_1 \dots X_d$  has strictly positive coefficient and is of maximum degree ( $d$ ), by Alon's Combinatorial Nullstellensatz such  $x_i$ 's can be picked for any choice of at least two values for each of them. So, we can colour the  $rv_i$ 's as desired, which concludes the proof.  $\square$

The upper bound in Theorem 4.2 is sometimes better than that of Theorem 4.3. For instance, nice subcubic graphs  $G$  (in general) verify  $\chi(G) \leq 3$  and  $\chi'_s(G) \leq 10$  (recall [3, 16]), so we obtain  $\chi'_{\text{NSDS}}(G) \leq 17$  by the former and  $\chi'_{\text{NSDS}}(G) \leq 60$  by the latter. However, in other cases the bound in Theorem 4.3 can be better than that of Theorem 4.2. For instance, nice planar graphs  $G$  with  $\Delta(G) = \Delta$  verify  $\chi(G) \leq 4$  and  $\chi'_s(G) \leq 4\Delta + 4$  (see [9]); so, we get that  $\chi'_{\text{NSDS}}(G) \leq 2\Delta^2 - 1$  by the former and  $\chi'_{\text{NSDS}}(G) \leq 32\Delta + 32$  by the latter.

## 5. Approaching Conjecture 4.1 for more classes of graphs

In this section, we provide better results towards Conjecture 4.1 for more classes of nice graphs. On the one hand, we provide a tight result for trees. On the other hand, we provide better results (yet probably not optimal) for subcubic graphs. In the latter case, we provide better bounds for both the general case, and sparse subcubic graphs.

### 5.1. Trees

We first prove that Conjecture 4.1 holds for trees, the additive term being here only 1. Let us recall that this is best possible in general, as there exist trees  $G$  with  $\chi'_{\text{NSDS}}(G) = \chi'_s(G) + 1$ , recall Theorem 3.1.

**Theorem 5.1.** *If  $G$  is a nice tree, then  $\chi'_{\text{NSDS}}(G) \leq \chi'_s(G) + 1$ .*

*Proof.* According to [9], because  $G$  is a tree, we have  $\chi'_s(G) = k = \max_{uv \in E(G)} d(u) + d(v) - 1$ . We prove that  $G$  admits a NSDS  $(k + 1)$ -edge-colouring, in the following way. We choose any vertex  $r$  of  $G$ , root  $G$  at  $r$ , and assign colours from the root towards the leaves. That is, we first assign colours to the edges incident to  $r$ , and then repeatedly consider a vertex  $v \neq r$  whose unique incident edge going to the parent  $u$  (closer to  $r$  than  $v$  is) has already been coloured, and assign colours to the other edges incident to  $v$ , *i.e.*, farther from  $r$ .

To begin with, we thus consider  $r$ , and assign to its  $d(r) \leq k$  incident edges any set of pairwise distinct colours from  $\{1, \dots, k + 1\}$ . As a result,  $\sigma(r)$  is fully determined, while, for every neighbour  $v$  of  $r$ , either  $d(v) = 1$  and  $\sigma(v)$  is fully determined, or  $d(v) \geq 2$  and other edges incident to  $v$  are not coloured yet. In the former case, we note that, since we are assigning strictly positive colours, we cannot have  $\sigma(r) = \sigma(v)$ . In the latter case, avoiding a conflict between  $r$  and  $v$  will be guaranteed later on, when treating  $v$ .

We now treat the general case, of a vertex  $v \neq r$  with parent  $u$ , such that  $uv$  has already been coloured, and  $d(v) - 1 \geq 1$  edges incident to  $v$  remain to be coloured. We denote by  $w_1, \dots, w_{d(v)-1}$  the children of  $v$ . So that we preserve the strongness of the edge-colouring we are designing, note that, among all already assigned colours, the colours we assign to  $vw_1, \dots, vw_{d(v)-1}$  must be different from those assigned to  $uv$  and to the  $d(u) - 1$  other edges incident to  $u$ . In total, this thus forbids the use of  $d(u)$  colours. However, we have  $|\{1, \dots, k + 1\}| \geq d(u) + d(v)$ ; thus, in  $\{1, \dots, k + 1\}$  there are  $d(v)$  colours  $\alpha_1, \dots, \alpha_{d(v)}$  which, when assigned to the  $vw_i$ 's, do not break the strongness of the edge-colouring obtained thus far. Note now that by assigning colours  $\alpha_1, \dots, \alpha_{d(v)-1}$  or  $\alpha_2, \dots, \alpha_{d(v)}$  to the  $vw_i$ 's, we get two distinct resulting values as  $\sigma(v)$ . We assign colours so that, for the eventual  $\sigma(v)$ , we obtain  $\sigma(u) \neq \sigma(v)$ . So, the partial edge-colouring remains strong at this point,  $u$  and  $v$  are not in conflict, and, regarding any  $w_i$ , either we have  $d(w_i) = 1$  and  $\sigma(w_i)$  is fully determined but we know that  $\sigma(v) \neq \sigma(w_i)$  since  $G$  is nice, or  $d(w_i) \geq 2$  and distinguishing  $v$  and  $w_i$  will be accomplished when treating  $w_i$  later on.

Once this process ends, we obtain a NSDS  $(k + 1)$ -edge-colouring of  $G$ .  $\square$

## 5.2. Subcubic graphs

Recall that a *subcubic graph* refers to a graph with maximum degree at most 3. If  $G$  is a subcubic graph, then  $\chi'_s(G) \leq 10$ , which is tight in general [3, 8, 16]. Regarding the NSDS chromatic index of subcubic graphs, the best bound we obtain from previous Theorems 4.2 and 4.3 is that  $\chi'_{\text{NSDS}}(G) \leq 17$  holds for every nice subcubic graph  $G$ . Recall that a tricky point is that, in nice subcubic graphs, the strong and NSDS chromatic indices are not always equal, by Theorem 3.1 (which indeed holds for subcubic graphs).

In what follows, we prove that  $\chi'_{\text{NSDS}}(G) \leq 16$  holds for every nice subcubic graph  $G$ , and that we even have  $\chi'_{\text{NSDS}}(G) \leq 10$  when  $G$  is somewhat sparse.

### 5.2.1. A better general bound

We first prove that  $\chi'_{\text{NSDS}}(G) \leq 16$  holds for all nice subcubic graphs  $G$ . Our upcoming proof borrows terminologies from previous works on the strong chromatic index of graphs with low maximum degree, namely [3, 7], which we recall now. Let  $u$  be any vertex of a graph  $G$ . For any  $v \in V(G)$ , we denote by  $\text{dist}_u(v)$  the distance from  $u$  to  $v$  in  $G$ . For an edge  $e = xy$ , we define  $\text{dist}_u(e)$  as  $\min\{\text{dist}_u(x), \text{dist}_u(y)\}$ . We say that an ordering  $(e_1, e_2, \dots)$  over the edges of  $G$  is *compatible with  $u$*  if  $\text{dist}_u(e_i) \geq \text{dist}_u(e_{i+1})$  for every  $i \geq 1$ . The same notions adapt if we consider any cycle  $C$  instead of  $u$ ; in particular, we directly infer the notations  $\text{dist}_C(v)$  and  $\text{dist}_C(e)$ , and of edge ordering compatible with  $C$ .

To prove our main result here, namely later Theorem 5.7, we mainly follow ideas from [7]. Namely, we first prove that if a nice subcubic graph  $G$  has a vertex  $u$  with low degree (1 or 2), then, following any ordering over the edges of  $G$  compatible with  $u$ , we can colour most edges of  $G$ , *i.e.*, all edges but those incident to  $u$ , in a desired way, with a “small” number of colours. We then prove that, perhaps with the use of a few additional colours, we can extend the edge-colouring to the edges incident to  $u$  in a correct way. Eventually, it remains to consider the cubic case, for which we essentially proceed the exact same way, except that it is a shortest cycle  $C$  (which can be assumed to exist, as otherwise Theorem 5.1 would apply) that plays the role of  $u$ .

