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Insurance Microeconomics: Problems and Solutions

Pierre Picard*

12 February 2025

Abstract

This paper brings together problems in insurance microeconomics that can be used in an intermediate level course. It starts with insurance demand and optimal insurance contracts, including coinsurance, straight deductible contracts and umbrella policy. The effects of a background risk and moral hazard on optimal insurance coverage are the subject of specific problems. Ex post moral hazard in health insurance, self-insurance and parametric insurance are also analyzed. Detailed solutions are provided.

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1 Insurance demand - 1

1.1 Questions

Consider an individual with initial wealth $w_0 = 40$. He may have an accident with probability $q = 1/2$ and loss $L = 20$. He maximizes the expected utility of his final wealth w_f , with a von Neumann–Morgenstern utility function

$$U(w_f) = \ln(w_f)$$

This individual may take out an insurance contract with premium P and indemnity I paid in the case of an accident.

Question 1

Consider the lottery on final wealth, where the individual has no insurance, that is

$$w_f = \begin{cases} 40 & \text{with probability } 1/2 \\ 20 & \text{with probability } 1/2 \end{cases} .$$

Compute the certainty equivalent (CE) and the risk premium (RP) of the lottery. What is the maximum insurance premium that the individual would accept to pay for a full coverage policy (that is $I = 20$) if the alternative is not taking any insurance?

Question 2

In the (P, I) plane, isoutility curves correspond to insurance contracts with the same expected utility for the individual and isoprofit curves correspond to insurance contracts with the same expected profit for the insurer. Compute the equation of an isoutility curve with expected utility u_0 and the equation of an isoprofit curve with expected profit Π_0 . Compute the slope of expected utility curves when $I = 20$.

Question 3

Which contract maximizes the expected utility of the individual under the constraint that the insurer's expected profit is zero? Illustrate the result with a figure.

Question 4

The insurer incurs a fixed cost equal to 1 for each written contract. The expected profit of a contract (P, I) is now

$$P - \frac{1}{2}I - 1.$$

What is the best contract for the individual under the constraint of zero expected profit? Illustrate the result with a figure.

Question 5

Show that the individual prefers to take out the contract characterized at question 4 rather than not being insured.

Question 6

There is no fix cost but there is a variable cost : each unit paid to the policyholder costs $1 + \frac{1}{6}$ to the insurer. Hence the expected profit is now

$$P - \frac{1}{2} \times \frac{7}{6} \times I$$

Zero expected profit is thus equivalent to a proportional loading factor $1/6$. What is the best insurance contract for the individual under the constraint of zero expected profit?

Question 7

We still assume $P = 7/12 \times I$ as in Question 6. Consider the two alternative situations $w_0 = 60, L = 20$ and $w_0 = 60, L = 30$, with still $q = 1/2$ in both cases. Compare the optimal contracts in these two cases to the optimal contract obtained at question 7.¹

¹This problem is drawn from *Éléments de Microéconomie, 2. Exercices et Corrigés*, by Bruno Julien and Pierre Picard, Montchrestien-Lextenso Editions, 2011.

1.2 Solution

Question 1

We have

$$\frac{1}{2} \ln 40 + \frac{1}{2} \ln 20 = \ln CE,$$

which gives

$$\ln(40 \times 20)^{1/2} = \ln CE,$$

and thus

$$CE = 20\sqrt{2} \simeq 28.3.$$

We deduce

$$RP = 30 - 20\sqrt{2} \simeq 1.7.$$

The maximum premium that the individual would accept for full coverage is

$$40 - CE = 40 - 20\sqrt{2} \simeq 11.7.$$

Question 2

The equation of an isoutility curve is

$$\frac{1}{2} \ln(40 - P) + \frac{1}{2} \ln(20 - P + I) = u_0,$$

which may be rewritten as

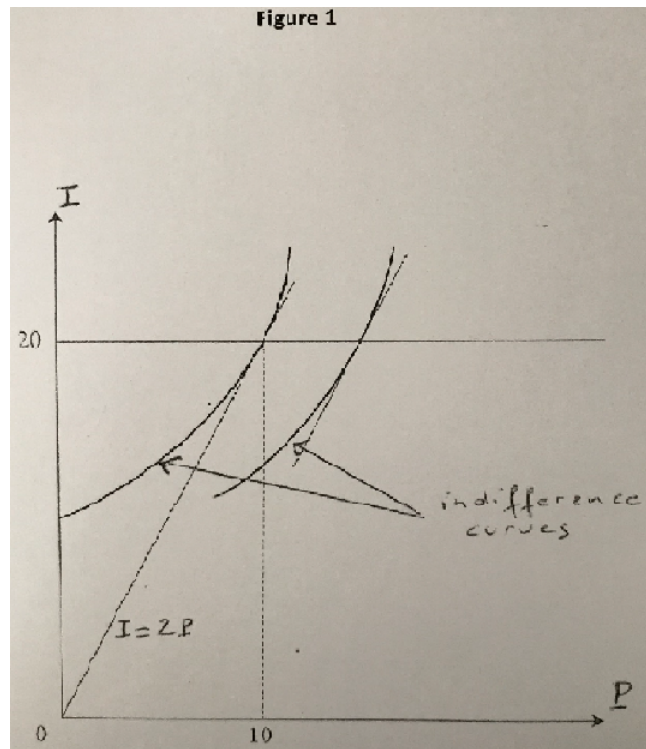
$$I = P + \frac{\exp(2u_0)}{40 - P} - 20.$$

The slope in the (P, I) plane is

$$\frac{dI}{dP} = 1 + \frac{\exp(2u_0)}{(40 - P)^2}.$$

When $I = 20$, we have $u_0 = 40 - P$ and thus

$$\frac{dI}{dP} = 2.$$



This is illustrated in Figure 1.

The equation of an isoprofit curve is

$$P - \frac{I}{2} = \Pi_0,$$

or

$$I = 2(P - \Pi_0).$$

Question 3

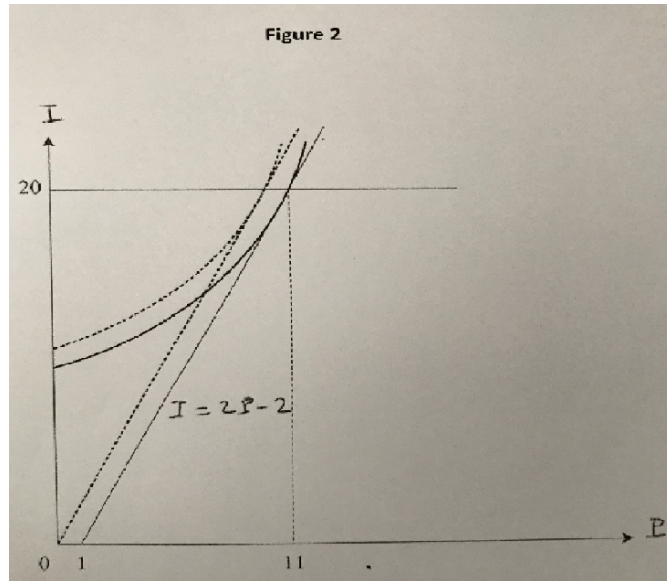
The maximum expected utility is reached at a tangency point between an indifference curve and the zero-profit line (see Figure 1), which gives

$$I = 20,$$

$$P = 10.$$

Thus full coverage is optimal in this case.

Question 4



The equation of an isoprofit curve is now written as

$$P - \frac{I}{2} - 1 = \Pi_0,$$

or

$$I = 2(P - 1 - \Pi_0)$$

The zero profit line shifts downwards with unchanged slope (see Figure 2) and the optimal contract still provide full coverage $I = 20$, with a higher premium $P = 11$.

Question 5

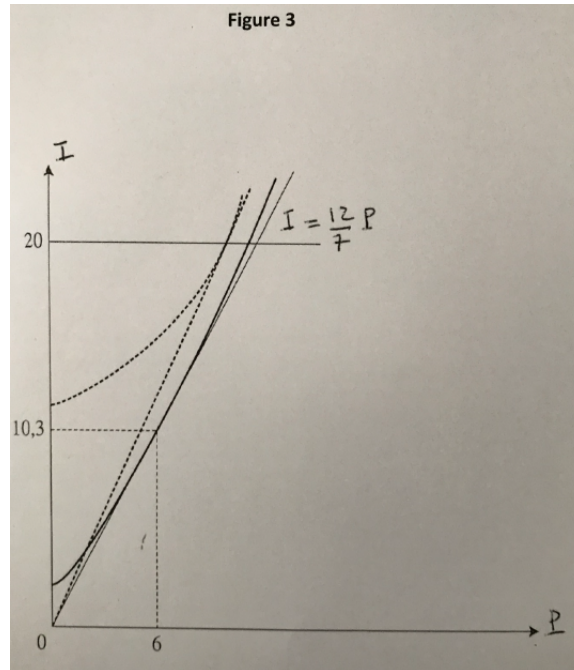
The expected utility is $\ln 29$ and we have $29 > 20\sqrt{2}$, which shows that the individuals still prefers purchasing insurance than not being insured.

Question 6

Now the zero profit line is given by

$$I = \frac{12}{7}P.$$

It goes through the origin of the axis in the (P, I) plane, but with lower slope than in the previous cases (see Figure 3).



The expected utility of the policyholder is written as

$$\begin{aligned}\mathbb{E}u(w_f) &= \frac{1}{2} \ln(40 - P) + \frac{1}{2} \ln(20 - P + I) \\ &= \frac{1}{2} \ln\left(40 - \frac{7}{12}I\right) + \frac{1}{2} \ln\left(20 + \frac{5}{12}I\right).\end{aligned}$$

The optimal coverage is such that

$$\begin{aligned}\frac{d\mathbb{E}u}{dI} &= -\frac{7}{2(480 - 7I)} + \frac{5}{2(240 + 5I)} \\ &= \frac{720 - 70I}{2(480 - 7I)(240 + 5I)} = 0,\end{aligned}$$

which gives

$$I = \frac{72}{7} \simeq 10.3.$$

Thus, because of the variable cost, the optimal insurance contract involves partial coverage (i.e. $I < 20$).

