



Understanding eigentheory: The modes of thinking in students' works

Yael Fleischmann, Emilie Lyse-Olsen

► To cite this version:

Yael Fleischmann, Emilie Lyse-Olsen. Understanding eigentheory: The modes of thinking in students' works. Fifth conference of the International Network for Didactic Research in University Mathematics, Centre de Recerca Matemàtica [CRM], Jun 2024, Barcelona, Spain. <hal-04944269>

HAL Id: hal-04944269

<https://hal.science/hal-04944269v1>

Submitted on 13 Feb 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Understanding eigentheory: The modes of thinking in students' works

Emilie Lyse-Olsen¹ and Yael Fleischmann¹

¹Norwegian University of Science and Technology, Faculty of Mathematics, Norway
yael.fleischmann@ntnu.no

In this study, we explore the understanding of eigenvectors and eigenvalues among university students by analysing their written responses to a task and conducting subsequent interviews. We make use of Tall and Vinner's notion of the concept image and Sierpinska's modes of thinking to assess their comprehension. Our findings show that the majority of students is able to engage with multiple modes of thinking when explaining the concepts of eigenvectors and eigenvalues. However, some students experience difficulties facing different modes. These insights are expected to inform our future work in designing tasks to further support students' learning of eigentheory.

Keywords: linear algebra, eigentheory, modes of thinking, concept image.

INTRODUCTION AND THEORY

Understanding linear algebra is an essential part of undergraduate mathematics education. To characterise the complex phenomenon of understanding, we make use of Tall and Vinner's notions of the concept image and, more specifically, the concept definition image. According to Vinner (2002), "To understand [...] means to have a concept image" (p. 69). The *concept image*, as defined by Tall and Vinner (1981), refers to the mental representation or internalisation of a mathematical concept that individuals develop through their experiences and interactions with mathematical ideas. Hence, for this study, we consider the concept image as referring to a part of an individual's subjective understanding of a concept. Closely linked to the concept image is the *concept definition*, which is a verbal explanation precisely characterising the mathematical object. Tall and Vinner (1981) further describe the *concept definition image*, as the part of the concept image generated by the concept definition (Figure 1).

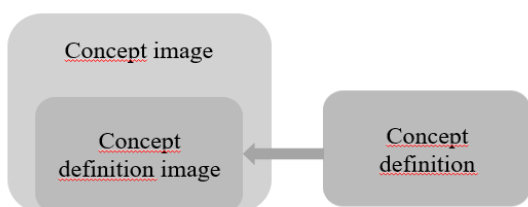


Figure 1: Diagram depicting the relationship between concept image, concept definition and concept definition image (adapted from Viholainen, 2008).

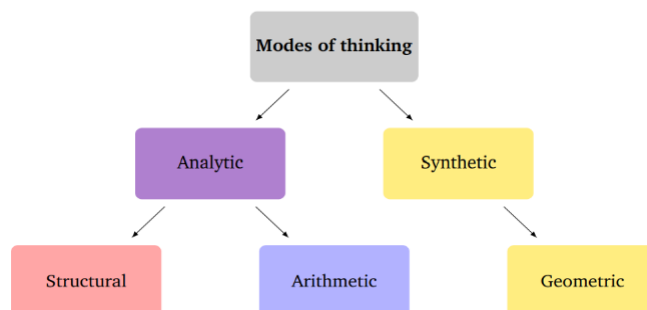


Figure 2: The hierarchic structure of the modes of thinking (based on Sierpinska, 2000).

From our perspective, the *modes of thinking* conceptualised by Sierpinska (2000) describe a part of students' understanding specific to linear algebra. In this study, we apply the modes of thinking as a theoretical lens to characterise certain parts of students' concept images of eigenvectors and eigenvalues, most of which align with the concept definition image. Sierpinska (2000) identifies three modes of thinking in linear algebra, namely the synthetic-geometric, analytic-arithmetic, and analytic-structural. These categories have a two-level hierarchy (see Figure 2). The first level distinguishes between synthetic and analytic modes of thinking, while the second level further separates them into geometric, arithmetic, and structural modes. Sierpinska (2000) describes the difference between *synthetic* and *analytic* modes of thinking as follows:

[I]n the synthetic mode the objects are, in a sense, given directly to the mind which then tries to describe them, while, in the analytic mode they are given indirectly: In fact, they are only constructed by the definition of the properties of their elements. (p. 233)

