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Fragmentations with erasure

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Abstract

We study sequences of partitions of the unit interval into subintervals, starting from the trivial partition, in which each partition is obtained from the one before by splitting its subintervals in two, according to a given rule, and then merging pairs of subintervals at the break points of the old partition. The n th partition then comprises $n + 1$ subintervals with n break points, and the empirical distribution of these points reveals a surprisingly rich structure, even when the splitting rule is completely deterministic. We consider both deterministic and randomized splitting rules and study the limiting behavior of the empirical distribution of the break points, from multiple angles.

Keywords: fragmentation, random walk in random environment, Markov chain, limit theorems, large deviations

AMS classification (2020): 60J05, 60F05.

1 Introduction

We consider sequences of partitions $(\mathcal{P}_n)_{n \geq 0}$ of the interval $(0, 1]$, of the form

$$\mathcal{P}_n = \{(a_{n,0}, a_{n,1}], \dots, (a_{n,n}, a_{n,n+1}]\}$$

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where $a_{n,0} = 0$ and $a_{n,n+1} = 1$ and the n break points $a_{n,1}, \dots, a_{n,n}$ satisfy

$$0 \leq a_{n,1} \leq \dots \leq a_{n,n} \leq 1.$$

Thus $\mathcal{P}_0 = (0, 1]$ and \mathcal{P}_n partitions $(0, 1]$ into $n + 1$ subintervals

$$(0, 1] = (a_{n,0}, a_{n,1}] \cup \dots \cup (a_{n,n}, a_{n,n+1}].$$

Suppose we are given a family of splitting proportions

$$\mathbf{p} = (p_{n,k} : n \geq 1, 1 \leq k \leq n)$$

with $p_{n,k} \in [0, 1]$ for all k . For $n \geq 1$, we define the break points of the partition \mathcal{P}_n recursively by

$$a_{n,k} = p_{n,k}a_{n-1,k-1} + (1 - p_{n,k})a_{n-1,k}, \quad k = 1, \dots, n. \quad (1)$$

Thus we split each interval $(a_{n-1,k-1}, a_{n-1,k}]$ of the partition \mathcal{P}_{n-1} into two subintervals, with proportions $1 - p_{n,k}$ on the left and $p_{n,k}$ on the right, and then we merge the resulting subintervals at the break points of \mathcal{P}_{n-1} . We call this process *fragmentation with erasure* because each interval of the partition \mathcal{P}_{n-1} is fragmented by adding a new point inside it, but then the break points of \mathcal{P}_{n-1} are erased. The cases where $p_{n,k} = 0$ or $p_{n,k} = 1$ for some n and k are allowed: in such cases some intervals of the partition will be empty, but we continue to list them ‘with multiplicity’. We use the same formula (1) for the dynamics of the break points in all cases.

See Figure 1 for an illustration.

We sometimes choose the family of splitting proportions randomly. In this case, we will use upper-case letters for both the the splitting proportions $\mathbf{P} = (P_{n,k} : n \geq 1, 1 \leq k \leq n)$ and the break points $A_{n,k}$, but their relation will be the same, namely $A_{n,0} = A_{n,n+1} = 0$ for all n and

$$A_{n,k} = P_{n,k}A_{n-1,k-1} + (1 - P_{n,k})A_{n-1,k}, \quad k = 1, \dots, n. \quad (2)$$

In this paper, we mainly consider the following three choices for the splitting proportions.

- **Deterministic stratified fragmentation:** we choose a deterministic sequence $(p_n)_{n \geq 1}$ in $[0, 1]$ and set $p_{n,k} = p_n$ for all k . An interesting special case is the choice $p_n \equiv p \in (0, 1)$.
- **Random stratified fragmentation:** we set $P_{n,k} = P_n$ for all k , where $(P_n)_{n \geq 1}$ are i.i.d. random variables in $[0, 1]$.

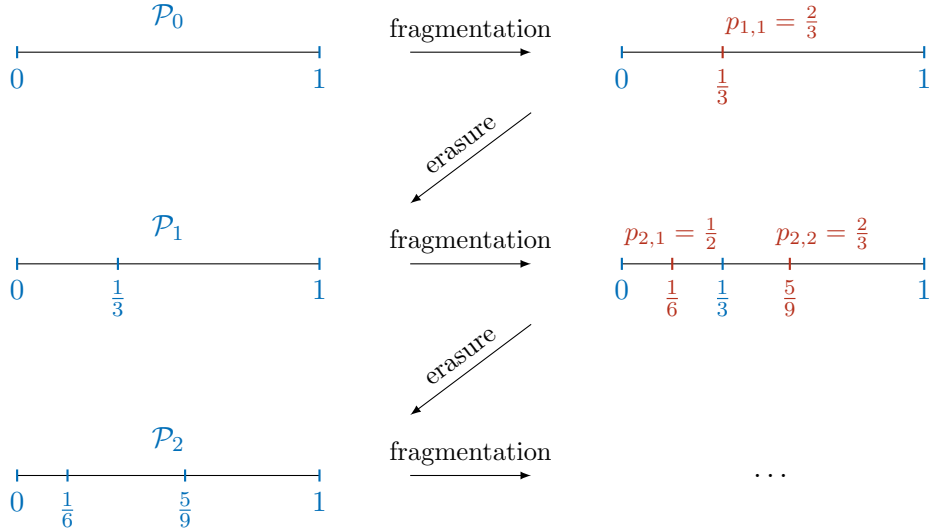


Figure 1: Illustration of the iterative definition of $(\mathcal{P}_n)_{n \geq 0}$.

- **Fully random fragmentation:** we take $(P_{n,k} : n \geq 1, 1 \leq k \leq n)$ to be an array of i.i.d. random variables in $[0, 1]$.

