



Supporting material for the paper: Robust semiparametric efficient estimator for time delay and doppler estimation

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Technical note for the paper:

Robust semiparametric efficient estimator for time delay and Doppler estimation

Lorenzo Ortega, Stefano Fortunati

I. COMPUTATION OF THE R -ESTIMATOR $\hat{\boldsymbol{\theta}}_R$

A. Signal model

Let us first remind the signal model used in this article. The received signal after the baseband demodulation can be expressed as [1]–[3]:

$$x(t; \bar{\boldsymbol{\eta}}) = \bar{\alpha} s(t - \bar{\tau}) e^{-j2\pi f_c(\bar{b}(t - \bar{\tau}))} + n(t), \quad (1)$$

with $\bar{\boldsymbol{\eta}} = (\bar{\tau}, \bar{b})^T$ and $\bar{\alpha}$ a complex gain. The discrete vector signal model is built from $N = N_1 - N_2 + 1$ samples at $T_s = 1/F_s = 1/B$,

$$\mathbf{x} = \bar{\alpha} \boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \quad (2)$$

with $\mathbf{x} = (\dots, x(kT_s), \dots)^T$, $\mathbf{n} = (\dots, n(kT_s), \dots)^T$, $\mathbf{s} = (\dots, s(kT_s), \dots)^T$, $N_1 \leq k \leq N_2$ signal samples and the i.i.d. noise samples $n(kT_s) \sim CES(0, \bar{\sigma}_n^2, \bar{g})$ are assumed to be CES-distributed with unknown noise power $\bar{\sigma}_n^2$ and unspecified density generator \bar{g} [4] and the location

$$\bar{\alpha} \boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) = (\dots, \bar{\alpha} s(kT_s - \bar{\tau}) e^{-j2\pi f_c(\bar{b}(kT_s - \bar{\tau}))} = f_k(\bar{\boldsymbol{\theta}}), \dots)^T. \quad (3)$$

The unknown deterministic parameters can be collected in vector $\bar{\boldsymbol{\epsilon}}^T = (\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\boldsymbol{\eta}}^T) = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^T)$, with $\bar{\alpha} = \bar{\rho} e^{j\bar{\Phi}}$ and $\bar{\rho} \in \mathbb{R}^+$, $0 \leq \bar{\Phi} \leq 2\pi$.

The objective of this supporting material is to derive the matrices and vectors necessary for the computation of the R estimator:

$$\hat{\boldsymbol{\theta}}_R = \boldsymbol{\theta}^* + (\sqrt{N} \hat{\alpha})^{-1} [\mathbf{K}(\boldsymbol{\theta}^*)]^{-1} \tilde{\Delta}_N(\boldsymbol{\theta}_n^*), \quad (4)$$

B. Misspecified ML estimator (MMLE) as preliminary estimator of $\boldsymbol{\theta}^*$, $(\sigma_n^*)^2$

As stated in Section IV.A of the letter, in order to implement an R -estimator, the preliminary estimator $(\boldsymbol{\theta}^*)^T = [\rho^*, \Phi^*, (\boldsymbol{\eta}^*)^T]$, that is a \sqrt{N} -consistent (but not necessarily efficient) estimator of $\bar{\boldsymbol{\theta}}$, is required. For the signal model at hand, a good candidate would be the Misspecified ML estimator (MMLE) derived in [5], [6]. The MMLE is derived under the assumption of a complex circular Gaussian distribution for the noise and it has been shown to be \sqrt{N} -consistent in [7]. The estimates of the MMLE $\boldsymbol{\theta}^*$ are given by:

$$\boldsymbol{\eta}^* = \arg \max_{\boldsymbol{\eta}} \|\mathbf{I} \boldsymbol{\mu}(\boldsymbol{\eta}) \mathbf{x}\|^2 \quad (5)$$

$$\rho^* = \left| [\boldsymbol{\mu}^H(\boldsymbol{\eta}^*) \boldsymbol{\mu}(\boldsymbol{\eta}^*)]^{-1} \boldsymbol{\mu}^H(\boldsymbol{\eta}^*) \mathbf{x} \right| \quad (6)$$

$$\Phi^* = \arg \left\{ [\boldsymbol{\mu}^H(\boldsymbol{\eta}^*) \boldsymbol{\mu}(\boldsymbol{\eta}^*)]^{-1} \boldsymbol{\mu}^H(\boldsymbol{\eta}^*) \mathbf{x} \right\} \quad (7)$$

where $\Pi_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$ is the orthogonal projector over S , with $S = (\text{Span})(\mathbf{A})$ and \mathbf{A} a matrix. Similarly, as preliminary estimator of the noise variance, we can use the MMLE, which yields to:

$$(\sigma_n^*)^2 = \|\mathbf{x} - \rho^* e^{j\Phi^*} \boldsymbol{\mu}(\boldsymbol{\eta}^*)\|^2 / N. \quad (8)$$