We thus start off with a crucial lemma, stating that any nice subcubic graph can be almost completely edge-coloured in a NSDS way with “few” colours. Note that the statements of some of the upcoming results are inspired by some results from [7].

**Lemma 5.2.** *If  $G$  is a subcubic graph, then  $G$  has a NSDS edge-colouring that uses at most 12 colours except that it leaves uncoloured those edges incident to a single vertex. Precisely, regarding the partial NSD requirement, the only possible sum conflicts involve at least one vertex incident to uncoloured edges. Likewise, if  $C$  is a cycle of  $G$ , then  $G$  has a NSDS edge-colouring that uses at most 12 colours except that it leaves uncoloured the edges of  $C$ . And, here as well, the only possible sum conflicts involve at least one vertex incident to uncoloured edges.*

*Proof.* We consider the first part of the claim first. Let  $u$  be any vertex of  $G$ , and consider any ordering compatible with  $u$  over the edges of  $G$  that are not incident to  $u$ . Suppose we are currently dealing with an edge  $e = xy$ , where  $\text{dist}_u(x) \leq \text{dist}_u(y)$  and all edges before  $e$  following the ordering have already been coloured. To ease upcoming counting arguments, we consider two main cases:

- Assume first  $\text{dist}_u(x) < \text{dist}_u(y)$ . In particular, since  $x \neq u$ , there is another uncoloured edge  $xx'$  incident to  $x$  (possibly  $x' = u$ ) where  $x'$  appears later in the ordering (that is,  $\text{dist}_u(x') < \text{dist}_u(x)$ ). Let us count the number of constraints we need to take into account, when assigning a colour to  $e$ . First, from  $y$ 's side, there are at most 6 coloured edges at distance at most 2, thus at most 6 colours to avoid. Still on  $y$ 's side, we want to guarantee that the resulting sum of  $y$  gets different from the sums of the at most 2 neighbours of  $y$  different from  $x$ ; thus, at most 2 other constraints. On  $x$ 's side, there are at most 3 coloured edges at distance at most 2 not incident to  $y$ , which thus bring at most 3 more constraints. This apart, there are no further constraints. Recall indeed that assigning a colour to  $e$  does not settle  $\sigma(x)$ . In particular, either  $x' \neq u$ , in which case making  $x$  distinguished from its neighbours will be achieved later on, when treating later edges; or  $x' = u$ , in which case, by the conditions of the statement, it is not required that  $x$  is distinguished from its neighbours. Hence, in total, there are thus at most 11 constraints for colouring  $e$ , which means that there is an open colour for  $e$  in  $\{1, \dots, 12\}$ .
- Assume second that  $\text{dist}_u(x) = \text{dist}_u(y)$ . This means that there are two edges  $xx'$  and  $yy'$  later in the ordering, where possible  $x' = y'$  (and even  $u = x' = y'$ ), and  $\text{dist}_u(x'), \text{dist}_u(y') < \text{dist}_u(x), \text{dist}_u(y)$ . Here, in terms of colouring constraints, there are at most 3 colours at distance at most 2 on  $x'$ 's side, and similarly on  $y'$ 's side; thus, 6 forbidden colours in total. Furthermore, each of  $x$  and  $y$  has at most one neighbour not in  $\{x, y, x', y'\}$ , which thus brings, in total, at most 2 additional sum constraints. So, again, we end up with at most 8 forbidden colours for  $e$ , and thus at least 4 open ones. In particular, in case  $u = x' = y'$ , then recall that by the conditions

of the statement, it is actually not required by  $x$  and  $y$  are distinguished from their neighbours. Otherwise, if  $u$  is farther, then making  $x$  and  $y$  distinguished from their neighbours will be achieved later on, when treating  $xx'$  and  $yy'$ .

This proves the first part of the statement. Note that the same arguments apply when considering  $C$  instead of  $u$ . In particular, when  $e$  is incident to vertices of  $C$ , note that there are even less constraints to take into account.  $\square$

Using Lemma 5.2, we can now prove that nice *strictly subcubic graphs*, i.e., connected subcubic graphs that are not cubic, admit NSDS 13-edge-colourings.

**Lemma 5.3.** *If  $G$  is a nice strictly subcubic graph, then  $\chi'_{\text{NSDS}}(G) \leq 13$ .*

*Proof.* Consider first when  $G$  has a vertex  $u$  with  $d(u) = 1$ . Denote by  $v$  the sole neighbour of  $u$ . By Lemma 5.2, there is a partial NSDS 12-edge-colouring where only  $uv$  is not coloured. When extending this edge-colouring to  $uv$ , we only need to make sure that  $uv$  is not assigned any of the at most 6 colours assigned to the other edges at distance at most 2 from  $v$ , and that the sum of  $v$  does not get equal to the sums of its at most two other neighbours (different from  $u$ ). In particular, since  $d(u) = 1$ , recall that we cannot get  $\sigma(u) = \sigma(v)$ . So there are at most 8 constraints, which means that at least 4 of the pool of 12 colours can freely be assigned to  $uv$  to get a NSDS 12-edge-colouring of  $G$ .

Similarly, if  $G$  has a vertex  $u$  with exactly two neighbours  $v$  and  $w$ , then, by Lemma 5.2, there is a partial NSDS 12-edge-colouring where only  $uv$  and  $vw$  are not coloured. Before treating the most general case, we first deal with pathological cases:

- Assume first  $d(u) = d(v) = d(w) = 2$ . As we are clearly done if  $G = C_3$ , we can assume  $v$  and  $w$  are not adjacent. We first assign a colour in  $\{1, \dots, 13\}$  to  $uv$ . Note that, taking the unique coloured edge incident to  $w$  into account, there are at most 4 already-coloured edges at distance at most 2 from  $uv$ . When assigning a colour to  $uv$ , we also need to guarantee that  $v$  does not get in conflict with its unique neighbour different from  $u$ , which yields a 5th constraint. So there are at least 8 open colours in  $\{1, \dots, 13\}$  for  $uv$ , and we assign any value that makes the resulting sum of  $u$  differ from the current sum of  $w$ . This guarantees that, whatever colour we assign to  $uv$ , we cannot get a conflict between  $u$  and  $w$ . Now, regarding colouring  $uw$ , there are now at most 5 already-coloured edges at distance at most 2 (recall  $uv$  is now coloured), and we must guarantee  $w$  does not get in conflict with its unique neighbour different from  $u$ , while  $u$  also does not get in conflict with  $v$ . So there are at most 7 constraints, and thus at least 6 open colours in  $\{1, \dots, 13\}$  for  $uw$ . We can thus obtain a NSDS 13-edge-colouring of  $G$ .
- Assume now  $d(v) = 2$  and  $d(w) = 3$ , w.l.o.g. In this case, we first assign a colour in  $\{1, \dots, 13\}$  to  $uw$  so that we do not violate the strong requirement, and  $w$  is not in conflict with its two neighbours other than  $u$ . Clearly, there are enough colours to achieve this. Also, note that the fact that  $uw$  and  $vw$  are assigned distinct colours guarantees  $u$  and  $v$  cannot get in conflict, whatever colour we assign to  $uv$ . It is now not too difficult to check that we can indeed assign a colour in  $\{1, \dots, 13\}$  to  $uv$  so that a NSDS 13-edge-colouring of  $G$  results.
- Assume last  $d(v) = d(w) = 3$  and  $vw$  is an edge. We denote by  $v'$  and  $w'$  the third neighbour of  $v$  and  $w$ , respectively. Possibly,  $v' = w'$ . Since  $vv'$  and  $ww'$  are at distance 2, they are assigned distinct colours; so, already, we have  $\sigma(v) \neq \sigma(w)$ .