Question 7

When $w_0 = 60$ we have

$$\mathbb{E}u = \frac{1}{2} \ln\left(60 - \frac{7}{12}I\right) + \frac{1}{2} \ln\left(40 + \frac{5}{12}I\right),$$

which gives

$$\begin{aligned}\frac{d\mathbb{E}u}{dI} &= -\frac{7}{2(720 - 7I)} + \frac{5}{2(480 + 5I)} \\ &= \frac{240 - 70I}{2(720 - 7I)(480 + 5I)} = 0,\end{aligned}$$

and thus

$$I = \frac{24}{7} \simeq 3.43.$$

Hence the individual purchase less insurance when his wealth is larger. This is because we are in the DARA case: a larger wealth makes the individual less risk averse, hence a lower insurance demand.

When $w_0 = 60$ and $L = 30$, we have

$$\mathbb{E}u(w_f) = \frac{1}{2} \ln(60 - \frac{7}{12}I) + \frac{1}{2} \ln(30 + \frac{5}{12}I),$$

which gives

$$\begin{aligned}\frac{d\mathbb{E}u}{dI} &= -\frac{7}{2(720 - 7I)} + \frac{5}{2(360 + 5I)} \\ &= \frac{1080 - 70I}{2(720 - 7I)(480 + 5I)} = 0,\end{aligned}$$

and thus

$$I = \frac{108}{7} \simeq 15.43.$$

Hence the optimal coverage increases with the risk exposure. We may check that when w_0 and L are multiplied by $3/2$ (increasing from 40 and 20 to 60 and 30 respectively) the optimal coverage is also multiplied by $3/2$ (it is increasing from $72/7$ to $108/7$). This proportionality results from the choice of the utility function $u(w_f) = \ln w_f$, which corresponds to constant relative risk aversion, but it may not hold for other (concave) utility function. However, we may keep in mind that the optimal insurance coverage depends at the same time from the initial wealth (in particular, we may consider as an acceptable assumption that a larger wealth makes individuals less risk averse, as in the case considered here) and from the risk exposure.

2 Insurance demand - 2

2.1 Questions

A risk averse individual with initial wealth w_0 faces an insurable risk with loss L incurred in the case of an accident, with $L < w_0$. An accident occurs with probability q . This individual may purchase insurance with indemnity I received if an accident occurs, and premium P . Let w_f the final wealth of the individual. He maximizes the expected utility of final wealth $Eu(w_f)$ where $u(\cdot)$ is a von Neumann-Morgenstern utility function, with $u' > 0, u'' < 0$.

Question 1

In this question and the following, it is assumed that the individual displays constant absolute risk aversion (CARA), with $\alpha > 0$ the index of absolute risk aversion. Compute the upper bound of the insurance premium P the individual would agree to pay for a full coverage, i.e. for an insurance policy such that $I = L$, if the only alternative choice were to remain uninsured.

Question 2

In this question and the following we assume $P = (1 + \sigma)qI$, where $\sigma \geq 0$ is the loading factor, with $(1 + \sigma)q < 1$. Compute the optimal insurance coverage I_1 . Comment on the result.

Question 3

We assume $u(w_f) = \ln(w_f)$. Compute the optimal insurance coverage I_2 and comment on the result.

2.2 Solution

Question 1

We may write $u(w_f) = -\exp\{-\alpha w_f\}$. If the individual does not purchase insurance, his expected utility is written as

$$\mathbb{E}u = -(1 - q) \exp\{-\alpha w_0\} - q \exp\{-\alpha(w_0 - L)\},$$

while it is

$$-\exp\{-\alpha(w_0 - P)\},$$

under full coverage with insurance premium P . The upper bound of P under which full insurance preferred to no insurance is obtained by equalizing these two expressions, which gives

$$P = \frac{1}{\alpha} \ln[1 - q + q \exp\{\alpha L\}].$$

Question 2

The individual's expected utility is written as

$$\begin{aligned} \mathbb{E}u &= -(1 - q) \exp\{-\alpha(w_0 - (1 + \sigma)qI)\} \\ &\quad - q \exp\{-\alpha(w_0 - L - (1 + \sigma)qI + I)\}, \end{aligned}$$

and maximizing this expected utility with respect to I yields

$$I_1 = L - \frac{1}{\alpha} \ln \left[\frac{(1 - q)(1 + \sigma)}{1 - q(1 + \sigma)} \right],$$

with $I_1 < L$ if $\sigma > 0$ and $I_1 = L$ if $\sigma = 0$. We observe that I_1 is increasing with L and α (if $\sigma > 0$) and it does not depend on w_0 : there is no wealth effect on insurance demand in the case of a CARA utility function.

Question 3

Now, the expected utility is written as

$$\begin{aligned} \mathbb{E}u &= (1 - q) \ln[w_0 - (1 + \sigma)qI] \\ &\quad + q \ln[w_0 - L - (1 + \sigma)qI + I], \end{aligned}$$

and maximizing this expected utility with respect to I gives

$$I_2 = \frac{L(1 + \sigma)(1 - q) - w_0\sigma}{(1 + \sigma)[1 - (1 + \sigma)q]}.$$

As in the previous question, we have $I_2 < L$ if $\sigma > 0$ and $I_2 = L$ if $\sigma = 0$. However, I_2 is decreasing with respect to w_0 : there is a wealth effect because we are in the DARA case when $u(w_f) = \ln(w_f)$.

3 Insurance demand - 3

3.1 Problem

An individual owns initial wealth w_0 and he faces an accident risk with loss L and probability $q \in (0, 1)$. He may purchase an insurance policy with indemnity I in the case of an accident, and premium P , such that $P = (1 + \sigma)qI$, where σ is the loading factor such that $\sigma > 0$ and $(1 + \sigma)q < 1$. His attitude toward risk corresponds to the expected utility criterion, with von Neumann-Morgenstern utility function

$$u(w_f) = (2aw_f + b)^{1/2},$$

where w_f is the individual's final wealth and parameters a and b are such that $a > 0, 0 \leq b < 1$.

Question 1

Let w_1 and w_2 denote the final wealth of the individual in the absence of accident, and in the case of an accident, respectively, that is $w_1 = w_0 - P$ and $w_2 = w_0 - P - L + I$. Write the linear relationship between w_1 and w_2 that defines the feasible lotteries. You may simplify by denoting $\alpha = (1 + \sigma)q \in (0, 1)$.

Question 2

What is the optimal lottery, i.e. the values of w_1 and w_2 that maximize the individual's expected utility. You may denote

$$K = \left[\frac{(1 - \alpha)q}{\alpha(1 - q)} \right]^2 \in (0, 1).$$

Deduce the values of P and I that define the optimal insurance contract. What is the limit of I and P when σ goes to 0 ?

Question 3

What is the change in the optimal insurance indemnity when initial wealth w_0 is increasing, all other parameters being unchanged. Comment on the result. Answer the same question when L is increasing.

Question 4

We now consider a population of individuals with attitude toward risk characterized by the same utility function as above. However, they do not own the same initial wealth w_0 and their losses in the case of an accident are different. We assume that the loss is related to initial wealth through the following equation

$$L = \beta w_0 + L_0,$$

with $\beta > 0$ and $L_0 \geq 0$. Hence, the larger the initial wealth, the larger the risk exposure.

Under which condition does insurance demand (measured through the optimal indemnity) increase when initial wealth increase. Under which condition, is it proportional to initial wealth.

3.2 Solution

Question 1

Straightforward calculation yields

$$(1 - \alpha)w_1 + \alpha w_2 = w_0 - \alpha L,$$

which defines the set of feasible lotteries.

Question 2

Maximizing the expected utility

$$(1 - q)u(w_1) + qu(w_2)$$

under the constraint defined at question 1 yields

$$\frac{(1 - q)u'(w_1)}{qu'(w_2)} = \frac{1 - \alpha}{\alpha},$$

which is obtained either through the Lagrange multiplier method, or directly from the equality between the marginal rate of substitution and the slope of the feasible lottery line. Using the definition of the utility function, we deduce:

$$w_1 = \frac{2a(w_0 - \alpha L) + \alpha b(1 - K)}{2a[1 - \alpha(1 - K)]},$$

$$w_2 = Kw_1 - \frac{b(1 - K)}{2a},$$

and thus

$$P = w_0 - w_1$$

$$= \frac{\alpha[2aL - b(1 - K)] - 2a\alpha w_0(1 - K)}{2a[1 - \alpha(1 - K)]}.$$

Consequently, we have

$$I = w_0 - w_1 + L$$

$$= L - \frac{(1 - K)[b + 2a(w_0 - \alpha L)]}{2a[1 - \alpha(1 - K)]},$$

with $I < L$ when $\sigma > 0$. Since $K \rightarrow 1$ and $\alpha \rightarrow q$ when $\sigma \rightarrow 0$, we deduce that $I \rightarrow L$ and $P \rightarrow qL$ when $\sigma \rightarrow 0$. In other words, there is partial coverage and the optimal contract tends toward full coverage at actuarial price when the loading factor σ goes to zero.

Question 3

We observe that the optimal coverage I obtained at question 2 is decreasing with respect to w_0 , which results from the fact that the individual displays DARA preferences, which can be easily verified. Furthermore, I is increasing with respect to L : the larger the loss in the case of an accident, the larger the optimal insurance indemnity.

Question 4

If $L = \beta w_0 + L_0$, then w_0 is increasing with respect to w_0 if $\alpha\beta > 1$ and I is proportional to w_0 if $L_0 = 0$, i.e. if the loss is proportional to initial wealth, and $b = 0$, which corresponds to the case of an index of relative risk aversion which is constant and equal to $1/2$.

4 Coinsurance

4.1 Problem

John and Paul have preferences that correspond to expected utility theory. John's utility function is

$$u_J(w_f) = \ln(w_f)$$

where w_f denotes final wealth. John's initial wealth is equal to 12. This initial wealth may be reduced by a loss equal to 8, which occurs with probability $1/4$. John can purchase a coinsurance contract that reimburses proportion $\alpha \in [0, 1]$ of losses. Let α_J^* be the optimal value of α , i.e. the value of α that maximizes John's expected utility of final wealth.

Question 1

What value of the insurance premium P should be paid by John if he chooses $\alpha = 1$ and the loading factor is $\sigma = 0.2$?

Question 2

Compute α_J^* when $\sigma = 0.2$.

Question 3

Compute John's expected utility for $\alpha = \alpha_J^*$ and compare this value with the expected utility for $\alpha = 0$ and $\alpha = 1$.

Question 4

How does α_J^* change when John's initial wealth increases from 12 to 16 without change in the risk exposure? Comment on this result.

