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Orientable burning number of graphs^{*}

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Abstract

In this paper, we introduce the problem of finding an orientation of a given undirected graph that maximizes the burning number of the resulting directed graph. We show that the problem is polynomial-time solvable on König–Egerváry graphs (and thus on bipartite graphs) and that an almost optimal solution can be computed in polynomial time for perfect graphs. On the other hand, we show that the problem is NP-hard in general and W[1]-hard parameterized by the target burning number. The hardness results are complemented by several fixed-parameter tractable results parameterized by structural parameters. Our main result in this direction shows that the problem is fixed-parameter tractable parameterized by cluster vertex deletion number plus clique number (and thus also by vertex cover number).

Keywords: burning number, graph orientation, fixed-parameter algorithm.

1. Introduction

The *burning number* of a directed or undirected graph G , denoted $\mathbf{b}(G)$, is the minimum number of steps for burning all vertices of G in the following way: in each step, we pick one vertex and burn it; and then between any two consecutive steps, the fire spreads to the neighbors (to the out-neighbors, in the directed setting) of the already burnt vertices. In other words, $\mathbf{b}(G)$ is the minimum integer b such that there exists a sequence $\langle w_0, \dots, w_{b-1} \rangle$ of vertices such that for each vertex v of G , there exists i ($0 \leq i \leq b-1$) such that the distance from w_i to v is at most i . Note that each w_i corresponds to the vertex that we picked in the $(b-i)$ th step.

The concept of burning number is introduced by Bonato, Janssen, and Roshanbin [6, 7] as a model of information spreading, while the same concept was studied already in 1992 by Alon [1]. The central question studied so far on this topic is the so-called *burning number conjecture*, which is about the worst case for a burning process and states that $\mathbf{b}(G) \leq \lceil \sqrt{n} \rceil$ for every connected undirected graph with n vertices. The conjecture has been studied intensively but it is still open (see [5] and the references therein). Recently, it has been announced that the conjecture holds asymptotically, that is, $\mathbf{b}(G) \leq (1 + o(1))\sqrt{n}$ [46]. For the directed case, the worst cases are completely understood in both weakly and strongly connected settings [33].

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17 Since the problem of computing the burning number is hard [4, 40, 44], several approximation
 18 algorithms [38, 39, 42] and parameterized algorithms [3, 34, 36] are studied.

19 In this paper, we investigate the worst case for a directed graph in the setting where we only
 20 know the underlying undirected graph. That is, given an undirected graph, which is assumed
 21 to be the underlying graph of a directed graph, we want to know how bad the original directed
 22 graph can be in terms of burning number. This concept is represented by the following new
 23 graph parameter: the *orientable burning number* of an undirected graph G , denoted $B(G)$, is
 24 the maximum burning number over all orientations of G ; that is,

$$25 \quad B(G) = \max_{\text{orientation } \vec{G} \text{ of } G} b(\vec{G}).$$

26 Recall that an orientation \vec{G} of an undirected graph G is a directed graph that gives exactly
 27 one direction to each edge of G . Now the main problem studied in this paper is formalized as
 28 follows.

Problem: ORIENTABLE BURNING NUMBER (OBN)

Input: An undirected graph $G = (V, E)$ and an integer b .

Question: Is $B(G) \geq b$?

29

30 In the setting of information spreading applications, this new problem can be seen as the
 31 one determining directions of links in a given underlying network structure to make the spread
 32 of something bad as slow as possible. Note that the dual problem of minimizing the burning
 33 number by an orientation is equivalent to the original graph burning problem on undirected
 34 graphs since $b(G) = \min_{\text{orientation } \vec{G} \text{ of } G} b(\vec{G})$. To see this equality, observe that each edge is
 35 used at most once and only in one direction to spread the fire.

36 See Fig. 1 for an example on the star graph $K_{1,n}$.

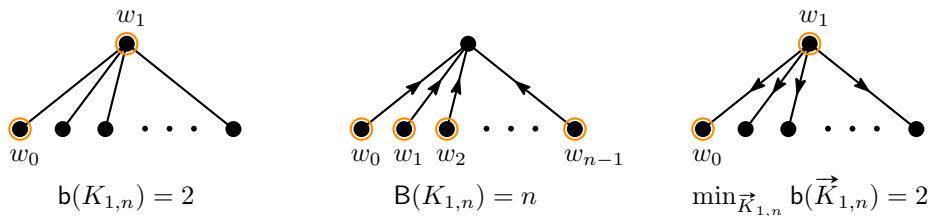


Figure 1: The star graph $K_{1,n}$ with $n \geq 2$.

37 1.1. Our results

38 We first present, in Section 3, several lower and upper bounds connecting the orientable
 39 burning number of a graph with other parameters such as the independence number. In par-
 40 ticular, for perfect graphs, we present almost tight lower and upper bounds that differ only
 41 by 2 and can be computed in polynomial time. We also consider Kónig–Egerváry graphs, which
 42 generalize bipartite graphs. Although our bounds for them are not exact, we show that the
 43 orientable burning number of a Kónig–Egerváry graph can be computed in polynomial time
 44 (see Section 3.1).

45 Next we consider the computational intractability of OBN in Section 4. We first show
 46 that OBN is W[1]-hard parameterized by the target burning number b . Although the proof of
 47 this result implies the NP-hardness of OBN for general graphs as well, we present another NP-
 48 hardness proof that can be applied to restricted graph classes that satisfy a couple of conditions.
 49 For example, this shows that OBN is NP-hard on planar graphs of maximum degree 3.

50 To cope with the hardness of OBN, we study structural parameterizations in Section 5.
51 We first observe that some sparseness parameters (e.g., treewidth) combined with b make the
52 problem fixed-parameter tractable. The main question there is the tractability of structural
53 parameterizations *not* combined with b . We show that OBN parameterized by cluster vertex
54 deletion number plus clique number is fixed-parameter tractable. As a corollary to this result,
55 we can see that OBN parameterized by vertex cover number is fixed-parameter tractable. We
56 believe that the techniques used there would be useful for studying other structural parameter-
57 izations of OBN as well.

58 1.2. Related problems

59 Although the problem studied in this paper is new, the concept of orientable number has
60 long history in the settings of some classical graph problems.

61 The most relevant is the orientable domination number. The *orientable domination number*
62 of an undirected graph G , denoted $\text{DOM}(G)$, is the maximum domination number over all
63 orientations \vec{G} of G . That is,

$$64 \quad \text{DOM}(G) = \max_{\text{orientation } \vec{G} \text{ of } G} \gamma(\vec{G}),$$

65 where $\gamma(\vec{G})$ is the size of a minimum dominating set of the directed graph \vec{G} .¹ Erdős [21]
66 initiated (under a different formulation) the study of orientable domination number by showing
67 that $\text{DOM}(K_n) \simeq \log_2 n$, where K_n is the complete graph on n vertices. Later, the concept of
68 orientable domination number is explicitly introduced by Chartrand et al. [13]. We can show
69 that orientable domination number (plus 1) is an upper bound of orientable burning number
70 (see Observation 3.1).