In terms of constraints, note that there are at most 5 already-coloured edges at distance at most 2 from any of  $uv$  and  $uw$ . Also, upon colouring  $uv$  (or  $uw$ ) we must guarantee that the resulting sum of  $v$  ( $w$ , resp.) does not get equal to  $\sigma(v')$  ( $\sigma(w')$ , resp.). Therefore, this far, there are at most 6 colours we cannot assign to any of  $uv$  or  $uw$ . Note now that, so that we do not get a conflict between  $u$  and  $v$  (or  $w$ ), the colour assigned to  $uv$  ( $uw$ , resp.) must be different from the current  $\sigma(v)$  ( $\sigma(w)$ , resp.). This leads us to deducing at most 1 additional constraint when colouring any of  $uv$  and  $uw$ ; thus, there are at most 7 constraints for each, or, the other way round, at least 6 open colours in  $\{1, \dots, 13\}$  for each. We first assign any such open colour to  $uv$ . This colour, now, cannot be assigned to  $uw$ , and, now that  $\sigma(v)$  is fixed, another open colour for  $uw$  might be forbidden to avoid the conflict between  $v$  and  $w$ . So there are, in total, at least 4 open colours in  $\{1, \dots, 13\}$  we can freely assign to  $uw$ . So, again, we can obtain a NSDS 13-edge-colouring of  $G$ .

We last deal with the case where  $d(v) = d(w) = 3$  and  $vw$  is not an edge of  $G$ . We denote by  $x_1, x_2$  the two other neighbours of  $v$ , and by  $y_1, y_2$  the two other neighbours of  $w$ . Possibly  $\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$ , but  $w \notin \{x_1, x_2\}$  and  $v \notin \{y_1, y_2\}$ .

Note that, at distance at most 2 from  $uv$ , there are at most 8 edges already coloured (in particular, observe that  $wy_1$  and  $wy_2$  are such edges), and similarly for  $uw$ ; thus we already have at most 8 constraints when colouring any of  $uv$  and  $uw$ . When colouring  $uv$ , we also need to guarantee that  $v$  gets in conflict with neither  $x_1$  nor  $x_2$ ; we thus deduce at most 2 more constraints for colouring  $uv$ , and these arguments apply to  $uw$  as well. For each of  $uv$  and  $uw$ , we have thus, thus far, at most 10 colouring constraints. We denote by  $L_{uv}, L_{uw} \subset \{1, \dots, 13\}$  the sets of colours we can freely assign to  $uv$  and  $uw$ , respectively, this far. Then, we have  $|L_{uv}|, |L_{uw}| \geq 3$ .

For the eventual sums  $\sigma(u), \sigma(v), \sigma(w)$ , we also need to guarantee that  $\sigma(u) \neq \sigma(v)$ ,  $\sigma(u) \neq \sigma(w)$ , and that  $uv$  and  $uw$  get assigned distinct colours. In particular, recall that  $v$  and  $w$  are assumed non-adjacent. As in the proof of Theorem 4.3, we can model the situation through a polynomial so that we can invoke the Combinatorial Nullstellensatz to get our conclusion. We denote by  $X_1$  a variable modelling a colour assigned to  $uv$ , by  $X_2$  a variable modelling a colour assigned to  $uw$ , and by  $n_1$  and  $n_2$  the current value of  $\sigma(v)$  and  $\sigma(w)$ , respectively. Then our constraints model through the polynomial

$$P(X_1, X_2) = (X_1 - n_2) \cdot (X_2 - n_1) \cdot (X_1 - X_2),$$

where the first two factors model the conditions guaranteeing  $u$  gets in conflict with neither  $w$  or  $v$ , respectively, while the last factor models that  $uv$  and  $uw$  must be assigned distinct colours. In particular,  $X_1^2 X_2$  is clearly a monomial with coefficient +1 of maximum degree in the expansion of  $P$ . By the Combinatorial Nullstellensatz, since  $|L_{uv}|, |L_{uw}| \geq 3$ , there are values  $x_1 \in L_{uv}$  and  $x_2 \in L_{uw}$  for which  $P(x_1, x_2) \neq 0$ . By then assigning colours  $x_1$  and  $x_2$  to  $uv$  and  $uw$ , respectively, we get a NSDS 13-edge-colouring of  $G$ , as desired.  $\square$

Due to Lemma 5.3, it remains to focus on cubic graphs. We first focus on those with small girth (length of the shortest cycles), before considering the more general case.

**Lemma 5.4.** *If  $G$  is a cubic graph with girth 3, then  $\chi'_{\text{NSDS}}(G) \leq 12$ .*

*Proof.* Let  $C = v_0 v_1 v_2 v_0$  be a triangle of  $G$ , and  $v'_0, v'_1, v'_2$  denote the last neighbour of  $v_0, v_1, v_2$ , respectively. By Lemma 5.2, there is a NSDS 12-edge-colouring of  $G$  where only the edges of  $C$  are uncoloured. Our goal is to extend this edge-colouring to the three edges of  $C$ . When colouring any edge of  $C$ , note that there are at most 7 coloured edges at distance

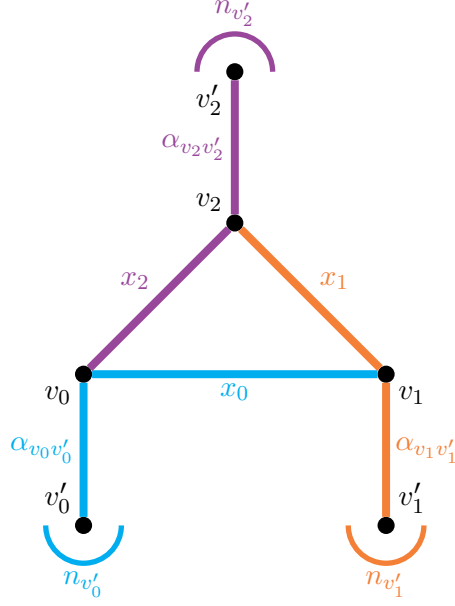


Figure 2: Terminology introduced in the proof of Lemma 5.4.

at most 2; thus, if we denote by  $L_{v_0 v_1}, L_{v_1 v_2}, L_{v_2 v_0} \subset \{1, \dots, 12\}$  the sets of open colours for  $v_0 v_1, v_1 v_2, v_2 v_0$ , respectively, w.r.t. these constraints, then  $|L_{v_0 v_1}|, |L_{v_1 v_2}|, |L_{v_2 v_0}| \geq 5$ .