Question 5

In this question, we assume that John's utility function is written as

$$u_J(w_f) = -\exp(-kw_f) \text{ avec } k > 0.$$

Answer once again to Question 4 with this utility function. How does α_J^* move when John's initial wealth increases from 12 to 16.

Question 6

In this question John and Paul have the same utility function

$$u_J(w_f) = u_P(w_f) = \ln(w_f)$$

but Paul's initial wealth is 24 with a risk of loss equal to 16 with probability 1/4, while John has an initial wealth 12 and a risk of loss equal to 8 with probability 1/4. Let α_P^* be the optimal value of α for Paul. Compare α_J^* et α_P^* .²

4.2 Solution

Question 1

We have

$$P = 1.2 \times \frac{1}{4} \times 8 = 2.4.$$

Question 2

The indemnity is 8α and the premium is equal to 2.4α when $\sigma = 0.2$. Hence, John's expected utility is written as

$$\begin{aligned} \mathbb{E}u_J(w_f) &= \mathbb{E}\ln(w_f) \\ &= \frac{1}{4}\ln(12 - 8 - 2.4\alpha + 8\alpha) + \frac{3}{4}\ln(12 - 2.4\alpha) \\ &= \frac{1}{4}\ln(4 + 5.6\alpha) + \frac{3}{4}\ln(12 - 2.4\alpha). \end{aligned}$$

Maximizing $\mathbb{E}u_J(w_f)$ w.r.t. α yields

$$\alpha_J^* = \frac{5}{7} \simeq 0.714.$$

Question 3

²This is inspired from a similar problem in *Economic and Financial Decisions under Risk*, by L. Eeckhoudt, C. Gollier and H. Schlesinger, 2005, Princeton University Press.

We have

$$\begin{aligned}
\mathbb{E}u_J(w_f)|_{\alpha=\alpha_J^*} &= \frac{1}{4} \ln\left(4 + \frac{5.6 \times 5}{7}\right) + \frac{3}{4} \ln\left(12 - \frac{2.4 \times 5}{7}\right) \\
&= \frac{1}{4} \ln(8) + \frac{3}{4} \ln\left(\frac{72}{7}\right) \simeq 2.268, \\
\mathbb{E}u_J(w_f)|_{\alpha=0} &= \frac{1}{4} \ln(4) + \frac{3}{4} \ln(12) \simeq 2.210, \\
\mathbb{E}u_J(w_f)|_{\alpha=1} &= \ln(12 - 2.4) = \ln(9.6) \simeq 2.262,
\end{aligned}$$

and thus

$$\mathbb{E}u_J(w_f)|_{\alpha=0} < \mathbb{E}u_J(w_f)|_{\alpha=1} < \mathbb{E}u_J(w_f)|_{\alpha=\alpha_J^*}.$$

Hence, the individual would be in a better situation with full insurance than without insurance, but partial insurance with coinsurance coefficient α_J^* is the optimal choice.

Question 4

When initial wealth is equal to 16, we have

$$\mathbb{E}u_J(w_f) = \frac{1}{4} \ln(8 + 5.6\alpha) + \frac{3}{4} \ln(16 - 2.4\alpha),$$

and maximizing $\mathbb{E}u_J(w_f)$ in that case yields

$$\alpha_J^* = \frac{100}{168} \simeq 0.595.$$

Hence, John's insurance demand decreases when his initial wealth increases. We are in the DARA case, and an increase in initial wealth makes John's less risk averse, hence a decrease in his optimal insurance demand.

Question 5

We have

$$\begin{aligned}
\mathbb{E}u_J(w_f) &= -\mathbb{E} \exp(-kw_f) \\
&= -\frac{1}{4} \exp\{-k(4 + 5.6\alpha)\} - \frac{3}{4} \exp\{-k(12 - 2.4\alpha)\}.
\end{aligned}$$

Maximizing $\mathbb{E}u_J(w_f)$ w.r.t. α gives

$$\alpha_J^* = 1 - \frac{1}{8k} \ln\left(\frac{9}{7}\right).$$

One may easily check that this optimal coinsurance rate is unchanged when the initial wealth is 16 instead of 12. There is no wealth effect on insurance demand because we are in the CARA case. The larger k (which is the index of absolute risk aversion), the larger α_J^* , and $\alpha_J^* \rightarrow 1$ when $k \rightarrow \infty$.

Question 6

In the case of Paul, we have

$$\begin{aligned} \mathbb{E}u_P(w_f) &= \frac{1}{4} \ln(24 - 16 - 4.8\alpha + 16\alpha) + \frac{3}{4} \ln(24 - 4.8\alpha) \\ &= \frac{1}{4} \ln(4 + 5.6\alpha) + \frac{3}{4} \ln(12 - 2.4\alpha) + \ln(2), \end{aligned}$$

which shows that $\alpha_P^* = \alpha_J^* = 5/7$. This is because John and Paul display constant relative risk aversion, and the ratio loss/initial wealth is the same for John and Paul.

5 Straight deductible vs coinsurance - 1

5.1 Questions

An individual may suffer an accident with probability q and in that case the monetary loss is X . This loss is uniformly distributed on $[0, a]$, with $a > 0$. The probability distribution of X (conditionnally on the fact that an accident occurs) is thus $f(x) = 1/a$ if $x \in [0, a]$ and $f(x) = 0$ otherwise. This individual is an expected utility maximizer and he is risk averse. He may take out an insurance contract that specifies an indemnity $I(x)$ as a function of damages x in case of accident, with premium P given by

$$P = (1 + \sigma)q\mathbb{E}I(X),$$

where σ is the loading factor (with $\sigma > 0$) and \mathbb{E} is the expected value operator.

We first consider a coinsurance contract C_1 with premium P_1 and indemnity schedule $I_1(x)$ such that

$$I_1(x) = \alpha x,$$

with $0 < \alpha < 1$. We also consider a straight deductible contract C_2 with premium P_2 and indemnity schedule $I_2(x)$ such that

$$I_2(x) = \max\{0, x - D\}$$

where D is the deductible such that $0 < D < a$.

Question 1

Compute the value of the deductible $D = D^*$ for which $P_1 = P_2$. Note that D^* is a function of α and a . Compute also the corresponding value of the premium, which will be denoted by P^* .

Question 2

Let w_f be the random final wealth of this individual, given by

$$w_f = w_0 - P - X + I(X)$$

where w_0 denotes the initial wealth, and denote $w = w_0 - P$. Determine and represent on a figure the cumulative distribution function of w_f for contract C_1 . This cumulative distribution function will be denoted by $F_1(w_f)$. Note that $F_1(w_f)$ has a discontinuity point.

Question 3

Determine and represent on a figure the cumulative distribution function of w_f when contract C_2 with $D = D^*$ has been taken out by the individual. This cumulative distribution function will be denoted by $F_2(w_f)$. Note that $F_2(w_f)$ has two discontinuity points.

Question 4

Show that the individual prefers contract C_2 with $D = D^*$ to contract C_1 .

5.2 Solution**Question 1**

We have $P_1 = P_2$ iff $\mathbb{E}I_1(X) = \mathbb{E}I_2(X)$, with

$$\begin{aligned}\mathbb{E}I_1(X) &= \int_0^a \alpha x \frac{dx}{a} = \frac{\alpha a}{2}, \\ \mathbb{E}I_2(X) &= \int_D^a (x - D) \frac{dx}{a} = \frac{1}{2a}(a - D)^2.\end{aligned}$$

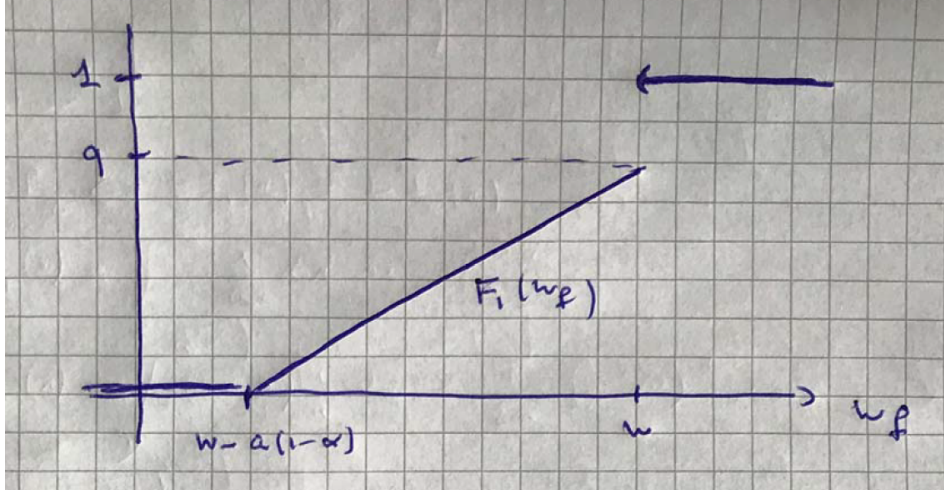
This gives

$$D^* = a(1 - \sqrt{\alpha}),$$

and

$$P^* = (1 + \sigma)q \frac{\alpha a}{2}.$$

Question 2



Using $w = w_0 - P$ gives $w_f = w - X + I(X)$. When $I(X) = I_1(X)$, we have $w_f = w - (1 - \alpha)X$. Since $X \in [0, a]$ and $\alpha \in (0, 1)$, we have $w_f \in [w - a(1 - \alpha), w]$, and thus $F_1(w_f) = 0$ if $w_f < w - a(1 - \alpha)$ and $F_1(w_f) = 1$ if $w_f \geq w$. If $w - a(1 - \alpha) < w_f < w$, then $F_1(w_f)$ is the probability of an accident with loss larger or equal to $(w - w_f)/(1 - \alpha)$, and thus in that case

$$\begin{aligned} F_1(w_f) &= q \int_{\frac{w-w_f}{1-\alpha}}^a \frac{dx}{a} \\ &= q \left[1 - \frac{w-w_f}{a(1-\alpha)} \right]. \end{aligned}$$

Hence $F_1(w_f)$ is discontinuous at $w_f = w$.