71 There are two other well-studied problems. One is to find an orientation that minimizes the
72 length of a longest path, which is equivalent to the graph coloring problem by the Gallai–
73 Hasse–Roy–Vitaver theorem [23, 29, 47, 48]. The other one is to find a strong orientation that
74 minimizes or maximizes the diameter. It is NP-complete to decide if a graph admits a strong
75 orientation with diameter 2 [14] and the maximum diameter of a strong orientation is equal to
76 the length of a longest path in the underlying 2-edge connected graph [27].

77 2. Preliminaries

78 *Terms in graph burning.* Let $D = (V, A)$ be a directed graph. By $N_{\ell, D}^+[v]$, we denote the set of
79 vertices with distance at most ℓ from v in D . We often omit D in the subscript and write $N_{\ell}^+[v]$
80 instead when D is clear from the context. A *burning sequence* of D with length b is a sequence
81 $\langle w_0, w_1, \dots, w_{b-1} \rangle \in V^b$ such that $\bigcup_{0 \leq i \leq b-1} N_i^+[w_i] = V$. Note that the burning number of D is
82 the minimum integer b such that D has a burning sequence of length b . We call the i th vertex
83 w_i in a burning sequence the *fire of radius i* and say that w_i *burns* the vertices in $N_i^+[w_i]$.

84 *Some basic graph terms.* Let $G = (V, E)$ be a graph. The *complement* of G , denoted $\bar{G} = (V, \bar{E})$,
85 is the graph in which two vertices are adjacent if and only if they are not adjacent in G . For
86 $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S . For $S \subseteq V$, let $G - S = G[V \setminus S]$. A
87 vertex set $S \subseteq V$ is an *independent set* in G if $G[S]$ contains no edge. The *independence number*
88 of G , denoted $\alpha(G)$, is the maximum size of an independent set in G . The *chromatic number*
89 of G , denoted $\chi(G)$, is the minimum integer c such that the vertices of G can be colored with
90 c colors in such a way that no two adjacent vertices have the same color. In other words, $\chi(G)$

¹In a directed graph, a vertex dominates itself and its out-neighbors.

91 is the minimum integer c such that V can be partitioned into c independent sets. A vertex set
 92 $S \subseteq V$ is a *clique* in G if $G[S]$ is a complete graph. The *clique number* of G , denoted $\omega(G)$,
 93 is the maximum size of a clique in G . The *clique cover number* of G , denoted $\theta(G)$, is the
 94 minimum integer t such that V can be partitioned into t cliques. An edge set $M \subseteq E$ is a
 95 *matching* in G if no two edges in M share an endpoint. The *matching number* of G , denoted
 96 $\mu(G)$, is the maximum size of a matching in G . Note that $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \theta(\overline{G})$ hold
 97 for every graph G .

98 A graph $G = (V, E)$ is a *perfect graph* if $\omega(G[S]) = \chi(G[S])$ holds for all $S \subseteq V$. Equivalently,
 99 G is a perfect graph if $\alpha(G[S]) = \theta(G[S])$ holds for all $S \subseteq V$ since the class of perfect graphs is
 100 closed under taking complements [41]. The class of perfect graphs contains several well-studied
 101 classes of graphs such as bipartite graphs and chordal graphs (see, e.g., [8]). A graph $G = (V, E)$
 102 is a *Kőnig–Egerváry graph* if $\alpha(G) = |V| - \mu(G)$. It is known that every bipartite graph is a
 103 Kőnig–Egerváry graph [19, 35].

104 *Structural parameters of graphs.* Let $G = (V, E)$ be a graph. A *vertex cover* of G is a set $S \subseteq V$
 105 such that $G - S$ has no edge. The *vertex cover number* of G , denoted $\text{vc}(G)$, is the minimum
 106 size of a vertex cover of G . A *cluster vertex deletion set* of G is a set $S \subseteq V$ such that each
 107 connected component of $G - S$ is a complete graph. The *cluster vertex deletion number* of G ,
 108 denoted $\text{cvd}(G)$, is the minimum size of a cluster vertex deletion set of G . In Section 5, we use
 109 $\text{cvd}(G) + \omega(G)$ as a parameter. This combined parameter can be related to other parameters
 110 as follows.

111 **Observation 2.1.** *For every graph G , the following inequalities hold:*

$$112 \quad \chi(G) \leq \text{cvd}(G) + \omega(G) \leq 2 \text{vc}(G) + 1.$$

113 *Proof.* Let $k = \text{cvd}(G) + \omega(G)$. To see that $\chi(G) \leq k$, one can construct a k -coloring by first
 114 coloring a cluster vertex deletion set S of size $\text{cvd}(G)$ with $\text{cvd}(G)$ colors, and then coloring each
 115 connected component of $G - S$ independently using at most $\omega(G)$ colors not used in S . For
 116 $k \leq 2 \text{vc}(G) + 1$, observe that a vertex cover is a cluster vertex deletion set and that the clique
 117 number cannot be larger than the vertex cover number plus 1. \square

118 We can see that $\text{cvd}(G) + \omega(G)$ is an upper bound of vertex integrity, and thus of treedepth,
 119 pathwidth, treewidth, and clique-width. We are not going to define these parameters as we do
 120 not explicitly use them in this paper. For their definitions, see [25, 30].

121 We assume that the readers are familiar with the theory of parameterized algorithms. (For
 122 standard concepts, see [17, 18, 22, 45].) Recall that a parameterized problem with input size n
 123 and parameter k is *fixed-parameter tractable* if there is a computable function f and a constant
 124 c such that the problem can be solved in time $O(f(k) \cdot n^c)$, while being $\text{W}[1]$ -hard means that
 125 the problem is unlikely to be fixed-parameter tractable.

126 3. General lower and upper bounds

127 In this section, we present lower and upper bounds of orientable burning number, which are
 128 useful in presenting algorithmic and computational results in the rest of the paper. We believe
 129 that the bounds would be of independent interest as well.

130 We start with a simple observation that orientable burning number is bounded from above
 131 by orientable domination number plus 1.

132 **Observation 3.1.** $\text{B}(G) \leq \text{DOM}(G) + 1$ for every graph G .

133 *Proof.* Let \vec{G} be an orientation of G . Let $\{w_1, \dots, w_\gamma\}$ be a minimum dominating set of \vec{G} .
 134 By arbitrarily setting w_0 , we construct a sequence $\sigma = \langle w_0, w_1, \dots, w_\gamma \rangle$ of length $\gamma + 1$. Since
 135 $\{w_1, \dots, w_\gamma\}$ is a dominating set, σ is a burning sequence of \vec{G} . Therefore, $\mathsf{B}(G) = \max_{\vec{G}} \mathsf{b}(\vec{G}) \leq$
 136 $\max_{\vec{G}} \gamma(\vec{G}) + 1 = \text{DOM}(G) + 1$. \square

137 Since $\text{DOM}(G) \in O(\alpha \cdot \log |V(G)|)$ [11, 28], Observation 3.1 implies that $\mathsf{B}(G) \in O(\alpha \cdot$
 138 $\log |V(G)|)$.

139 For orientable domination number, it is known that

$$140 \quad \alpha(G) \leq \text{DOM}(G) \leq n - \mu(G) \quad (1)$$

141 for every n -vertex graph G [12, 13]. The following counterpart for orientable burning number
 142 can be shown in almost the same way.

