Supporting material for the paper: Robust semiparametric efficient estimator for time delay and doppler estimation

Lorenzo Ortega, Stefano Fortunati

Technical note for the paper:

Robust semiparametric efficient estimator for time delay and Doppler estimation

Lorenzo Ortega, Stefano Fortunati

I. Computation of the R-estimator $\hat{\boldsymbol{\theta}}_R$

A. Signal model

Let us first remind the signal model used in this article. The received signal after the baseband demodulation can be expressed as [1]–[3]:

$$x\left(t;\bar{\boldsymbol{\eta}}\right)=\bar{\alpha}s\left(t-\bar{\tau}\right)e^{-j2\pi f_c(\bar{b}(t-\bar{\tau}))}+n\left(t\right), \tag{1}$$

with $\bar{\boldsymbol{\eta}}=\left(\bar{\tau},\bar{b}\right)^T$ and $\bar{\alpha}$ a complex gain. The discrete vector signal model is built from $N=N_1-N_2+1$ samples at $T_s=1/F_s=1/B$,

$$\mathbf{x}=\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})+\mathbf{n}, \tag{2}$$

with $\mathbf{x}=\left(\ldots,x\left(kT_s\right),\ldots\right)^\top$, $\mathbf{n}=\left(\ldots,n\left(kT_s\right),\ldots\right)^\top$, $\mathbf{s}=\left(\ldots,s\left(kT_s\right),\ldots\right)^\top$, $N_1\leq k\leq N_2$ signal samples and the i.i.d. noise samples $n(kT_s)\sim CES(0,\bar{\sigma}_n^2,\bar{g})$ are assumed to be CES-distributed with unknown noise power $\bar{\sigma}_n^2$ and unspecified density generator $\bar{g}$ [4] and the location

$$\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})=\left(\ldots,\bar{\alpha}s(kT_s-\bar{\tau})e^{-j2\pi f_c(\bar{b}(kT_s-\bar{\tau})}=f_k(\bar{\boldsymbol{\theta}}),\ldots\right)^\top. \tag{3}$$

The unknown deterministic parameters can be collected in vector $\bar{\boldsymbol{\epsilon}}^\top=(\bar{\sigma}_n^2,\bar{\rho},\bar{\Phi},\bar{\boldsymbol{\eta}}^\top)=(\bar{\sigma}_n^2,\bar{\boldsymbol{\theta}}^\top)$, with $\bar{\alpha}=\bar{\rho}e^{j\bar{\Phi}}$ and $\bar{\rho}\in\mathbb{R}^+,0\leq\bar{\Phi}\leq 2\pi$.

The objective of this supporting material is to derive the matrices and vectors necessary for the computation of the R estimator:

$$\hat{\boldsymbol{\theta}}_R=\boldsymbol{\theta}^\star+(\sqrt{N}\hat{\alpha})^{-1}[\mathbf{K}(\boldsymbol{\theta}^\star)]^{-1}\widetilde{\Delta}_N(\boldsymbol{\theta}_n^\star), \tag{4}$$

B. Misspecified ML estimator (MMLE) as preliminary estimator of $\boldsymbol{\theta}^\star$, $(\sigma_n^\star)^2$

As stated in Section IV.A of the letter, in order to implement an R-estimator, the preliminary estimator $(\boldsymbol{\theta}^\star)^\top=[\rho^\star,\Phi^\star,(\boldsymbol{\eta}^\star)^\top]$, that is a $\sqrt{N}$-consistent (but not necessarily efficient) estimator of $\bar{\boldsymbol{\theta}}$, is required. For the signal model at hand, a good candidate would be the Misspecified ML estimator (MMLE) derived in [5], [6]. The MMLE is derived under the assumption of a complex circular Gaussian distribution for the noise and it has been shown to be $\sqrt{N}$-consistent in [7]. The estimates of the MMLE $\boldsymbol{\theta}^\star$ are given by:

$$\boldsymbol{\eta}^\star=\arg\max_{\boldsymbol{\eta}}\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\mathbf{x}\right\|^2 \tag{5}$$