Question 3

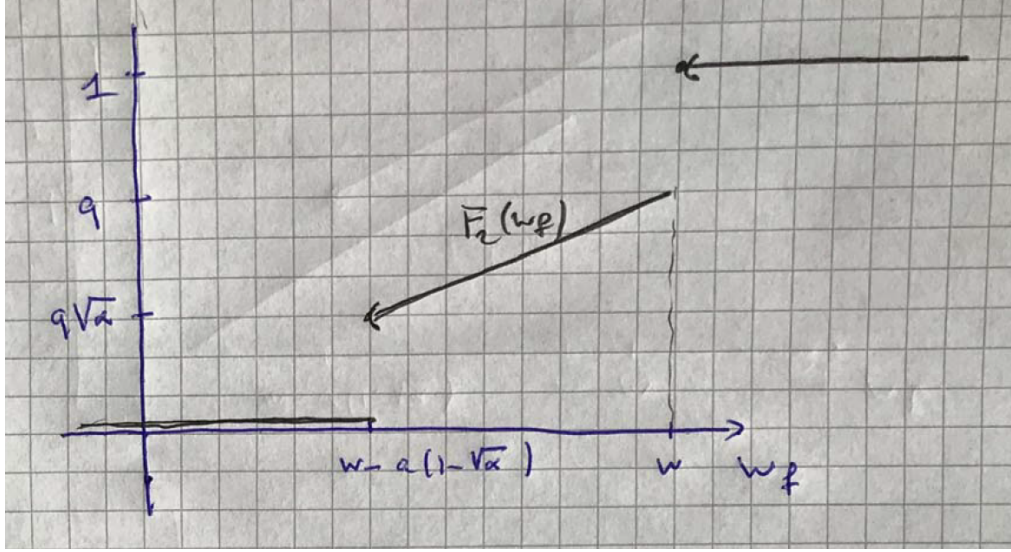
We still denote $w = w_0 - P$ and we have $w_f = w - X + \max\{0, X - D\}$ when $I(X) = I_2(X)$. In that case, we have $w_f \in [w - D, w]$, and thus $F_2(w_f) = 0$ if $w_f < w - D$ and $F_2(w_f) = 1$ if $w_f \geq w$.

We have $w_f = w - D$ when there is an accident with loss X larger or equal to D , and thus

$$F_2(w - D) = q \int_D^a \frac{dx}{a} = q \frac{a - D}{a}.$$

When $w_f \in (w - D, w)$ there is an accident with loss $X = w - w_f$ and thus

$$\begin{aligned} F_2(w_f) &= \Pr(X \geq w - w_f) \\ &= q \int_{w-w_f}^a \frac{dx}{a} = q \frac{a - w + w_f}{a}. \end{aligned}$$



Hence $F_2(w_f)$ is discontinuous at $w_f = w - D$ and $w_f = w$.

In the figure, function $F_2(w_f)$ is represented when $D = D^* = a(1 - \sqrt{\alpha})$ and thus $w = w_0 - P^*$.

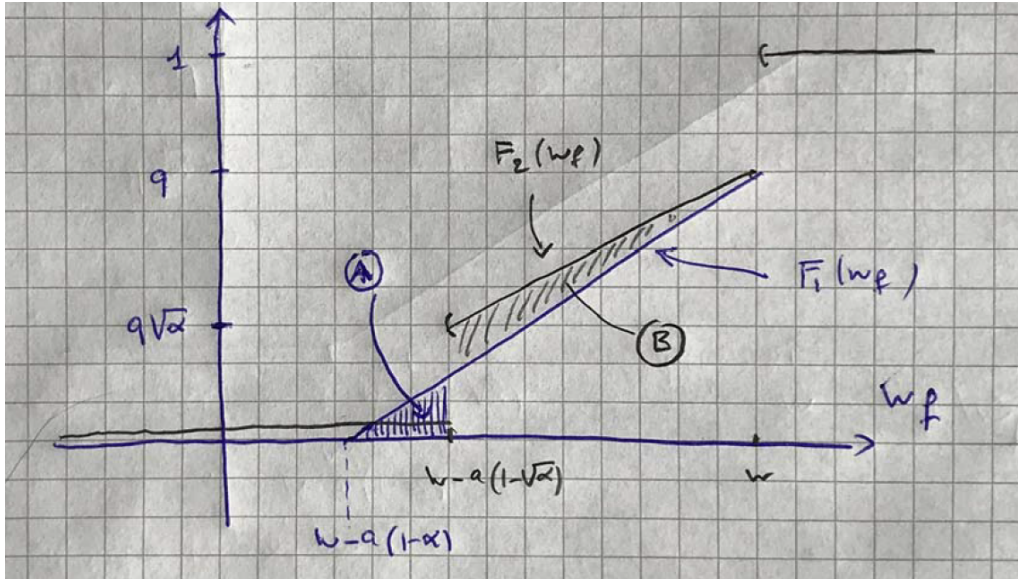
Question 4

If $D = D^*$ and $w = w_0 - P^*$, then $F_1(w_f) \geq F_2(w_f)$ if $w_f \leq w - a(1 - \sqrt{\alpha})$ and $F_1(w_f) < F_2(w_f)$ if $w_f \geq w - a(1 - \sqrt{\alpha})$. Furthermore, dashed areas **A** and **B** are both equal to $aq(\sqrt{\alpha} - \alpha)^2/2(1 - \alpha)$ in that case. Consequently, we have

$$\int_{-\infty}^t [F_1(w_f) - F_2(w_f)] dw_f \geq 0 \text{ for all } t,$$

$$\int_{-\infty}^t [F_1(w_f) - F_2(w_f)] dw_f = 0 \text{ if } t \geq w.$$

Hence, F_1 is more risky than F_2 in the sense of Rothschild and Stiglitz, and thus the risk-averse individual prefers insurance contract C_2 with deductible D^* to coinsurance contract C_1 .



6 Straight deductible vs coinsurance - 2

6.1 Problem

An individual with initial wealth w_0 faces the risk of an accident with probability $3/4$, and in that case the loss X is equal to 20, 40 or 60 with equal probability $1/3$. He is risk averse with von Neumann-Morgenstern utility function $u(w_f)$ such that $u' > 0$ and $u'' < 0$, where w_f denotes final wealth. His loss may be covered by an insurance contract that specifies an indemnity $I(X)$ and a premium P with loading 20%. We consider coinsurance contracts where $I(X) = \alpha X$ and $P = P_1(\alpha)$ with $\alpha \in [0, 1]$, and straight-deductible contracts where $I(X) = \max\{0, X - D\}$ and $P = P_2(D)$ where $D \in [0, 60]$ is the deductible. The individual owns an initial wealth w_0 and his final wealth is $w_f = w_0 - P - X + I(X)$.

Question 1

Define functions $P_1(\alpha)$ and $P_2(D)$.

Question 2

Let $D^*(\alpha)$ be the value of deductible D such that $P_1(\alpha) = P_2(D)$. Calculate $D^*(\alpha)$ for $\alpha = 0.1$, $\alpha = 0.2$ and $\alpha = 0.5$.

Question 3

Consider the coinsurance contract with $\alpha = 0.5$ and the straight deductible contract with $D = D^*(0.5)$, and denote w_f^1 and w_f^2 the random final wealth for these contracts, respectively. Show that the individual prefers the straight-deductible policy to the coinsurance policy, by using the fact that $u(w_f)$ is a concave function.

6.2 Solution

Question 1

Straightforward calculations yield

$$P_1(\alpha) = 36\alpha,$$

and

$$P_2(D) = \begin{cases} 36 - 0.9D & \text{if } D \in [0, 20], \\ 30 - 0.6D & \text{if } D \in [20, 40], \\ 18 - 0.3D & \text{if } D \in [40, 60]. \end{cases}$$

Question 2

We obtain

$$D^*(0.1) = 48,$$

$$D^*(0.2) = 38,$$

$$D^*(0.5) = 20.$$

Obviously, the larger the coinsurance coefficient α , the lower the deductible D for which both premiums are equal.

Question 3

Using $D^*(0.5) = 20$ and $P^1(0.5) = P^2(20) = 18$ yields

$$\begin{aligned}\mathbb{E}u(w_f^1) &= \frac{1}{4}u(w_0 - 18) \\ &\quad + \frac{3}{4}\left[\frac{1}{3}u(w_0 - 28) + \frac{1}{3}u(w_0 - 38) + \frac{1}{3}u(w_0 - 48)\right] \\ &< \frac{1}{4}u(w_0 - 18) \\ &\quad + \frac{3}{4}u(w_0 - \frac{1}{3}(28 + 38 + 48)) \\ &= \frac{1}{4}u(w_0 - 18) + \frac{3}{4}u(w_0 - 38) \\ &= \mathbb{E}u(w_f^2),\end{aligned}$$

where the inequality follows from the concavity of function $u(w_f)$.

7 Umbrella policy

7.1 Problem

A firm owns two buildings. Each building has been valued at 40 and can be destroyed by fire with probability $1/4$. In case of a fire, the building is fully destroyed with no residual value. Because these buildings are located far away from each other, the fire risks are independent. The firm can purchase fire insurance at actuarial price (no loading). The firm's managers display risk aversion about the net value of the buildings W , defined as the sum of the value of the buildings not destroyed by fire, minus the insurance premium plus the insurance indemnity received in case of fire.

Question 1

The firm purchases a fire insurance coverage that pays the value of damages above a deductible equal to 20 for each building that is destroyed by fire. In other words, for each building, the insurer pays 20 in case of fire. This contract is called a policy with "deductible per risk" (here each building corresponds to a risk). Compute the insurance premium of the insurance policy with deductible per risk. Let P^* denote this premium. What is the cumulative distribution function $F_1(W)$ of the random variable W_1 that corresponds to the net value of the firm's buildings under the deductible per risk policy?

Question 2

The insurer suggests an "umbrella policy" where the firm receives an indemnity equal to the share of the aggregate loss (i.e., the total loss for the two buildings) over a deductible D . We may assume that the most advantageous umbrella policy is such that $20 < D < 40$. In other words, if both buildings are destroyed by fire, the indemnity payment is $80 - D$ and if only one building is destroyed the indemnity is $40 - D$. Compute the value of the deductible D^* such that the umbrella policy induces the same premium P^* as the policy with deductible per risk considered in question 1. What is the cumulative distribution function $F_2(W)$ of the random variable W_2 that corresponds to the net value of the buildings under the umbrella policy with deductible D^* .