143 **Lemma 3.2.** *For every n -vertex graph G , $\alpha(G) \leq \mathsf{B}(G) \leq n - \mu(G) + 1$.*

144 *Proof.* The second inequality follows from the corresponding one in Eq. (1) and Observation 3.1.

145 To show the first inequality, let I be a maximum independent set of G . Since I is indepen-
 146 dent, there is an orientation \vec{G} of G such that all vertices in I are of in-degree 0. Since every burn-
 147 ing sequence has to contain all vertices of in-degree 0, we have $\mathsf{B}(G) \geq \mathsf{b}(\vec{G}) \geq |I| = \alpha(G)$. \square

148 Eq. (1) and the equality $\alpha(G) = n - \mu(G)$ for König–Egerváry graphs imply that $\text{DOM}(G) =$
 149 $\alpha(G)$ for them [12]. On the other hand, because of the additive factor $+1$ in Lemma 3.2, we
 150 only know that $\mathsf{B}(G) \in \{\alpha(G), \alpha(G) + 1\}$ for König–Egerváry graphs. In Section 3.1, we present
 151 a polynomial-time algorithm that determines which is the case.

152 A *tournament* is an orientation of a complete graph. A *king* of a tournament $T = (V, A)$ is
 153 a vertex $v \in V$ such that $N_2^+[v] = V$ [43]. The following fact due to Landau [37] is well known.

154 **Proposition 3.3** ([37]). *In a tournament, every vertex with the maximum out-degree is a king.*

155 By using Proposition 3.3, we can show the following upper bound of orientable burning
 156 number in terms of clique cover number.

157 **Lemma 3.4.** *For every graph G , $\mathsf{B}(G) \leq \theta(G) + 2$.*

158 *Proof.* Let $\mathcal{C} = \{C_2, \dots, C_{\theta+1}\}$ be a minimum clique cover of $G = (V, E)$. (Notice the shift in
 159 the numbering.) Given an orientation \vec{G} of G , we construct a sequence $\langle w_0, w_1, \dots, w_{\theta+1} \rangle$ by
 160 setting w_0 and w_1 to arbitrary vertices and setting w_i with $i \geq 2$ to a king of $\vec{G}[C_i]$. Since
 161 $C_i \subseteq N_2^+[w_i]$ for $2 \leq i \leq \theta + 1$, it holds that $\bigcup_{0 \leq i \leq \theta+1} N_i^+[w_i] = V$. Thus, $\langle w_0, w_1, \dots, w_{\theta(G)+1} \rangle$
 162 is a burning sequence of \vec{G} with length $\theta(G) + 2$. \square

163 Recall that $\theta(G) = \alpha(G)$ holds for every perfect graph G . Hence, Lemmas 3.2 and 3.4 imply
 164 the following almost tight bounds for perfect graphs.

165 **Corollary 3.5.** *For every perfect graph G , $\alpha(G) \leq \mathsf{B}(G) \leq \alpha(G) + 2$.*

166 Since the independence number of a perfect graph G can be computed in polynomial
 167 time [26], one can compute in polynomial time a value b such that $b \leq \mathsf{B}(G) \leq b + 2$. We
 168 left the complexity of ORIENTABLE BURNING NUMBER on perfect graphs unsettled.

169 Corollary 3.5 implies that $\mathsf{B}(K_n) \leq 3$. As $\text{DOM}(K_n) \simeq \log_2 n$, this example shows that the
 170 gap between $\mathsf{B}(G)$ and $\text{DOM}(G)$ can be arbitrarily large. It is easy to see that the lower bound
 171 $\alpha(G)$ is not always tight. For example, $\mathsf{B}(P_2) = 2 = \alpha(P_2) + 1$, where P_n is the path on n
 172 vertices. To give examples of graphs G with $\mathsf{B}(G) = \alpha(G) + 2$, we show that $\mathsf{B}(K_n) = 3$ for
 173 $n \geq 5$. To this end, we need the following simple observation.

174 **Observation 3.6.** *If H is an induced subgraph of G , then $B(H) \leq B(G)$.*

175 *Proof.* It suffices to show the statement for the case where $V(G) = V(H) \cup \{v\}$ with $v \notin V(H)$.
 176 Let \vec{H} be an orientation of H that satisfies $B(H) = b(\vec{H})$. Let \vec{G} be an orientation of G such
 177 that $A(\vec{H}) \subseteq A(\vec{G})$ and that the arcs in $A(\vec{G}) \setminus A(\vec{H})$ are oriented toward the new vertex v .
 178 Since $b(\vec{G}) \leq B(G)$, it suffices to show that $b(\vec{H}) \leq b(\vec{G})$. Let $\sigma = \langle w_0, \dots, w_{b-1} \rangle$ be a burning
 179 sequence of \vec{G} . Since v has out-degree 0 in \vec{G} , we have $N_{i,\vec{H}}^+[w_i] = N_{i,\vec{G}}^+[w_i] \setminus \{v\}$ for all i with
 180 $w_i \neq v$. Thus, if $v \notin \{w_0, \dots, w_{b-1}\}$, then σ is a burning sequence of \vec{H} . Assume that $w_j = v$
 181 for some j . We obtain a sequence σ' from σ by replacing the j th vertex with an arbitrary vertex
 182 in \vec{H} . Since $N_{j,\vec{G}}^+[w_j] = \{v\}$, σ' is a burning sequence of \vec{H} . \square

183 **Lemma 3.7.** $B(K_1) = 1$, $B(K_n) = 2$ for $2 \leq n \leq 4$, and $B(K_n) = 3$ for $n \geq 5$.

184 *Proof.* Clearly, $B(K_1) = 1$. Assume first that $2 \leq n \leq 4$. We can see that $B(K_n) > 1$ as a fire
 185 of radius 0 can burn only one vertex. Let \vec{K}_n be an orientation of K_n . Observe that \vec{K}_n has
 186 a vertex of out-degree at least $\lceil (n-1)/2 \rceil$ as there are $n(n-1)/2$ arcs. Hence we can burn at
 187 least $1 + \lceil (n-1)/2 \rceil$ vertices by placing a fire of radius 1 at a vertex of maximum out-degree.
 188 This is already enough for $n = 2$. If $n \in \{3, 4\}$, there can be one unburnt vertex, which can be
 189 burned by a fire of radius 0.

190 Next assume that $n \geq 5$. By Corollary 3.5 and Observation 3.6, it suffices to show that
 191 $B(K_5) > 2$. Let $V(K_5) = \{0, 1, 2, 3, 4\}$. Let \vec{K}_5 be the orientation of K_5 that has $A(\vec{K}_5) =$
 192 $\{(i, i+1), (i, i+2) \mid 0 \leq i \leq 4\}$, where the addition is modulo 5. See Fig. 2. This orientation
 193 shows that $B(K_5) > 2$ since a fire of radius 1 burns only three vertices. \square

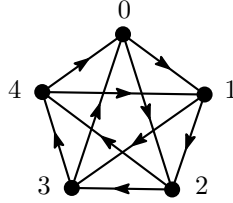


Figure 2: An optimal orientation of K_5 .

