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# The study of functional equations to highlight the role of order in proof and proving at the interface between algebra and analysis

Viviane Durand-Guerrier and Gaëtan Planchon

University of Montpellier, IMAG, CNRS-UM, Montpellier, France,

viviane.durand-guerrier@umontpellier.fr

*Abstract: Functional equations are mathematical objects that are defined within the framework of algebra of functions (i.e. equation involving only the four arithmetical operations) for which the establishment of solutions most often requires recourse to methods of proof and proving specific to analysis. In this paper, we focus on the first Cauchy functional equation  $f(x + y) = f(x) + f(y)$  to highlight the role of order in such process, and to argue that the study of such functional equations is an effective means of simultaneously developing proof and proving skills and an understanding of the concepts involved when working with ordered sets of numbers, at university.*

*Keywords: Teaching and learning of logic, reasoning, and proof; Teaching and learning of specific topics in university mathematics; Epistemology and didactics; Order in Mathematics; Cauchy functional equations.*

## INTRODUCTION

Functional equations are mathematical objects that are defined within the framework of algebra of functions (i.e. equations involving only the four arithmetical operations: addition, subtraction, multiplication, and division) for which the establishment of solutions most often requires recourse to methods of proof and proving specific to analysis. Working on these equations is therefore relevant to address issues of interaction between algebra and analysis. In the cases that we will examine in this paper, we will highlight the possible role of order in this process. This presentation falls in a broader project to make the role of order in the French curriculum more visible. Indeed, order plays a role in many areas of mathematics, as Sinaceur points out.

In the French educational system, order is nearly never studied for itself neither at secondary level nor in early university courses. Moreover, in analysis, its role is often hidden by the recourse to limits for defining objects (such as Integral) or for proof and proving theorems (such as the Intermediate value theorem).

The main goal of this paper is to develop an epistemological analysis to address the following research question: “In which respect is the first Cauchy functional equation  $f(x + y) = f(x) + f(y)$  a good candidate for shedding light on the possible role of order at the interface between algebra and analysis?”. Such analysis is a first step for developing a didactical engineering (Artigue, 2020). In a preliminary section, we briefly remind the definition of irrational numbers by Dedekind who, according to Sinaceur (1992) “reduced continuity to order”. In a second section we present some epistemological issues concerning functional equations in Cauchy (1821) by Jean

Dhombres. In a third section, we focus on discontinuous solutions of the first Cauchy functional equation  $f(x + y) = f(x) + f(y)$ . In a fourth section, we present three proofs that solutions of this Cauchy equation in the class of monotonic functions are linear functions to enhance the relevance of this functional equation to put on the scene with undergraduates the role of order at the interface between algebra and analysis. Finally, we discuss didactic implications and present briefly a forthcoming experiment.

## **REDUCTION OF CONTINUITY TO ORDER: DEDEKIND'S CREATION OF IRRATIONAL NUMBERS**

The title of this section is borrowed from Sinaceur (1992) who claims in a section entitled "Reduction of continuity to order" that:

It was undoubtedly Dedekind who, by wishing to provide "a purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal analysis", highlighted the structure of the ordered set of  $\mathbb{R}$ . For him, this means finding a true definition of the nature of continuity. (ibid, p.110) [1].

We briefly remind here that Dedekind (1872) defines a cut in the set of rational numbers as a pair  $(A, B)$  such that

$$A \cup B = \mathbb{Q}, A \cap B = \emptyset, \text{ and } \forall x \in A, \forall y \in B, x \leq y.$$

After having shown that there are infinitively many cuts that are not operated by a rational number (which he names its incompleteness), he defines completeness as the property that every cut of a given ordered set be operated by an element of the set. Consequently, to complete the set of rational numbers, for every cut not operated by a rational, he creates a new number, an *irrational* one. He then proves that the new set is a complete (in the sense above) ordered set. Based on this construction, the definition of a least upper bound (a supremum) comes: an upper bound for a given subset  $M$  of an ordered set  $E$  is an element that is greater than or equal to any element of  $M$ . A least upper bound for  $M$ , if it exists, is the smaller among the upper bounds of  $M$ . Given a cut  $(A, B)$  in the sense of Dedekind, the unique element operating the cut, if it exists, is the supremum of the subset  $A$ , and the infimum of the subset  $B$ . In a complete ordered set  $E$ , every bounded above (resp. below) subset of  $E$  admits a supremum (resp. infimum) that is unique. Due to this close relation between cuts and supremum (resp. infimum), it is not seldom that they are used concomitantly in a proof: creating a cut, assuming the existence of the supremum (resp. the infimum), showing that this element is a candidate to have the desired property, and proving it with order consideration. An example can be found below in this paper for the second and third proofs that "if a solution of the Cauchy equation  $f(x + y) = f(x) + f(y)$  is monotonic, then it is a linear function".