$$\rho^\star=\left|\left[\boldsymbol{\mu}^H\left(\boldsymbol{\eta}^\star\right)\boldsymbol{\mu}\left(\boldsymbol{\eta}^\star\right)\right]^{-1}\boldsymbol{\mu}^H\left(\boldsymbol{\eta}^\star\right)\mathbf{x}\right| \tag{6}$$

$$\Phi^\star=\arg\left\{\left[\boldsymbol{\mu}^H\left(\boldsymbol{\eta}^\star\right)\boldsymbol{\mu}\left(\boldsymbol{\eta}^\star\right)\right]^{-1}\boldsymbol{\mu}^H\left(\boldsymbol{\eta}^\star\right)\mathbf{x}\right\} \tag{7}$$

where $\mathbf{\Pi_A} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)\mathbf{A}^H$ is the orthogonal projector over S, with $S = span\left(\mathbf{A}\right)$ and $\mathbf{A}$ a matrix. Similarly, as preliminary estimator of the noise variance, we can used the MMLE, which yields to:

$$(\sigma_n^\star)^2 = \|\boldsymbol{x} - \rho^\star e^{j\Phi^\star}\boldsymbol{\mu}(\boldsymbol{\eta}^\star)\|^2/N. \tag{8}$$

C. Explicit calculation of $\mathbf{K}(\boldsymbol{\theta}^\star)$

Let us start by recalling the definition of $\mathbf{K}(\boldsymbol{\theta}^\star)$ as:

$$\mathbf{K}(\bar{\boldsymbol{\theta}}) = \frac{2}{\bar{\sigma}_n^2}\Re\left\{\left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)^H\left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)\right\}, \tag{9}$$

where $\bar{\boldsymbol{\theta}}$ indicates the true parameter vector characterizing the signal model in (1). Following the outcomes presented in [2] a closed-form expression for $\mathbf{K}(\boldsymbol{\theta}^\star)$ can be obtained as:

$$\mathbf{K}(\bar{\boldsymbol{\theta}}) \triangleq \frac{2F_s}{\bar{\sigma}_n^2}\Re\left\{\mathbf{QWQ}^H\right\} \tag{10}$$

where the entries of the matrix

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_{2,2} & w_4^* \\ w_3 & w_4 & w_{3,3} \end{bmatrix} \tag{11}$$

are given by $w_1 = \frac{1}{F_s}\mathbf{s}^H\mathbf{s}$, $w_2 = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{Ds}$, $w_3 = \mathbf{s}^H\mathbf{\Lambda s}$, $w_4 = \frac{1}{F_s}\mathbf{s}^H\mathbf{D\Lambda s}$, $w_{2,2} = \frac{1}{F_s^3}\mathbf{s}^H\mathbf{D}^2\mathbf{s}$, $w_{3,3} = F_s\mathbf{s}^H\mathbf{Vs}$. Moreover, the matrix $\mathbf{Q}$ in (15) is as follows:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ j\bar{\rho} & 0 & 0 \\ j\bar{\rho}2\pi f_c\bar{b} & 0 & -\bar{\rho} \\ 0 & -j\bar{\rho}2\pi f_c & 0 \end{bmatrix}. \tag{12}$$

The vector $\mathbf{s} = (\ldots,\, s(kT_s),\, \ldots)^\top_{N_1\le k\le N_2}$ is the baseband samples vector, $\mathbf{D} = \text{diag}\,(\ldots,\, k,\, \ldots)_{N_1\le k\le N_2}$ and the entries of the matrices $\mathbf{\Lambda}$ and $\mathbf{V}$ are defined as

$$(\mathbf{\Lambda})_{k,k'} = \begin{vmatrix} k'\neq k : \frac{(-1)^{|k-k'|}}{k-k'} \\ k' = k : 0 \end{vmatrix} \tag{13}$$

and

$$(\mathbf{V})_{k,k'} = \begin{vmatrix} k'\neq k : (-1)^{|k-k'|}\,\frac{2}{(k-k')^2} \\ k' = k : \frac{\pi^2}{3} \end{vmatrix}. \tag{14}$$