In each of these three cases, the sequence $(\mathcal{P}_n)_{n \geq 0}$ forms a Markov chain whose state-space is the set of finite partitions of the unit interval into subintervals. The motivation for our work comes from an analysis of the asymptotic behavior, in a certain regime, of a Markov chain on $\{-1, 1\}^K$ with K large, considered in [6]. This latter Markov chain moves from a state $a \in \{-1, 1\}^K$ to a state $b \in \{-1, 1\}^K$ according to the following rule: choose, uniformly at random, a subsequence J of $\{1, \dots, K\}$ from the set of subsequences $j = (j(1), \dots, j(\ell))$ of $\{1, \dots, K\}$ such that $a_{j(i+1)} \neq a_{j(i)}$ for all i . Once a subsequence J is sampled, the Markov chain moves to the state $b \in \{-1, 1\}^K$ defined by $b_k = a_k$ if k is not in the range of J , and $b_k = -a_k$ otherwise. Starting from the initial state $a = (1, \dots, 1)$, the first transition leads to a state b that has at most one digit equal to -1 , and this b is chosen uniformly among the $K + 1$ possible choices; furthermore, for $n \ll \log K$ the behavior of this Markov chain can be approximated by the fully random fragmentation model we consider in the present paper, where the proportions have the standard uniform distribution. For more related results concerning this Markov chain, please refer to [5].

Fragmentation processes generally form an interesting and well studied class of stochastic processes; see for example [3, 4, 2], but the erasure feature

prominent in our model does not seem to have been studied yet. Our model also bears some resemblance to a multiplicative cascade (see e.g. [8]), but in the latter models the number of points in the n th generation is typically of the order 2^n instead of the order n , as in our model. Our model can be viewed as a type of a cascade where the points of the previous generation are erased at each step.

We are interested in the behavior of the empirical distribution of the break points of the partition \mathcal{P}_n as $n \rightarrow \infty$. We will set

$$g_n = \frac{1}{n} \sum_{k=1}^n \delta_{a_{n,k}}, \quad G_n = \frac{1}{n} \sum_{k=1}^n \delta_{A_{n,k}} \quad (3)$$

in the deterministic and random cases respectively. Here δ_a denotes the unit mass at point a . Note that, when the splitting proportions \mathbf{P} are random, the empirical distribution G_n is a random probability measure. We will study both the overall behavior of the empirical distribution and its restriction to certain parts of the unit interval.

We begin in Section 2 by establishing a key representation of the break points in terms of the cumulative distribution function of a certain random walk, whose steps are governed by the splitting proportions. We denote this walk by $(x_n)_{n \geq 0}$ when we restrict to the case of deterministic splitting proportions, and by $(X_n)_{n \geq 0}$ in the general random case. Of course the walk is random in both cases. Interestingly, this representation is useful even in the case of a deterministic fragmentation. In the case of the random stratified fragmentation, $(X_n)_{n \geq 0}$ becomes a random walk with a random environment in time, where the environment consists of the random splitting proportions $(P_n)_{n \geq 1}$. Similarly, in the case of the fully random fragmentation, $(X_n)_{n \geq 0}$ becomes a random walk with a random environment in time and space.

In Section 3, weak convergence of the empirical distribution is established, as a consequence of the law of large numbers satisfied by the random walk $(X_n)_{n \geq 0}$. The weak limit turns out always to be supported by the endpoints of the unit interval. Nonetheless, the number of points of \mathcal{P}_n in any given open subset of $(0, 1)$ diverges, and we proceed in Section 4.1 to investigate the behavior of the empirical distribution away from the endpoints. This study relies on the central limit theorem satisfied by $(X_n)_{n \geq 0}$ and we will show vague convergence, under suitable renormalization, to an infinite measure on $(0, 1)$, whose density is the derivative of the quantile function of a standard normal distribution. The exact scaling is however sensitive to the specific type of fragmentation. Finally, in Section 4.2, we investigate more precisely the behavior of the empirical distribution near the endpoints,

by studying for example the number of break points of \mathcal{P}_n in intervals of the type $[0, x^n]$ for $0 < x < 1$. This behavior turns out to be related to the large deviation principle for the random walk $(X_n)_{n \geq 0}$.

2 Representation of the break points

Set

$$I = \{(n, k) : n \geq 1, 1 \leq k \leq n\}$$

and fix

$$\mathbf{p} = (p_{n,k} : (n, k) \in I)$$

with $p_{n,k} \in [0, 1]$ for all n and k . Let $(U_{n,k} : (n, k) \in I)$ be a family of independent random variables, all uniformly distributed on $[0, 1]$. Set

$$Y_{n,k} = 1_{\{U_{n,k} \leq p_{n,k}\}}$$

so that $Y_{n,k}$ is a Bernoulli random variable with success probability $p_{n,k}$. Write Ω_0 for the set of sequences $(y_{n,k} : (n, k) \in I)$ with $y_{n,k} \in \{0, 1\}$ for all n and k . Denote by $\mathbb{P}^{\mathbf{p}}$ the law of $(Y_{n,k} : (n, k) \in I)$ on Ω_0 . Define a random process $(x_n)_{n \geq 0}$ on Ω_0 by setting $x_0 = 0$ and then, recursively for $n \geq 1$,

$$x_n = x_{n-1} + y_{n, x_{n-1}+1}. \quad (4)$$

Note that $x_n \in \{0, 1, \dots, n\}$ for all n and $(x_n)_{n \geq 0}$ is a time-inhomogeneous Markov chain under $\mathbb{P}^{\mathbf{p}}$. The following proposition provides a useful formula for the break points of the deterministic fragmentation-erasure process with splitting proportions \mathbf{p} .