Before moving to the next session, we would like to remind that the property of density (in-itself) of an ordered set is an important issue when considering the elaboration of the theory of real numbers: between the discrete set of integers, and the continuous set of real numbers, there is, among others, the dense (in-itself) incomplete set of rational

numbers; the dichotomy discrete-continuous does not capture the mathematical fact that there are dense ordered sets that are not continuous (Durand-Guerrier, 2016). Note that the property “To be dense (in-itself)” for an ordered set is different of the relation “To be dense in...” between a subset of an ordered set and this set.

## THE ROLE OF FUNCTIONAL EQUATIONS IN CAUCHY’S ALGEBRAIC ANALYSIS

Jean Dhombres, a French historian of mathematics, studied functional equations as a mathematician at the beginning of his career and published with J. Aczel a treatise (Aczel & Dhombres, 1989) which deals with modern theory of functional equations in several variables and their applications to mathematics, information theory, and the natural and social sciences. In a paper published in 1992, he examines the role of the four fundamental functional equations studied by Cauchy in the first part of his *Course of Analysis of the Ecole royale Polytechnique* (Cauchy, 1821). In a chapter of his course entitled “*Determination of a continuous function of a single variable verifying certain conditions*” [2], Cauchy treats simultaneously the four functional equations conserving or exchanging addition and multiplication:

$$\begin{array}{ll} \text{(A)} \ \Phi(x + y) = \Phi(x) + \Phi(y) & \text{(B)} \ \Phi(x + y) = \Phi(x)\Phi(y) \\ \text{(C)} \ \Phi(xy) = \Phi(x) + \Phi(y) & \text{(D)} \ \Phi(xy) = \Phi(x)\Phi(y) \end{array}$$

The first one (A) is the Cauchy equation that we will study in the next sections. In the class of continuous functions considered by Cauchy, the solutions of (A) are the linear functions those of (B) are the exponential functions; those of (C) are the logarithmic functions composed with the absolute value, and those of (D) are the power function with arbitrary real exponent composed with the absolute value. The resolution by Cauchy of the first functional equation (A) is made in two times: first algebraic manipulation leading to the image by a continuous function solution of (A) of numbers in form  $ra$ :  $\Phi(ra) = r \Phi(a)$ ; second using the fact that  $\mathbb{Q}$  is dense in the set of real numbers  $\mathbb{R}$  and the continuity of the searched functions, he proves that the only solutions defined and continuous on  $\mathbb{R}$  of equation (A) are the linear functions. [3] Although the functional equation had already been studied before Cauchy, Dhombres (1992, p.28) underlines the novelty and the fecundity of this method that Cauchy then successfully applied to equations (B), (C) and (D). In addition, this method, together with the use of the density in  $\mathbb{R}$  of the set of dyadic numbers, allows him to solve completely in the class of continuous functions, the functional equation:

$$\Phi(x + y) + \Phi(x - y) = 2\Phi(x)\Phi(y).$$

Considering this, Dhombres claims that relying on the set of rational numbers for solving the first four equations was motivated by the fact that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , as is the set of dyadic numbers with the standard order. This allowed him to fully justify, in the case of continuous functions, that the form established for rational numbers holds for real numbers, that was previously, and even later often taken for granted by mathematicians. Dhombres, at the beginning of the paper, wondered why Cauchy paid

attention to the four functional equations (A), (B), (C) and (D). In the conclusion, he considers that for Cauchy, they were only a transitory step, not goal in themselves (ibid, p. 48). He also pointed the relevance of solving these equations in a delimited class, here the class of continuous functions. This choice provides the regular solutions that we are used dealing with at the secondary-tertiary transition, with proofs that are accessible at this level. Considering here the class of continuous functions, Cauchy embeds the solutions in the domain of analysis. It seems that Cauchy did not search solutions in class of functions else than the continuous ones. This will be done later by G. Hamel in a paper published in 1905, that we present in the next section.