When colouring  $v_0 v_1, v_1 v_2$ , and  $v_2 v_0$ , we also need to guarantee these three edges are assigned pairwise distinct colours. Also, denoting (see Figure 2)

- by  $\alpha_{v_0 v'_0}, \alpha_{v_1 v'_1}, \alpha_{v_2 v'_2}$  the colours assigned to  $v_0 v'_0, v_1 v'_1, v_2 v'_2$ , respectively,
- by  $n_{v'_0}, n_{v'_1}, n_{v'_2}$  the values  $\sigma(v'_0) - \alpha_{v_0 v'_0}, \sigma(v'_1) - \alpha_{v_1 v'_1}, \sigma(v'_2) - \alpha_{v_2 v'_2}$ , respectively, and
- for every  $i \in \{0, 1, 2\}$ , by  $x_i$  some colour assigned to  $v_i v_{i+1}$  (where, here and further, operations over subscripts are modulo 3),

then, so that  $\sigma(v_i) \neq \sigma(v'_i)$  for all  $i \in \{0, 1, 2\}$  we must have  $x_{i-1} + x_i \neq n_{v'_i}$ , and, so that  $\sigma(v_i) \neq \sigma(v_{i+1})$  for all  $i \in \{0, 1, 2\}$  we must have  $x_{i-1} + \alpha_{v_i v'_i} \neq x_{i+1} + \alpha_{v_{i+1} v'_{i+1}}$ . From this, denoting by  $X_i$  a variable modelling a colour assigned to  $v_i v_{i+1}$  for all  $i \in \{0, 1, 2\}$ , the modelling polynomial to consider is here

$$P(X_0, X_1, X_2) = \prod_{i=0}^2 (X_i - X_{i+1}) \cdot \prod_{i=0}^2 (X_{i-1} + X_i - n_{v'_i}) \cdot \prod_{i=0}^2 (X_{i-1} + \alpha_{v_i v'_i} - X_{i+1} - \alpha_{v_{i+1} v'_{i+1}}).$$

Because the  $n_{v'_i}$ 's and  $\alpha_{v_i v'_i}$ 's are constant terms, we can restrict ourselves to studying the monomials of the polynomial (still called  $P$  for convenience, abusing the notations)

$$\begin{aligned} P(X_0, X_1, X_2) &= (X_0 - X_1) \cdot (X_1 - X_2) \cdot (X_2 - X_0) \\ &\quad \cdot (X_0 + X_1) \cdot (X_1 + X_2) \cdot (X_2 + X_0) \\ &\quad \cdot (X_2 - X_1) \cdot (X_0 - X_2) \cdot (X_1 - X_0). \end{aligned}$$

It can be checked that, in the expansion of  $P$ , the monomial  $X_0^4 X_1^4 X_2$  has coefficient  $-2$ , and is of maximum degree. By the Combinatorial Nullstellensatz, since  $|L_{v_0 v_1}|, |L_{v_1 v_2}|, |L_{v_2 v_0}| \geq 5$ , there are values  $x_0 \in L_{v_0 v_1}, x_1 \in L_{v_1 v_2}, x_2 \in L_{v_2 v_0}$  for which  $P(x_0, x_1, x_2) \neq 0$ . Thus, by assigning colours  $x_0, x_1, x_2$  to  $v_0 v_1, v_1 v_2, v_2 v_0$ , we get a NSDS 12-edge-colouring of  $G$ .  $\square$

**Lemma 5.5.** *If  $G$  is a cubic graph with girth 4, then  $\chi'_{\text{NSDS}}(G) \leq 13$ .*

*Proof.* The proof goes similarly as that of Lemma 5.4, with a few differences. Let  $C = v_0v_1v_2v_3v_0$  be a square of  $G$ , where  $v_iv'_i$  denotes the edge incident to  $v_i$  not in  $C$  for all  $i \in \{0, \dots, 3\}$ . By Lemma 5.2, there is a NSDS 12-edge-colouring of  $G$  where only the edges of  $C$  are not assigned colours. Again, our goal is to properly edge-colour  $C$ . In what follows, we reuse and generalise the terminology from Lemma 5.4 in the obvious way.

There are two important points to raise. First off, note that, again, any two edges of  $C$  are at distance at most 2; thus, we must assign pairwise distinct colours to the  $v_iv_{i+1}$ 's. More importantly, since  $G$  has no triangle due to its girth being 4, there is no  $i \in \{0, 1\}$  such that  $v_iv_{i+2}$  is an edge; this implies that, when trying to make some of the  $\sigma(v_i)$ 's pairwise distinct, we need to care only about consecutive  $\sigma(v_i)$ 's along  $C$ . Last, we note that, here, for every  $v_iv_{i+1}$  there are at most 8 coloured edges at distance at most 2; however, since we here edge-colour with  $\{1, \dots, 13\}$ , we still have that all  $L_{v_iv_{i+1}}$ 's have cardinality at least 5.

By these arguments, alike the proof of Lemma 5.4, all colouring constraints are here modelled by the polynomial

$$\begin{aligned} P(X_0, \dots, X_3) &= (X_0 - X_1) \cdot (X_0 - X_2) \cdot (X_0 - X_3) \cdot (X_1 - X_2) \cdot (X_1 - X_3) \cdot (X_2 - X_3) \\ &\quad \cdot (X_0 + X_1) \cdot (X_1 + X_2) \cdot (X_2 + X_3) \cdot (X_3 + X_0) \\ &\quad \cdot (X_3 - X_1) \cdot (X_0 - X_2) \cdot (X_1 - X_3) \cdot (X_2 - X_0). \end{aligned}$$

This time, it can be checked that, in the expansion of  $P$ , the monomial  $X_0^4X_1^4X_2^3X_3^3$  is of maximum degree and has coefficient  $-24$ . Again, this implies we can properly assign some colour  $x_i \in L_{v_iv_{i+1}}$  to every edge  $v_iv_{i+1}$  for every  $i \in \{0, \dots, 3\}$ , so that a NSDS 13-edge-colouring of  $G$  results eventually.  $\square$

We now get to the general case.

**Lemma 5.6.** *If  $G$  is a cubic graph with girth at least 5, then  $\chi'_{\text{NSDS}}(G) \leq 16$ .*

*Proof.* Let  $C$  be a shortest (thus, induced, chordless) cycle of  $G$ , thus with length at least 5. By the choice of  $C$ , note that  $C$  has no chord, which implies that any two edges of  $C$  at distance at most 2 in  $G$  are actually at distance at most 2 along  $C$ . Hence, any edge of  $C$  is at distance at most 2 from at most four other edges of  $C$  (the two next ones, and the two previous ones).

We set  $C = v_0v_1 \dots v_{p-1}v_0$ , where  $p \geq 5$ , and, for every  $i \in \{0, \dots, p-1\}$ , set  $e_i = v_iv_{i+1}$ . As in the proofs of Lemmas 5.4 and 5.5, we start from a NSDS 12-edge-colouring  $\phi$  of  $G$  where only the edges of  $C$  are not coloured, which exists by Lemma 5.2. In what follows, we reuse the notation and terminology from Lemmas 5.4 and 5.5, adapting them in the obvious way. Our goal is to extend the given edge-colouring into a NSDS 16-edge-colouring of  $G$ , by assigning colours to the edges of  $C$ .

Recall that the  $v_iv'_i$ 's are assigned colours in  $\{1, \dots, 12\}$ . Free to relabel the vertices and edges of  $C$ , we can assume  $\phi(v_3v'_3) > \phi(v_2v'_2)$  (recall that  $v_2v'_2$  and  $v_3v_3v'$  are at distance 2; thus,  $\phi(v_2v'_2) \neq \phi(v_3v'_3)$ ). We first extend  $\phi$  to  $e_2, \dots, e_{p-1}$ , following this order, assigning colours in  $\{13, 14, 15, 16\}$ . An important point is that, because all edges not in  $C$  are assigned colours in  $\{1, \dots, 12\}$ , for any  $i \in \{0, \dots, p-1\}$ , if  $e_{i-1}$  and  $e_i$  (where operations, here and further, are modulo  $p$ ) are assigned colours in  $\{13, 14, 15, 16\}$ , then we necessarily obtain  $\sigma(v_i) > \sigma(v'_i)$ . We start by assigning colours 16 and 15 to  $e_2$  and  $e_3$ , respectively. By earlier arguments, and because only  $e_2$  and  $e_3$  have been coloured, the strong and NSD requirements are met thus far. Now consider  $i = 4, \dots, p-1$  in order, and assume that all edges  $e_2, \dots, e_{i-1}$  have been coloured properly. We claim we can also colour  $e_i$  this way.