194 3.1. Polynomial-time algorithm for König–Egerváry graphs

195 We now present a polynomial-time algorithm for König–Egerváry graphs. Recall that
 196 $B(G) \in \{\alpha(G), \alpha(G) + 1\}$ for every König–Egerváry graph G . Intuitively, we show that the
 197 fire of radius 0 (often called w_0 in our exposition) is useful in most of the cases and the case
 198 $B(G) = \alpha(G) + 1$ rarely happens.

199 **Theorem 3.8.** *Let G be a König–Egerváry graph with more than four vertices, then*

$$200 \quad B(G) = \begin{cases} \alpha(G) + 1 & \text{if } G = mP_2, \\ \alpha(G) & \text{otherwise,} \end{cases}$$

201 where $m = |E|$ and mP_2 is the disjoint union of m edges.

202 This immediately gives the complexity of ORIENTABLE BURNING NUMBER for König–
 203 Egerváry graphs.

204 **Corollary 3.9.** ORIENTABLE BURNING NUMBER on König–Egerváry graphs is solvable in
 205 polynomial time.

206 *Proof of Theorem 3.8.* Let $G = (V, E)$ be an n -vertex König–Egerváry graph. Recall that
 207 $\alpha(G) = n - \mu(G)$ and $B(G) \in \{\alpha(G), \alpha(G) + 1\}$.

208 First assume that $G = mP_2$. In this case, $\alpha(G) = m$ holds. Suppose to the contrary that
 209 $B(G) = \alpha(G)$. To burn the whole graph G , each connected component has to contain a fire.
 210 Since we have $\alpha(G)$ connected components and $\alpha(G)$ fires, each connected component contains
 211 exactly one fire. However, since each connected component contains two vertices, the one with
 212 the fire of radius 0 is not completely burned. Thus, $B(G) \geq \alpha(G) + 1$.

213 Next assume that $G \neq mP_2$ and G has no perfect matching. Let $M = \{e_1, \dots, e_{\mu(G)}\}$ be
 214 a maximum matching of G . By the non-existence of a perfect matching, $2\mu(G) < n$ holds,
 215 and thus there exist $n - 2\mu(G) = \alpha(G) - \mu(G) > 0$ vertices not covered by M . Let \vec{G} be
 216 an orientation of G . We set $w_0, \dots, w_{\alpha(G)-\mu(G)-1}$ to the $\alpha(G) - \mu(G)$ vertices not covered by
 217 M , and set $w_{\alpha(G)-\mu}, \dots, w_{\alpha(G)-1}$ to the tails² of the arcs corresponding to $e_1, \dots, e_{\mu(G)}$. The
 218 constructed sequence $\langle w_0, \dots, w_{\alpha(G)-1} \rangle$ is a burning sequence of \vec{G} with length $\alpha(G)$.

219 Finally, we consider the case where $G \neq mP_2$ and G has a perfect matching M . Observe that
 220 in this case $\mu(G) = |V|/2$, and thus $\alpha(G) = n - \mu(G) = |V|/2$. We set $M = \{e_0, \dots, e_{\alpha(G)-1}\}$.
 221 Note that $|M| \geq 3$ as $|V| > 4$. Since $G \neq mP_2$, G has another edge $f \notin M$. Without loss of
 222 generality, we assume that f has one endpoint in e_0 and the other in e_2 . That is, e_0, f , and e_2
 223 together form P_4 . Observe that every orientation of P_4 can be burned by fires of radii 0 and
 224 2 (see Fig. 3). For an orientation \vec{G} of G , we construct a sequence $\sigma = \langle w_0, w_1, \dots, w_{\alpha(G)-1} \rangle$
 225 by setting w_0 and w_2 appropriately in the P_4 formed by e_0, f, e_2 , and setting w_i to the tail of
 226 the arc corresponding to e_i for each $i \in \{0, \dots, \alpha(G) - 1\} \setminus \{0, 2\}$. The sequence σ is a burning
 227 sequence of \vec{G} with length $\alpha(G)$. \square

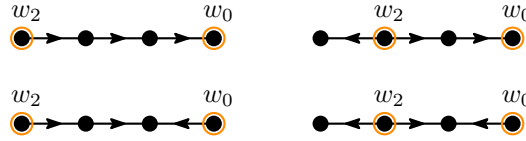


Figure 3: All orientations of P_4 can be burned by fires of radii 0 and 2. (By symmetry, the direction of the central edge is fixed.)

228 4. Hardness of the problem

229 Now we demonstrate that ORIENTABLE BURNING NUMBER is intractable in general. We
 230 present two reductions implying that

- 231 • OBN is NP-hard, and
- 232 • OBN is W[1]-hard parameterized by the target burning number b .

233 The reductions are similar and they follow the same idea: reduce the problem to INDEPENDENT
 234 SET by adding a sufficiently large number of isolated vertices.

235 We can see that our reduction showing the W[1]-hardness also shows NP-hardness in gen-
 236 eral. However, we present the separate reduction for NP-hardness as it has a wider range of
 237 applications. Basically, our reduction for W[1]-hardness works only for dense graphs, while the
 238 one for NP-hardness works also for sparse graphs like planar graphs.

239 Technically, the reduction for W[1]-hardness is a little more involved as it has to control the
 240 number of additional isolated vertices to keep the target burning number small. Thus, we first
 241 prove W[1]-hardness and then show a similar proof for NP-hardness.

²The tail of an arc $a = (u, v)$ is the vertex u , which has a as an out-going arc.

242 4.1. $W[1]$ -hardness parameterized by b

243 We reduce the following problem, which is known to be $W[1]$ -complete parameterized by
 244 the solution size k [17], to OBN parameterized by the target burning number b .

245 **Problem:** MULTICOLORED INDEPENDENT SET (MCIS)

246 **Input:** An undirected graph $G = (V, E)$ and a partition (V_1, \dots, V_k) of V into cliques.

247 **Question:** Does G contain an independent set of size k ?

248 **Theorem 4.1.** ORIENTABLE BURNING NUMBER on connected graphs is $W[1]$ -hard parameter-
 249 ized by the target burning number b .

250 *Proof.* Let (G, V_1, \dots, V_k) be an instance of MCIS. Let H be the connected graph obtained
 251 from G by first adding a set I of three isolated vertices and then adding a universal vertex u .³
 252 We prove that $(H, k + 3)$ is a yes-instance of OBN if and only if (G, V_1, \dots, V_k) is a yes-instance
 253 of MCIS. The if direction follows from Lemma 3.2, which says $B(H) \geq \alpha(H) \geq \alpha(G) + 3$.

254 In the following, we show the only-if direction. Assume that $(H, k + 3)$ is a yes-instance of
 255 OBN and an orientation \vec{H} of H satisfies $b(\vec{H}) \geq k + 3$.

256 We construct a sequence $\sigma = \langle w_0, \dots, w_{k+2} \rangle$ as follows. If all vertices in I are of in-degree 0,
 257 then we set w_0, w_1, w_2 to the vertices in I . Otherwise, we set w_2 to u and set w_0, w_1 to two
 258 vertices of I including the ones of in-degree 0 (if any exist). For $1 \leq i \leq k$, we set w_{i+2} to a
 259 king of the tournament $\vec{H}[V_i]$. Recall that every tournament has a king (see Proposition 3.3)
 260 and a king of a tournament can reach the other vertices in the tournament in at most two steps.
 261 We can see that σ is a burning sequence of \vec{H} (with length $k + 3$) as follows. Each vertex of
 262 in-degree 0 in $\{u\} \cup I$, if any exists, is burned by itself, and the other vertices in $\{u\} \cup I$ are
 263 burned by w_2 . For $1 \leq i \leq k$, w_{i+2} burns V_i because w_{i+2} is a king of $\vec{H}[V_i]$. Since $b(\vec{H}) \geq k + 3$,
 264 σ is a shortest burning sequence of \vec{H} .