Proposition 2.1. *For all $(n, k) \in I$, we have*

$$a_{n,k} = \mathbb{P}^{\mathbf{p}}(x_n \leq k - 1).$$

Proof. Define for $n \geq 0$ and $0 \leq k \leq n + 1$

$$\alpha_{n,k} = \mathbb{P}^{\mathbf{p}}(x_n \leq k - 1).$$

Then $\alpha_{n,0} = 0$ and $\alpha_{n,n+1} = 1$ for all n . For $(n, k) \in I$, we have

$$\begin{aligned} \alpha_{n,k} &= \mathbb{P}^{\mathbf{p}}(x_n \leq k - 1) \\ &= \mathbb{P}^{\mathbf{p}}(x_{n-1} \leq k - 2) + \mathbb{P}^{\mathbf{p}}(x_{n-1} = k - 1, y_{n,k} = 0) \\ &= p_{n,k} \alpha_{n-1, k-1} + (1 - p_{n,k}) \alpha_{n-1, k}. \end{aligned}$$

The same recursion (1) defines $a_{n,k}$ so $\alpha_{n,k} = a_{n,k}$ for all n and k . □

We turn now to the more general case where the splitting proportions \mathbf{P} are random. By augmenting our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if necessary, we can assume that Ω supports a family of independent random variables $(U_{n,k} : (n,k) \in I)$ as above, which is moreover independent of \mathbf{P} . Set

$$Y_{n,k} = 1_{\{U_{n,k} \leq P_{n,k}\}}$$

and define a random process $X = (X_n)_{n \geq 0}$ on Ω by setting $X_0 = 0$ and then, recursively for $n \geq 1$,

$$X_n = X_{n-1} + Y_{n, X_{n-1}+1}.$$

We note the following straightforward proposition.

Proposition 2.2. *Given a bounded measurable function Φ on $(\mathbb{Z}^+)^{\mathbb{Z}^+}$, define $\phi : [0, 1]^I \rightarrow \mathbb{R}$ by*

$$\phi(\mathbf{p}) = \mathbb{E}^{\mathbf{P}}(\Phi(x))$$

where $x = (x_n)_{n \geq 0}$ is the time-inhomogeneous Markov chain defined at (4). Then ϕ is measurable and we have, almost surely,

$$\mathbb{E}(\Phi(X) | \mathbf{P}) = \phi(\mathbf{P}).$$

It will be convenient sometimes to regard (1) and (3) as defining measurable functions $a_{n,k}(\mathbf{p})$ and $g_n(\mathbf{p})$ of \mathbf{p} , so we can write

$$A_{n,k} = a_{n,k}(\mathbf{P}), \quad G_n = g_n(\mathbf{P}).$$

Note that, on taking $\Phi(x) = 1_{\{x_n \leq k-1\}}$, we have $\phi(\mathbf{p}) = a_{n,k}(\mathbf{p})$ by Proposition 2.1. Hence, the random break points $(A_{n,k} : (n,k) \in I)$ associated with the random splitting proportions \mathbf{P} satisfy, almost surely,

$$A_{n,k} = a_{n,k}(\mathbf{P}) = \mathbb{P}(X_n \leq k-1 | \mathbf{P})$$

We remark on two special cases where $(X_n)_{n \geq 0}$ takes a simple form.

- In the case where $p_{n,k} \equiv p \in [0, 1]$ for all n and k , which is a special case of the deterministic stratified fragmentation, $(X_n)_{n \geq 0}$ is a simple (biased) random walk. In particular X_n has the binomial distribution with parameters n and p .
- In the fully random case, the process $(X_n)_{n \geq 0}$ is a Random Walk in Dynamic Random Environment (RWDRE). See for example [10]. We will be interested not only in its annealed distribution, which is a simple random walk, but also in its quenched distribution, conditional on the splitting proportions \mathbf{P} .

3 Weak convergence of the empirical distributions

Our first use of the representation of the partitions \mathcal{P}_n developed in the previous section shows that, perhaps somewhat unexpectedly, under quite general conditions, the sequence of empirical distributions of break points $(g_n)_{n \geq 0}$ (or $(G_n)_{n \geq 0}$ in the random case) has a weak limit supported on $\{0, 1\}$. This is the content of the main result of this section, Theorem 3.2 below.

We start with a key lemma stating that this weak convergence is a consequence of a law of large numbers for the auxiliary walk $(x_n)_{n \geq 0}$.

Lemma 3.1. *Fix a family of splitting proportions $\mathbf{p} = (p_{n,k} : (n, k) \in I)$ and consider the associated random process $(x_n)_{n \geq 0}$ defined on Ω_0 by (4). Assume that, for some $\bar{p} \in [0, 1]$, under $\mathbb{P}^{\mathbf{p}}$, as $n \rightarrow \infty$,*

$$x_n/n \rightarrow \bar{p} \quad \text{in probability.} \quad (5)$$

Then, as $n \rightarrow \infty$,

$$g_n \Rightarrow \bar{p}\delta_0 + (1 - \bar{p})\delta_1 \quad \text{weakly on } [0, 1].$$

Proof. It follows from the assumption that, for any fixed $\alpha \in (0, 1)$, by Proposition 2.1,

$$a_{n, [\alpha n]} = \mathbb{P}^{\mathbf{p}}(x_n \leq [\alpha n] - 1) \rightarrow \begin{cases} 0 & \text{if } \alpha < \bar{p}, \\ 1 & \text{if } \alpha > \bar{p}, \end{cases}$$

and so $g_n([0, x]) \rightarrow \bar{p}$ for all $x \in (0, 1)$, which implies the claimed weak convergence. \square