### **DISCONTINUOUS SOLUTION OF THE CAUCHY FUNCTIONAL EQUATION: $f(x + y) = f(x) + f(y)$**

In a paper published in 1905, Hamel considered discontinuous solutions of the Cauchy functional equation  $f(x + y) = f(x) + f(y)$  (A). It is known since Cauchy that looking for continuous solutions, the solutions are the linear functions. In addition, it is easy to prove that if the solutions are searched among functions defined on the sets of rational numbers, then the solutions are linear functions in form  $f(x) = Kx$  without any additional hypothesis on the functions. The question raised by Hamel is: “And what happens if we don't assume that the solutions defined on the set of real numbers are necessarily continuous functions?”. In his paper of 1905, Hamel proves the existence of discontinuous functions solutions of the Cauchy equation (A); he did this by introducing a basis for the real numbers (named today Hamel Basis) that in modern terms would be expressed as: “the set  $\mathbb{R}$  of real numbers is a linear space over the field  $\mathbb{Q}$  of rational numbers” (Aczel & Dhombres, 1989, p.19). Moreover, Hamel establishes that such functions are totally discontinuous:

Each of these discontinuous solutions of the functional equation is totally discontinuous; in any neighbourhood of any point of the  $(x, y)$ -plane there are points of the "curve"  $y = f(x)$  [4]. (Hamel, 1905, pp.461- 462)

A consequence of this theorem is that when considering a graphical representation on a real interval of a discontinuous solution of (A) (that is not in the form  $g(x) = Kx$ ), given a point of the plan with a rational abscissa  $\alpha$ , and an ordinate different of  $K\alpha$  there will be points of the graphical representation in every neighbourhood of this point. Because of the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , on the graphical representation, there will not be only one point that will appear on the vertical line corresponding to the point of rational coordinates  $(\alpha, g(\alpha))$  that lies on the line with equation  $y = Kx$ . [5] More precisely the graph of such a totally discontinuous solution is dense in  $\mathbb{R} \times \mathbb{R}$ . In Durand-Guerrier et al. (2019), we analyse a similar phenomenon in the case of the functional equation for exponential, which we show relevant for a discussion with undergraduates on the  $\mathbb{Q}$ -incompleteness versus the  $\mathbb{R}$ -completeness, and related issues with graphical representations. Relying on this experience, we hypothesise that the Cauchy functional equation (A) would be a good candidate for designing an activity at the secondary-tertiary transition and in teacher training program aiming at shedding

light on the crucial role of completeness/incompleteness of the standard order on the numbers sets at the interface between algebra and analysis.

In the next section, we focus on proofs that every monotonic function with domain of definition  $\mathbb{R}$  and solution of the Cauchy functional equation (A) is a linear function.

**THREE PROOFS THAT EVERY MONOTONIC FUNCTION SOLUTION OF THE CAUCHY FUNCTIONAL EQUATION (A) IS A LINEAR FUNCTION.**

Hewitt and Zuckerman (1969, p.121) underline that a consequence of the theorem above established by Hamel is that: “If  $f$  satisfies (A) [6] and is continuous at some point, or is bounded above or below on some interval, then  $f(x)$  has the form  $kx$ .”

It is also the case if  $f$  is monotonic (Aczel & Dhombres, 1989, p.15).

*Theorem: if a function defined on  $\mathbb{R}$  satisfies equation (A) and is monotonic on  $\mathbb{R}$ , then there exists a real  $k$  such that  $\forall x \in \mathbb{R} f(x) = kx$ .*

We provide below three proofs of the theorem above shedding light on the role of order in the study of the Cauchy functional equation (A). The first proof relies on the fact that every real number is the limit of a pair of adjacent rational sequences; the second and the third ones on the definition of the set of real numbers by Dedekind’s cut method. The proofs are done in the class of *increasing functions from  $\mathbb{R}$  to  $\mathbb{R}$* ; in the three proofs,  $f$  denotes a function of this class.