In the *analytic-arithmetic mode*, an object is defined by the formula enabling its computation. Hence, the arithmetic mode concerns n-tuples of specific numbers satisfying equations or inequalities (p. 235). The *analytic-structural mode*, on the other hand, is more general, concerning the characteristic properties defining the mathematical objects. Finally, the *synthetic-geometric mode* employs the vocabulary of geometric figures, such as points, lines and planes (p. 234). This mode is concerned with the geometric characteristics of the objects and their visual representations. In a study by Gol Tabaghi and Sinclair (2013), the modes of thinking were used to analyse students' reasoning. Their analysis led to the conceptualisation of an additional mode of thinking, the *dynamic-synthetic-geometric mode*, emerging from students' emphasis on the dynamic aspects of eigenvectors.

According to Sierpinska (2000), each of the three modes of thinking correspond to a specific "system of representation" (p. 234), which we interpret in the sense of Duval (2006). Sierpinska (2000) highlights a shared characteristic between the synthetic-geometric and analytic-structural modes of thinking, namely the independence of a coordinate system. However, the analytic-structural mode is characterised by schematic representations, illustrating the inherent properties and the abstract relations between objects, whereas the synthetic-geometric mode portrays the mathematical objects in a more concrete manner.

METHOD

In this study, we employ the modes of thinking and concept image to elucidate specific facets of students' understanding of eigentheory. It is our perspective that eigentheory,

the field of linear algebra concerning eigenvectors, eigenvalues, and eigenspaces, can prove to be particularly challenging for students to grasp. In their 2011 study, Thomas and Stewart observed that while many students demonstrated proficiency in performing the arithmetic computations related to eigenvectors and eigenvalues, they encountered difficulties in associating these calculations with their geometric interpretations. Moreover, within the context of eigentheory, students must adeptly navigate several key concepts simultaneously, including linear transformation, vector space, and span (Wawro et al., 2018, p. 275).

This paper is based on the master's project of the first author (Lyse-Olsen, 2023), aiming to explore aspects of students' understanding of eigenvectors and eigenvalues through the following research question:

What parts of students' concept images can be described through the modes of thinking identified in their characterisations of eigenvectors and eigenvalues?

The presented study took place in a first linear algebra course of a Norwegian university during the fall of 2022, attended by ca. 700 students from various engineering study programs, most of which were in their second year of study and aged in their early 20s. The instructional approach of the course employed a flipped classroom style, with interactive lectures, exercise classes and exercise lectures. To prepare for these activities, students were expected to watch a set of short lecture videos and read the course materials. The following definition was used to introduce eigenvalues and eigenvectors in the written materials:

Let $T: V \rightarrow V$ be a linear transformation. A scalar λ is called an eigenvalue of T if there exists a vector $\vec{v} \neq \vec{0}$ in V such that $T(\vec{v}) = \lambda\vec{v}$. The vector \vec{v} is called an eigenvector of T corresponding to the eigenvalue λ . When T is given as an $n \times n$ matrix A , λ is called an eigenvalue of A and \vec{v} is an eigenvector of A corresponding to the eigenvalue λ . (NTNU, n.d.)

The preparational material also emphasised geometric aspects of eigenvectors and eigenvalues, including visual representations of two-dimensional eigenvectors. As all activities and preparational material was voluntary, it is likely that different students were exposed to different modes of thinking prior to this study, depending on their decisions to engage with or omit certain preparations or activities. To come to know about students' concept images and modes of thinking, four tasks were designed and implemented. In this paper, we will only present the results of the analysis of one of these tasks (see Figure

Task 9: Explain in your own words:

- a) What is an eigenvector?
- b) What is an eigenvalue?

You may use sketches to illustrate.

Figure 3: The task given to the students (translated from Norwegian).

3), where students were asked to explain eigenvectors and eigenvalues in their own terms and encouraged to produce an illustrating sketch. The task was designed with an open phrasing to challenge students with a different task than they were used to and to prompt reflection upon the concepts, thereby allowing our exploration of their understanding. The results are based on the written answers of 170 students and additional semi-structured interviews with five individual students lasting up to 45 minutes each. The interviews were conducted and audio-recorded by the first author of this paper, further exploring their reasoning in completing the tasks and their overall experiences with the course, approximately 3-5 weeks after submitting their homework. The students were selected based upon a preliminary analysis of the students' written homework and the selection aimed to include students exhibiting different modes of thinking and various concept images.