Finally, $\mathbf{K}(\boldsymbol{\theta}^\star)$ can be expressed as function of the preliminary estimator $\boldsymbol{\theta}^\star$ as:

$$\mathbf{K}(\boldsymbol{\theta}^\star) \triangleq \frac{2F_s}{(\sigma^\star)_n^2}\Re\left\{\mathbf{Q}^\star\mathbf{W}\mathbf{Q}^{\star H}\right\} \tag{15}$$

with

$$\mathbf{Q}^\star = \begin{bmatrix} 1 & 0 & 0 \\ j\rho^\star & 0 & 0 \\ j\rho^\star 2\pi f_c b^\star & 0 & -\rho^\star \\ 0 & -j\rho^\star 2\pi f_c & 0 \end{bmatrix}. \tag{16}$$

D. Explicit calculation of $\widetilde{\Delta}_N(\boldsymbol{\theta}^\star)$

Along with $\mathbf{K}(\boldsymbol{\theta}^\star)$, the other term in (4) to be calculated is the random vector :

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) \triangleq \frac{-2}{\sqrt{N}\sigma_n^\star} \sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right) \Re\left[(u_k^\star)^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star)\right]. \tag{17}$$

Note that (17) is a ranks-measurable approximation of the central sequence $\Delta_N(\boldsymbol{\theta}^\star)$ with

$$\Delta_N(\boldsymbol{\theta}) \triangleq N^{-1/2} \sum\nolimits_{k=1}^{N} \mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_k), \tag{18}$$

where the score vector $\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_k)$ is given by [8]:

$$\mathbf{s}_{\boldsymbol{\theta}}(x_k) =_d -2\sigma_n^{-1}\sqrt{Q_k}\bar{\psi}(Q_k)\Re\left\{u_k^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta})\right\}. \tag{19}$$

Clearly, the score vector in (19) cannot be calculated for the following three reasons:

1) The real and positive random variables $\{Q_k \sim p_{\mathcal{Q}}\}_{k=N_1}^{N_2}$ (distributed as the second order modular variate of the CES noise) are not observable,
2) The complex random variables $\{u_k \sim \mathcal{U}(\mathbb{C}S^1)\}_{k=N_1}^{N_2}$ (uniformly distributed on $\mathbb{C}S^1$) are not observable,
3) Since the functional form of the true distribution is unknown, the function $\bar{\psi}(\cdot)$ cannot be explicitly defined.

Then, following the outcomes of the seminal work [9], we rely on a ranks-measurable approximation for the central sequence (18). Ranks owe their importance in robust statistics to the property of being "distribution-free". In particular, ranks-measurable functions can be used to derive non-parametric estimators that show a remarkable efficiency even when the the data distribution is unknown. In particular, for in the considered application, the unknown term if the score vector (19) can be approximated by means of the so-called "score function". Specifically, it can be shown that [10, Ch. 13], [11], for a given "score function" $M(\cdot)$, [1] we have[2]:

$$M\left(\frac{r_k}{N+1}\right) = \sqrt{\mathcal{Q}}\bar{\psi}(\mathcal{Q}) + o_P(1). \tag{20}$$

with $\{r_k\}_{k=N_1}^{N_2}$ the *ranks* of the (continuous) real random variables $\{Q_k^\star\}_{k=N_1}^{N_2}$ that represent a sort of preliminary estimates of the unobservable random variables $\{Q_k \sim p_{\mathcal{Q}}\}_{k=N_1}^{N_2}$

$$Q_k^\star \triangleq |x_k - f_k(\boldsymbol{\theta}^\star)|^2/(\sigma_n^\star)^2. \tag{21}$$

Finally, preliminary estimates of the unobservable random variables $\{u_k\}_{k=N_1}^{N_2}$ can be obtained as:

$$u_k^\star \triangleq \left(x_k - f_k(\boldsymbol{\theta}^\star)\right) / \left(\sigma_n^\star \sqrt{Q_k^\star}\right). \tag{22}$$

Regarding the "score function" $M(\cdot)$ many choices are possible (see e.g. [11]). However, the one that provide a good trade of between semiparametric efficiency and robustness is the complex *van der Waerden* score function:

$$M_{vdW}(t) \triangleq \sqrt{\Phi_G^{-1}(t)}, \quad t \in (0,1), \tag{23}$$

where Φ_G^{-1} indicates the inverse function of the cdf of a Gamma random variable with parameters $(1,1)$.