265 Now we show that $\{w_3, \dots, w_{k+2}\}$ is an independent set of G , which implies that (G, V_1, \dots, V_k)
 266 is a yes-instance of MCIS. Suppose to the contrary that G has an edge between vertices
 267 $w_p, w_q \in \{w_3, \dots, w_{k+2}\}$. By symmetry, we may assume that $(w_p, w_q) \in A(\vec{H})$. Let $\sigma' =$
 268 $\langle w'_0, \dots, w'_{k+1} \rangle$ be any sequence satisfying that $\langle w'_0, w'_1, w'_2 \rangle = \langle w_0, w_1, w_2 \rangle$ and $\{w'_3, \dots, w'_{k+1}\} =$
 269 $\{w_3, \dots, w_{k+2}\} \setminus \{w_q\}$. We show that σ' is a burning sequence of \vec{H} , which contradicts that σ
 270 is shortest.

271 Clearly, $\{u\} \cup I \subseteq \bigcup_{0 \leq i \leq 2} N_i^+[w'_i]$ ($= \bigcup_{0 \leq i \leq 2} N_i^+[w_i]$). For each $i \in \{3, \dots, k + 2\} \setminus \{q\}$,
 272 there exists $j \in \{3, \dots, k + 1\}$ such that $w_i = w'_j$. Since w'_j ($= w_i$) is a king of $\vec{H}[V_{i-2}]$, we have
 273 $V_{i-2} \subseteq N_j^+[w'_j]$. The discussion so far implies that σ' burns $V(H) \setminus V_{q-2}$.

274 Since w_q is a king of $\vec{H}[V_{q-2}]$, we have $V_{q-2} \subseteq N_2^+[w_q]$. As $(w_p, w_q) \in A(\vec{H})$, it holds that
 275 $N_2^+[w_q] \subseteq N_3^+[w_p]$. Let $r \in \{3, \dots, k + 1\}$ be the index such that $w'_r = w_p$. Since $r \geq 3$,
 276 $N_3^+[w_p] \subseteq N_r^+[w_r]$ holds. Hence, $V_{q-2} \subseteq N_r^+[w_r]$, and thus σ' burns V_{q-2} as well. \square

277 4.2. NP-hardness

278 We reduce the following NP-complete problem to OBN.

279 **Problem:** INDEPENDENT SET

280 **Input:** An undirected graph $G = (V, E)$ and an integer k .

281 **Question:** Does G contain an independent set of size k ?

³If we do not need the connectivity, we can omit the universal vertex and the proof becomes slightly simpler.

282 INDEPENDENT SET is quite well studied and known to be NP-complete on many restricted
 283 graph classes such as planar graphs of maximum degree 3 [24]. Theorem 4.2 below shows that
 284 the hardness of INDEPENDENT SET on a graph class can be translated to the hardness of OBN
 285 on the same graph class, under an assumption that the class does not change by additions of
 286 isolated vertices.

287 The proof of Theorem 4.2 is quite similar to that of Theorem 4.1. The main difference is that
 288 here we can increase the target burning number as much as we like. This makes the discussion
 289 very simple; e.g., we do not need the concept of kings any more, and thus the hardness can be
 290 proved for almost all graph classes for which INDEPENDENT SET is hard.

291 **Theorem 4.2.** *Let \mathcal{G} be a graph class such that INDEPENDENT SET is NP-complete on \mathcal{G} . If*
 292 *\mathcal{G} is closed under additions of isolated vertices, then ORIENTABLE BURNING NUMBER on \mathcal{G} is*
 293 *NP-hard.*

294 *Proof.* Let (G, k) be an instance of INDEPENDENT SET, where $G \in \mathcal{G}$ and $|V(G)| = n$. Let
 295 H be the graph obtained from G by adding a set I of n isolated vertices. Since \mathcal{G} is closed
 296 under additions of isolated vertices, $H \in \mathcal{G}$ holds. To prove the theorem, it suffices to show that
 297 $(H, k+n)$ is a yes-instance of OBN if and only if (G, k) is a yes-instance of INDEPENDENT SET.

298 If G has an independent set S of size k , then Lemma 3.2 implies that $B(H) \geq |S \cup I| = k+n$.

299 Conversely, assume that $(H, k+n)$ is a yes-instance of OBN and there is an orientation \vec{H} of
 300 H such that $b(\vec{H}) \geq k+n$. Let $\sigma = \langle w_0, \dots, w_{b-1} \rangle$ be a shortest burning sequence of \vec{H} , where
 301 $b \geq k+n$. Necessarily, $I \subset \{w_0, \dots, w_{b-1}\}$, let $S = \{w_{i_0}, \dots, w_{i_{b-1-n}}\} = \{w_0, \dots, w_{b-1}\} \setminus I$. It
 302 suffices to show that S is an independent set of G . Suppose to the contrary that G has an edge
 303 between vertices $w_p, w_q \in S$. By symmetry, we may assume that $(w_p, w_q) \in A(\vec{H})$. Let $\sigma' =$
 304 $\langle w'_0, \dots, w'_{b-2} \rangle$ be any sequence such that: $\{w'_0, \dots, w'_{n-1}\} = I$ and $\{w'_n, \dots, w'_{b-2}\} = S \setminus \{w_q\}$.

305 We show that σ' is a burning sequence of \vec{H} , which contradicts that σ is a shortest one.
 306 Observe that since a longest directed path in \vec{H} has length less than n , for any $w_i \in S$ and
 307 $n \leq j \leq b-2$, we have $N_i^+[w_i] \subseteq N_j^+[w_i]$. Furthermore, added to the fact that $(w_p, w_q) \in A(\vec{H})$,
 308 it holds that $N_q^+[w_q] \subseteq N_{q+1}^+[w_p] \subseteq N_j^+[w_p]$ for any $n \leq j \leq b-2$. As a consequence,

$$\begin{aligned}
 309 \quad \bigcup_{0 \leq i \leq b-1} N_i^+[w_i] &= I \cup N_q^+[w_q] \cup \bigcup_{w_i \in S \setminus \{w_q\}} N_i^+[w_i] \\
 310 \quad &\subseteq I \cup \bigcup_{n \leq j \leq b-2} N_j^+[w'_j] \\
 311 \quad &\subseteq \bigcup_{0 \leq j \leq b-2} N_j^+[w'_j].
 \end{aligned}$$

312 This implies that σ' is a burning sequence of \vec{H} . □

313 As an application of Theorem 4.2, we can show the following corollary. (Recall that INDE-
 314 PENDENT SET is NP-complete on planar graphs of maximum degree 3 [24].)