Data analysis: The students' written answers were analysed in a two-step coding process using the qualitative data analysis software NVivo. The first level codes were based on the words or short phrases used by the students, aiming to capture the essence of their ideas. In the second level of coding, we created codes corresponding to the modes of thinking and categorised the first level codes accordingly. It should be noted that the answers to parts a) and b) were analysed and coded together due to the interconnectedness between the concepts of eigenvectors and eigenvalues. The transcriptions from the audio-recordings of the interviews were employed as supportive data in this study.

To effectively characterise students' modes of thinking, we must first establish what the modes entail in the field of eigentheory, as Sierpinska (2000) does not specify her framework for this particular context. First, it is our perspective that an analytic-structural mode of thinking about eigenvectors and eigenvalues would involve a general description true for all eigenvectors, not just particular examples. Thus, this mode of thinking could present as characterising eigenvectors as preserving their span when imaged by a linear transformation or providing a schematic sketch illustrating these properties. An analytic-arithmetic mode of thinking, on the other hand, could manifest as characterising eigenvectors and eigenvalues as the solutions of the equations allowing their computation, $(A - \lambda I)\vec{x} = \vec{0}$ or

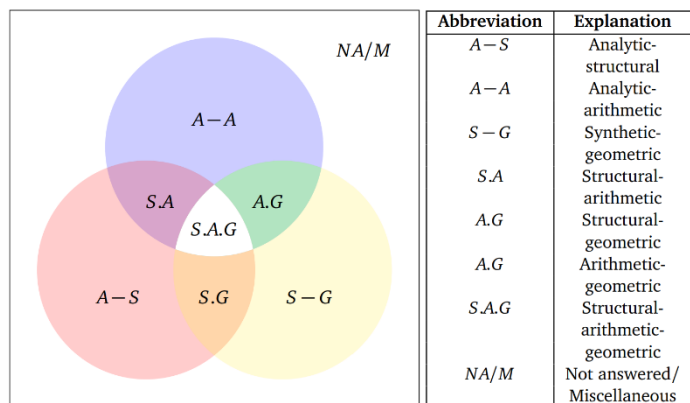


Figure 4: Overview of modes of thinking exhibited in the students' answers to Task 9.

$\det(A - \lambda I) = 0$, respectively. Finally, a synthetic-geometric mode of thinking about eigenvectors and eigenvalues may involve geometric descriptions of the representations of particular examples, such as characterising eigenvectors as remaining on the same line or preserving their direction under a linear transformation (or matrix multiplication). From our understanding, descriptions in the synthetic-geometric mode describe certain characteristic properties of *some* eigenvectors and eigenvalues, but they do not define them.

As we shall see, we quickly came to realise that many of the students' answers aligned with multiple modes of thinking. To capture the nuances of answers incorporating multiple modes of thinking or falling between modes, we created four intersectional modes of thinking, which we call the *structural-arithmetic*, *structural-geometric*, *arithmetic-geometric* and *structural-arithmetic geometric* modes of thinking (see Figure 4). These modes will be elaborated in the upcoming section.

RESULTS AND ANALYSIS

In this section, we first present examples of students' work aligning with the original modes of thinking described by Sierpiska (2000), and later present examples of answers that extend beyond the initial categorisation.

Analytic-structural mode of thinking:

The following work of one student (see Figure 5), who we shall call Alex, was identified as an example of a structural representation. The sketch shows two V s

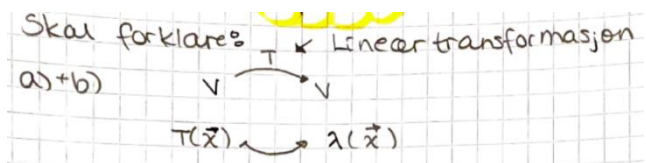


Figure 5: Alex's sketch.

connected by an arrow labelled T , which we interpreted as symbolising a linear transformation within a vector space denoted V . Below, the linear transformation T is depicted as operating on a vector \vec{x} and mapping it to $\lambda(\vec{x})$. Hence, we understand this as a schematic sketch of the relations between the mathematical object, depicting how a *general* linear transformation T acts upon a corresponding eigenvector \vec{x} by scaling it with a factor of λ , the eigenvalue. Other answers, describing eigenvectors and eigenvalues in relation to the notion of span or the image of linear transformation were also interpreted as exhibiting elements of an analytic-structural mode of thinking.