[1] The family of score functions is defined in [12, Sect. 2.2], [10, Ch. 13]

[2] We write: $x_l = o_P(1)$ if $\lim_{l\to\infty} \Pr\{|x_l| \geq \epsilon\} = 0, \forall \epsilon > 0$ (*convergence in probability to 0*)

The next step to calculate $\widetilde{\Delta}_N(\boldsymbol{\theta}^\star)$ is to compute $\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star)$, which can be shown to be

$$\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star) = \boldsymbol{Q}^\star \boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) e^{j\Phi^\star} e^{-j\omega_c b^\star (kT_s-\tau^\star)}, \tag{24}$$

with $\boldsymbol{Q}^\star$ computed in (16) and

$$\boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) = \begin{bmatrix} s(kT_s-\tau^\star) \\ (t-\tau^\star)s(kT_s-\tau^\star) \\ s^{(1)}(kT_s-\tau^\star) \end{bmatrix}, \tag{25}$$

with $s^{(1)}(t) = \frac{\partial s(t)}{\partial t}$. Then, through explicit calculation

$$\frac{\partial f_k(\boldsymbol{\theta}^\star)}{\partial \rho^\star} = e^{j\Phi^\star} s(kT_s-\tau^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}, \tag{26}$$

$$\frac{\partial f_k(\boldsymbol{\theta}^\star)}{\partial \Phi^\star} = j\rho^\star e^{j\Phi^\star} s(kT_s-\tau^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}, \tag{27}$$

$$\frac{\partial f_k(\boldsymbol{\theta}^\star)}{\partial \tau^\star} = -\rho^\star e^{j\Phi^\star} s^{(1)}(kT_s-\tau^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)} + j\omega_c b^\star \rho^\star e^{j\Phi^\star} s(kT_s-\tau^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}, \tag{28}$$

$$\frac{\partial f_k(\boldsymbol{\theta}^\star)}{\partial b^\star} = -j\omega_c (kT_s-\tau^\star)\rho^\star e^{j\Phi^\star} s(kT_s-\tau^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}. \tag{29}$$

Since $M\left(\frac{r_k}{N+1}\right)$ are real values, we can compute (17) by computing the following sum

$$\Re\left[\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right)(u_k^\star)^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star)\right] = \Re\left[\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right)(u_k^\star)^* \boldsymbol{Q}^\star \boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) e^{j\Phi^\star} e^{-j\omega_c b^\star (kT_s-\tau^\star)}\right]. \tag{30}$$

Unfortunately, due to the term $s^{(1)}(kT_s-\tau^\star)$, the sum is not trivial to resolve. In order to provide closed-form expression of the previous sum, we can assume that the signal $s(t)$ is band-limited and apply the Nyquist-Shannon theorem.

A band-limited signal assumption, $s(t)$ can be expressed as:

$$\begin{aligned} s(t) &= \sum_{k=N_1}^{N_2} s(kT_s)\,\mathrm{sinc}\left(\pi B(t-kT_s)\right) \leftrightharpoons \\ s(f) &= \left(\frac{1}{F_s}\sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi f k T_s}\right) 1_{\left[-\frac{F_s}{2};\frac{F_s}{2}\right]} \end{aligned} \tag{31}$$

where $T_s = \frac{1}{B} = \frac{1}{F_s}$ is the sampling period. Moreover, by definition of band-limitied signal, we can show that

$$\lim_{(N_1,N_2)\to(-\infty,+\infty)} T_s \sum\nolimits_{k=N_1}^{N_2} s\left(nT_s\right) = \int_{-\infty}^{+\infty} s\left(t\right) dt.$$

Then, considering enough samples and defining $\boldsymbol{\mathcal{U}}^\star = (\cdots, \mathcal{U}_k^\star, \cdots)^T$, $\mathcal{U}_k^\star = M\left(\frac{r_k}{N+1}\right)(u_k^\star)^*$, we have that

$$\lim_{(N_1,N_2)\to(-\infty,\infty)} T_s \left(\sum_{k=N_1}^{N_2} e^{j\Phi^\star} \mathbf{Q}^\star \mathcal{U}_k^* \boldsymbol{\vartheta}(kT_s;\boldsymbol{\theta}^\star) e^{-j\omega_c b^\star (kT_s-\tau^\star)}\right) =$$