Regarding the strong requirement, because only edges of  $C$  have been assigned colours in  $\{13, 14, 15, 16\}$ , and  $C$  is a chordless cycle of  $G$ , we only need to avoid, as  $\phi(e_i)$ , the colours assigned to  $e_{i-1}$  and  $e_{i-2}$ . In terms of sums, note that colouring  $e_i$  settles the value of  $\sigma(v_i)$ . As mentioned earlier, we cannot get  $\sigma(v_i) = \sigma(v'_i)$ , while  $v_{i+1}$  is incident to  $e_{i+1}$  which will be coloured in a next step. So we here only need to guarantee that  $\sigma(v_i) \neq \sigma(v_{i-1})$ . There are thus, in total, at most 3 constraints we need to take into account when colouring  $e_i$ , and at least 1 open colour in  $\{13, 14, 15, 16\}$  we can safely set as  $\phi(e_i)$ .

Once all edges  $e_2, \dots, e_{p-1}$  have been assigned a colour this way, we are left with assigning one to  $e_0$  and  $e_1$ . To these two edges, we will allow ourselves to assign colours in  $\{1, \dots, 16\}$ . We first count some constraints. First off, regarding the strong requirement, note that there are at most 11 distinct colours assigned to edges at distance at most 2 from any of  $e_0$  and  $e_1$ . Regarding the NSD requirement, note that colouring  $e_0$  settles  $\sigma(v_0)$ , which must be different from the sums of  $v_{p-1}$  and  $v'_0$ , thus at most two other constraints. Colouring  $e_1$  settles  $\sigma(v_2)$ , and we essentially deduce the same constraints, saved that we cannot obtain  $\sigma(v_2) = \sigma(v_3)$ , as  $\phi(v_3v'_3) > \phi(v_2v'_2)$  and the two edges incident to  $v_3$  along  $C$  have been assigned colours 15 and 16. Thus, whatever allowed colour we assign to  $e_1$ , we will necessarily get  $\sigma(v_3) > \sigma(v_2)$ . So, if we denote by  $L_0$  and  $L_1$  the subsets of allowed colours in  $\{1, \dots, 16\}$  for  $e_0$  and  $e_1$ , respectively, then  $|L_0| \geq 3$  and  $|L_1| \geq 4$ .

To be done, it suffices to show that we can assign a colour in  $L_0$  to  $e_0$ , one in  $L_1$  to  $e_1$ , and this so that we eventually obtain  $\sigma(v_0) \neq \sigma(v_1)$ ,  $\sigma(v_1) \neq \sigma(v_2)$ , and  $\sigma(v_1) \neq \sigma(v'_1)$ . To achieve this, first assign any colour in  $L_0$  to  $e_0$  so that the resulting current sum of  $v_1$  gets different from that of  $v_2$ . Since  $|L_0| \geq 3$  this is possible, and this guarantees that  $v_1$  cannot get in conflict with  $v_2$  whatever colour we assign to  $e_1$ . Now, taking into account the fact that the colour assigned to  $e_0$  cannot be assigned to  $e_1$ , since  $|L_1| \geq 4$  there is a colour in  $L_1$  we can assign to  $e_1$  so that, also,  $v_1$  gets in conflict with neither  $v_0$  nor  $v'_1$ . We eventually obtain a NSDS 16-edge-colouring of  $G$ , thus our conclusion.  $\square$

From Lemmas 5.3 to 5.6, we now deduce the following refined bound:

**Theorem 5.7.** *If  $G$  is a nice subcubic graph, then  $\chi'_{\text{NSDS}}(G) \leq 16$ .*

### 5.2.2. Sparse subcubic graphs

We now focus on sparse subcubic graphs, where the notion of sparseness is w.r.t. the following notions. For a graph  $G$ , we denote by  $\text{mad}(G)$  the *maximum average degree* of  $G$ , being the largest average degree over all subgraphs of  $G$  (i.e., the largest value  $\frac{2|E(H)|}{|V(H)|}$  for some  $H \subseteq G$ ). Previous works have highlighted that there is a strong relationship between the mad of a subcubic graph and its strong chromatic index, see [13]. More precisely, the lower the mad of a subcubic graph is, the lower its strong chromatic index is. For comparison with the upcoming result, let us recall that subcubic graphs  $G$  with  $\text{mad}(G) < \frac{5}{2}$  have strong chromatic index at most 7, see [13].

We prove a similar result regarding the NSDS chromatic index. That is, we prove that any nice subcubic graph with sufficiently low mad admits NSDS 10-edge-colourings. Recall that focusing on 10 colours only here, is primarily motivated by Conjecture 4.1, and the fact that the maximum strong chromatic index of a subcubic graph is 10 (recall [3, 16]). Let us remind as well that the construction in the proof of Theorem 3.1 provides subcubic graphs, so we do not always have  $\chi'_{\text{NSDS}}(G) = \chi'_s(G)$  for a nice subcubic graph  $G$ .

**Theorem 5.8.** *If  $G$  is a nice subcubic graph with  $\text{mad}(G) < \frac{5}{2}$ , then  $\chi'_{\text{NSDS}}(G) \leq 10$ .*

*Proof.* We prove the result through the so-called *discharging method*. That is, assuming, towards a contradiction, that the claim is wrong, we consider  $G$ , a minimum counterexample to the claim. We first prove that, since  $G$  is a counterexample to the claim, it cannot

contain certain configurations. Eventually, we will prove that  $G$  cannot exist at all, given  $G$  is sparse and does not contain any of these (sparse) configurations.

So,  $G$  is a nice subcubic graph with  $\text{mad}(G) < \frac{5}{2}$  that does not admit any NSDS 10-edge-colouring. In particular,  $G$  must be connected (since it is a minimum counterexample to the claim). It is worth recalling that removing any set of vertices and/or edges from  $G$  preserves the maximum average degree (in the sense that it cannot grow larger). This means that if  $S$  is any set of vertices/edges of  $G$ , then either  $G - S$  is nice and admits NSDS 10-edge-colourings (by the minimality of  $G$ ), or  $G - S$  is not nice, *i.e.*, contains connected components isomorphic to  $K_2$ .

In what follows, for any  $k \geq 1$ , a  $k$ -vertex is a vertex of degree  $k$ , while a  $k^-$ -vertex is a vertex of degree at most  $k$ .

**Claim 5.9.**  $G$  cannot contain any of the following configurations:

- (C1) a 1-vertex;
- (C2) two adjacent 2-vertices;
- (C3) a 3-vertex adjacent to three 2-vertices.

*Proof of the claim.* We treat the three configurations sequentially, one by one.

- (C1) Assume  $G$  contains a 1-vertex  $u$  with a unique neighbour  $v$ . Note that  $d(v) \geq 2$  since  $G$  is nice. Also, note that if  $G - u$  is not nice, then, necessarily,  $G$  must be isomorphic to the path of order 3, which clearly admits NSDS 10-edge-colourings, a contradiction. So, we can assume  $G - u$  is nice.