**Proof 1, with adjacent rational sequences**

Given a real number  $\alpha$ ,  $u$  and  $v$  two adjacent rational sequences converging to  $\alpha$ , with  $u$  an increasing sequence and  $v$  a decreasing sequence with  $u \leq v$ , we have:

$$\forall n \in \mathbb{N}, u_n \in \mathbb{Q}, \wedge v_n \in \mathbb{Q} \wedge u_n \leq \alpha \leq v_n \text{ and } \lim_{\infty} u = \lim_{\infty} v = \alpha$$

$f$  is increasing on  $\mathbb{R}$ ,  $\forall n \in \mathbb{N} f(u_n) \leq f(\alpha) \leq f(v_n)$  (\*)

As  $\forall n \in \mathbb{N} u_n \in \mathbb{Q}, v_n \in \mathbb{Q}$ , then  $(\forall n \in \mathbb{N} f(u_n) = u_n f(1)) \wedge (f(v_n) = v_n f(1))$

Then we have:  $\forall n \in \mathbb{N} u_n f(1) \leq f(\alpha) \leq v_n f(1)$  (\*\*)

As  $u_n$  and  $v_n$  converge to  $\alpha$ , and  $\forall n \in \mathbb{N} u_n \leq \alpha \leq v_n$ , we have:

$$\alpha f(1) \leq f(\alpha) \leq \alpha f(1) (***)$$

Finally, we conclude that  $f(\alpha) = \alpha f(1)$ , from which follows:

$\forall x \in \mathbb{R} f(x) = xf(1)$ , i.e.  $f$  is linear.

**Proofs 2.1 & 2.2, using the Dedekind’s cuts.**

Given a real number  $\alpha$ , there is a cut  $(A_1, A_2)$  of the set of the rational numbers for which  $\alpha$  is the only real number operating this cut; i.e.  $\alpha$  is the supremum of  $A_1$  and the infimum of  $A_2$ . By the definition of  $A_1, A_2$  et  $\alpha$ ,  $\forall x \in A_1 \forall y \in A_2 x \leq \alpha \leq y$  (\*)

**Proof 2.1.**

Given  $b \in A_1$  and  $c \in A_2$  we have  $b \leq \alpha \leq c$  (from \*); then, as  $f$  is increasing:

$$f(b) \leq f(\alpha) \leq f(c) (**).$$

As  $A_1$  and  $A_2$  are subsets of the set of rational numbers, we have:

$$f(b) = bf(1) \wedge f(c) = cf(1)$$

Then, by substitution in (\*\*), we have  $bf(1) \leq f(\alpha) \leq cf(1) (***)$

1<sup>st</sup> case:  $f(1) = 0$ ; then  $f(\alpha) = 0$  hence  $\forall x \in \mathbb{R}, f(x) = 0$ .

2<sup>nd</sup> case:  $f(1) > 0$  [7]; by dividing by  $f(1)$  in \*\*\* we have  $b \leq \frac{f(\alpha)}{f(1)} \leq c$  \*\*\*\*

We deduced that  $\forall x \in A_1 \forall y \in A_2 x \leq \frac{f(\alpha)}{f(1)} \leq y$  \*\*\*\*\*

This proves that  $\frac{f(\alpha)}{f(1)}$  is operating the cut  $(A_1, A_2)$ .

Because there is a unique real number operating a cut, we conclude that:

$\frac{f(\alpha)}{f(1)} = \alpha$ , and finally,  $f(\alpha) = \alpha f(1)$ , from which follows:  $\forall x \in \mathbb{R} f(x) = xf(1)$ , i.e.  $f$  is linear.

### **Proof 2.2**

We first prove that  $(f(A_1), f(A_2))$  is a cut of  $f(\mathbb{Q})$  operated by  $f(\alpha)$ .

Let us consider  $e \in \mathbb{R}^{+*}$ .

As  $(A_1, A_2)$  is a cut of  $\mathbb{Q}$ , there exist  $x \in A_1$  and  $y \in A_2$ , such that  $0 \leq y - x \leq \frac{e}{f(1)}$

Let us consider  $c$  and  $d$  two such elements.