$$\int_{-\infty}^{\infty} e^{j\Phi^\star} \mathbf{Q}^\star \mathcal{U}^*(t) \boldsymbol{\vartheta}(t;\boldsymbol{\theta}^\star) e^{-j\omega_c b^\star (t-\tau^\star)} dt = e^{j\Phi^\star} \mathbf{Q}^\star \int_{-\infty}^{\infty} \mathcal{U}^*(t) \boldsymbol{\vartheta}(t;\boldsymbol{\theta}^\star) e^{-j\omega_c b^\star (t-\tau^\star)} dt = e^{j\Phi^\star} \mathbf{Q}^\star \boldsymbol{w}_e. \tag{32}$$

yielding to

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) = (2F_s)/(\sqrt{N}\sigma^\star)\Re\left\{e^{j\Phi^\star}\mathbf{Q}^\star\boldsymbol{w}_e\right\}. \tag{33}$$

Now, we have to compute $\boldsymbol{w}_e = (w_{e_1}, w_{e_2}, w_{e_3})^T$. To do that, we have to use the Parseval theorem and the expression of the Fourier transform of band-limited signals as done in [13]:

$$\begin{aligned}
w_{e1} &= \int_{-\infty}^{\infty} \mathcal{U}^*(t)s(t-\tau^\star)e^{-j\omega_c b^\star(t-\tau^\star)}dt = \int_{-\infty}^{\infty} \mathcal{U}^*(t+\tau^\star)s(t)e^{-j\omega_c b^\star t}dt \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\mathcal{U}(f)e^{j2\pi f\tau^\star}\right)^* s(f+f_c b^\star)\,df \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s}\sum_{k=N_1}^{N_2} s(kT_s)e^{-j2\pi(f+f_c b^\star)kT_s}\right) e^{-j2\pi f\tau^\star} \left(\frac{1}{F_s}\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s)e^{-j2\pi fkT_s}\right)^* df \\
&= \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=N_1}^{N_2} s(kT_s)e^{-j2\pi fk}e^{-j2\pi\frac{f_c b^\star}{F_s}k}\right) e^{-j2\pi f\frac{\tau^\star}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s)e^{-j2\pi fk}\right)^* df \\
&= \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{\nu}(f)^*\right) e^{-j2\pi f\frac{\tau^\star}{T_s}} \left(\mathcal{U}^H\boldsymbol{\nu}(f)\right) df = \frac{1}{F_s}\boldsymbol{\mathcal{U}}^H\boldsymbol{V}^{\Delta,0}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{s}
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{\nu}(f) &= \Big(\ \ldots,\quad e^{j2\pi fk},\quad \ldots\ \Big)^T_{N_1\le k\le N_2} \\
\mathbf{V}^{\Delta,0}(q) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{-j2\pi fq}df.
\end{aligned}$$

Moreover, $\mathbf{V}^{\Delta,0}(q)$ yields to

$$\left[\mathbf{V}^{\Delta,0}(q)\right]_{k,l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-q)}df = \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\sin(\pi(k-l-q))}{\pi(k-l-q)} = \mathrm{sinc}(k-l-q).$$

and

$$\boldsymbol{U}(p) = \mathrm{diag}\left(\ldots, e^{-j2\pi pk}, \ldots\right)_{N_1\le k\le N_2}.$$

Note that in order to compute w_{e1}, we have apply the following Fourier transform property:

$$s(t)e^{j2\pi f_c bt} \rightleftharpoons \mathrm{FT}\left\{s(t)e^{j2\pi f_c bt}\right\}(f) \triangleq s(f-f_c b)$$