315 **Corollary 4.3.** *ORIENTABLE BURNING NUMBER is NP-hard on planar graphs of maximum*
 316 *degree 3.*

317 5. Structural parameterizations

318 In this section, we consider some structural parameterizations of ORIENTABLE BURNING
 319 NUMBER. Given Theorem 4.1, which shows that OBN is intractable when parameterized by
 320 the target burning number b , it is natural to consider the problem parameterized by some
 321 structural parameters of the input graph.

322 *5.1. Parameterizations combined with b*

323 The first observation is that some sparseness parameters combined with b make the problem
324 tractable.

325 **Observation 5.1.** *Let \mathcal{G} be a class of graphs with a constant $c_{\mathcal{G}} > 1$ such that every graph*
326 *$G \in \mathcal{G}$ satisfies $\alpha(G) \geq |V(G)|/c_{\mathcal{G}}$. When parameterized by $c_{\mathcal{G}}$ plus the target burning number*
327 *b , ORIENTABLE BURNING NUMBER on \mathcal{G} is fixed-parameter tractable.*

328 *Proof.* Let G be an n -vertex graph in \mathcal{G} . If $b \leq n/c_{\mathcal{G}}$, then $\alpha(G) \geq b$ and thus (G, b) is a
329 yes-instance of OBN. If $b > n/c_{\mathcal{G}}$, then $n < b \cdot c_{\mathcal{G}}$, and thus (G, b) itself is a kernel with less
330 than $b \cdot c_{\mathcal{G}}$ vertices. \square

331 It is known that $\alpha(G) \geq n/(d+1)$ for every n -vertex graph G with average degree d [10, 49].
332 Thus Observation 5.1 implies the following corollary.

333 **Corollary 5.2.** *ORIENTABLE BURNING NUMBER is fixed-parameter tractable parameterized by*
334 *b plus average degree.*

335 Corollary 5.2 implies that OBN is fixed-parameter tractable parameterized by $b + \text{treewidth}$,
336 and OBN on planar graphs is fixed-parameter tractable parameterized by b . Recall that OBN is
337 NP-hard on planar graphs even if the maximum degree is 3 (Corollary 4.3). On the other hand,
338 the parameterized complexity of OBN parameterized solely by treewidth remains unsettled.

339 *5.2. Parameterizations without b*

340 Now we consider structural parameterizations *not* combined with b . As the first step in this
341 direction, we consider parameters less general than treewidth such as vertex cover number. In
342 some sense, vertex cover number is one of the most restricted parameters that is always larger
343 than or equal to treewidth (see, e.g., [25]).

344 We show that OBN parameterized solely by vertex cover number is fixed-parameter tractable.
345 Our proof is actually for a slightly more general case, where the parameter is cluster vertex
346 deletion number plus clique number. In the rest of this section, we prove the following theorem.

347 **Theorem 5.3.** *ORIENTABLE BURNING NUMBER is fixed-parameter tractable parameterized by*
348 *cluster vertex deletion number plus clique number.*

349 Theorem 5.3 and Observation 2.1 imply the fixed-parameter tractability parameterized by
350 vertex cover number.

351 **Corollary 5.4.** *ORIENTABLE BURNING NUMBER is fixed-parameter tractable parameterized by*
352 *vertex cover number.*

353 *5.2.1. Proof of Theorem 5.3.*

354 In the proof, we use the theory of monadic second-order logic on graphs (MSO_2), which will
355 be introduced right before we use it. If we allow an MSO_2 formula to have length depending on
356 b , it is not difficult to express OBN. However, this only implies the fixed-parameter tractability
357 of OBN parameterized by a parameter combined with b . To avoid the dependency on b , we
358 have to bound the length of an MSO_2 formula with a function not depending on b . To this
359 end, we make a series of observations to bound the number of *parts* not used in a *good* burning
360 sequence, then represent the problem by expressing the unused parts instead of the used parts.

361 *Useful assumptions.* In the following, let (G, b) be an instance of OBN. Let ω be the clique
362 number of G ; that is, $\omega = \omega(G)$. Let S be a cluster vertex deletion set of G with size $s = \text{cvd}(G)$.
363 Our parameter is $k := \omega + s$. Note that finding S is fixed-parameter tractable parameterized
364 by s [31], and thus by k as well. Let C_1, \dots, C_p be the connected components of $G - S$, which
365 are complete graphs. When we are dealing with an orientation \vec{G} of G , we sometimes mean by
366 C_i the tournament $\vec{G}[V(C_i)]$. For example, we may say “a king of C_i .”

367 *Claim 5.5.* If $b \leq p$, then (G, b) is a yes-instance.

368 *Proof.* By picking arbitrarily one vertex from each C_i , we can construct an independent set of
369 size p . By Lemma 3.2, $B(G) \geq \alpha(G) \geq p \geq b$. \square

370 *Claim 5.6.* If $b > p + s + 2$, then (G, b) is a no-instance.

371 *Proof.* Let \vec{G} be an orientation of G . It suffices to show that $b(\vec{G}) \leq p + s + 2$. For each C_i ,
372 we place a fire of radius at least 2 at a king of C_i . For each vertex in S , we place a fire of
373 arbitrary radius. If we have not used the fires of radii 0 and 1, then we place them at arbitrary
374 vertices. \square

375 By Claims 5.5 and 5.6, we may assume that

$$376 \quad p < b \leq p + s + 2.$$

377 Let ℓ be the length of a longest path in G . We assume that $\ell \geq 1$ since otherwise G cannot
378 have any edge and the problem becomes trivial. Note that in every orientation \vec{G} or G , the
379 length of a longest directed path is at most ℓ .

380 *Claim 5.7.* $\ell \leq s\omega + s + \omega - 1$.

381 *Proof.* Let P be a longest path in G . Since P can visit at most $|S| + 1$ connected components
382 of $G - S$, we have $|V(P)| \leq |S| + (|S| + 1)\omega$ as each C_i is a complete graph. The claim follows
383 as $|S| = s$ and $|E(P)| = |V(P)| - 1$. \square

384 In a burning sequence of an orientation of G , we call a fire of radius at least ℓ a *large fire*.
385 Note that a large fire w burns all vertices that can be reached from w in the orientation as no
386 directed path in the orientation is longer than ℓ .

387 In the following, we focus on burning sequences of length $b - 1$ since we are going to express
388 the *non-existence* of such sequences. Let $L = \max\{0, b - 1 - \ell\}$; that is L is the number of large
389 fires in a sequence of length $b - 1$. Observe that $L \leq p + s$ as $b - 1 - \ell \leq b - 2 \leq p + s$.

390 A burning sequence of an orientation of G is *good* if the following conditions are satisfied:

- 391 1. two large fires do not have the same position;
- 392 2. each C_j contains at most one large fire;
- 393 3. if a large fire is placed in some C_h , then it is placed at a king of C_h .

394 *Claim 5.8.* Let \vec{G} be an orientation of G . If \vec{G} admits a burning sequence with length $b - 1$,
395 then there is a good burning sequence of \vec{G} with the same length.

396 *Proof.* From a burning sequence σ of \vec{G} with length $b - 1$, we first construct a sequence σ_1 that
397 satisfies the first condition of the goodness. We repeatedly find two large fires placed at the
398 same vertex and then replace arbitrary one of them with another vertex not occupied by any
399 large fire. The replacement is possible as L is not larger than the number of vertices. Since two
400 large fires placed at the same vertex burn the same set of vertices, the obtained sequence is still

401 a burning sequence of \vec{G} . When there is no pair of large fires occupying the same vertex, we
 402 stop this phase and call the resultant sequence σ_1 .