Theorem 3.2. (a) **Deterministic stratified fragmentation.** *Assume that the averages $\frac{1}{n} \sum_{m=1}^n p_m$ converge as $n \rightarrow \infty$, with limit \bar{p} say. Then, as $n \rightarrow \infty$,*

$$g_n \Rightarrow \bar{p}\delta_0 + (1 - \bar{p})\delta_1 \quad \text{weakly on } [0, 1].$$

(b) **Random stratified fragmentation and fully random fragmentation.** *Define $\bar{p} = \mathbb{E}(P_1)$ or $\bar{p} = \mathbb{E}(P_{1,1})$ according to the case in hand. Then, almost surely, as $n \rightarrow \infty$,*

$$G_n \Rightarrow \bar{p}\delta_0 + (1 - \bar{p})\delta_1 \quad \text{weakly on } [0, 1].$$

Proof. For part (a), under $\mathbb{P}^{\mathbf{p}}$, the process $(x_n)_{n \geq 0}$ is a sum of independent Bernoulli random variables, with success probabilities p_1, p_2, \dots . Then, by Hoeffding's inequality, for all $\varepsilon > 0$,

$$\mathbb{P}^{\mathbf{p}} \left(\left| \frac{x_n}{n} - \frac{1}{n} \sum_{m=1}^n p_m \right| \geq \varepsilon \right) \leq 2 \exp\{-2n\varepsilon^2\}.$$

Hence $x_n/n \rightarrow \bar{p}$ in probability and Lemma 3.1 applies.

For part (b), note that, in both cases, the process $(X_n)_{n \geq 0}$ is a random walk with mean step size \bar{p} , so $X_n/n \rightarrow \bar{p}$ almost surely by the strong law of large numbers. Set

$$\Phi(x) = 1_{\{x_n/n \rightarrow \bar{p} \text{ as } n \rightarrow \infty\}}$$

then

$$\phi(\mathbf{p}) = \mathbb{E}^{\mathbf{p}}(\Phi(x)) = \mathbb{P}^{\mathbf{p}}(x_n/n \rightarrow \bar{p} \text{ as } n \rightarrow \infty).$$

By Proposition 2.2, almost surely,

$$\phi(\mathbf{P}) = \mathbb{P}(X_n/n \rightarrow \bar{p} \text{ as } n \rightarrow \infty | \mathbf{P}).$$

Hence $\phi(\mathbf{P}) = 1$ almost surely. Consider the event $\Omega^* = \{\mathbf{P} \in B\}$, where

$$B = \{\mathbf{p} : x_n/n \rightarrow \bar{p} \text{ in probability as } n \rightarrow \infty \text{ under } \mathbb{P}^{\mathbf{p}}\}.$$

By Lemma 3.1, for $\mathbf{p} \in B$, we have $g_n(\mathbf{p}) \Rightarrow \bar{p}\delta_0 + (1-\bar{p})\delta_1$ weakly as $n \rightarrow \infty$. Then, since convergence almost surely implies convergence in probability,

$$\mathbb{P}(\Omega^*) \geq \mathbb{P}(\phi(\mathbf{P}) = 1) = 1$$

and on Ω^* we have

$$G_n = g_n(\mathbf{P}) \Rightarrow \bar{p}\delta_0 + (1-\bar{p})\delta_1 \quad ,$$

weakly as $n \rightarrow \infty$. □

4 Taking a closer look

The main result of the previous section, Theorem 3.2, shows that, under general conditions, the empirical probability distribution of break points G_n , defined at (3), converges weakly on $[0, 1]$ (on an event of probability 1 in the random cases) to a limit supported by the endpoints of the interval. That is, most of the break points $A_{n,k}$ 'move' either to 0 or to 1 as the

fragmentation process proceeds. Still, it is clear that this result does not provide a completely satisfactory picture of the distribution of the break points, even for large n . For example, as $n \rightarrow \infty$, there will be more and more break points away from the endpoints of the interval. How are these points distributed? Furthermore, even though most of the mass of G_n moves towards the endpoints, how fast does this movement happen? We address these questions in this section.

4.1 Away from the endpoints

The results in this subsection will be expressed in terms of the quantile function

$$Q = \Phi^{-1} : (0, 1) \rightarrow \mathbb{R}$$

associated with the standard normal distribution function Φ .

Lemma 4.1. *Fix a family of splitting proportions $\mathbf{p} = (p_{n,k} : (n, k) \in I)$ and write $(g_n)_{n \geq 0}$ for the associated sequence of empirical distributions of the break points. Consider the probability measure $\mathbb{P}^{\mathbf{p}}$ on Ω_0 defined in Section 2 and the random process $(x_n)_{n \geq 0}$ on Ω_0 defined at (4). Assume that there exist sequences $(m_n)_{n \geq 0}$ in \mathbb{R} and $(\sigma_n)_{n \geq 0}$ in $(0, \infty)$, such that, as $n \rightarrow \infty$, we have $\sigma_n \rightarrow \infty$ and, for all $t \in \mathbb{R}$,*

$$\mathbb{P}^{\mathbf{p}}\left(\frac{x_n - m_n}{\sigma_n} \leq t\right) \rightarrow \Phi(t). \quad (6)$$

Then, for all $x, y \in (0, 1)$ with $x \leq y$, as $n \rightarrow \infty$,

$$\frac{ng_n([x, y])}{\sigma_n} \rightarrow Q(y) - Q(x).$$

That is, the sequence (ng_n/σ_n) of Radon measures on $(0, 1)$ converges vaguely on that space to a Radon measure with the density Q' with respect to the Lebesgue measure.

Proof. For $x \in (0, 1)$, define

$$k_n(x) = \min\{k \in \{0, 1, \dots, n+1\} : a_{n,k} \geq x\}.$$

Then

$$\frac{ng_n([0, x])}{\sigma_n} = \frac{k_n(x)}{\sigma_n}. \quad (7)$$

We now focus on estimating the asymptotic behavior of $k_n(x)$.