Question 3

Represent $F_1(W)$ and $F_2(W)$ in the same figure, and deduce the policy preferred by the firm's managers

7.2 Solution

Question 1

We have

$$P^* = \frac{1}{4} \times 20 + \frac{1}{4} \times 20 = 10.$$

There is no fire with probability $9/16$, one fire with probability $6/16$, and two fires with probability $1/16$, and thus

$$W_1 = \begin{cases} 70 & \text{with probability } 9/16, \\ 50 & \text{with probability } 6/16, \\ 30 & \text{with probability } 1/16, \end{cases}$$

which gives

$$F_1(W) = \begin{cases} 0 & \text{if } W < 30, \\ 1/16 & \text{if } 30 \leq W < 50, \\ 7/16 & \text{if } 50 \leq W < 70, \\ 1 & \text{if } W \geq 70. \end{cases}$$

Question 2

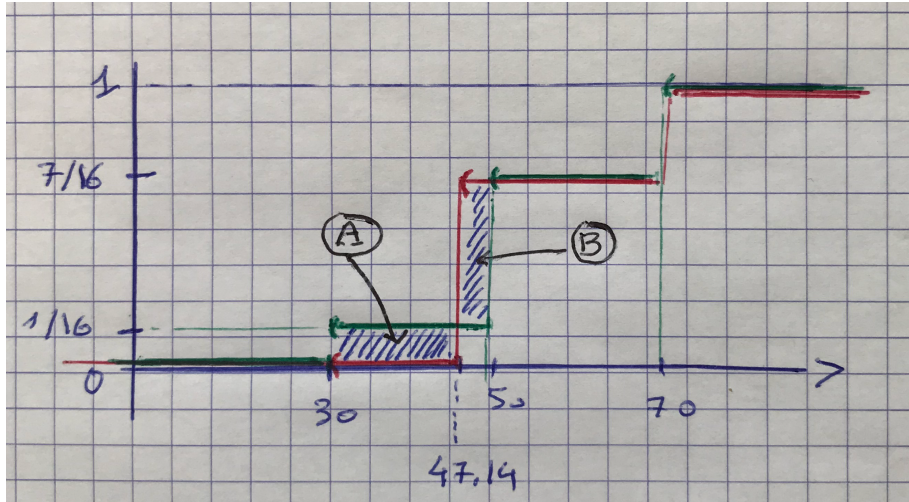
When $20 < D < 40$, the insurance premium of the umbrella policy is

$$P = \frac{6}{16}(40 - D) + \frac{1}{16}(80 - D),$$

and using $P = P^* = 10$ gives $D = D^* = 160/7 \simeq 22.86$.

We have $W_1 = 70$ if no fire occurs and $W_2 = 70 - (160/7) = 330/7 \simeq 47.14$ if there is one or two fires. Hence, we have

$$F_2(W) = \begin{cases} 0 & \text{if } W < 330/7, \\ 1/16 & \text{if } 330/7 \leq W < 70, \\ 1 & \text{if } W \geq 70. \end{cases}$$



Question 3

When $D = D^*$ and $P = P^*$, we have $F_1(W) \geq F_2(W)$ if $W \leq 330/7$ and $F_1(W) < F_2(W)$ if $W \geq 330/7$. Furthermore, dashed areas **A** and **B** are both equal to $15/14$. Consequently, we have

$$\int_{-\infty}^t [F_1(W) - F_2(W)] dW \geq 0 \text{ for all } t,$$

$$\int_{-\infty}^t [F_1(W) - F_2(W)] dW = 0 \text{ if } t \geq 70.$$

Hence, F_1 is more risky than F_2 in the sense of Rothschild and Stiglitz, and thus the risk-averse manager prefers the umbrella policy with deductible $D^* = 160/7$ to the policy with deductible per risk equal to 20.

8 Background risk

8.1 Problem

An individual owns a random initial wealth W defined by

$$W = w_0 + \varepsilon \text{ with probability } 1/2,$$

$$W = w_0 - \varepsilon \text{ with probability } 1/2,$$

where ε and w_0 are parameters such that $0 \leq \varepsilon < 1 \leq w_0$. The individual also receives a random income $\tilde{y} \geq 0$. W and \tilde{y} are independent random variables. Thus, the final wealth W_f is

$$W_f = w_0 + \varepsilon + \tilde{y} \text{ with probability } 1/2,$$

$$W_f = w_0 - \varepsilon + \tilde{y} \text{ with probability } 1/2.$$

The individual is an expected-utility maximizer with von Neumann-Morgenstern utility function

$$u(W_f) = \ln(W_f).$$

Let

$$\tilde{x} = w_0 + \tilde{y}$$

and denote

$$v(x) = \frac{1}{2}u(x + \varepsilon) + \frac{1}{2}u(x - \varepsilon).$$

Hence, the individual's expected utility is $\mathbb{E}v(\tilde{x})$.

Question 1

Find function $\phi(\cdot)$ such that

$$v(x) = \phi(u(x)) \text{ for all } x$$

Hint : use the change of variable $y = \ln(x)$. Check that $\phi' > 0$, $\phi'' < 0$ if $\varepsilon > 0$, $\phi'' = 0$ if $\varepsilon = 0$.

Question 2

Income \tilde{y} is such that

$$\tilde{y} = y_0 \text{ with probability } 1 - q,$$

$$\tilde{y} = y_0 - L \text{ with probability } q.$$

y_0 is a non-random income and L is a loss in the case of an accident that occurs with probability q , with $0 < q < 1$ and $0 < L < y_0$. The individual can cover the loss L through an insurance contract with indemnity I and actuarial premium $P = qI$. What is the optimal insurance coverage for this individual?

Question 3

We now assume $P = (1 + \sigma)qI$, where $\sigma > 0$ is a loading factor. Compare the optimal insurance demand when $\varepsilon > 0$ and when $\varepsilon = 0$.

Question 4

Show that you could anticipate the answer to question 3 by using a result on the effect of a background risk on risk aversion.

8.2 Solution**Question 1**

Let $y = \ln(x)$. This gives

$$\begin{aligned} v(x) &= \frac{1}{2} \ln(x + \varepsilon) + \frac{1}{2} \ln(x - \varepsilon) \\ &= \frac{1}{2} \ln(e^y + \varepsilon) + \frac{1}{2} \ln(e^y - \varepsilon) \\ &= \frac{1}{2} \ln(e^{2y} - \varepsilon^2) \\ &= \frac{1}{2} \ln(e^{2u(x)} - \varepsilon^2) \end{aligned}$$

Hence, we have

$$v(x) = \phi(u(x)),$$

where function $\phi(\cdot)$ is defined by

$$\phi(y) = \frac{1}{2} \ln(e^{2y} - \varepsilon^2),$$

with

$$\begin{aligned} \phi'(y) &= \frac{e^{2y}}{e^{2y} - \varepsilon^2} > 0, \\ \phi''(y) &= -\frac{2e^{2y}\varepsilon^2}{(e^{2y} - \varepsilon^2)^2}, \end{aligned}$$

and $\phi'' < 0$ if $\varepsilon > 0$ and $\phi'' = 0$ if $\varepsilon = 0$.

Question 2

The individual's expected utility is written as

$$\mathbb{E}u(W_f) = (1 - q)v(w_0 + y_0 - P) + qv(w_0 + y_0 - L - P + I).$$

This corresponds to the expected utility of an individual with utility function $v(\cdot)$ who owns a non-random initial wealth $w_0 + y_0$ and may suffer a loss L with probability q , this individual being covered with an insurance contract with indemnity I and premium P . We have $v'(x) = \phi'(u(x))u'(x) > 0$ and $v''(x) = \phi''(u(x))u'(x)^2 + \phi'(u(x))u''(x) < 0$, and thus this individual displays risk aversion. When $P = qI$, insurance is purchased at actuarial price, and full insurance $I = L$ is optimal for this individual.

Question 3

When $\varepsilon > 0$ we have $\phi'' < 0$ and thus function $v(\cdot)$ is "more concave" than function $u(\cdot)$. In other words, $v(\cdot)$ is a concave transformation of $u(\cdot)$. This implies that the individual with utility function $v(\cdot)$ displays more risk aversion (i.e., has a larger index of absolute risk aversion) than with utility function $u(\cdot)$, when $\varepsilon > 0$, while $u(\cdot) \equiv v(\cdot)$ if $\varepsilon = 0$. We deduce that the insurance demand (i.e., the optimal indemnity I when $\sigma > 0$) is larger when $\varepsilon > 0$ than when $\varepsilon = 0$.

Question 4

Random variable $\pm\varepsilon$ with equal probability is a zero-mean background risk, which is uncorrelated with the insurable loss L . This background risk makes the individual more risk averse w.r.t. the insurable loss when he is "risk vulnerable". The fact that the index of absolute risk aversion $A(w) = -u''(w)/u'(w)$ is a decreasing convex function of the individual's wealth is a sufficient condition for risk vulnerability. This is the case here because $A(w) = 1/w$ when $u(w) = \ln(w)$.

9 Health insurance

9.1 Questions

Consider an individual whose von Neumann-Morgenstern utility U depends on two variables: final wealth R and health state measured by real variable H . The larger H , the better the individual's health state. We assume

$$U(R, H) = u(R) + H,$$

with

$$u(R) = \frac{1 - \exp\{-\alpha R\}}{\alpha}, \alpha > 0.$$

We have $R = w_0 - T$ where w_0 is the initial wealth and T stands for the net payment made by the individual for health care. Let m denote his health care expenditures, i.e. what he pays to health care providers (doctors, hospitals, etc...) independently from insurance-linked money transfers. In what follows, we assume that care units are such that $m \in [0, 1]$.

The health state of the individual is characterized by an insitial state h_0 , by an illness that may occur and whose severity is defined by a random variable X with support $[a, b]$, $a > 0$, and by health care expenditures m , with

$$H = h_0 - X(1 - m).$$

Hence, the larger the health care expenditures, the lower the consequences of the illness on the individual's health state.