403 Next we modify σ_1 to obtain a sequence σ_2 that satisfies the first and second conditions.
 404 Assume that two large fires w_i and w_j are placed in the same connected component C_h of $G - S$
 405 and that $(w_i, w_j) \in A(\vec{G})$. (Recall that C_h is a complete graph.) Since w_i is a large fire, it
 406 burns every vertex that can be reached from w_i . In particular, w_i burns all vertices reachable
 407 from w_j . Hence, w_j is useless for burning the graph. We replace w_j with another vertex v such
 408 that v is not occupied by any large fire and if v belongs to some $C_{h'}$, then there is no large fire
 409 belonging to $C_{h'}$ prior to the replacement. This is always possible as $L \leq p + s$. We exhaustively
 410 apply this replacement procedure and get σ_2 , which satisfies the first and second conditions of
 411 the goodness.

412 Finally, we obtain a sequence σ_3 from σ_2 by replacing each large fire that is placed in some
 413 C_h with a king of C_h . We can see that σ_3 is a burning sequence of \vec{G} since the new large fire
 414 placed at a king of C_h burns all vertices reachable from the king and the king can reach all
 415 vertices in C_h . Since σ_3 satisfies all conditions of goodness and has the same length as σ , the
 416 claim holds. \square

417 If σ is a good burning sequence of an orientation \vec{G} of G with length $b - 1$, then the sum of
 418 the following two numbers is $p + s - L$:

- 419 • the number of vertices in S not occupied by the large fires of σ , and
- 420 • the number of connected components of $G - S$ not including large fires of σ .

421 Since $L = \max\{0, b - 1 - \ell\}$ and $p < b$, it holds that $p + s - L < s + \ell + 1$. Since ℓ can be
 422 bounded from above by a function of $k = s + \omega$, so is $p + s - L$. Thus our MSO_2 formula can
 423 have length depending on $p + s - L$, the number of *unused parts*.

424 *MSO₂ expressions.* We now express the problem in the monadic second-order logic on graphs.
 425 A formula in the monadic second-order logic on graphs, denoted MSO_2 , can have variables
 426 representing vertices, vertex sets, edges, and edge sets. As atomic formulas, we can use the
 427 equality $x = y$, the inclusion $x \in X$, the adjacency relation $\text{adj}(x, y)$ meaning that vertices x
 428 and y are adjacent, and the incidence relation $\text{inc}(e, x)$ meaning that a vertex x is an endpoint
 429 of an edge e . Atomic formulas can be recursively combined by the usual Boolean connectives \neg ,
 430 \wedge , \vee , \Rightarrow , and \Leftarrow to form an MSO_2 formula. Furthermore, variables in an MSO_2 formula can be
 431 quantified by \exists and \forall . If an MSO_2 formula $\phi(X)$ with one free (vertex-set or edge-set) variable
 432 X is evaluated to true on a graph G and a subset S of $V(G)$ or $E(G)$, we write $G \models \phi(S)$.
 433 It is known that, given an MSO_2 formula $\phi(X)$, a graph G , and a subset S of $V(G)$ or $E(G)$,
 434 the problem of deciding whether $G \models \phi(S)$ is fixed-parameter tractable parameterized by the
 435 length of ϕ plus the treewidth of G [2, 9, 15].

436 In the following, we focus on an instance (G, b) of OBN given with a minimum cluster
 437 vertex deletion set S of G such that $s = \text{cvd}(G) = |S|$, $\omega = \omega(G)$, p is the number of connected
 438 components in $G - S$, and L is the number of large fires in a sequence of length $b - 1$. As we
 439 justified above, we assume that

- 440 • $p < b \leq p + s + 2$, and
- 441 • the sum of the numbers of unused vertices in S and unused connected components of
 442 $G - S$ is $p + s - L$.

443 Under these conditions, we construct an MSO_2 formula $\phi(X)$ such that $G \models \phi(S)$ if and only if
 444 (G, b) is a yes-instance of OBN. We also show that the length of $\phi(X)$ is bounded from above

445 by a function of $k = s + \omega$. Since the treewidth of G is at most $s + \omega$ (see [25]), this implies
 446 Theorem 5.3.

447 The formula $\phi(X)$ asks whether there exists an orientation \vec{G} of G such that no sequence
 448 $\langle w_0, \dots, w_{b-2} \rangle$ of length $b - 1$ is a good burning sequence of \vec{G} . In MSO_2 , we can handle
 449 orientations of k -colorable graphs by first assuming a *default* orientation using a k -coloring and
 450 then represent the *reversed* edges by a set of edges [16, 32].⁴ More precisely, such a formula first
 451 expresses a k -coloring and a set of reversed edge, and then it considers each edge as oriented
 452 from the vertex with a smaller label to the one with a larger label if and only if the edge is not
 453 a reversed one. Note that Observation 2.1 implies that G is k -colorable. We use this technique
 454 and thus $\phi(X)$ has the following form:

$$455 \quad \phi(X) = \exists V_1, \dots, V_k \subseteq V, \exists F \subseteq E: \text{proper-coloring}(V_1, \dots, V_k) \wedge \phi_1,$$

456 where $\text{proper-coloring}(V_1, \dots, V_k)$ expresses that V_1, \dots, V_k is a proper k -coloring of G . The
 457 formula $\text{proper-coloring}(V_1, \dots, V_k)$ can be defined as

$$458 \quad \text{proper-coloring}(V_1, \dots, V_k) = \text{part}(V_1, \dots, V_k) \wedge \bigvee_{1 \leq i \leq k} \text{ind}(V_i),$$

459 where $\text{part}(V_1, \dots, V_k)$ and $\text{ind}(V_i)$ are defined as follows

$$460 \quad \text{part}(V_1, \dots, V_k) = \forall v \in V: \bigvee_{1 \leq i \leq k} (v \in V_i) \wedge \bigwedge_{1 \leq i < j \leq k} \neg(v \in V_i \wedge v \in V_j),$$

$$461 \quad \text{ind}(V_i) = \forall u, v \in V_i: \neg \text{adj}(u, v).$$

462 The subformula ϕ_1 can use the formula $\text{arc}(u, v)$ expressing that there is an arc from u to
 463 v in the orientation defined by V_1, \dots, V_k and F , which can be defined as follows:

$$464 \quad \text{arc}(u, v) = \text{adj}(u, v) \wedge (((u < v) \wedge \neg \text{rev}(u, v)) \vee (\neg(u < v) \wedge \text{rev}(u, v))),$$

465 where $(u < v)$ and $\text{rev}(u, v)$ are defined as

$$466 \quad (u < v) = \bigvee_{1 \leq i < j \leq k} (u \in V_i \wedge v \in V_j),$$

$$467 \quad \text{rev}(u, v) = \exists e \in F: \text{inc}(e, u) \wedge \text{inc}(e, v).$$