Analytic-arithmetic mode of thinking: Some students explained the concept of eigenvalue according to the procedure for computing them, namely by computing the roots of the characteristic polynomial. For instance, a student, Tyler, stated that: "Eigenvalue or characteristic value is a solution of the characteristic equation $\det(A - \lambda I) = 0$ ". In describing an eigenvalue as the solution of an equation, the answer was deemed as aligning with an analytic-arithmetic mode of thinking.

Synthetic-geometric mode of thinking: Several students gave geometric descriptions of eigenvectors and eigenvalues using words and phrases like “stretching”, “shrinking”, “changing length” or “preserving direction”. For example, Sam stated that “[The] Eigenvalue is how much the vector is stretched.”. Hereby, Sam gave a visual description of a particular type of eigenvalues, namely real eigenvalues with an absolute value greater than 1, suggesting a synthetic-geometric mode of thinking. In fact, several other students gave similar descriptions of eigenvalues being factors of stretching or eigenvectors being stretched, thus excluding the possibilities of shrinking or preserving the length. Notably, the definition presented to students (see Method) does not imply such restrictions. In this study, none of the 170 participating students gave a comprehensive description of all possible effects on eigenvectors by linear transformations or matrix multiplications, including stretching, shrinking, rotating by 180 degrees, preserving its length or direction, or combinations thereof.

While the previous example of Sam concerned a change in *length*, other students focused on the effect of the linear transformation on the eigenvector’s *direction*. For instance, Robin explained an eigenvector as “[A] vector which does not change direction”. However, characterising eigenvectors as always preserving their direction is inaccurate as real eigenvectors can reverse direction with negative eigenvalues, and eigenvectors with complex eigenvalues can undergo both scaling and rotation. Thus, Robin’s answer is a geometric interpretation of *specific* examples of eigenvectors, which we consider evidence of a synthetic-geometric mode of thinking.

Robin was selected for an interview to further explore their concept image. When asked to explain the concept of eigenvectors and eigenvalues in the interview, Robin gave a verbal rephrasing of the eigenequation (the following is our own translation from Norwegian):

Robin: [long pause] Yes, that, if you have a... a matrix, then you can... And you multiply it with the eigenvalue, then you will have the same as if you multiply the eigenvector... an eigenvector with the eigenvalue. Is actually the only thing I know about that [...]. So, basically, an eigenvalue is a value which you can multiply by the matrix and a vector and obtain the same result.

Thus, while the written task evoked one part of Robin’s concept definition image associated with a synthetic-geometric mode, the interview appeared to trigger another. In describing eigenvectors and eigenvalues as fulfilling an equation, specifically the faulty equality $A\lambda = \vec{x}\lambda$, Robin’s oral response aligns with an arithmetic mode of thinking. To support Robin in linking the different modes of thinking expressed in their written and oral response and restore their error, Robin was reminded of their written description:

Interviewer: It seems to me that you know another thing because you wrote that.... In [Task 9] a) you wrote: “A vector which does not change direction”?

Robin: Oh... Yes... Uhm... [long pause]. I don't really know what I meant by that. If I... Maybe I meant, if I... That I multiplied by a number and then... No... Now I'm not quite sure what that means.

While Robin's written and oral characterisations do not entirely contradict each other (scaling a vector \vec{x} by a number λ can preserve its direction), Robin's apparent confusion could suggest they *perceive* their answers as conflicting. Nevertheless, when prompted to recall their written explanation of *eigenvalues* affecting the length of the eigenvector, Robin appeared to align their oral and written answer:

Interviewer: In [Task 9] b) you wrote: “The scalar which determines the length of the eigenvector”.

Robin: That makes a bit more sense. Maybe if you take a vector and you multiply it by a number, it would change length.

Hence, Robin seems able to visually interpret scalar multiplication as a means of altering the length of a vector (in this case, an eigenvector), thereby navigating between the synthetic-geometric mode and the more arithmetic thinking of scalar multiplication. Thus, Robin's interview illustrates a development of their concept image.