By Proposition 2.1, we have

$$a_{n,k_n(x)} = \mathbb{P}^{\mathbf{P}}(x_n \leq k_n(x) - 1) = \mathbb{P}^{\mathbf{P}}\left(\frac{x_n - m_n}{\sigma_n} \leq \frac{k_n(x) - 1 - m_n}{\sigma_n}\right).$$

Moreover, the convergence in (6) actually holds uniformly in $t \in \mathbb{R}$ by Pólya's theorem (because the cumulative distribution functions are non-decreasing). So we get

$$\left|a_{n,k_n(x)} - \Phi\left(\frac{k_n(x) - 1 - m_n}{\sigma_n}\right)\right| \rightarrow 0. \quad (8)$$

On the other hand, by definition of $k_n(x)$, we have $x \leq a_{n,k_n(x)} \leq x + \ell_n$, where $\ell_n = \max_{i=0}^n (a_{n,i+1} - a_{n,i})$ is the length of the longest interval in the partition \mathcal{P}_n . Moreover, by Proposition 2.1 again

$$\begin{aligned} a_{n,i+1} - a_{n,i} &\leq \Phi\left(\frac{i - m_n}{\sigma_n}\right) - \Phi\left(\frac{i - 1 - m_n}{\sigma_n}\right) \\ &\quad + 2 \sup_{t \in \mathbb{R}} \left| \mathbb{P}^{\mathbf{P}}\left(\frac{X_n - m_n}{\sigma_n} \leq t\right) - \Phi(t) \right|. \end{aligned} \quad (9)$$

Using (6) and that Φ is Lipschitz on \mathbb{R} together with $\sigma_n \rightarrow \infty$, we deduce that $a_{n,i+1} - a_{n,i} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in i , which implies that $\ell_n \rightarrow 0$. In particular, we get that $a_{n,k_n(x)} \rightarrow x$ as $n \rightarrow \infty$ and, coming back to (8), this yields

$$\left|x - \Phi\left(\frac{k_n(x) - 1 - m_n}{\sigma_n}\right)\right| \xrightarrow{n \rightarrow \infty} 0. \quad (10)$$

Using continuity of Q at x and the fact that $Q = \Phi^{-1}$, we get

$$\left|Q(x) - \frac{k_n(x) - 1 - m_n}{\sigma_n}\right| \rightarrow 0 \quad (11)$$

and therefore $k_n(x) = m_n + Q(x)\sigma_n + o(\sigma_n)$. Coming back to (7), we get, in the limit $n \rightarrow \infty$,

$$\frac{ng_n([0, x])}{\sigma_n} = \frac{m_n}{\sigma_n} + Q(x) + o(1) \quad (12)$$

It follows that, for any $x, y \in (0, 1)$ with $x \leq y$, we have

$$\frac{ng_n([x, y])}{\sigma_n} \rightarrow Q(y) - Q(x).$$

□

Remark 4.2. Consider the following strengthening of (6): there is a constant $C < \infty$ such that, for all n , we have

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}^{\mathbf{p}} \left(\frac{x_n - m_n}{\sigma_n} \leq t \right) - \Phi(t) \right| \leq \frac{C}{\sigma_n} \quad (13)$$

Such an assumption would correspond to a Berry–Esseen bound for the walk $(x_n)_{n \geq 0}$ and is known under the assumptions of Theorems 4.3 and 4.4. Then, the same proof gives a rate of convergence: for any $0 < \delta < 1/2$,

$$\frac{ng_n([x, y])}{\sigma_n} = Q(y) - Q(x) + O\left(\frac{1}{\sigma_n}\right)$$

uniformly in $\delta \leq x < y \leq 1 - \delta$. The restriction to the interval $[\delta, 1 - \delta]$ is required to have Q Lipschitz in the step from (10) to (11). Moreover, (9) shows that under this stronger assumption $\ell_n = O(1/\sigma_n)$.

We now state and prove convergence results for our three main cases of fragmentation-erasure. The first result addresses the case of a deterministic stratified fragmentation. Note that the assumptions of this result may hold even when the assumption in part (a) of Theorem 3.2 does not hold.

Theorem 4.3. Deterministic stratified fragmentation: *Set*

$$s_n = \sum_{k=1}^n p_k(1 - p_k)$$

and assume that $s_n \rightarrow \infty$ as $n \rightarrow \infty$. Then the sequence $(ng_n/\sqrt{s_n})$ of Radon measures on $(0, 1)$ converges vaguely on that space to a Radon measure with the density Q' with respect to the Lebesgue measure.

Proof. In this case, under $\mathbb{P}^{\mathbf{p}}$, x_n is a sum of independent Bernoulli random variables with success probabilities p_1, \dots, p_n , so (6) holds by Lindeberg’s central limit theorem, with

$$m_n = \sum_{k=1}^n p_k, \quad \sigma_n = \sqrt{s_n}.$$

Hence the result follows from Lemma 4.1. □

We obtain as a corollary the following result for the random stratified fragmentation. This can alternatively be shown directly from Lemma 4.1.

Theorem 4.4. Random stratified fragmentation: *Set*

$$s = \mathbb{E}(P_1(1 - P_1))$$

and assume that $\mathbb{P}(P_1 \in (0, 1)) > 0$. Then, almost surely, the sequence $(\sqrt{n}G_n/\sqrt{s})$ of Radon measures on $(0, 1)$ converges vaguely on that space to a Radon measure with the density Q' with respect to the Lebesgue measure.