C. Explicit calculation of $\mathbf{K}(\boldsymbol{\theta}^*)$

Let us start by recalling the definition of $\mathbf{K}(\boldsymbol{\theta}^*)$ as:

$$\mathbf{K}(\bar{\boldsymbol{\theta}}) = \frac{2}{\bar{\sigma}_n^2} \Re \left\{ \left(\frac{\partial \bar{\alpha} \boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial \bar{\boldsymbol{\theta}}} \right)^H \left(\frac{\partial \bar{\alpha} \boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial \bar{\boldsymbol{\theta}}} \right) \right\}, \quad (9)$$

where $\bar{\boldsymbol{\theta}}$ indicates the true parameter vector characterizing the signal model in (1). Following the outcomes presented in [2] a closed-form expression for $\mathbf{K}(\boldsymbol{\theta}^*)$ can be obtained as:

$$\mathbf{K}(\bar{\boldsymbol{\theta}}) \triangleq \frac{2F_s}{\bar{\sigma}_n^2} \Re \{ \mathbf{Q} \mathbf{W} \mathbf{Q}^H \} \quad (10)$$

where the entries of the matrix

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_{2,2} & w_4^* \\ w_3 & w_4 & w_{3,3} \end{bmatrix} \quad (11)$$

are given by $w_1 = \frac{1}{F_s} \mathbf{s}^H \mathbf{s}$, $w_2 = \frac{1}{F_s^2} \mathbf{s}^H \mathbf{D} \mathbf{s}$, $w_3 = \mathbf{s}^H \boldsymbol{\Lambda} \mathbf{s}$, $w_4 = \frac{1}{F_s} \mathbf{s}^H \mathbf{D} \boldsymbol{\Lambda} \mathbf{s}$, $w_{2,2} = \frac{1}{F_s^3} \mathbf{s}^H \mathbf{D}^2 \mathbf{s}$, $w_{3,3} = F_s \mathbf{s}^H \mathbf{V} \mathbf{s}$. Moreover, the matrix \mathbf{Q} in (15) is as follows:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ j\bar{\rho} & 0 & 0 \\ j\bar{\rho} 2\pi f_c \bar{b} & 0 & -\bar{\rho} \\ 0 & -j\bar{\rho} 2\pi f_c & 0 \end{bmatrix}. \quad (12)$$

The vector $\mathbf{s} = (\dots, s(kT_s), \dots)_{N_1 \leq k \leq N_2}^\top$ is the baseband samples vector, $\mathbf{D} = \text{diag}(\dots, k, \dots)_{N_1 \leq k \leq N_2}$ and the entries of the matrices $\boldsymbol{\Lambda}$ and \mathbf{V} are defined as

$$[\boldsymbol{\Lambda}]_{k,k'} = \begin{cases} k' \neq k : \frac{(-1)^{|k-k'|}}{k-k'} \\ k' = k : 0 \end{cases} \quad (13)$$

and

$$[\mathbf{V}]_{k,k'} = \begin{cases} k' \neq k : (-1)^{|k-k'|} \frac{2}{(k-k')^2} \\ k' = k : \frac{\pi^2}{3} \end{cases}. \quad (14)$$

Finally, $\mathbf{K}(\boldsymbol{\theta}^*)$ can be expressed as function of the preliminary estimator $\boldsymbol{\theta}^*$ as:

$$\mathbf{K}(\boldsymbol{\theta}^*) \triangleq \frac{2F_s}{(\sigma^*)^2_n} \Re \{ \mathbf{Q}^* \mathbf{W} \mathbf{Q}^{*H} \} \quad (15)$$

with

$$\mathbf{Q}^* = \begin{bmatrix} 1 & 0 & 0 \\ j\rho^* & 0 & 0 \\ j\rho^* 2\pi f_c b^* & 0 & -\rho^* \\ 0 & -j\rho^* 2\pi f_c & 0 \end{bmatrix}. \quad (16)$$

D. Explicit calculation of $\tilde{\Delta}_N(\boldsymbol{\theta}^*)$

Along with $\mathbf{K}(\boldsymbol{\theta}^*)$, the other term in (4) to be calculated is the random vector :

$$\tilde{\Delta}_N(\boldsymbol{\theta}^*) \triangleq \frac{-2}{\sqrt{N}\sigma_n^*} \sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right) \Re[(u_k^*)^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^*)]. \quad (17)$$