The insurance indemnity schedule specifies an indemnity schedule $I(m)$ and an insurance premium P . Hence, we have

$$T = m + P - I(m),$$

$$R = w_0 - T = w_0 - P - m + I(m)$$

When the severity of illness is equal to x , the individual chooses the health care expenditures $\hat{m}(x)$ that maximize his utility. In other words, $\hat{m}(x)$ maximizes

$$u(w - P - m + I(m)) + h_0 - x(1 - m)$$

with respect to $m \geq 0$, or equivalently $\hat{m}(x)$ maximizes

$$u(w - P - m + I(m)) + xm,$$

with respect to $m \geq 0$. His final wealth and final health care then become random variable:

$$R(X) = w - P - \hat{m}(X) + I(\hat{m}(X)),$$

$$H(X) = h_0 - X[1 - \hat{m}(X)].$$

According to the terms of the insurance contract, health expenditures are reimbursed in a given proportion θ with actuarial premium, which gives

$$I(m) = \theta m,$$

$$P = \theta \mathbb{E}[\hat{m}(X)].$$

,

We also assume that parameter a is such that $\hat{m}(a) > 0$ and thus $\hat{m}(x) > 0$ for all x in $[a, b]$ since $\hat{m}(x)$ will be an increasing function.

Question 1

Write the first-order optimality condition for $\hat{m}(x)$ to be an optimal choice for illness severity x , and deduce $R(x)$.

Question 2

Deduce $\hat{m}(x)$ when the insurance contract is characterized by θ and P , and deduce P .

Question 3

The expected utility is written as

$$\mathbb{E}[u(R(X))] + h_0 - \mathbb{E}[X(1 - \hat{m}(X))].$$

which depends on $\theta \in [0, 1]$. Rewrite this expected utility as $V(z)$ by using the change of variable $z = 1/(1 - \theta) \in [1, +\infty)$.

Question 4

What is the optimal value of θ ? You may assume $\mathbb{E}[X \ln(X)] - \mathbb{E}[X]E[\ln(X)] < \mathbb{E}[X]$. Comment on the result.

9.2 Solution

Question 1

When the severity of his illness is equal to x , the individual chooses m by maximizing his utility

$$-\frac{1}{\alpha} \exp\{-\alpha(w - (1 - \theta)m - P)\} + xm.$$

His optimal choice $\hat{m}(x)$ satisfies the following first-order optimality condition

$$-(1 - \theta) \exp\{-\alpha R(x)\} + x = 0,$$

where $R(x) = w - (1 - \theta)\hat{m}(x) - P$. This gives

$$R(x) = \frac{1}{\alpha} \ln \left(\frac{1 - \theta}{x} \right).$$

Question 2

The result obtained at question 1 gives

$$\hat{m}(x) = \frac{\alpha(w - P) + \ln \left(\frac{x}{1 - \theta} \right)}{\alpha(1 - \theta)}.$$

Using $P = \theta \mathbb{E}[\hat{m}(X)]$ then yields

$$\begin{aligned} \mathbb{E}[R(X)] &= w - (1 - \theta)\mathbb{E}[\hat{m}(X)] - P \\ &= w - \frac{P}{\theta} \\ &= \frac{1}{\alpha} \mathbb{E} \left[\ln \left(\frac{1 - \theta}{X} \right) \right], \end{aligned}$$

which gives

$$P = \theta w + \frac{\theta}{\alpha} \mathbb{E} \left[\ln \left(\frac{X}{1 - \theta} \right) \right].$$

Question 3

The results at question 2 allow us to write

$$\begin{aligned} \hat{m}(x) &= \frac{w - P - R(x)}{1 - \theta} \\ &= w + \frac{1}{\alpha} \ln \left(\frac{1}{1 - \theta} \right) + \frac{\ln(x) - \theta \mathbb{E}[\ln(x)]}{\alpha(1 - \theta)} \end{aligned}$$

and the optimal indemnity schedule maximizes

$$\begin{aligned}
& -\frac{1}{\alpha} \mathbb{E}[\exp\{-\alpha R(X)\}] + \mathbb{E}[X \hat{m}(X)] \\
& = -\frac{\mathbb{E}[X]}{\alpha(1-\theta)} \\
& + \left[\mathbb{E}[X]w + \frac{\mathbb{E}[X]}{\alpha} \ln\left(\frac{1}{1-\theta}\right) + \frac{\mathbb{E}[X \ln(X)] - \theta \mathbb{E}[X] \mathbb{E}[\ln(X)]}{\alpha(1-\theta)} \right]
\end{aligned}$$

with respect to $\theta \in [0, 1]$.

Let us consider $z = 1/(1-\theta)$ as the choice variable. Hence z maximizes

$$V(z) \equiv \mathbb{E}[X] \ln(z) + z[\Delta - \mathbb{E}[X]],$$

in $[1, +\infty)$, where

$$\Delta = \mathbb{E}[X \ln(X)] - \mathbb{E}[X] \mathbb{E}[\ln(X)] > 0.$$

Question 4

We have

$$\begin{aligned}
V'(z) &= \Delta - E[X] + \frac{E[X]}{z}, \\
V''(z) &= -\frac{E[X]}{z^2} < 0.
\end{aligned}$$

and

$$V'(1) = \Delta > 0$$

If $\Delta < E[X]$ then $V(z)$ reaches a maximum in $[1, +\infty)$ when

$$z = \frac{E[X]}{E[X] - \Delta} > 1,$$

which gives

$$\theta = \frac{\Delta}{E[X]} \in (0, 1).$$

We observe that the possibility for the policyholder to choose his health care expenditures without being monitored by the insurer leads to partial insurance (i.e. $\theta < 1$) although insurance is purchased at actuarial price. Providing incentives to the policyholder in this

situation of ex post moral hazard is the motive for such a partial coverage. We may also note that the optimal coverage coefficient θ does not depend on the policyholder's absolute risk aversion (measured by parameter α) contrary to what is obtained in usual insurance demand problems.

10 Moral hazard

10.1 Questions

Consider a risk-averse individual with initial wealth equal to 40 and loss equal to 20 in the case of an accident. Such an accident occurs with probability 0.25 or 0.5 according to whether the individual is cautious or not, respectively. The von Neumann - Morgenstern utility function of this individual is $\ln(w_f) - \ln(\beta)$ if he is cautious, and $\ln(w_f)$ otherwise, where w_f denotes his final wealth and β is a parameter larger than 1, but close to 1 (we may, for instance, assume $\beta = 1.01$). Hence, $\ln(\beta)$ may be viewed as the disutility of cautiousness (because being cautious requires effort). An insurance contract is characterized by indemnity I paid by the insurer in the case of an accident, and premium P paid by the policyholder to the insurer. Hence, we have

$$\begin{aligned}w_f &= 40 - P \text{ if there is no accident,} \\w_f &= 20 - P + I \text{ in the case of an accident.}\end{aligned}$$

Question 1

Assume that the insurer can verify whether the individual is cautious or not. Under this assumption, an insurance contract may be written as (I, P, a) where $a = 1$ if the insurer requires that the policyholder is cautious, and $a = 0$ otherwise. What are the contracts with zero expected profit for the insurer? Among these contracts, what is the best one for the policyholder?

Question 2

In what follows, we assume that the insurer cannot verify whether the policyholder is cautious or not. Hence, the insurance contract is just written as (I, P) and the policyholder is free either to be cautious ($a = 1$) or not ($a = 0$). What is the behaviour of the policyholder when $I = 20$? Deduce the insurance premium P that provides zero expected profit to the insurer for such a contract. What is the optimal choice of the individual when he has only two options: either to choose this contract with $I = 20$ and zero expected profit for the insurer, or remaining uninsured?

Question 3

What is the behaviour of the policyholder and what is the expected profit of the insurer when $(I, P) = (16, 4)$. What is the optimal choice of the individual when he may either choose this partial insurance contract or remain uninsured?³

10.2 Solution

Question 1

The expected profit is $P - I/2$ if $a = 0$ and $P - I/4$ if $a = 1$. The best contract maximizes the individual's expected utility with zero expected profit (i.e., the insurance premium is the actuarial premium), and we know that this optimal contract provides full coverage to the policyholder. Hence, we have just to compare $(I, P, a) = (20, 10, 0)$ to $(I, P, a) = (20, 5, 1)$. For these full-insurance contracts, we have

$$\mathbb{E} \ln(w_f) = \ln(30) \text{ when } a = 0,$$

$$\mathbb{E} \ln(w_f) = \ln(35) - \ln(\beta) \text{ when } a = 1.$$

For $\beta < 7/6 \simeq 1.17$, the optimal choice is $a = 1$.

Question 2

When $I = 20$, i.e. when there is full coverage, we have

$$\mathbb{E} \ln(w_f) = \ln(40 - P) \text{ when } a = 0,$$

$$\mathbb{E} \ln(w_f) = \ln(40 - P) - \ln(\beta) \text{ when } a = 1,$$

and obviously the policyholder chooses $a = 0$. Hence, for such a contract the zero-expected profit premium is $P = I/2 = 10$, with expected utility $\mathbb{E} \ln(w_f) = \ln(30)$.

If the individual remains uninsured, his expected utility is

$$\frac{1}{2} \ln(40) + \frac{1}{2} \ln(20)$$

if he chooses $a = 0$, and

$$\frac{3}{4} \ln(40) + \frac{1}{4} \ln(20) - \ln(\beta)$$

³This problem is drawn from *Eléments de Microéconomie, 2. Exercices et Corrigés*, by Bruno Julien and Pierre Picard, Montchrestien-Lextenso Editions, 2011.

if he chooses $a = 1$. Choosing $a = 1$ is the optimal choice if

$$\ln(\beta) < \frac{1}{4} \ln(40) - \frac{1}{4} \ln(20),$$

or equivalently if $\beta^4 < 2$, which holds when β is close to 1. Hence, the individual chooses $a = 1$ when he is uninsured, and his expected utility is

$$\frac{3}{4} \ln(40) + \frac{1}{4} \ln(20) - \ln(\beta) \simeq 3.516 - \ln(\beta).$$

Hence, he prefers to remain uninsured if

$$3.516 - \ln(\beta) > \ln(30).$$

This inequality holds if $\beta < 1.12$, hence if β is close to 1.

Question 3

When $(I, P) = (16, 4)$, we have

$$\begin{aligned} \mathbb{E} \ln(w_f) &= \frac{1}{2} \ln(36) + \frac{1}{2} \ln(32) \text{ if } a = 0, \\ \mathbb{E} \ln(w_f) &= \frac{3}{4} \ln(36) + \frac{1}{4} \ln(32) - \ln(\beta) \text{ if } a = 1, \end{aligned}$$

and thus the policyholder chooses $a = 1$ if $\ln(\beta) < \frac{1}{4} \ln(\frac{36}{32})$ or $\beta < 1.03$. Then, his expected utility is equal to $3.554 - \ln(\beta)$, which is larger than $3.516 - \ln(\beta)$, the no-insurance expected utility. The partial coverage $I = 16$ provides enough incentives to the policyholder for him to choose to be cautious, and this is preferable to remaining uninsured.