Intersectional modes of thinking: It is noteworthy that a substantial proportion of the students participating in this study demonstrated engagement with multiple modes of thinking in their written answers to Task 9. In particular, we identified an overwhelming majority of 123 answers expressing the symbolic eigenequation of the linear transformation ($T(\vec{x}) = \lambda\vec{x}$) or matrix ($A\vec{x} = \lambda\vec{x}$), or a verbal rephrasing of it. For instance, Jordan gave the following explanation of eigenvectors and eigenvalues: “If one has a matrix A and a vector \vec{x} , the product will give a number λ multiplied by \vec{x} . Then λ will be an eigenvalue and \vec{x} an eigenvector: $A\vec{x} = \lambda\vec{x}$.” It is our perspective that characterising eigenvectors as fulfilling the eigenequation aligns with both an analytic-structural mode of thinking (by expressing a defining property of eigenvectors and eigenvalues), as well as an analytic-arithmetic mode (in stating an equation allowing the computation of eigenvectors and eigenvalues). To capture these nuances, we introduced the intersectional mode *structural-arithmetic*, and categorised such answers accordingly. Nevertheless, Jordan's answer is brief and therefore provides little other information regarding their concept images of eigenvectors and eigenvalues. For instance, there is no mention of the relation between eigenvectors and eigenvalues to other key concepts like linear transformation or span, and there is no sketch or geometric interpretation of eigenvectors. Other answers incorporating elements from analytic-structural, analytic-arithmetic and/or synthetic-geometric modes of thinking

were classified as structural-geometric, arithmetic-geometric, and structural-arithmetic-geometric. However, due to space limitations, we are unable to elaborate and exemplify all categories here.

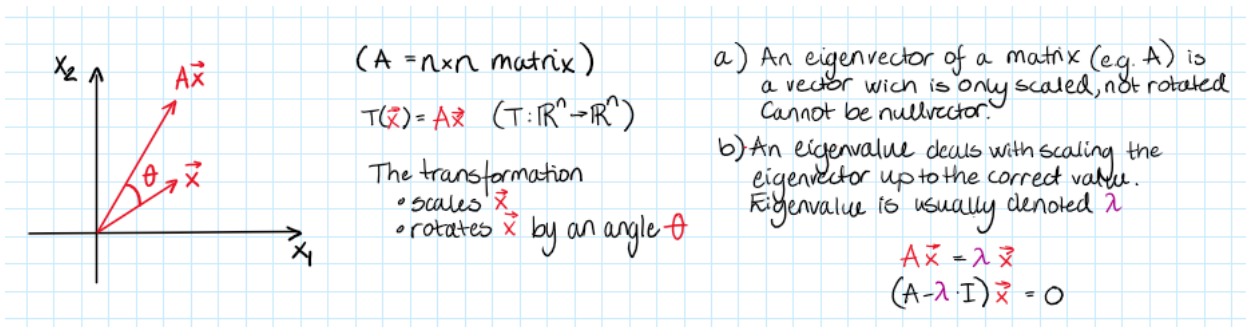


Figure 6: Excerpt (translated from Norwegian) of Riley's written answer.

Figure 6 shows a particularly interesting example of the work of student Riley, who incorporated all three modes of thinking in their answers. Our interpretation of Riley's work is that they first explain and illustrate how a general linear transformation may act upon any vector \vec{x} , and in Task 9 a) the student specifies that the vectors which are "only scaled" are eigenvectors, so long as they are not the nullvector. By relating eigenvectors to the notion of linear transformation, the answer can be characterised as incorporating an analytic-structural mode of thinking. By further stating that eigenvectors are scaled, not rotated, Riley provides a geometric description of typical examples of eigenvectors, aligning with a synthetic-geometric mode of thinking. Moreover, in providing the equation $(A - \lambda I)\vec{x} = \vec{0}$, the answer could indicate an analytic-arithmetic mode of thinking as well. Consequently, our analysis suggests this student possesses a concept image encompassing multiple characteristics of eigenvectors, suggesting a rich understanding of them.

Summary: Initially, based on Sierpinska's framework, our expectation was that most students would adhere to one mode of thinking in their responses. However, it became evident that the majority of the students incorporated more than one mode of thinking in their answers, implying a richer concept image. As shown in Figure 7, only few answers were characterised as belonging to one of the original categories described by Sierpinska (2000), while combining two modes of thinking was much more prevalent (98 students). Interestingly, as many as 54 students gave answers which encompassed all three modes of thinking. In Figure 7, the number 9 outside the Venn diagram represent the written answers that could not be classified into either

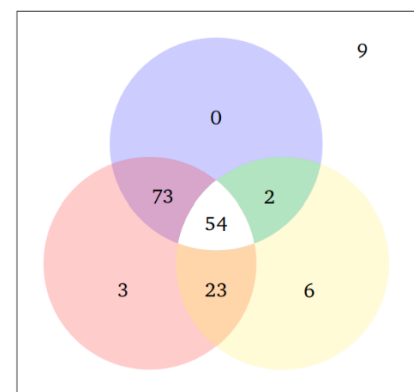


Figure 7: Quantitative overview with the observed modes of thinking.

mode of thinking (task unanswered: 4 students; aligns with none of the modes: 5 students).