Note that (17) is a ranks-measurable approximation of the central sequence $\Delta_N(\boldsymbol{\theta}^*)$ with

$$\Delta_N(\boldsymbol{\theta}) \triangleq N^{-1/2} \sum_{k=1}^N \mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_k), \quad (18)$$

where the score vector $\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_k)$ is given by [8]:

$$\mathbf{s}_{\boldsymbol{\theta}}(x_k) =_d -2\sigma_n^{-1} \sqrt{Q_k} \bar{\psi}(Q_k) \Re\{u_k^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta})\}. \quad (19)$$

Clearly, the score vector in (19) cannot be calculated for the following three reasons:

- 1) The real and positive random variables $\{Q_k \sim p_Q\}_{k=N_1}^{N_2}$ (distributed as the second order modular variate of the CES noise) are not observable,
- 2) The complex random variables $\{u_k \sim \mathcal{U}(\mathbb{C}S^1)\}_{k=N_1}^{N_2}$ (uniformly distributed on $\mathbb{C}S^1$) are not observable,
- 3) Since the functional form of the true distribution is unknown, the function $\bar{\psi}(\cdot)$ cannot be explicitly defined.

Then, following the outcomes of the seminal work [9], we rely on a ranks-measurable approximation for the central sequence (18). Ranks owe their importance in robust statistics to the property of being “distribution-free”. In particular, ranks-measurable functions can be used to derive non-parametric estimators that show a remarkable efficiency even when the the data distribution is unknown. In particular, for in the considered application, the unknown term if the score vector (19) can be approximated by means of the so-called “score function”. Specifically, it can be shown that [10, Ch. 13], [11], for a given “score function” $M(\cdot)$,¹ we have²:

$$M\left(\frac{r_k}{N+1}\right) = \sqrt{Q_k} \bar{\psi}(Q_k) + o_P(1). \quad (20)$$

with $\{r_k\}_{k=N_1}^{N_2}$ the *ranks* of the (continuous) real random variables $\{Q_k^*\}_{k=N_1}^{N_2}$ that represent a sort of preliminary estimates of the unobservable random variables $\{Q_k \sim p_Q\}_{k=N_1}^{N_2}$

$$Q_k^* \triangleq |x_k - f_k(\boldsymbol{\theta}^*)|^2 / (\sigma_n^*)^2. \quad (21)$$

Finally, preliminary estimates of the unobservable random variables $\{u_k\}_{k=N_1}^{N_2}$ can be obtained as:

$$u_k^* \triangleq (x_k - f_k(\boldsymbol{\theta}^*)) / (\sigma_n^* \sqrt{Q_k^*}). \quad (22)$$

Regarding the “score function” $M(\cdot)$ many choices are possible (see e.g. [11]). However, the one that provide a good trade of between semiparametric efficiency and robustness is the complex *van der Waerden* score function:

$$M_{vdW}(t) \triangleq \sqrt{\Phi_G^{-1}(t)}, \quad t \in (0, 1), \quad (23)$$

where Φ_G^{-1} indicates the inverse function of the cdf of a Gamma random variable with parameters $(1, 1)$.

¹The family of score functions is defined in [12, Sect. 2.2], [10, Ch. 13]

²We write: $x_l = o_P(1)$ if $\lim_{l \rightarrow \infty} \Pr\{|x_l| \geq \epsilon\} = 0, \forall \epsilon > 0$ (convergence in probability to 0)

The next step to calculate $\tilde{\Delta}_N(\boldsymbol{\theta}^*)$ is to compute $\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^*)$, which can be shown to be

$$\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^*) = \mathbf{Q}^* \boldsymbol{\vartheta}(kT_s; \boldsymbol{\theta}^*) e^{j\Phi^*} e^{-j\omega_c b^*(kT_s - \tau^*)}, \quad (24)$$

with \mathbf{Q}^* computed in (16) and

$$\boldsymbol{\vartheta}(kT_s; \boldsymbol{\theta}^*) = \begin{bmatrix} s(kT_s - \tau^*) \\ (t - \tau^*) s(kT_s - \tau^*) \\ s^{(1)}(kT_s - \tau^*) \end{bmatrix}, \quad (25)$$

with $s^{(1)}(t) = \frac{\partial s(t)}{\partial t}$. Then, through explicit calculation

$$\frac{\partial f_k(\boldsymbol{\theta}^*)}{\partial \rho^*} = e^{j\Phi^*} s(kT_s - \tau^*) e^{-j\omega_c b^*(kT_s - \tau^*)}, \quad (26)$$