Proof. Condition on the stratified family of splitting proportions $(P_n)_{n \geq 1}$ and write

$$\frac{\sqrt{n}G_n}{\sqrt{s}} = \frac{nG_n}{\sqrt{\sum_{k=1}^n P_k(1 - P_k)}} \times \sqrt{\frac{\sum_{k=1}^n P_k(1 - P_k)}{ns}}.$$

Theorem 4.3 applies to the first factor on the right, while the second factor tends to 1 almost surely by the strong law of large numbers. \square

Our final result in this subsection describes what happens in the case of fully random fragmentation.

Theorem 4.5. Fully random fragmentation: *Set*

$$\bar{p} = \mathbb{E}(P_{1,1}), \quad s = \bar{p}(1 - \bar{p})$$

and assume that $\mathbb{P}(P_{1,1} \in (0, 1)) > 0$. Then, almost surely, the sequence $(\sqrt{n}G_n/\sqrt{s})$ of Radon measures on $(0, 1)$ converges vaguely on that space to a Radon measure with the density Q' with respect to the Lebesgue measure.

Proof. Set

$$m_n = np, \quad \sigma_n = \sqrt{ns}$$

and consider the event $\Omega^* = \{\mathbf{P} \in B\}$, where

$$B = \{\mathbf{p} : \mathbb{P}^{\mathbf{p}}((x_n - m_n)/\sigma_n \leq t) \rightarrow \Phi(t) \text{ for all } t \in \mathbb{R}\}.$$

Then

$$\Omega^* = \{\mathbb{P}((X_n - m_n)/\sigma_n \leq t | \mathbf{P}) \rightarrow \Phi(t) \text{ for all } t \in \mathbb{Q}\}$$

so, by the almost sure central limit theorem for the RWDRE, which has been proved in [9], we have $\mathbb{P}(\Omega^*) = 1$. In particular, the assumption $\mathbb{P}(P_{1,1} \in (0, 1)) > 0$ implies the ellipticity assumption, Hypothesis (ME), in that paper. Then, by Lemma 4.1, on Ω^* , for all $x, y \in (0, 1)$ with $x \leq y$,

$$\frac{\sqrt{n}G_n([x, y])}{\sqrt{s}} = \frac{ng_n(\mathbf{P})([x, y])}{\sigma_n} \rightarrow Q(y) - Q(x),$$

which concludes the proof. \square

Remark 4.6. The limits in Theorem 3.2 depend only on the ‘average’ fragmentation proportions. Comparing with Theorems 4.3, 4.4 and 4.5, on the other hand, shows that the speed of convergence to the limits in Theorem 3.2 depends on other factors. For example:

- choosing $p_n = 1/2$ for all n in the deterministic stratified case, we get $s_n = n/4$, so the scaling factor in Theorem 4.3 becomes $2\sqrt{n}$;
- choosing P_1 to be uniformly distributed on $[0, 1]$ in the random stratified case, we get $s = 1/6$, so the scaling factor in Theorem 4.4 becomes $\sqrt{6n}$;
- choosing $P_{1,1}$ to be uniformly distributed on $[0, 1]$ in the fully random case, we get $s = 1/4$, so the scaling factor in Theorem 4.5 becomes $2\sqrt{n}$.

This indicates that there is a smaller residual mass in G_n away from the endpoints of the unit interval in the random stratified case than in the two other cases, hence a higher speed of ‘escape’ of the mass to the endpoints, though not by an order of magnitude. Furthermore, it can look surprising that the fully random case has exactly the same multiplicative factor as the case $p_n = 1/2$: this can be explained by the fact that, for later fragmentation steps, a large number of random variables are involved, resulting in an averaging effect.

4.2 Near the endpoints

We now look more closely at how the mass of the empirical distribution of break points moves towards the endpoints of the interval $[0, 1]$. We make statements for the behavior close to 0. Analogous statements near 1 can be obtained by flipping the interval.

Consider the transformed empirical distributions \tilde{g}_n defined by

$$\tilde{g}_n = \frac{1}{n} \sum_{k=1}^n \delta_{\tilde{a}_{n,k}}, \quad \tilde{a}_{n,k} = a_{n,k}^{1/n}.$$

As usual, we will change to upper-case and write \tilde{G}_n when the splitting proportions are random. Note that \tilde{g}_n is simply the pushforward of g_n by $x \mapsto x^{1/n}$ and the following relation holds, for $x, y \in [0, 1]$ with $x \leq y$,

$$\tilde{g}_n([x, y]) = g_n([x^n, y^n]).$$

We start with a lemma relating the limiting behavior of \tilde{g}_n to large deviation estimates on the lower tail for the associated random walk.

Lemma 4.7. Fix a family of splitting proportions $\mathbf{p} = (p_{n,k} : (n, k) \in I)$. Consider the probability measure $\mathbb{P}^{\mathbf{p}}$ on Ω_0 defined in Section (2) and the random process $(x_n)_{n \geq 0}$ on Ω_0 defined at (4). Assume that there exists $\bar{p} \in (0, 1]$ and a function $I : (0, \bar{p}] \rightarrow [0, \infty)$, continuous and decreasing with $I(\bar{p}) = 0$, and such that, for all $\alpha \in (0, \bar{p})$,

$$\frac{1}{n} \log \mathbb{P}^{\mathbf{p}}(x_n \leq \alpha n) \rightarrow -I(\alpha), \quad \text{as } n \rightarrow \infty. \quad (14)$$

Define $I(0) \in (0, \infty]$ and $x_* \in [0, 1)$ by

$$I(0) = \lim_{\alpha \rightarrow 0^+} I(\alpha), \quad x_* = e^{-I(0)}.$$

For $x \in [x_*, 1]$, let $\alpha_I(x)$ be the unique solution $\alpha \in [0, \bar{p}]$ of the equation $I(\alpha) = \log(1/x)$. Write \tilde{g} for the probability measure on $[0, 1]$ with distribution given by

$$\tilde{g}([0, x]) = \begin{cases} 0, & \text{if } x \in [0, x_*], \\ \alpha_I(x), & \text{if } x \in (x_*, 1). \end{cases} \quad (15)$$