DISCUSSION AND CONCLUSION

Through our analysis, we have seen the participants effectively connecting eigenvectors and eigenvalues with linear algebra concepts such as span, linear transformations, and vector spaces – a skill emphasized by Harel (1997) as key to understanding linear algebra. Diverging from the findings of Thomas and Stewart (2011), our study revealed that many students demonstrated an ability to describe eigenvectors and eigenvalues in terms of their geometric properties. Interestingly, we have seen that the majority of students engaged with multiple modes of thinking or gave answers falling between Sierpiska's (2000) categories in their characterisations of eigenvectors and eigenvalues. In doing so, we consider that the students demonstrated rich concept images and an ability to navigate multiple modes of thinking. This prompted us to introduce intersectional modes of thinking, thus refining Sierpiska's (2000) framework in the context of eigentheory, similar to the approach taken by Gol Tabaghi and Sinclair (2013). Our analysis further aligns with Sierpiska's (2000) observation that there is no empirical basis for a general preference among students for one mode of thinking over another. Instead, we propose that certain types of tasks may trigger specific modes of thinking in students. While most participants expressed multiple modes of thinking in their answers to the presented task, the majority exclusively employed the analytic-structural mode in their answers to subsequent tasks not covered in this paper (see Lyse-Olsen, 2023).

In our quest to describe students' understanding of eigenvectors and eigenvalues, it is important to acknowledge that as students express their ideas in sentences (whether written or oral) and illustrations, there is a potential for meaning to be lost, transformed, or even added to their utterances. While our methods do not allow us to identify *all aspects* of students' understanding, our theoretical lenses have enabled us to describe *certain aspects* of their concept images (closely aligning with the concept definition image) of eigenvectors and eigenvalues and their modes of thinking. Moreover, it is important to recognise that absence of specific expressions of knowledge does not necessarily imply that students lack awareness of them. For instance, students' concept images may contain several different ways in which a linear transformation (or matrix) can transform eigenvectors, even though not all these ways are explicit in their answers. We expect these findings to inform our future work, which aims to develop tasks that effectively address students' challenges and foster a deeper understanding of eigentheory.

LITERATURE

- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics* 61(1), 103–131. <https://doi.org/10.1007/s10649-006-0400-z>
- Gol Tabaghi, S., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: The emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning*, 18, 149–164. <https://doi.org/10.1007/s10758-013-9206-0>
- Harel, G. (1997). The linear algebra curriculum study group recommendations: Moving beyond concept definition. In D. Carlson, R. C. Johnson, D. C. Lay, A. D. Porter & W. Watkins (Eds.), *Resources for teaching linear algebra* (pp. 107–126). The Mathematical Association of America.
- Lyse-Olsen, E. (2023). *On eigenvectors and eigenvalues: An examination of university students' concept images of eigenvectors and eigenvalues, and their modes of thinking* (Master's thesis). Norwegian University of Science and Technology.
- NTNU (n.d.). *Kapittel 10: Egenvektorer og egenverdier*. Norwegian University of Science and Technology. <https://www.math.ntnu.no/emner/TMA4115/2022v/notater/10-egenverdier-og-egenvektorer.pdf>
- Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). Springer Netherlands. https://doi.org/10.1007/0-306-47224-4_8
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12. <https://doi.org/10.1007/BF00305619>
- Thomas, M., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal*, 23, 275–296. <https://doi.org/10.1007/s13394-011-0016-1>
- Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Mathematics Enthusiast*, 5(2), 77–82. <https://doi.org/10.54870/1551-3440.1104>
- Vinner, S. (2002). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking*. Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47203-1_5
- Wawro, M., Zandieh, M., & Watson, K. (2018). Delineating aspects of understanding eigentheory through assessment development. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild & N. M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018, 5-7 April 2018)* (pp. 275–284).