$$\frac{\partial f_k(\boldsymbol{\theta}^*)}{\partial \Phi^*} = j\rho^* e^{j\Phi^*} s(kT_s - \tau^*) e^{-j\omega_c b^*(kT_s - \tau^*)}, \quad (27)$$

$$\frac{\partial f_k(\boldsymbol{\theta}^*)}{\partial \tau^*} = -\rho^* e^{j\Phi^*} s^{(1)}(kT_s - \tau^*) e^{-j\omega_c b^*(kT_s - \tau^*)} + j\omega_c b^* \rho^* e^{j\Phi^*} s(kT_s - \tau^*) e^{-j\omega_c b^*(kT_s - \tau^*)}, \quad (28)$$

$$\frac{\partial f_k(\boldsymbol{\theta}^*)}{\partial b^*} = -j\omega_c (kT_s - \tau^*) \rho^* e^{j\Phi^*} s(kT_s - \tau^*) e^{-j\omega_c b^*(kT_s - \tau^*)}. \quad (29)$$

Since $M\left(\frac{r_k}{N+1}\right)$ are real values, we can compute (17) by computing the following sum

$$\Re \left[\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right) (u_k^*)^* \nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^*) \right] = \Re \left[\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right) (u_k^*)^* \mathbf{Q}^* \boldsymbol{\vartheta}(kT_s; \boldsymbol{\theta}^*) e^{j\Phi^*} e^{-j\omega_c b^*(kT_s - \tau^*)} \right]. \quad (30)$$

Unfortunately, due to the term $s^{(1)}(kT_s - \tau^*)$, the sum is not trivial to resolve. In order to provide closed-form expression of the previous sum, we can assume that the signal $s(t)$ is band-limited and apply the Nyquist-Shannon theorem.

A band-limited signal assumption, $s(t)$ can be expressed as:

$$\begin{aligned} s(t) &= \sum_{k=N_1}^{N_2} s(kT_s) \text{sinc}(\pi B(t - kT_s)) \Leftrightarrow \\ s(f) &= \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi f k T_s} \right) 1_{[-\frac{F_s}{2}, \frac{F_s}{2}]} \end{aligned} \quad (31)$$

where $T_s = \frac{1}{B} = \frac{1}{F_s}$ is the sampling period. Moreover, by definition of band-limited signal, we can show that

$$\lim_{(N_1, N_2) \rightarrow (-\infty, +\infty)} T_s \sum_{k=N_1}^{N_2} s(nT_s) = \int_{-\infty}^{+\infty} s(t) dt.$$

Then, considering enough samples and defining $\mathbf{u}^* = (\dots, \mathcal{U}_k^*, \dots)^T$, $\mathcal{U}_k^* = M\left(\frac{r_k}{N+1}\right) (u_k^*)^*$, we have that

$$\begin{aligned} \lim_{(N_1, N_2) \rightarrow (-\infty, \infty)} T_s \left(\sum_{k=N_1}^{N_2} e^{j\Phi^*} \mathbf{Q}^* \mathcal{U}_k^* \boldsymbol{\vartheta}(kT_s; \boldsymbol{\theta}^*) e^{-j\omega_c b^*(kT_s - \tau^*)} \right) &= \\ \int_{-\infty}^{\infty} e^{j\Phi^*} \mathbf{Q}^* \mathcal{U}^*(t) \boldsymbol{\vartheta}(t; \boldsymbol{\theta}^*) e^{-j\omega_c b^*(t - \tau^*)} dt &= e^{j\Phi^*} \mathbf{Q}^* \int_{-\infty}^{\infty} \mathcal{U}^*(t) \boldsymbol{\vartheta}(t; \boldsymbol{\theta}^*) e^{-j\omega_c b^*(t - \tau^*)} dt = e^{j\Phi^*} \mathbf{Q}^* \mathbf{w}_e. \end{aligned} \quad (32)$$

yielding to

$$\tilde{\Delta}_N(\boldsymbol{\theta}^*) = (2F_s)/(\sqrt{N}\sigma^*)\Re\{e^{j\Phi^*}\mathbf{Q}^*\mathbf{w}_e\}. \quad (33)$$