For simplicity, in what follows we deal only with the most complicated scenario (constraints-wise), which is when  $d(v) = 3$  (but most of the counting arguments below would apply when  $d(v) = 2$ ). Let thus  $w_1$  and  $w_2$  denote the two neighbours of  $v$  different from  $u$  (possibly,  $w_1w_2$  might be an edge). We here consider  $G' = G - u$ . Since  $G'$  is nice, by minimality of  $G$ , it admits a NSDS 10-edge-colouring  $\phi'$ . Our goal is to extend  $\phi'$  to the whole of  $G$  by assigning an open colour to  $uv$ . In  $G$ , note that there are at most 6 edges at distance at most 2 from  $uv$ , namely those incident to either  $w_1$  or  $w_2$ ; these at most 6 edges thus forbid at most 6 colours for  $uv$ . Also, assigning a colour to  $uv$  modifies  $\sigma(v)$ , and we need to guarantee, upon doing so, that  $\sigma(v) \notin \{\sigma(w_1), \sigma(w_2)\}$ , which forbids at most 2 other colours to be assigned to  $uv$ . In total, there are thus at most 8 colours that cannot be assigned to  $uv$ ; since we are dealing with 10 colours here, there are thus at least 2 colours in  $\{1, \dots, 10\}$  we can assign to  $uv$  that will result in a NSDS 10-edge-colouring of  $G$ , a contradiction. In particular, let us recall that, since we are assigning colours with strictly positive values, we cannot have  $\sigma(u) = \sigma(v)$ , since  $d(u) = 1$  and  $d(v) \geq 2$ .

- (C2) Assume  $G$  contains two adjacent 2-vertices  $u$  and  $v$ . We denote by  $u'$  and  $v'$  the other neighbour of  $u$  and  $v$ , respectively. Possibly,  $u' = v'$ . Since  $G$  does not contain Configuration C1, we have  $d(u'), d(v') \geq 2$ . In particular, this implies that  $G - uv$  must be nice.

We here consider  $G' = G - uv$ , which, thus, is nice, and, by minimality of  $G$ , admits a NSDS 10-edge-colouring  $\phi'$ , which we wish to extend to  $uv$ . As earlier, for simplicity we assume that  $d(u') = d(v') = 3$ , but the upcoming arguments would also hold if  $2 \in \{d(u'), d(v')\}$ . First off, by the exact same counting arguments we used to deal with Configuration C1, note that we can, if necessary, recolour  $vv'$  so that

$\phi'(uu') \neq \phi'(vv')$ . This is, indeed, because we have  $d(u) = d(v) = 1$  in  $G'$ . This guarantees that  $\phi'$  remains strong in  $G$  (as, remark,  $uu'$  and  $vv'$  are at distance 2 in  $G$ , but this is not necessarily the case in  $G'$ ), and, whatever colour we assign to  $uv$  now, we cannot get  $\sigma(u) = \sigma(v)$ . Let us now count the number of constraints. Note first that, in  $G$ , the edges at distance at most 2 from  $uv$  are those incident to  $u'$  and  $v'$ , thus at most 6 edges in total, and hence at most 6 colours that cannot be assigned to  $uv$ . When assigning a colour to  $uv$ , we also modify both  $\sigma(u)$  and  $\sigma(v)$ , and we need to make sure they do not get equal to  $\sigma(u')$  and  $\sigma(v')$ , respectively, which forbids at most 2 other colours. Thus, in total, at most 8 colours in  $\{1, \dots, 10\}$  cannot be assigned to  $uv$ , so there are at least 2 open colours we can freely assign to  $uv$  to obtain a NSDS 10-edge-colouring of  $G$ . This is a contradiction.

- (C3) Assume  $G$  contains a 3-vertex  $u$  adjacent to three 2-vertices  $v_1, v_2, v_3$ . We denote by  $v'_1, v'_2, v'_3$  the other neighbour of  $v_1, v_2, v_3$ , respectively. Since  $G$  does not contain Configurations C1 and C2, we must have  $d(v'_1) = d(v'_2) = d(v'_3) = 3$ . In particular, this implies  $G - u$  is nice.

We consider  $G' = G - u$ , which admits a NSDS 10-edge-colouring  $\phi'$ . Our goal is to extend  $\phi'$  to the edges incident to  $u$ , so that we get a NSDS 10-edge-colouring of  $G$ , thus a contradiction. To every edge  $uv_i$ , we note that we can assign, as a colour, neither the 3 colours assigned to the edges incident to  $v'_i$ , nor the at most 2 other colours assigned to  $v_j v'_j$ , for all  $j \in \{1, 2, 3\} \setminus \{i\}$ . Likewise, any colour assigned to  $uv_i$  must guarantee  $v_i$  and  $v'_i$  do not get in conflict. Thus, in total, not taking into account the possible conflicts between  $u$  and the  $v_i$ 's, and the fact that all of  $uv_1, uv_2, uv_3$  must be assigned pairwise distinct colours, we obtain that there are at least 4 colours in  $\{1, \dots, 10\}$  that can freely be assigned to each  $uv_i$ . For every  $i \in \{1, 2, 3\}$ , we denote by  $L_i$  the set of these at least 4 open colours for  $uv_i$ .

- If there is, say,  $x \in L_1$  such that  $x \notin L_2$ , then we first assign colour  $x$  to  $uv_1$ . Recall that this does not create a conflict between  $v_1$  and  $v'_1$ . Since  $x \notin L_2$ , if we update  $L_2$  and  $L_3$  by removing  $x$  from  $L_2$  and  $L_3$  (if these lists indeed contain  $x$ ), then  $|L_2| \geq 4$  and  $|L_3| \geq 3$ . We next assign a colour  $y$  in  $L_3$  to  $uv_3$  so that  $x + y \neq \phi'(v_2 v'_2)$ ; this is possible since  $|L_3| \geq 3$ , and this guarantees that, whatever colour we assign to  $uv_2$ , we cannot get a conflict between  $u$  and  $v_2$ . We now update  $L_2$  by removing  $y$  from  $L_2$ , if  $L_2$  indeed contains  $y$ ; then,  $|L_2| \geq 3$ . We last assign a colour  $z \in L_2$  to  $uv_2$ , chosen so that we do not get a conflict between  $u$  and neither  $v_1$  nor  $v_3$ . This is possible, since  $|L_2| \geq 3$ . As a result, we obtain a NSDS 10-edge-colouring of  $G$ , a contradiction.
- If, say,  $|L_1| \geq 5$ , then, upon assigning any colour  $x$  in  $L_2$  to  $uv_2$ , and after updating  $L_1$  and  $L_3$  by removing  $x$  from these (if they do contain it), we have  $|L_1| \geq 4$  and  $|L_3| \geq 3$ . Arguments we used in the previous case then apply.
- It remains to consider when  $L_1 = L_2 = L_3 = \{c_1, c_2, c_3, c_4\}$  for some pairwise distinct  $c_1, c_2, c_3, c_4 \in \{1, \dots, 10\}$ . We can suppose that  $c_1 < c_2 < c_3 < c_4$ . For this, set  $\phi'(uv_1) = c_1$ ,  $\phi'(uv_2) = c_2$ , and  $\phi'(uv_3) = c_3$ . We have the following cases.

1. We have three different colours on the edges  $v_i v'_i$ , say  $a, b, c$ . We can suppose that  $\phi'(v_1 v'_1) = a$ ,  $\phi'(v_2 v'_2) = b$ ,  $\phi'(v_3 v'_3) = c$  and  $a < b < c$ . We show that there are no conflicts between  $\sigma(u)$  and  $\sigma(v_i)$ , for  $i \in \{1, 2, 3\}$ . By contradiction, suppose there is a conflict.