From  $0 \leq c - b \leq \frac{e}{f(1)}$ , and  $f(1) > 0$ , we get:  $0 \leq cf(1) - bf(1) \leq e$ ;

as  $b \in A_1$ ,  $f(b) = bf(1)$ ; as  $c \in A_2$ ,  $f(c) = cf(1)$ ; then we have:  $0 \leq f(c) - f(b) \leq e$ . It follows that:

$$\forall \varepsilon \in \mathbb{R}^{+*} \exists w \in f(A_1) \exists z \in f(A_2), 0 \leq z - w \leq \varepsilon$$

This proves that:  $(f(A_1), f(A_2))$  is a cut of  $f(\mathbb{Q})$  (\*)

As  $\alpha$  is operating the cut  $(A_1, A_2)$ , and  $f$  is an increasing function, we have:

$$\forall x \in A_1 \forall y \in A_2 f(x) \leq f(\alpha) \leq f(y)$$

By definition of  $f(A_1)$  and  $f(A_2)$ , we have:  $\forall w \in f(A_1) \forall z \in f(A_2) w \leq f(\alpha) \leq z$

This proves that  $f(\alpha)$  operates the cut  $(f(A_1), f(A_2))$  (\*\*)

From (\*) and (\*\*) we conclude that  $(f(A_1), f(A_2))$  is a cut of  $f(\mathbb{Q})$  operated by  $f(\alpha)$ .

We now prove that  $\alpha f(1)$  is also operating the cut.

As  $A_1$  and  $A_2$  are subsets of the set of rational numbers, we have:

$$\forall x \in A_1 \forall y \in A_2 f(x) = xf(1) \wedge f(y) = yf(1)$$

Given  $d \in f(A_1)$ , and  $b \in A_1$  such that  $d = f(b)$  we have  $d = bf(1)$ , and given  $h \in f(A_2)$  and  $g \in A_2$  such that  $h = f(g)$  we have  $h = gf(1)$ .

As  $f(1) > 0$ , and  $b \leq \alpha \leq g$ , we have  $bf(1) \leq \alpha f(1) \leq gf(1)$ .

From which follows:  $d \leq \alpha f(1) \leq h$  and finally:

$$\forall z \in f(A_1) \forall w \in f(A_2) \quad z \leq \alpha f(1) \leq w.$$

This last assertion means that  $\alpha f(1)$  is operating the cut  $(f(A_1), f(A_2))$  (\*\*\*)

Thanks to the uniqueness of the real number operating the cut, we conclude that:

$f(\alpha) = \alpha f(1)$ , from which follows:  $\forall x \in \mathbb{R} f(x) = xf(1)$ , i.e.  $f$  is linear.

In these three proofs that *any increasing function solution of the functional equation (A) is a linear function*, the role of order is highlighted. In the second and third proofs, we refer only to properties related to order, without involving limits of sequences This is an illustration of the claim by Sinaceur that Dedekind reduced the continuity [of the set of real numbers] to order.

## **DIDACTIC IMPLICATION**

From the above, there are two main points of interest from our didactic perspective.

*The first* concerns the important and surprising result that there are totally discontinuous functions among the solutions of the functional equation (A). In university courses, when such functions are introduced, it is common for the professor to give examples that, for the students, seems to be constructed *for this purpose*, except for the Dirichlet function, the indicator function of  $\mathbb{Q}$  in  $\mathbb{R}$ , whose usefulness can be easily demonstrated. Such a presentation does not highlight the rationale for considering totally discontinuous functions, which might appear as pathological monsters, that should be relegated, as suggested by Lakatos (1976). However, our experience with the case of the Cauchy functional equation (B) whose continuous solutions are exponential functions, shows that this provides a rich opportunity to highlight the role of completeness/incompleteness, and allow graphical proofs to be questioned, justifying Bolzano and Dedekind's concerns that geometry-based proofs are not appropriate when moving on to analysis (Durand-Guerrier, 2022a). Starting with equation (A), instead of equation (B) could allow the emphasis to be placed on the topological properties, because the algebraic calculations are easier.

*The second* is that when solving the equation in the class of monotonic functions, the solutions are necessarily linear. In the French syllabus, this kind of results are seldom taken into consideration. This, together with the usual practice of working mostly in the set of real numbers, leaves in the shadow the role of order and the topological properties relevant to ordered sets (completeness/incompleteness; connectivity/non-connectivity; compactness/non-compactness, etc.). This is likely reinforced by the usual practice in first-year university courses of giving privilege to the following characterization of the Supremum:



$$(\forall x \in F, x \leq M) \wedge (\forall \varepsilon > 0 \exists x \in F, M - x < \varepsilon)$$