Now, we have to compute $\mathbf{w}_e = (w_{e1}, w_{e2}, w_{e3})^T$. To do that, we have to use the Parseval theorem and the expression of the Fourier transform of band-limited signals as done in [13]:

$$\begin{aligned} w_{e1} &= \int_{-\infty}^{\infty} \mathcal{U}^*(t)s(t-\tau^*)e^{-j\omega_c b^*(t-\tau^*)}dt = \int_{-\infty}^{\infty} \mathcal{U}^*(t+\tau^*)s(t)e^{-j\omega_c b^*t}dt \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\mathcal{U}(f)e^{j2\pi f\tau^*}\right)^* s(f+f_c b^*)df \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} s(kT_s)e^{-j2\pi(f+f_c b^*)kT_s}\right) e^{-j2\pi f\tau^*} \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} \mathcal{U}(kT_s)e^{-j2\pi f kT_s}\right)^* df \\ &= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=N_1}^{N_2} s(kT_s)e^{-j2\pi f k} e^{-j2\pi \frac{f_c b^*}{F_s} k}\right) e^{-j2\pi f \frac{\tau^*}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s)e^{-j2\pi f k}\right)^* df \\ &= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{U} \left(\frac{f_c b^*}{F_s}\right) \boldsymbol{\nu}(f)^*\right) e^{-j2\pi f \frac{\tau^*}{T_s}} (\mathcal{U}^H \boldsymbol{\nu}(f)) df = \frac{1}{F_s} \mathbf{U}^H \mathbf{V}^{\Delta,0} \left(\frac{\tau^*}{T_s}\right) \mathbf{U} \left(\frac{f_c b^*}{F_s}\right) \mathbf{s} \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{\nu}(f) &= \left(\dots, e^{j2\pi f k}, \dots \right)_{N_1 \leq k \leq N_2}^T \\ \mathbf{V}^{\Delta,0}(q) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} df. \end{aligned}$$

Moreover, $\mathbf{V}^{\Delta,0}(q)$ yields to

$$[\mathbf{V}^{\Delta,0}(q)]_{k,l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-q)} df = \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\sin(\pi(k-l-q))}{\pi(k-l-q)} = \text{sinc}(k-l-q).$$

and

$$\mathbf{U}(p) = \text{diag} \left(\dots, e^{-j2\pi p k}, \dots \right)_{N_1 \leq k \leq N_2}.$$

Note that in order to compute w_{e1} , we have apply the following Fourier transform property:

$$s(t)e^{j2\pi f_c b t} \rightleftharpoons \text{FT} \left\{ s(t)e^{j2\pi f_c b t} \right\} (f) \triangleq s(f - f_c b)$$

$$\begin{aligned}
w_{e2} &= \int_{-\infty}^{\infty} (t - \tau^*) s(t - \tau^*) \mathcal{U}^*(t) e^{-j\omega_c b^* (t - \tau^*)} dt = \int_{-\infty}^{\infty} ts(t) \mathcal{U}^*(t + \tau^*) e^{-j\omega_c b^* t} dt \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{j}{2\pi} \frac{d}{df} s(f - f_c b^*) \right) \left(\mathcal{U}(f) e^{j2\pi f \tau^*} \right)^* df \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \frac{j}{2\pi} (-j2\pi T_s) \sum_{k=N_1}^{N_2} s(kT_s) k e^{-j2\pi(f + f_c b^*) k T_s} \right) e^{-j2\pi f \tau^*} \times \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k T_s} \right)^* df \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=N_1}^{N_2} s(kT_s) k e^{-j2\pi f k} e^{-j2\pi \frac{f_c b^*}{F_s} k} \right) e^{-j2\pi f \frac{\tau^*}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k} \right)^* df \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D} \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{-j2\pi f \frac{\tau^*}{T_s}} (\mathbf{U}^H \boldsymbol{\nu}(f)) df = \frac{1}{F_s} \mathbf{U}^H \mathbf{V}^{\Delta,0} \left(\frac{\tau}{T_s} \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{D} \mathbf{s},
\end{aligned}$$

with

$$\mathbf{D} = \text{diag}(\dots, k, \dots)_{N_1 \leq k \leq N_2}.$$

Note that in order to compute w_{e2} , we can apply the following property of the Fourier transform:

$$ts(t) e^{j2\pi f_c b t} \Leftrightarrow \frac{j}{2\pi} \frac{d}{df} (s(f - f_c b)).$$