11 Self-insurance

11.1 Problem

An individual with initial wealth w_0 may suffer from an accident that occurs with probability q with loss L . Market insurance is unavailable but the loss in case of an accident may be reduced by expenditures made beforehand and called "self-insurance". Let x be the self-insurance expenditures. Thus, should an accident occur, the loss is a function of x : it is written as $L(x)$ with $x \geq 0$. We assume $L(0) > 0$, $L'(x) \leq 0$ and $L''(x) \geq 0$. We also assume $L'(x) < -1$ for x small enough, for otherwise the individual would never make self-insurance expenditures.

The final wealth of the individual is denoted by w_f with

$$\begin{aligned}w_f &= w_0 - x \quad \text{with probability } 1 - q, \\w_f &= w_0 - x - L(x) \quad \text{with probability } q.\end{aligned}$$

The individual chooses his self-insurance expenditures to maximize the expected utility of final wealth $Eu(w_f)$, which is written as

$$Eu(w_f) = (1 - q)u(w_0 - x) + qu(w_0 - x - L(x))$$

where $u(\cdot)$ is a von Neumann-Moregenstern utility function.

Question 1

In this question and the following, we assume :

$$\begin{aligned}L(x) &= \ell_0 - \ell_1 x \quad \text{if } 0 \leq x \leq \ell_0/\ell_1, \\L(x) &= 0 \quad \text{if } x \geq \ell_0/\ell_1,\end{aligned}$$

with $\ell_0 > 0$ and $1 < \ell_1 < 1/q$, and we may restrict attention to the case where $0 \leq x \leq \ell_0/\ell_1$ since there is no loss anymore when $x \geq \ell_0/\ell_1$,

Compute the self-insurance expenditures x^* if the individual is risk-neutral.

Question 2

Let $\hat{x}(\alpha)$ be the self-insurance expenditures if the individual displays constant absolute risk aversion $\alpha > 0$. Give a necessary and sufficient condition for $\hat{x}(\alpha) > 0$.

Question 3

Compute $\hat{x}(\alpha)$ when $\hat{x}(\alpha) > 0$.

In the two following questions, we compare the self-insurance expenditures x^* under risk neutrality and \hat{x} under risk aversion, without restricting ourselves neither to the case of CARA preferences, nor to the particular $L(x)$ function used in questions 1 to 3. We assume $L(x) > 0$ and $L'(x) < 0$ for all $x \geq 0$.

Question 4

Derive a necessary and sufficient condition for $x^* > 0$ and a condition that defines x^* if $x^* > 0$. Note that probability q and function $L(x)$ are used in these conditions.

Question 5

We here consider the case where $x^* > 0$. When the individual displays risk aversion, let $g(x) = Eu(w_f)$ denote his expected utility as a function of x . Derive the sign of $g'(x^*)$ when $x^* > 0$. Deduce a comparison between x^* and \hat{x} . Comment on the result.

11.2 Solution**Question 1**

When the individual is risk-neutral, we may assume $u(w_f) = w_f$ w.l.o.g, and thus

$$\begin{aligned}\mathbb{E}u(w_f) &= w_0 - x - qL(x) \\ &= w_0 - x - q[\ell_0 - \ell_1 x] \\ &= w_0 - q\ell_0 - x(1 - q\ell_1),\end{aligned}$$

which is decreasing w.r.t. x because $q\ell_1 < 1$. This gives $x^* = 0$. The individual abstains from any self-insurance when he is risk-neutral.

Question 2

When the individual displays constant absolute risk aversion at level α , we may assume $u(w_f) = -\exp\{-\alpha w_f\}$, which gives

$$\begin{aligned}Eu(w_f) &= -(1 - q) \exp\{-\alpha(w_0 - x)\} - q \exp\{-\alpha(w_0 - x - L(x))\} \\ &= -(1 - q) \exp\{-\alpha(w_0 - x)\} - q \exp\{-\alpha(w_0 - x - \ell_0 + \ell_1 x)\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{\partial \mathbb{E}u(w_f)}{\partial x} &= -\alpha(1-q)\exp\{-\alpha(w_0-x)\} - \alpha q(1-\ell_1)\exp\{-\alpha(w_0-x-\ell_0+\ell_1x)\}, \\ \frac{\partial^2 \mathbb{E}u(w_f)}{\partial x^2} &= \alpha^2(1-q)\exp\{-\alpha(w_0-x)\} + \alpha^2 q(1-\ell_1)^2 \exp\{-\alpha(w_0-x-\ell_0+\ell_1x)\} < 0,\end{aligned}$$

Thus $\mathbb{E}u(w_f)$ is a concave function of x and $\hat{x}(\alpha) > 0$ iff

$$\frac{\partial \mathbb{E}u(w_f)}{\partial x} \Big|_{x=0} = -\alpha(1-q)\exp\{-\alpha w_0\} - \alpha q(1-\ell_1)\exp\{-\alpha(w_0-\ell_0)\} > 0.$$

A short calculation shows that this condition is equivalent to

$$\alpha > \frac{1}{\ell_0} \ln \left(\frac{1-q}{q(\ell_1-1)} \right) = \alpha_0.$$

In other words, the individual makes some self-insurance expenses iff his degree of absolute risk aversion is larger than threshold α_0 .

Question 3

When $\alpha > \alpha_0$ the optimal level of self-insurance $\hat{x}(\alpha)$ is given by

$$\frac{\partial \mathbb{E}u(w_f)}{\partial x} = -\alpha(1-q)\exp\{-\alpha(w_0-x)\} - \alpha q(1-\ell_1)\exp\{-\alpha(w_0-x-\ell_0+\ell_1x)\} = 0.$$

A short calculation then gives

$$\hat{x}(\alpha) = \frac{\ell_0}{\ell_1} - \frac{1}{\alpha \ell_1} \ln \left(\frac{1-q}{q(\ell_1-1)} \right).$$

Using $q\ell_1 < 1$ shows that $\hat{x}'(\alpha) > 0$: the larger the individual's risk aversion, the larger his optimal self-insurance expenditures.

Question 4

Using $L''(x) \geq 0$ shows that $\mathbb{E}u(w_f) = w_0 - x - qL(x)$ is a concave function of x . Its maximum is reached at $x^* > 0$ iff

$$\frac{\partial \mathbb{E}u(w_f)}{\partial x} \Big|_{x=0} = -1 - qL'(0) > 0,$$

hence $qL'(0) < 1$ is a necessary and sufficient condition for $x^* > 0$. In that case, x^* is given by

$$\frac{\partial \mathbb{E}u(w_f)}{\partial x} \Big|_{x=x^*} = -1 - qL'(x^*) = 0.$$

Question 5

We have

$$g(x) = (1 - q)u(w_0 - x) + qu(w_0 - x - L(x)),$$

and

$$\begin{aligned} g'(x) &= -(1 - q)u'(w_0 - x) - qu'(w_0 - x - L(x))[1 + L'(x)], \\ g''(x) &= (1 - q)u''(w_0 - x) + qu''(w_0 - x - L(x))[1 + L'(x)]^2 \\ &\quad - qu'(w_0 - x - L(x))L''(x), \end{aligned}$$

and thus $g''(x) < 0$. Hence $g(x)$ is a concave function. When $x^* > 0$, we have $L'(x^*) = -1/q$ and

$$g'(x^*) = -(1 - q)u'(w_0 - x^*) + qu'(w_0 - x^* - L(x^*))\left[\frac{1}{q} - 1\right].$$

Using $u'' < 0$ yields

$$u'(w_0 - x^* - L(x^*)) > u'(w_0 - x^*),$$

which implies $g'(x^*) > 0$. Since function $g(x)$ is concave and increasing at $x = x^*$, we may deduce that its maximum is reached at $x = \hat{x} > x^*$. In other words, the self-insurance expenditures are larger when the individual is risk-averse than when he is risk-neutral.

12 Self-insurance and market insurance

We consider the case of a risk-averse who may protect himself against the risk of an accident by self-insurance and by market insurance. We denote:

- w_0 his initial wealth,
- q the probability of an accident, with $0 < q < 1$.
- $L(x)$ the monetary value of the damages incurred in the case of an accident, where $x \geq 0$ denotes self-insurance expenditures (paid whether an accident occurs or not), with $L'(x) \leq 0$, $L''(x) \geq 0$ and $L'(0) < -1$,
- I the insurance indemnity paid in the case of an accident,
- P the insurance premium.

Final wealth $w_f > 0$ is denoted w_1 if there is no accident, and w_2 in the case of an accident, with

$$\begin{aligned}w_1 &= w_0 - P - x, \\w_2 &= w_0 - P - x - L(x) + I,\end{aligned}$$

and the insurance premium P is related to the indemnity I through

$$P = (1 + \sigma)qI,$$

where $\sigma \geq 0$ denotes the loading factor. We assume $(1 + \sigma)q < 1$.

The individual's expected utility is written as

$$\begin{aligned}\mathbb{E}u(w_f) &= (1 - q)u(w_1) + qu(w_2). \\ &= (1 - q)u(w_0 - P - x) + qu(w_0 - P - x - L(x) + I),\end{aligned}$$

where $u(w_f)$ is the utility function, with $u' > 0$, $u'' < 0$.

We denote $s = I - P$ the net indemnity paid in the case of an accident. Hence, we have

$$P = (1 + \sigma)q(s + P),$$

and we may write

$$P = \pi(\sigma)s,$$

where we denote

$$\pi(\sigma) = \frac{(1 + \sigma)q}{1 - (1 + \sigma)q} > 0 \text{ avec } \pi'(\sigma) > 0.$$

with $\pi(0) = q/(1 - q)$ and $\pi(\sigma) > q/(1 - q)$ if $\sigma > 0$.