468 Given an orientation defined above, the subformula ϕ_1 expresses that there is no good
 469 burning sequence of length $b - 1$ for this orientation. Indeed, we set $\phi_1 = \neg \phi_2$ and give a
 470 definition of ϕ_2 that expresses there is a good burning sequence of length $b - 1$. We assume
 471 that $b - 1 > \ell$ since the other case can be easily obtained from the expression of this case. The
 472 subformula ϕ_2 has the following form

$$473 \quad \phi_2 = \exists w_0, \dots, w_{\ell-1} \in V, \exists u_1, \dots, u_{p+s-L} \in V:$$

$$474 \quad \phi_3 \wedge \bigwedge_{1 \leq i < j \leq p+s-L} (u_i \neq u_j) \wedge \bigwedge_{1 \leq i < j \leq p+s-L} ((u_i \notin X) \wedge (u_j \notin X) \Rightarrow \neg \text{adj}(u_i, u_j))$$

475 where $w_0, \dots, w_{\ell-1}$ simply correspond to the first ℓ fires in a (good) burning sequence and
 476 u_1, \dots, u_{p+s-L} correspond to the representatives of unused parts. More precisely, if $u_i \in X$,

⁴There is another way for handling orientation by using a variant of MSO_2 defined for directed graphs, where we can fix an arbitrary orientation first (without using a k -coloring) and then represent reversed edges by an edge set. See e.g., [20].

477 then it means that u_i is not used by any large fire; if $u_i \notin X$ and thus u_i belongs to some
 478 connected component C of $G - X$, then it means that no vertex in C is used by large fires.
 479 Note that the second line of the formula forces that u_1, \dots, u_{p+s-L} are distinct and not chosen
 480 multiple times from one connected component of $G - X$. (Recall that X is promised to be a
 481 cluster vertex deletion set.) Now ϕ_3 expresses that every vertex is burned. Hence, it can be
 482 expressed as follows

$$483 \quad \phi_3 = \forall v \in V: \text{burned}(v),$$

484 where the definition of $\text{burned}(v)$ is given below.

485 To define $\text{burned}(v)$, observe that v is burned if and only if one of the following conditions
 486 is satisfied:

- 487 1. some w_i ($0 \leq i \leq \ell - 1$) has a directed path of length at most i to v ;
- 488 2. some large fire has a directed path to v .

489 We express the first case as $\text{burned-small}(v)$ and the second as $\text{burned-large}(v)$, and thus
 490 $\text{burned}(v) = \text{burned-small}(v) \vee \text{burned-large}(v)$. The first case is easy to state as

$$491 \quad \text{burned-small}(v) = \bigvee_{0 \leq i \leq \ell - 1} \text{reachable}_i(w_i, v),$$

492 where $\text{reachable}_d(x, y)$ means that there is a directed path of length at most d from x to y ,
 493 which can be defined as

$$494 \quad \text{reachable}_d(x, y) = \exists z_0, \dots, z_d \in V: \\ 495 \quad (z_0 = x) \wedge (z_d = y) \wedge \bigwedge_{0 \leq j \leq d-1} ((z_j = z_{j+1}) \vee \text{arc}(z_j, z_{j+1})).$$

496 On the other hand, the second case is a bit tricky as the large fires are not explicitly handled.
 497 Recall that the vertices u_1, \dots, u_{p+s-L} tell us which vertices in X are not large fires and which
 498 connected components of $G - X$ include no large fires. From this information, we can determine
 499 whether a vertex x is used as a large fire by setting $\text{large-fire}(x) = \neg \text{unused}(x)$, where
 500 $\text{unused}(x)$ is defined as

$$501 \quad \text{unused}(x) = \left(\bigvee_{1 \leq i \leq p+s-L} x = u_i \right) \vee \left((x \notin X) \wedge \bigvee_{1 \leq i \leq p+s-L} ((u_i \notin X) \wedge \text{adj}(x, u_i)) \right).$$

502 Note that the correctness of the right side depends on the assumption that each connected
 503 component of $G - X$ is a complete graph. Now $\text{burned-large}(v)$ can be expressed as follows.

$$504 \quad \text{burned-large}(v) = \exists x \in V: \text{large-fire}(x) \wedge \text{reachable}_\ell(x, v).$$

505 The length of the entire formula $\phi(X)$ depends only on k , ℓ , and $p + s - L$, where ℓ and
 506 $p + s - L$ can be bounded from above by function depending only on k . Therefore, the length
 507 of $\phi(X)$ depends only on k . This completes the proof of Theorem 5.3.

508 6. Concluding remarks

509 In this paper, we initiated the study of ORIENTABLE BURNING NUMBER (OBN), which is
 510 the problem of finding an orientation of a graph that maximizes the burning number. We first
 511 observed some graph-theoretic bounds and then showed algorithmic and complexity results.

512 We showed that OBN is NP-hard even on some classes of sparse graphs (Theorem 4.2). On
513 the other hand, we do not know whether it belongs to NP. We can see that OBN belongs
514 to Σ_2^P since it is an $\exists\forall$ -problem that asks for the existence of an orientation of a given graph
515 such that all short sequences of fires are not burning sequences of the oriented graph (see [50]
516 for a friendly introduction to Σ_2^P). It would be natural to suspect that the problem is indeed
517 Σ_2^P -hard.

518 **Question 6.1.** Does OBN belong to NP, or is it Σ_2^P -complete?

519 In contrast to the NP-hardness of the general case, we showed that the problem is solvable
520 in polynomial time on bipartite graphs or more generally on König–Egerváry graphs (Corol-
521 lary 3.9). We also showed that for perfect graphs, which form a large superclass of bipartite
522 graphs, we can compute the orientable burning number with an additive error of 2 (Corol-
523 lary 3.5). Given these facts, we would like to ask whether the problem can be solved in poly-
524 nomial time on perfect graphs or on some of its subclasses such as chordal graphs.

525 **Question 6.2.** Is OBN polynomial-time solvable on perfect graphs, or on some of its (non-
526 bipartite) subclasses such as chordal graphs?

527 In the parameterized setting, we showed that OBN parameterized by the target burning
528 number b is W[1]-hard in general (Theorem 4.1), while it is fixed-parameter tractable on some
529 sparse graphs such as planar graphs (Corollary 5.2). We then studied the setting where b is
530 not part of the parameter. In this case, we showed that OBN parameterized solely by vertex
531 cover number (or more generally by cluster vertex deletion number plus clique number) is
532 fixed-parameter tractable (Theorem 5.3). It would be interesting to study the complexity of
533 parameterizations by more general parameters, e.g., vertex integrity [25].

534 **Question 6.3.** Is OBN fixed-parameter tractable when parameterized solely by treewidth,
535 pathwidth, treedepth, vertex integrity, or other related parameters?

536 Finally, we ask a graph-theoretic question. Most of the algorithmic and complexity results
537 in this paper directly or indirectly used the relations between the orientable burning number
538 and the independence number shown in Section 3. As shown there, we have $\alpha(G) \leq B(G)$ and
539 $B(G) \in O(\alpha(G) \cdot \log n)$. Now the question would be the maximum difference between $\alpha(G)$ and
540 $B(G)$. At this point, we only know that the maximum gap is at least 2 as $B(K_n) = 3 = \alpha(K_n) + 2$
541 for $n \geq 5$.

542 **Question 6.4.** Is there a graph G with $B(G) > \alpha(G) + 2$? Is there a function f such that
543 $B(G) \leq f(\alpha(G))$ for every graph G ?

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