$$\begin{aligned}
w_{e3} &= \int_{-\infty}^{\infty} s^{(1)}(t - \tau) \mathcal{U}^*(t) e^{-j\omega_c b^* (t - \tau)} dt = \int_{-\infty}^{\infty} s^{(1)}(t) \mathcal{U}^*(t + \tau^*) e^{-j\omega_c b^* t} dt \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} (j2\pi(f + f_c b^*) s(f + f_c b^*)) \left(\mathcal{U}(f) e^{j2\pi f \tau^*} \right)^* df \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi(f + f_c b^*) \frac{1}{F_s} \sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi(f + f_c b^*) k T_s} \right) e^{-j2\pi f \tau^*} \left(\frac{1}{F_s} \sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k T_s} \right)^* df \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi(f F_s + f_c b^*) \sum_{k=N_1}^{N_2} s(kT_s) e^{-j2\pi f k} e^{-j2\pi \frac{f_c b^*}{F_s} k} \right) e^{-j2\pi f \frac{\tau^*}{T_s}} \left(\sum_{k=N_1}^{N_2} \mathcal{U}(kT_s) e^{-j2\pi f k} \right)^* df \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi(f F_s + f_c b^*) \mathbf{s}^T \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \boldsymbol{\nu}(f)^* \right) e^{-j2\pi f \frac{\tau^*}{T_s}} (\mathbf{U}^H \boldsymbol{\nu}(f)) df \\
&= \mathbf{U}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f \frac{\tau^*}{T_s}} df \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{s} + \frac{j2\pi f_c b^*}{F_s} \mathbf{U}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f \frac{\tau^*}{T_s}} df \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{s} \\
&= \mathbf{U}^H \mathbf{V}^{\Delta,1} \left(\frac{\tau^*}{T_s} \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{s} + \frac{j2\pi f_c b^*}{F_s} \mathbf{U}^H \mathbf{V}^{\Delta,0} \left(\frac{\tau^*}{T_s} \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{s} \\
&= \mathbf{U}^H \mathbf{V}^{\Delta,1} \left(\frac{\tau^*}{T_s} \right) \mathbf{U} \left(\frac{f_c b^*}{F_s} \right) \mathbf{s} + jw_c b^* w_{e1},
\end{aligned}$$

with

$$\mathbf{V}^{\Delta,1}(q) = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} df$$

and

$$\begin{aligned}
[\mathbf{V}^{\Delta,1}(q)]_{k,l} &= j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-q)} df = j2\pi \left(\left[\frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} df \right) \\
&= \frac{j2\pi}{j2\pi(k-l-q)} \left(\left[\frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2} \right) e^{-j\pi(k-l-q)} \right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \frac{1}{k-l-q} (\cos(\pi(k-l-q)) - \text{sinc}(k-l-q)).
\end{aligned}$$

Note that in order to compute w_{e3} , we have apply the following Fourier transform property:

$$s^{(1)}(t) e^{j2\pi f_c b t} \Rightarrow j2\pi (f - f_c b) s(f - f_c b).$$

To conclude, the last term to compute for the implementation of (4) is $\hat{\alpha}$, which represents a non-parametric estimator of $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$. The derivation of this estimator is a delicate step [14], [15] and is based on technical results of LeCam theory and its rank-based extension to semiparametric models. For this reason it is impossible to give here an explanation, however intuitive and tutorial, of its final expression, which will be given below as:

$$\hat{\alpha}_N = \frac{(\sigma_n^*)^2}{N} \frac{\|\tilde{\Delta}_N(\boldsymbol{\theta}^* + N^{-1/2}\mathbf{v}^0) - \tilde{\Delta}_N(\boldsymbol{\theta}^*)\|}{\|\mathbf{K}(\boldsymbol{\theta}^*)\mathbf{v}^0\|}, \quad (34)$$

Where we introduced the “small perturbation” vector $\mathbf{v}^0 \sim \mathcal{N}(\mathbf{0}, \varrho \mathbf{I})$. It can be shown that such $\hat{\alpha}$ satisfies the following \sqrt{N} -consistency property:

$$E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \hat{\alpha}_N + o_P(1). \quad (35)$$

It is worth to underline here that, even if $\hat{\alpha}$ can be shown to be consistent, nothing can be said about its efficiency. In particular, the derivation of a consistent and efficient estimator of $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$ is still an open problem.