$$w_{e2} = \int_{-\infty}^{\infty} (t-\tau^\star) s(t-\tau^\star) \mathcal{U}^*(t) e^{-j\omega_c b^\star (t-\tau^\star)} dt = \int_{-\infty}^{\infty} t s(t) \mathcal{U}^*(t+\tau^\star) e^{-j\omega_c b^\star t} dt$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{j}{2\pi} \frac{d}{df} s(f - f_c b^\star) \right) \left(\mathcal{U}(f) e^{j2\pi f \tau^\star} \right)^* df$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \frac{j}{2\pi} (-j2\pi T_s) \sum_{k=N_1}^{N_2} s(kT_s) k e^{-j2\pi (f+f_c b^\star) k T_s} \right) e^{-j2\pi f \tau^\star} \times \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k T_s} \right)^* df$$
$$= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=N_1}^{N_2} s(kT_s) k e^{-j2\pi f k} e^{-j2\pi \frac{f_c b^\star}{F_s} k} \right) e^{-j2\pi f \frac{\tau^\star}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k} \right)^* df$$
$$= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D} \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{-j2\pi f \frac{\tau^\star}{T_s}} \left(\boldsymbol{\mathcal{U}}^H \boldsymbol{\nu}(f) \right) df = \frac{1}{F_s} \boldsymbol{\mathcal{U}}^H \boldsymbol{V}^{\Delta,0} \left(\frac{\tau}{T_s} \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \boldsymbol{D} \boldsymbol{s},$$

with

$$\mathbf{D} = \operatorname{diag}\left(\ldots, k, \ldots \right)_{N_1 \le k \le N_2}.$$

Note that in order to compute w_{e2}, we can apply the following property of the Fourier transform:

$$t s(t) e^{j2\pi f_c b t} \rightleftharpoons \frac{j}{2\pi} \frac{d}{df} \left(s(f - f_c b) \right).$$

$$w_{e3} = \int_{-\infty}^{\infty} s^{(1)}(t-\tau) \mathcal{U}^*(t) e^{-j\omega_c b^\star (t-\tau)} dt = \int_{-\infty}^{\infty} s^{(1)}(t) \mathcal{U}^*(t+\tau^\star) e^{-j\omega_c b^\star t} dt$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi (f + f_c b^\star) s(f + f_c b^\star) \right) \left(\mathcal{U}(f) e^{j2\pi f \tau^\star} \right)^* df$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi (f + f_c b^\star) \frac{1}{F_s} \sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi (f+f_c b^\star) k T_s} \right) e^{-j2\pi f \tau^\star} \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k T_s} \right)^* df$$
$$= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi (f F_s + f_c b^\star) \sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi f k} e^{-j2\pi \frac{f_c b^\star}{F_s} k} \right) e^{-j2\pi f \frac{\tau^\star}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k} \right)^* df$$
$$= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi (f F_s + f_c b^\star) \mathbf{s}^T \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{-j2\pi f \frac{\tau^\star}{T_s}} \left(\boldsymbol{\mathcal{U}}^H \boldsymbol{\nu}(f) \right) df$$
$$= \boldsymbol{\mathcal{U}}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f \frac{\tau^\star}{T_s}} df \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \mathbf{s} + \frac{j2\pi f_c b^\star}{F_s} \boldsymbol{\mathcal{U}}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f \frac{\tau^\star}{T_s}} df \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \mathbf{s}$$
$$= \boldsymbol{\mathcal{U}}^H \mathbf{V}^{\Delta,1} \left(\frac{\tau^\star}{T_s} \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \mathbf{s} + \frac{j2\pi f_c b^\star}{F_s} \boldsymbol{\mathcal{U}}^H \mathbf{V}^{\Delta,0} \left(\frac{\tau^\star}{T_s} \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \mathbf{s}$$
$$= \boldsymbol{\mathcal{U}}^H \boldsymbol{V}^{\Delta,1} \left(\frac{\tau^\star}{T_s} \right) \boldsymbol{U} \left(\frac{f_c b^\star}{F_s} \right) \mathbf{s} \; + j w_c b^\star w_{e_1},$$

with

$$\mathbf{V}^{\Delta,1}(q) = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} df$$

and

$$\begin{aligned}\left[\mathbf{V}^{\Delta,1}\left(q\right)\right]_{k,l} &= j2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-q)}df = j2\pi\left(\left[\frac{fe^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}}\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}df\right)\\ &= \frac{j2\pi}{j2\pi(k-l-q)}\left(\left[\frac{1}{2}e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right)e^{-j\pi(k-l-q)}\right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}}\right)\\ &= \frac{1}{k-l-q}\left(\cos\left(\pi(k-l-q)\right) - \operatorname{sinc}\left(k-l-q\right)\right).\end{aligned}$$

Note that in order to compute w_{e3}, we have apply the following Fourier transform property:

$$s^{(1)}(t)e^{j2\pi f_c bt} \rightleftharpoons j2\pi\left(f - f_c b\right)s(f - f_c b).$$