- (a)  $\sigma(u) = \sigma(v_1)$ . That is,  $c_1 + c_2 + c_3 = a + c_1$ , therefore  $a = c_2 + c_3$ . We swap  $c_1$  and  $c_2$ , that is,  $\phi'(uv_1) = c_2$ ,  $\phi'(uv_2) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_1)$ . Now,
- \* If  $\sigma(u) = \sigma(v_2)$ , then  $c_1 + c_2 + c_3 = b + c_1$ , therefore  $b = c_2 + c_3 = a$ . A contradiction.
  - \* If  $\sigma(u) = \sigma(v_3)$ , then  $c_1 + c_2 + c_3 = c + c_3$ , therefore  $c = c_1 + c_2 < c_2 + c_3 = a$ . A contradiction.
- (b)  $\sigma(u) = \sigma(v_2)$ . That is,  $c_1 + c_2 + c_3 = b + c_2$ , therefore  $b = c_1 + c_3$ . We swap  $c_2$  and  $c_3$ , that is,  $\phi'(uv_2) = c_3$ ,  $\phi'(uv_3) = c_2$ . Clearly  $\sigma(u) \neq \sigma(v_2)$ . Now,
- \* If  $\sigma(u) = \sigma(v_1)$ , then  $c_1 + c_2 + c_3 = a + c_1$ , therefore  $a = c_2 + c_3 > c_1 + c_3 = b$ . A contradiction.
  - \* If  $\sigma(u) = \sigma(v_3)$ , then  $c_1 + c_2 + c_3 = c + c_2$ , therefore  $c = c_1 + c_3 = b$ . A contradiction.
- (c)  $\sigma(u) = \sigma(v_3)$ . That is,  $c_1 + c_2 + c_3 = c + c_3$ , therefore  $c = c_1 + c_2$ . We swap  $c_3$  and  $c_1$ , that is,  $\phi'(uv_1) = c_3$ ,  $\phi'(uv_3) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_3)$ . Now,
- \* If  $\sigma(u) = \sigma(v_1)$ , then  $c_1 + c_2 + c_3 = a + c_3$ , therefore  $a = c_1 + c_2 = c$ . A contradiction.
  - \* If  $\sigma(u) = \sigma(v_2)$ , then  $c_1 + c_2 + c_3 = b + c_2$ , therefore  $b = c_1 + c_3 > c_1 + c_2 = c$ . A contradiction.
2. We have two different colours on the edges  $v_i v'_i$ , say  $a, b$ , where  $a < b$ . Suppose first that  $\phi'(v_1 v'_1) = a$ ,  $\phi'(v_2 v'_2) = a$ ,  $\phi'(v_3 v'_3) = b$ . As in the previous case, by contradiction, suppose there is a conflict.
- (a)  $\sigma(u) = \sigma(v_1)$  (similar for  $\sigma(u) = \sigma(v_2)$ ). That is,  $c_1 + c_2 + c_3 = a + c_1$ , therefore  $a = c_2 + c_3$ . Note that  $\sigma(u) \neq \sigma(v_2)$ . We swap  $c_1$  and  $c_3$ , that is,  $\phi'(uv_1) = c_3$ ,  $\phi'(uv_3) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_1)$ . Now, if  $\sigma(u) = \sigma(v_3)$ , then  $c_1 + c_2 + c_3 = b + c_1$ , therefore  $b = c_2 + c_3 = a$ . A contradiction.
- (b)  $\sigma(u) = \sigma(v_3)$ . That is,  $c_1 + c_2 + c_3 = b + c_3$ , therefore  $b = c_1 + c_2$ . We swap  $c_1$  and  $c_3$ , that is,  $\phi'(uv_1) = c_3$ ,  $\phi'(uv_3) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_3)$ . Now,
- \* If  $\sigma(u) = \sigma(v_1)$ , then  $c_1 + c_2 + c_3 = a + c_3$ , therefore  $a = c_1 + c_2 = b$ . A contradiction.
  - \* If  $\sigma(u) = \sigma(v_2)$ , then  $c_1 + c_2 + c_3 = a + c_2$ , therefore  $a = c_1 + c_3 > c_1 + c_2 = b$ . A contradiction.
- Suppose last that  $\phi'(v_1 v'_1) = a$ ,  $\phi'(v_2 v'_2) = b$ ,  $\phi'(v_3 v'_3) = b$ . By contradiction, suppose there is a conflict.
- (a)  $\sigma(u) = \sigma(v_1)$ . That is,  $c_1 + c_2 + c_3 = a + c_1$ , therefore  $a = c_2 + c_3$ . We swap  $c_1$  and  $c_2$ , that is,  $\phi'(uv_1) = c_2$ ,  $\phi'(uv_2) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_1)$ . Now,
- \* If  $\sigma(u) = \sigma(v_2)$ , then  $c_1 + c_2 + c_3 = b + c_1$ , therefore  $b = c_2 + c_3 = a$ . A contradiction.
  - \* If  $\sigma(u) = \sigma(v_3)$ , then  $c_1 + c_2 + c_3 = b + c_3$ , therefore  $b = c_1 + c_2 < c_2 + c_3 = a$ . A contradiction.
- (b)  $\sigma(u) = \sigma(v_2)$  (similar for  $\sigma(u) = \sigma(v_3)$ ). That is,  $c_1 + c_2 + c_3 = b + c_2$ , therefore  $b = c_1 + c_3$ . Note that  $\sigma(u) \neq \sigma(v_3)$ . We swap  $c_1$  and  $c_2$ , that is,  $\phi'(uv_1) = c_2$ ,  $\phi'(uv_2) = c_1$ . Clearly  $\sigma(u) \neq \sigma(v_2)$ . Now, if  $\sigma(u) = \sigma(v_1)$ , then  $c_1 + c_2 + c_3 = a + c_2$ , therefore  $a = c_1 + c_3 = b$ . A contradiction.

3. All edges  $v_i v'_i$  have the same colour. We can recolour an edge, say  $v_1 v'_1$ , so that we can apply a previous case. To see this is true, recall that  $d(v'_1) = 3$ , so, when recolouring  $v_1 v'_1$ , the number of constraints is at most 8 (at most 6 constraints from the strong requirement, and at most 2 from the NSD one); so at least 2 colours in  $\{1, \dots, 10\}$  can be assigned to  $v_1 v'_1$ . We can then choose one of these two and apply the previous configuration.

By previous arguments, we obtain in all cases that  $\sigma(u) \neq \sigma(v_i)$ , for every  $i \in \{1, 2, 3\}$ , and therefore we can extend  $\phi'$  to  $G$ , a contradiction.

This concludes the proof in all cases.

The whole claim thus holds.  $\diamond$

Back to the proof of Theorem 5.8, we are now ready to contradict the existence of  $G$ . To every vertex  $v$  of  $G$ , we assign an initial charge  $\omega(v)$  with value  $\omega(v) = d(v) - \frac{5}{2}$ . Now, since

$$\sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} \left( d(v) - \frac{5}{2} \right) = \sum_{v \in V(G)} d(v) - \frac{5}{2} \cdot |V(G)|$$

and

$$\sum_{v \in V(G)} d(v) \leq |V(G)| \cdot \text{mad}(G) < \frac{5}{2} \cdot |V(G)|,$$

the total amount of charges is strictly negative. Without creating or deleting charges, we move charges from neighbours to neighbours according to the following unique rule:

- Every 3-vertex sends  $\frac{1}{4}$  to each of its  $2^-$ -neighbours.

For every vertex  $v$  of  $G$ , we denote by  $\omega^*(v)$  the eventual charge of  $v$  once the rule above has been applied to all vertices. We claim that, because  $G$  cannot contain Configurations C1 to C3 by Claim 5.9, by  $\omega^*$  the total amount of charges must be positive, which is a final contradiction to the existence of  $G$  since charges are preserved by the rule. We prove this by considering every vertex  $v$ , and showing that  $\omega^*(v) \geq 0$  whatever  $d(v)$  is.

- $d(v) = 1$ .

This case does not have to be treated, as  $G$  cannot contain a 1-vertex by Claim 5.9.

- $d(v) = 2$ .

Observe that  $\omega(v) = -\frac{1}{2}$ . Note first that  $v$  did not send any charge by the unique rule above. By Claim 5.9, since  $G$  does not contain any of Configurations C1 and C2, the two neighbours of  $v$  must be 3-vertices, from each of which  $v$  received  $\frac{1}{4}$  by the unique rule. Then  $\omega^*(v) = \omega(v) + 2 \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{2} = 0$ .