E. Extra Results

In this subsection, we include additional results that could not be included in the main article due to space limitations. Once again, we describe the simulation parameters. We consider a scenario where a GPS L1 C/A signal [16] is received by a GNSS receiver. We set a true signal model where the noise is sampled from a complex centered t -distribution [17, Sec. 4.6.1.1] with $v = \{1.2, 1.5, 2\}$ degrees of freedom (or *shape parameter*) that control the level of non-Gaussianity and a scale parameter μ . The second-order modular variate \mathcal{Q} of a t -distribution is an F -distributed random variable with parameter 2 and $2v$, i.e., $\mathcal{Q} \sim \mu^{-1} F_{2,2v}$ [4, Sec. IV.A]. Then, to meet the constraint $E\{\mathcal{Q}\} = 1$, the scale must be set as $\mu = \frac{v}{(v-1)}$. Moreover, for the t -distribution $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \frac{\mu(v+1)}{(v+2)}$ as it has been shown in [8] and $SNR_{out} = |\bar{\alpha}|^2 \mathbf{s}^H \mathbf{s} / \bar{\sigma}_n^2$. Figures 3 and 4 present the root mean square error (RMSE) results of the MMLE and the R-estimator derived in (4) for the parameters of interest $\bar{\boldsymbol{\eta}}^T = [\bar{\tau}, \bar{b}]$, as a function of the SNR_{out} . This analysis considers a setup with a GNSS receiver operating at a sampling frequency $F_s = 4\text{MHz}$ and an integration time of 1 ms. The results are based on 1000 Monte Carlo iterations. The results show that the RMSE (\sqrt{MSE}) of the MMLE asymptotically converge to the MCRB, which matches with the Complex-Gaussian CRB. This results was already shown in [18]. Moreover, we show the RMSE of the estimator derived in (4) asymptotically converge to the SCRB. We emphasize that the implementation of the R estimator is independent of the degrees of freedom. *Finally, it is important to emphasize that the theoretical results presented apply to any CES-distributed true noise model, not just the t -distribution.* Therefore, we also set a true signal model where the noise is distributed according to a complex centered Generalized Gaussian

(GG) distribution, [17, Sec. 4.6.1.2] with exponent $s > 0$ and scale $b > 0$, where s is a parameter controlling the level of non-Gaussianity. The second-order modular variate \mathcal{Q} of a GG distribution is given by $\mathcal{Q} =_d G^{1/s}$ where G is a Gamma distributed random variable with parameter $1/s$ and b , i.e. $G \sim \text{Gam}(1/s, b)$ [4, Sec. IV.B]. In order to satisfy the constraint $E\{\mathcal{Q}\} = 1$, we set $b = \left(\frac{\Gamma(1/s)}{\Gamma(2/s)}\right)^s$. Moreover, $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \frac{\Gamma(2)\Gamma(2/s)}{(\Gamma(1+1/s))^2}$ as it has been shown in [19]. Again, $\bar{\sigma}_n^2$ set the SNR_{out} . The RMSE results for the R-est for the parameters of interest $\boldsymbol{\eta}^T = [\tau, b]$ are shown in Figs. 1 and 2 w.r.t. the SNR_{out} . In the simulation, complex centered Generalized Gaussian distribution with $s = \{0.25, 0.5\}$ are set. Again, we show that the RMSE (\sqrt{MSE}) of the MMLE asymptotically converge to the MCRB, which matches with the Complex-Gaussian CRB. However, in this particular case, we observe that although the RMSE of the R-estimator improves upon that of the MMLE, it does not asymptotically converge to the SCRB. The main reason for this is that, unlike the Student-t distribution, the complex van der Waerden score function is less well-adapted, which leads to a suboptimal estimation of in the estimator of α in (34). As a consequence, our estimator cannot converge to the SCRB.

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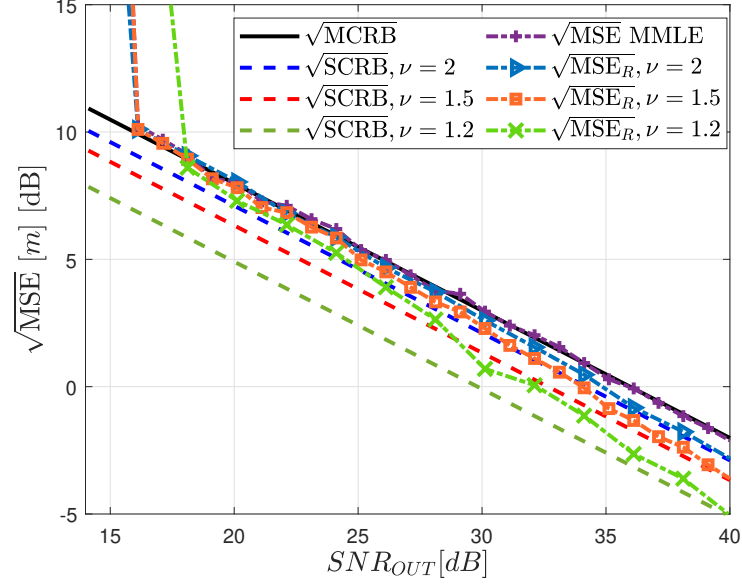


Fig. 1: RMSE of the MMLE and R-estimator in (4) of the time-delay considering complex centered t-dist. with $\nu = \{1.2, 1.5, 2\}$.