Then, as $n \rightarrow \infty$, we have

$$\tilde{g}_n \Rightarrow \tilde{g} \quad \text{weakly on } [0, 1].$$

Remark 4.8. Under the assumptions of the lemma, the map $I : [0, \bar{p}] \rightarrow [0, \log(1/x_*)]$ is one-to-one, which implies that $\alpha_I(x)$ is well-defined for $x \in [x_*, 1]$. Moreover, α_I is continuous and increasing on $[x_*, 1]$ with $\alpha_I(x_*) = 0$ and $\alpha_I(1) = \bar{p}$. Hence $x \mapsto \tilde{g}([0, x])$ is continuous on $[0, 1)$, so \tilde{g} is atomless on $[0, 1)$, and

$$\tilde{g}(\{1\}) = 1 - \bar{p}.$$

Proof. Since \tilde{g}_n and \tilde{g} are supported on $[0, 1]$, and α_I is continuous on $[x_*, 1]$ with $\alpha_I(x_*) = 0$, it will suffice to show that $\tilde{g}_n([0, x]) \rightarrow \alpha_I(x)$ for all $x \in (x_*, 1)$. Fix $x \in (x_*, 1)$ and note that

$$\tilde{g}_n([0, x]) = \frac{1}{n} \max \left\{ k \geq 0 : a_{n,k}^{1/n} \leq x \right\}.$$

We use Proposition 2.1 and (14), together with continuity of I , to see that, for all $\alpha \in (0, \bar{p})$, as $n \rightarrow \infty$,

$$\log a_{n, \lfloor \alpha n \rfloor}^{1/n} = \frac{1}{n} \log a_{n, \lfloor \alpha n \rfloor} = \frac{1}{n} \log \mathbb{P}^{\mathbf{p}}(x_n \leq \lfloor \alpha n \rfloor - 1) \rightarrow -I(\alpha)$$

so

$$a_{n, \lfloor \alpha n \rfloor}^{1/n} \rightarrow e^{-I(\alpha)}.$$

This implies the desired limit for $\tilde{g}_n([0, x])$. \square

We now apply this lemma to the case of deterministic stratified fragmentation.

Theorem 4.9. Deterministic stratified fragmentation: *Assume that, for some probability measure H on $[0, 1]$ with $H(\{0, 1\}) < 1$, the following weak limit holds on $[0, 1]$ as $n \rightarrow \infty$,*

$$\frac{1}{n} \sum_{m=1}^n \delta_{p_m} \Rightarrow H. \quad (16)$$

Set

$$\bar{p} = \int_0^1 t H(dt).$$

Then $\bar{p} \in (0, 1)$ and, for all $\alpha \in (0, \bar{p})$, there is a unique $\theta = \theta(\alpha) \in \mathbb{R}$ such that

$$\int_0^1 \frac{te^\theta}{1-t+te^\theta} H(dt) = \alpha. \quad (17)$$

Define, for $\alpha \in (0, \bar{p})$,

$$I(\alpha) = \alpha\theta(\alpha) - \int_0^1 \log(1-t+te^{\theta(\alpha)}) H(dt).$$

Then I is continuous and decreasing on $(0, \bar{p})$ and $I(\alpha) \rightarrow 0$ as $\alpha \rightarrow \bar{p}$. Moreover, in the limit $n \rightarrow \infty$, we have

$$\tilde{g}_n \Rightarrow \tilde{g} \quad \text{weakly on } [0, 1]$$

where \tilde{g} is defined as in Lemma 4.7.

Remark 4.10. The hypothesis (16) implies in particular that $\frac{1}{n} \sum_{m=1}^n p_m$ converges as $n \rightarrow \infty$ with limit \bar{p} , and hence, by Theorem 3.2, that

$$g_n \Rightarrow \bar{p}\delta_0 + (1-\bar{p})\delta_1 \quad \text{weakly on } [0, 1].$$

Moreover (16) is also symmetric under flipping the interval $[0, 1]$, so (16) implies an analogous limiting statement for $g_n([1-x^n, 1])$.

Remark 4.11. It is straightforward to check from (17) that as $\alpha \rightarrow 0$ we have $\theta(\alpha) \rightarrow -\infty$ and $\alpha\theta(\alpha) \rightarrow 0$, so

$$I(\alpha) \rightarrow - \int_0^1 \log(1-t) H(dt).$$

Hence, in the notation of Lemma 4.7, we have

$$x_* = \exp \left\{ \int_0^1 \log(1-t) H(dt) \right\} \in [0, 1-\bar{p}]. \quad (18)$$

Proof. We check the assumptions of Lemma 4.7. Note that, for $\theta \in \mathbb{R}$,

$$\Lambda_n(\theta) := \log \mathbb{E}^{\mathbf{P}}(e^{\theta x_n/n}) = \sum_{i=1}^n \log(1 - p_i + p_i e^{\theta/n})$$

so it follows by the assumption (16) that

$$\frac{1}{n} \Lambda_n(n\theta) \xrightarrow{n \rightarrow \infty} \int_0^1 \log(1 - x + x e^\theta) H(dx) =: \Lambda(\theta).$$