To conclude, the last term to compute for the implementation of (4) is $\hat{\alpha}$, which represents a non-parametric estimator of $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$. The derivation of this estimator is a delicate step [14], [15] and is based on technical results of LeCam theory and its rank-based extension to semiparametric models. For this reason it is impossible to give here an explanation, however intuitive and tutorial, of its final expression, which will be given below as:

$$\hat{\alpha} = \frac{(\sigma_n^\star)^2}{N}\frac{||\widetilde{\Delta}_N(\boldsymbol{\theta}^\star + N^{-1/2}\mathbf{v}^0) - \widetilde{\Delta}_N(\boldsymbol{\theta}^\star)||}{||\mathbf{K}(\boldsymbol{\theta}^\star)\mathbf{v}^0||}, \tag{34}$$

Where we introduced the "small perturbation" vector $\mathbf{v}^0 \sim \mathcal{N}(\mathbf{0}, \varrho\mathbf{I})$. It can be shown that such $\hat{\alpha}$ satisfies the following $\sqrt{N}$-consistency property:

$$E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \hat{\alpha} + o_P(1). \tag{35}$$

In is worth to underline here that, even if $\hat{\alpha}$ can be shown to be consistent, nothing can be said about its efficiency. In particular, the derivation of a consist and efficient estimator of $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$ is still an open problem.

REFERENCES

[1] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with an active array," *IEEE Trans. Signal Process.*, vol. 49, no. 6, pp. 1122–1137, June 2001.

[2] D. Medina, L. Ortega, J. Vilà-Valls, P. Closas, F. Vincent, and E. Chaumette, "Compact CRB for delay, doppler and phase estimation - application to GNSS SPP & RTK performance characterization," *IET Radar, Sonar & Navigation*, vol. 14, no. 10, pp. 1537–1549, Sep. 2020.

[3] M. I. Skolnik, *Radar Handbook*, 3rd ed. New York, USA: McGraw-Hill, 1990.

[4] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5597–5625, 2012.

[5] C. Lubeigt, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Untangling first and second order statistics contributions in multipath scenarios," *Signal Processing*, vol. 205, p. 108868, 2023.

[6] L. Ortega, C. Lubeigt, J. Vilà-Valls, and E. Chaumette, "On gnss synchronization performance degradation under interference scenarios: Bias and misspecified cramér-rao bounds," *NAVIGATION: Journal of the Institute of Navigation*, vol. 70, no. 4, 2023.

[7] L. Ortega and S. Fortunati, "Misspecified time-delay and doppler estimation over non gaussian scenarios," in *ICASSP 2024-2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2024, pp. 9346–9350.

[8] S. Fortunati, F. Gini, M. S. Greco, A. Zoubir, and M. Rangaswamy, "Semiparametric crb and slepian-bangs formulas for complex elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 67, no. 20, pp. 5352–5364, 2019.

[9] M. Hallin and B. J. M. Werker, "Semi-parametric efficiency, distribution-freeness and invariance," *Bernoulli*, vol. 9, no. 1, pp. 137–165, 2003.

[10] A. W. van der Vaart, *Asymptotic Statistics*, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998.

[11] S. Fortunati, A. Renaux, and F. Pascal, "Robust semiparametric efficient estimators in complex elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 68, pp. 5003–5015, 2020.

[12] M. Hallin, D. Paindaveine, and T. Verdebout, "Optimal rank-based testing for principal components," *The Annals of Statistics*, vol. 38, no. 6, pp. 3245–3299, 12 2010.

[13] C. Lubeigt, L. Ortega, J. Vilà-Valls, and E. Chaumette, "Developments for MCRB Computation in Multipath Scenarios," 2022.

[14] M. Hallin and D. Paindaveine, "Semiparametrically efficient rank-based inference for shape I. Optimal rank-based tests for sphericity," *The Annals of Statistics*, vol. 34, no. 6, pp. 2707–2756, 2006.

[15] M. Hallin, H. Oja, and D. Paindaveine, "Semiparametrically efficient rank-based inference for shape II. Optimal *R*-estimation of shape," *The Annals of Statistics*, vol. 34, no. 6, pp. 2757–2789, 2006.