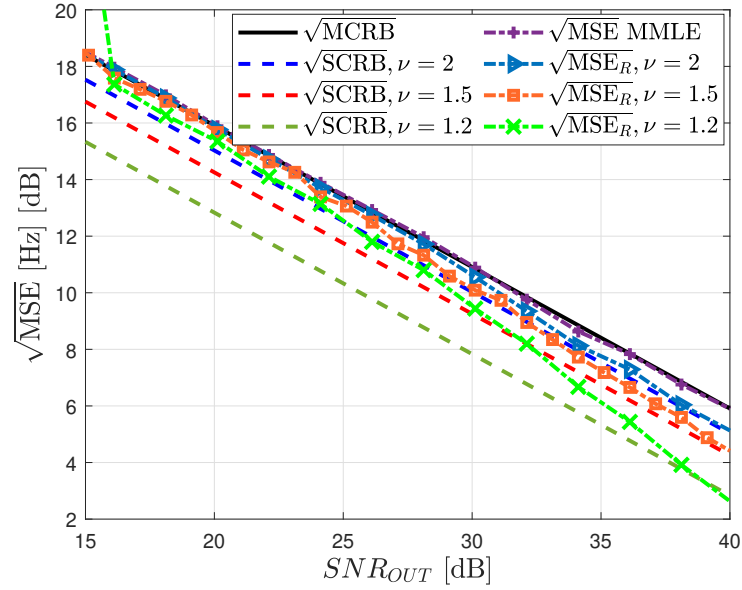


Fig. 2: RMSE of the MMLE and R-estimator in (4) of the Doppler considering complex centered Generalized Gaussian distribution with $s = \{0.25, 0.5\}$.

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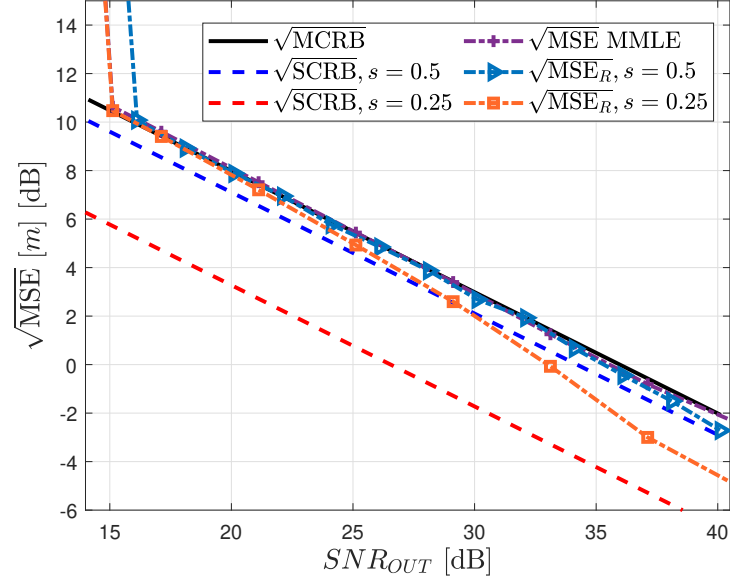


Fig. 3: RMSE of the MMLE and R-estimator in (4) of the time-delay considering complex centered Generalized Gaussian distribution with $s = \{0.25, 0.5\}$.

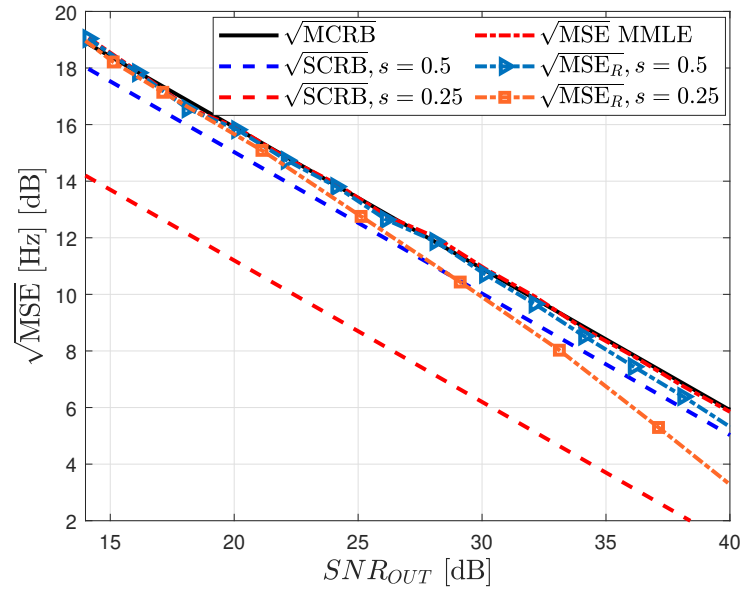


Fig. 4: RMSE of the MMLE and R-estimator in (4) of the Doppler considering complex centered t-dist. with $s = \{0.25, 0.5\}$.