and the corresponding sequential characterisation, which favour a point of view linked with limits. Although these characterisations are useful in many cases, for some proofs it may be more efficient to use the definition of Supremum (resp. Infimum) as the minimum (resp. maximum), if any, of the upper bounds (resp. lower bounds). In Durand-Guerrier (2016) we report the case of Master students in a teacher training program in France working on a fixed-point theorem for an increasing function, who initially thought the continuity of the function in the interval  $[0,1]$  of domain  $\mathbb{R}$  was a necessary condition. Once they realised that this was not the case, they looked for a proof using the sequential characterisation; none of them search for a proof consisting in considering the supremum of a well-chosen subset as a candidate for a fixed point and proving that this is the case. This proof is efficient and holds as soon as we are in a complete lattice (Tarski, 1955). This is not to say that proofs using the sequential characterisation should be replaced by proof using the definition; rather, we consider that multiple proofs activities should be proposed and discussed with students at the secondary-tertiary transition and in teacher training programs as an efficient means of simultaneously increasing skills in proof and proving, as well as understanding of concepts. (Durand-Guerrier, 2022b). This is particularly important in the case of order which, as mentioned above, is a transversal notion at the interface of several areas of mathematics (e.g. Algebra, Geometry, Analysis, Combinatorics, etc.).

### **A FORTHCOMING EXPERIMENT WITH CAUCHY EQUATION (A)**

The second author of this paper has for years proposed activities based on functional equations, including the Cauchy equation (A). Naturalistic observations support the conjecture of their relevance to address some of the issues developed in the previous sections regarding order. The next step is to design an experiment around the Cauchy functional equation (A) to test our conjecture. The population we will consider for this experiment will be made of small groups of volunteer students following a teacher training program in different contexts (third year university, master's degree, preparation to the French *Agrégation*), leaving for other experiments the suitability for the secondary-tertiary transition. This choice is based on the hypotheses that working on the Klein's second transition for these students moving from university to secondary education shed light on the transition from secondary to tertiary education (Winsløw & Grønbaek, 2014). The experiment is planned in the spring fall 2024. We will follow the methodology of didactical engineering (Gonzales-Martin & al. 2014), with an initial open question as:

The goal is to solve the functional equation  $f(x + y) = f(x) + f(x)$  under various hypothesis on the domain and the property of the function. You are asked to formulate your hypotheses and to prove the assertions done under these hypotheses.

Our didactic organisation will comprise two sessions. The first will consist of a period of individual research followed first by a discussion in small groups and then by a group discussion. We will collect the questions and the answers produced by students,

both written and oral during this session. The second will be collaborative work in small groups, starting with a few questions that did not emerge during the first session to carry out specific work on the concepts of continuity, completeness, and monotony. We will also conduct interviews with students having participated at the two sessions.

## CONCLUSION

In this paper we provide motivations for studying functional equations as a means of highlighting the role of order in proof and proving at the interface between algebra and analysis. We show that even the simplest functional equation has unexpected solutions in the set of real numbers as soon as we look for solutions without assuming continuity of the functions, whereas the solutions in the set of integers or of rational numbers are exactly what we expect, i.e. linear functions. We then give three proofs, one using sequences, the two others using Dedekind's cuts, that solutions in the class of monotonic functions are linear. We consider that, from a didactic perspective, this highlights the relevance of introducing multiple proofs activities at university as a means of simultaneously developing proof and proving skills and an understanding of concepts involved. In the case of ordered sets, we consider that this could contribute to a better appropriation by undergraduate students of the general topological concepts that they will encounter later, and which are known to be difficult.

1. Our translation from French.
2. Our translation from French.
3. We will discuss below what happens if we do not impose to the function to be continuous.
4. Our translation in English from German: Jede dieser unstetigen Lösungen der Functional gleichung ist total unstetig; in jeder beliebigen Nähe eines jeden Punktes der  $(x, f)$  –Ebene liegen Punkte de "Kurve"  $f = f(x)$ .
5. Since the restriction of the function on the set of rational numbers is in all cases of the form  $g(x) = Kx$ , the point of coordinates  $(\alpha, g(\alpha))$  with  $\alpha$  rational are on the straight line with equation  $y = Kx$ , whatever the solution continuous or discontinuous.
6. In the original text, the author refers to this equation by (1). For being homogeneous along the text, we changed (1) in (A) everywhere.
7. For  $f$  an increasing function solution of the Cauchy functional equation (A),  $f(1) \geq 0$ .

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