- $d(v) = 3$ .

Observe that  $\omega(v) = \frac{1}{2}$ . First off,  $v$  did not receive any charge by the rule. On the other hand,  $v$  sent  $\frac{1}{4}$  to each of its  $2^-$ -neighbours. By Claim 5.9, since  $G$  does not contain Configurations C1 and C3,  $v$  cannot be adjacent to a 1-vertex, and  $v$  is adjacent to at most two 2-vertices. Then  $\omega^*(v) \geq \omega(v) - 2 \times \frac{1}{4} = \frac{1}{2} - \frac{1}{2} = 0$ .

Thus, the total amount of charges, once the unique rule above has been applied, is non-negative, which is impossible. This contradicts the existence of  $G$ , and its minimality implies the claim holds true. This concludes the proof.  $\square$

## 6. Conclusion

In this work, we introduced NSDS edge-colourings as a combination of strong edge-colourings and NSD edge-colourings. By definition, NSDS edge-colourings are very close to the latter two notions, recall Observation 2.2. However, given that strong edge-colourings require more colours than NSD edge-colourings, quite naturally, we turned mostly to comparing NSDS edge-colourings and strong edge-colourings. Although we proved that the strong and NSDS chromatic indices coincide in many cases (recall our results from Section 2), we also came up with examples of graphs showing that equality does not always hold (Section 3). This led us to raise Conjecture 4.1, and to establish general bounds on the NSDS chromatic index (Theorems 4.2 and 4.3). We improved these bounds in Section 5 for some classes of graphs, namely trees (for which we provided a tight result) and subcubic graphs (for which there is more room for improvement). An interesting fact also is that we came up with connections with other types of distinguishing labellings, namely locally antimagic labellings (recall Observation 2.5).

These results lead us to establish that NSDS edge-colourings are definitely closer to strong edge-colourings, which are so constraining that the contribution of the distinguishing condition from NSD edge-colourings does not seem to bring that many more constraints. It is surprising to us, however, that there are graphs  $G$  with  $\chi'_{\text{NSDS}}(G) > \chi'_s(G)$ . Our initial guess was that, perhaps, there are so many ways to permute colours in strong edge-colourings that we should always be possible to make the distinguishing condition hold.

Our results open the way to many questions, which we think might be worth considering further in later works. In particular:

- We were not able to come up with constructions different from the one in the proof of Theorem 3.1, to establish that there are other, different graphs  $G$  with  $\chi'_{\text{NSDS}}(G) > \chi'_s(G)$ . In particular, we are pretty sure that there are ways to modify our construction slightly to provide graphs with the same edge-colouring properties that are very close structure-wise. Our question, here, is rather whether there are graphs  $G$  for which the fact that  $\chi'_{\text{NSDS}}(G) > \chi'_s(G)$  holds is because of arguments that have nothing to do with those we imagined in the proof of Theorem 3.1.
- In the same line, we have no evidence that there are graphs  $G$  with  $\chi'_{\text{NSDS}}(G) > \chi'_s(G) + 1$ . As a result, we wonder if  $c$  in Conjecture 4.1 could be 1.
- Among the results we provided, the ones for which there is definitely room for improvement are those for subcubic graphs, namely Theorems 5.7 and 5.8. In the proof of Theorem 5.7, from experimentations through computer programs, it seems that, using the Combinatorial Nullstellensatz, it could be possible to reduce a whole cycle at once (whatever its length) provided with use at least 14 colours. Thus, it seems that, through a better analysis of the monomials of some polynomials' expansions, we could be able to improve our upper bound from 16 to 14. Regarding Theorem 5.8, we believe it could be interesting to study subcubic graphs with larger  $\text{mad}$ , or more generally to investigate the connection between the girth of a graph (which relates to its density) and its NSDS chromatic index.
- It could be interesting to wonder about more classes of graphs. For instance, planar graphs and degenerate graphs have been quite investigated for both strong edge-colourings and NSD edge-colourings, so such classes of graphs would be natural to consider here as well. Recall, as mentioned by the end of Section 4, that our bounds from Theorems 4.2 and 4.3 already provide first steps in that direction.

## References

- [1] N. Alon. Combinatorial Nullstellensatz. *Combinatorics, Probability and Computing*, 8:7–29, 1999.
- [2] N. Alon, G. Kaplan, A. Lev, Y. Roditty, R. Yuster. Dense graphs are antimagic. *Journal of Graph Theory*, 47(4):297–309, 2004.
- [3] L.D. Andersen. The strong chromatic index of a cubic graph is at most 10. *Discrete Mathematics*, 108:231–252, 1992.
- [4] S. Arumugam, K. Premalatha, M. Bača, A. Semaničová-Feňovčíková. Local antimagic vertex coloring of a graph. *Graphs and Combinatorics*, 33:275–285, 2017.
- [5] J. Bensmail, H. Hocquard, D. Lajou. On the algorithmic complexity of determining the AVD and NSD chromatic indices of graphs. *Theoretical Computer Science*, 939:105–118, 2023.
- [6] J. Bensmail, M. Senhaji, K. Szabo Lyngsie. On a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture. *Discrete Mathematics and Theoretical Computer Science*, 19(1), 2017, #22.
- [7] D.W. Cranston. Strong edge-coloring of graphs with maximum degree 4 using 22 colors. *Discrete Mathematics*, 306(21):2772–2778, 2006.
- [8] P. Erdős, J. Nešetřil. *Irregularities of partitions*. G. Halász, V.T. Sós, Eds., [Problem], 162–163, 1989.
- [9] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza. The strong chromatic index of graphs. *Ars Combinatoria*, 29B:205–211, 1990.
- [10] E. Flandrin, A. Marczyk, J. Przybyło, J-F. Sacle, M. Woźniak. Neighbor sum distinguishing index. *Graphs and Combinatorics*, 29(5):1329–1336, 2013.
- [11] J.L. Fouquet, J.L. Jolivet. Strong edge-colorings of graphs and applications to multi- $k$ -gons. *Ars Combinatoria*, 16A:141–150, 1983.
- [12] J. Haslegrave. Proof of a local antimagic conjecture. *Discrete Mathematics and Theoretical Computer Science*, 20(1), 2018, #18.
- [13] H. Hocquard, M. Montassier, A. Raspaud, P. Valicov. On strong edge-colouring of subcubic graphs. *Discrete Applied Mathematics*, 161(16-17):2467–2479, 2013.
- [14] H. Hocquard, P. Ochem, P. Valicov. Strong edge-colouring and induced matchings. *Information Processing Letters*, 113(19-21):836–843, 2013.
- [15] I. Holyer. The NP-Completeness of Edge-Colouring. *SIAM Journal on Computing*, 10(4):718–720, 1981.
- [16] P. Horák, H. Qing, W.T. Trotter. Induced matchings in cubic graphs. *Journal of Graph Theory*, 17:151–160, 1993.
- [17] M. Huang, M. Santana, G. Yu. Strong. Strong Chromatic Index of Graphs With Maximum Degree Four. *Electronic Journal of Combinatorics*, 25(3):P3.31, 2018.

- [18] K. Szabo Lyngsie, L. Zhong. A Generalized Version of a Local Antimagic Labelling Conjecture. *Graphs and Combinatorics*, 34(6):1363–1369, 2018.
- [19] V.G. Vizing. On an estimate of the chromatic class of a  $p$ -graph. *Diskret. Analiz.*, 3:25–30, 1964.
- [20] B. Vučković. Edge-partitions of graphs and their neighbor-distinguishing index. *Discrete Mathematics*, 340(12):3092–3096, 2017.