Since Λ is finite and differentiable on \mathbb{R} , it follows from the Gärtner-Ellis theorem (see e.g. Theorem 2.3.6 in [7]) that, under $\mathbb{P}^{\mathbf{P}}$, the sequence $(x_n/n)_{n \geq 1}$ satisfies a large deviation principle with speed n and rate function

$$\Lambda^*(\alpha) = \sup_{\theta \in \mathbb{R}} \left(\theta \alpha - \int_0^1 \log(1 - x + x e^\theta) H(dx) \right)$$

which is the Fenchel–Legendre transform of Λ . In particular, the function $\Lambda^* : \mathbb{R} \rightarrow [0, \infty]$ is convex and has a unique minimum value equal to 0 at $\Lambda'(0) = \bar{p}$. If $\alpha \in [0, \bar{p}]$, then the supremum in the definition of Λ^* is attained at $\theta = \theta(\alpha)$ and therefore $I(\alpha) = \Lambda^*(\alpha)$. It follows that I satisfies the assumptions of Lemma 4.7: the function $I : (0, \bar{p}] \rightarrow [0, \infty)$ is continuous, decreasing, and satisfies $I(\bar{p}) = 0$ and (14) as a consequence of the large deviation principle mentioned above. Therefore, we can apply Lemma 4.7 which yields the result. \square

Theorem 4.9 immediately allows us to deduce the following result.

Theorem 4.12. Random stratified fragmentation: *Consider a random stratified fragmentation with splitting proportions $(P_n)_{n \geq 1}$. Denote by H the distribution of P_1 and assume that $H(\{0, 1\}) < 1$. Then $\tilde{G}_n \Rightarrow \tilde{g}$ weakly on $[0, 1]$, almost surely as $n \rightarrow \infty$, where \tilde{g} is defined as in Theorem 4.9.*

Proof. It is enough to note that, by Glivenko–Cantelli theorem, as $n \rightarrow \infty$, almost surely,

$$\frac{1}{n} \sum_{m=1}^n \delta_{P_m} \Rightarrow H \quad \text{weakly on } [0, 1].$$

Then Theorem 4.9 applies on the same set of full probability. \square

We finish this section with a discussion of the fully random fragmentation. The overall picture of the movement of the mass of G_n towards the endpoints of the interval $[0, 1]$ remains the same as in the deterministic

stratified fragmentation and the random stratified fragmentation. However, the assumptions are stronger and less information about the limiting distribution is available.

Theorem 4.13. Fully random fragmentation: *Consider a fully random fragmentation with splitting proportions $(P_{n,k} : (n, k) \in I)$. Assume that for some $p > 2$ we have*

$$\mathbb{E}(|\log(P_{1,1})|^p) < \infty \quad \text{and} \quad \mathbb{E}(|\log(1 - P_{1,1})|^p) < \infty. \quad (19)$$

Then, for some probability distribution \tilde{g} on $[0, 1]$, we have $\tilde{G}_n \Rightarrow \tilde{g}$ weakly on $[0, 1]$, almost surely as $n \rightarrow \infty$. Moreover \tilde{g} is supported on $[x_, 1]$ for some $x_* \in (0, 1)$, \tilde{g} has no atoms in $[0, 1)$ and $\tilde{g}(\{1\}) = 1 - \mathbb{E}(P_{1,1})$.*

Proof. Set $\bar{p} = \mathbb{E}(P_{1,1})$. It follows from e.g. Corollary 2.3 and Proposition 2.4 in [10] that there is a function $I_q : [0, \bar{p}] \rightarrow [0, \infty)$, continuous and decreasing with $I_q(\bar{p}) = 0$, and such that, almost surely, for all $\alpha \in (0, \bar{p})$ as $n \rightarrow \infty$,

$$\frac{1}{n} \log \mathbb{P}(X_n \leq \alpha n | \mathbf{P}) \rightarrow -I_q(\alpha). \quad (20)$$

We use here the assumption (19). The subscript q here is for ‘quenched’. Hence, on the same event of full probability, we can apply Lemma 4.7 to deduce the claimed weak convergence of \tilde{G}_n . The fact that $x_* > 0$ in this case follows from the finiteness of $I_q(0)$ guaranteed in [10]. Stated properties of \tilde{g} follow from Remark 4.8. \square

Remark 4.14. The limit distribution \tilde{g} in Theorem 4.13 is given in Lemma 4.7 in terms of I_q . Since a full description of the rate function I_q is not available, neither is a full description of \tilde{g} . Nonetheless, some information is available in [10]. Under \mathbb{P} , that is in the annealed case, $(X_n)_{n \geq 0}$ is a random walk with i.i.d. Bernoulli(\bar{p}) jumps. So $(X_n/n)_{n \geq 1}$ satisfies a large deviation principle with rate function I_a , and for $\alpha \in [0, 1]$,

$$I_a(\alpha) = \alpha \log \frac{\alpha}{\bar{p}} + (1 - \alpha) \log \frac{1 - \alpha}{1 - \bar{p}}.$$

Proposition 2.4 in [10] says that $I_a(\alpha) \leq I_q(\alpha)$ for $0 \leq \alpha \leq 1$, and that the two functions share the same unique global minimum, at the point \bar{p} , where they both vanish. Let

$$x_{*,a} = e^{-I_a(0)} = 1 - \bar{p} \in (0, 1)$$

and define a probability law supported by $[x_{*,a}, 1]$ by the cumulative distribution function

$$F_a(x) = \frac{\alpha_a(x)}{\bar{p}}, \quad x \in [x_{*,a}, 1]$$

where $\alpha_a(x)$ is the unique solution in $[0, \bar{p}]$ of the equation

$$I_a(\alpha) = \log(1/x),$$

then the law F_a stochastically dominates the limiting distribution in Theorem 4.13. Furthermore, unless $P_{1,1}$ is non-random, according to the same reference we have $I_a(0) < I_q(0)$, and so we have a strict ordering of the left endpoints of the support: $x_* < 1 - \bar{p}$.

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