In what follows, the size of the insurance cover is measured by s and the insurance price by $\pi(\sigma)$. With these notations, we have

$$w_1 = w_0 - \pi(\sigma)s - x,$$

$$w_2 = w_0 - x - L(x) + s.$$

Question 1

In this question, and in question 3 as well, self-insurance expenditures are considered as fixed: hence, x is a fixed parameter, with loss $L(x)$ in the case of an accident. Write the equation that defines the set of feasible lotteries, i.e. the vectors (w_1, w_2) that can be reached through an insurance cover $s \geq 0$. Write this equation in a simple way by denoting

$$\widehat{w}(x, \sigma) = (w_0 - x)[1 + \pi(\sigma)] - \pi(\sigma)L(x).$$

Question 2

From now on, we assume $u(w_f) = \ln(w_f)$. What is the optimal lottery (w_1, w_2) , i.e. the lottery which maximizes $\mathbb{E}\ln(w_f)$ in the set of feasible lotteries characterized at question 1? Deduce the optimal insurance choice, i.e. the optimal net indemnity s . This optimal indemnity may be written as a function of $\pi(\sigma)$ and $\widehat{w}(x, \sigma)$, and we will denote it as $\widehat{s}(x, \sigma)$. How can we interpret $\widehat{s}(x, 0)$?

Question 3

We denote $\bar{U}(x)$ the individual's expected utility when he chooses his optimal insurance cover $\hat{s}(x, \sigma)$ and his self-insurance expenditures are equal to x . Optimal self-insurance expenditures maximize $\bar{U}(x)$. Show that these optimal expenditures maximize $\hat{w}(x, \sigma)$ with respect to x . What are these optimal expenditures when function $L(x)$ is defined by

$$L(x) = \frac{a}{x+b},$$

where parameters a and b are such that $a > b^2 > 0$? These optimal self-insurance expenditures depend on σ and will be written as $x^*(\sigma)$.

Question 4

How does an increase in σ affect $x^*(\sigma)$? What is illustrated in this example?

12.1 Solution

Question 1

We obtain

$$s = \frac{w_0 - x - w_1}{\pi(\sigma)} = w_2 - w_0 + x + L(x),$$

which gives

$$w_1 + \pi(\sigma)w_2 = \hat{w}(x, \sigma),$$

where

$$\hat{w}(x, \sigma) = (w_0 - x)[1 + \pi(\sigma)] - \pi(\sigma)L(x).$$

Question 2

Maximizing

$$\mathbb{E}(w_f) = (1 - q) \ln(w_1) + q \ln(w_2),$$

with respect to w_1 and w_2 , under constraint

$$w_1 + \pi(\sigma)w_2 = \hat{w}(x, \sigma),$$

leads to the following first-order optimality conditions

$$\begin{aligned}\frac{1-q}{w_1} - \lambda &= 0, \\ \frac{q}{w_2} - \lambda\pi(\sigma) &= 0,\end{aligned}$$

where λ is a Lagrange multiplier. Equalizing the two values of λ provided by these equations yields

$$\frac{(1-q)\pi(\sigma)}{w_1} = \frac{q}{w_2}.$$

We deduce

$$\begin{aligned}w_1 &= (1-q)\widehat{w}(x, \sigma), \\ w_2 &= \frac{q\widehat{w}(x, \sigma)}{\pi(\sigma)},\end{aligned}$$

which gives

$$\begin{aligned}\widehat{s}(x, \sigma) &= w_2 - w_0 + x + L(x) \\ &= \frac{q\widehat{w}(x, \sigma)}{\pi(\sigma)} - w_0 + x + L(x).\end{aligned}$$

We note that $\widehat{s}(x, 0) = (1-q)L(x)$, which means that the optimal net indemnity is equal to the difference between the losses $L(x)$ and the actuarial insurance premium $qL(x)$, when $\sigma = 0$. In other words, full coverage is optimal when insurance is purchased at actuarial price.

Question 3

We obtain

$$\begin{aligned}\overline{U}(x) &= (1-q)\ln[(1-q)\widehat{w}(x, \sigma)] + q\ln(q\widehat{w}(x, \sigma)/\pi(\sigma)) \\ &= \ln(\widehat{w}(x, \sigma)) + (1-q)\ln(1-q) + q\ln(q/\pi(\sigma)).\end{aligned}$$

Hence, maximizing $\overline{U}(x)$ is equivalent to maximizing $\widehat{w}(x, \sigma)$. When $L(x) = a/(x+b)$, we have

$$\widehat{w}(x, \sigma) = (w_0 - x)[1 + \pi(\sigma)] - \pi(\sigma)\frac{a}{x+b}.$$

Optimal self-insurance is such that

$$\frac{\partial \widehat{w}(x, \sigma)}{\partial x} = -[1 + \pi(\sigma)] + a\pi(\sigma)(x+b)^{-2} = 0,$$

which gives

$$x^*(\sigma) = \left(\frac{a\pi(\sigma)}{1 + \pi(\sigma)} \right)^{1/2} - b.$$

Question 4

We have

$$x^{*\prime}(\sigma) = \frac{a}{2}\pi(\sigma)^{-1/2}[1 + \pi(\sigma)]^{-3/2}\pi'(\sigma) > 0.$$

Hence, in this setting, self-insurance and market insurance are substitutable: an increase in the insurance price (associated with an increase in the loading factor) leads the individual to increase his self-insurance expenditures.

13 Parametric insurance

13.1 Problem

A farmer owns an initial wealth w_0 , which includes the monetary value of its crop in the absence of any weather hazard (hail, storm...). These hazards are called "accidents" below. Such an accident occurs with probability q , and in that case the farmer's crop is reduced by an amount L , and his wealth then become $w_0 - L$. The farmer displays constant absolute risk aversion, with index of absolute risk aversion $\alpha > 0$, and thus we may write his von Neumann-Morgenstern utility function as $u(w_f) = -\exp(-\alpha w_f)$, where w_f is the farmer's final wealth.

Weather conditions in the region are characterized by index $Y \in \{0, 1\}$ which is a random variable correlated with the actual weather conditions of the farm, with $Y = 1$ in the case of very unfavorable weather conditions and $Y = 0$ otherwise. We denote Z the random variable defined by $Z = L$ in the case of an accident (affecting the farm's crop) and $Z = 0$, otherwise. The joint probability distribution of Z and Y is defined in the following table:

$Z \backslash Y$	0	1	
0	π_{00}	π_{01}	$1 - q$
L	π_{10}	π_{11}	q
	$1 - p$	p	

In this table, we denote $\pi_{00} = \mathbb{P}(Z = 0, Y = 0)$, $\pi_{01} = \mathbb{P}(Z = 0, Y = 1)$, etc..., with $\pi_{00} + \pi_{01} + \pi_{10} + \pi_{11} = 1$, and with the marginal probability distributions in the last row and the last column. In particular, the weather index takes the value 1 with probability p . We assume

$$\mathbb{P}(Z = L \mid Y = 1) > \mathbb{P}(Z = L \mid Y = 0),$$

which means that the crop is affected by an accident more frequently when the weather index is bad than when it is good. This holds if

$$\frac{\pi_{11}}{\pi_{01}} > \frac{\pi_{10}}{\pi_{00}}.$$

We denote w_{00} the farmer's final wealth if $Z = 0, Y = 0$, w_{01} his final wealth if $Z = 0, Y = 1$, etc... Hence, his expected utility is

$$\mathbb{E}u(w_f) = \pi_{00}u(w_{00}) + \pi_{01}u(w_{01}) + \pi_{10}u(w_{10}) + \pi_{11}u(w_{11}).$$

Insurers offer parametric insurance contracts with indemnity I when $Y = 1$, whatever the actual loss of the farmer. The insurance premium P is such that

$$P = (1 + \sigma)pI,$$

where $p = \pi_{01} + \pi_{11}$ is the probability of an insurance payment made to the farmer, and $\sigma > 0$ is the loading factor. Hence, we have

$$w_{00} = w_0 - P,$$

$$w_{01} = w_0 - P + I,$$

$$w_{10} = w_0 - P - L,$$

$$w_{11} = w_0 - P - L + I.$$

Question 1

Compute the optimal parametric indemnity I .

Question 2

Comment on the effect of parameters α, σ, L and w on this optimal parametric insurance when $\pi_{01} = 0$.

13.2 Solution

Question 1

Using $P = (1 + \sigma)pI$ allows us to write $\mathbb{E}u(w_f)$ as a function of I only, and the first-order

optimality condition is written as

$$\begin{aligned}
\frac{\partial \mathbb{E}u(w_f)}{\partial I} &= -\pi_{00}\alpha(1+\sigma)p \exp(-\alpha w_{00}) \\
&\quad - \pi_{01}\alpha[(1+\sigma)p-1] \exp(-\alpha w_{01}) \\
&\quad - \pi_{10}\alpha(1+\sigma)p \exp(-\alpha w_{10}) \\
&\quad - \pi_{11}\alpha[(1+\sigma)p-1] \exp(-\alpha w_{11}) \\
&= 0.
\end{aligned}$$

This yields

$$\begin{aligned}
&-\pi_{00}\alpha(1+\sigma)p - \pi_{01}\alpha[(1+\sigma)p-1] \exp(-\alpha I) \\
&-\pi_{10}\alpha(1+\sigma)p \exp(\alpha L) - \pi_{11}\alpha[(1+\sigma)p-1] \exp(\alpha L - \alpha I) \\
&= 0,
\end{aligned}$$

which gives

$$I = \frac{1}{\alpha} \left\{ \ln \left[\frac{1 - (1+\sigma)p}{(1+\sigma)p} \right] + \ln \left[\frac{\pi_{01} + \pi_{11} \exp(\alpha L)}{\pi_{00} + \pi_{10} \exp(\alpha L)} \right] \right\}.$$

Question 2

If $\pi_{01} = 0$, then the farmer always incur a loss if the weather index is bad. He may also incur a loss if the index is good because $\pi_{10} > 0$. We have

$$I = \frac{1}{\alpha} \ln \left\{ \frac{[1 - (1+\sigma)p] \exp(\alpha L)}{(1+\sigma)[\pi_{00} + \pi_{10} \exp(\alpha L)]} \right\}.$$

Hence, the optimal parametric coverage I is decreasing w.r.t. σ and increasing w.r.t. L . Note that $\pi_{00} + \pi_{10} \exp(\alpha L) > \pi_{00} + \pi_{10} = 1 - p$ implies $I < L$ even if $\sigma = 0$. There is no effect of initial wealth w on this optimal coverage (which reflects the absence of wealth effect in the CARA case). The effect α on I is ambiguous: hence, a larger degree of risk aversion does not necessarily mean a larger demand for parametric insurance.