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Structuralist praxeologies in the perspective of Klein: the case of connectedness in analysis

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This article is a continuation of our previous work on structuralist aspects of analysis at university in the framework of the Anthropological Theory of the Didactic. We now examine how the teaching of abstract structures (in particular metric spaces and topology) may contribute to Klein's perspective of "Elementary Mathematics from a higher standpoint". Precisely, on the basis of a real and abstract analysis textbook used in the transition between Bachelor and Master degree programs in mathematics and in the light of the notion of structuralist praxeology and its dialectics, we discuss whether we can defend teaching the notion of connectedness to teacher students as a means to link real and abstract analyses in the spirit of Klein.

Keywords: Teaching and learning of analysis and calculus; Transition to, across and from university mathematics; Curricular and institutional issues concerning the teaching of mathematics at university level; Structuralist praxeologies; Anthropological Theory of the Didactic.

INTRODUCTION

At the beginning of the last century, Felix Klein developed material for university lectures for teacher students in the form of three volumes *Elementary Mathematics from a higher standpoint*. Still today, his seminal ideas and methodological orientations continue to inspire mathematics education research (Weigand, McCallum, Menghini, Neubrand and Schubring, 2019). In particular, Klein posed the issue of the relation between school mathematics and academic mathematics, in other words, how a future teacher can be introduced to further advances in mathematics so that this knowledge is useful for his or her role as a secondary school teacher. A main general principle in this endeavour is to underline the mutual connections between problems in the various sub-disciplines of mathematics, offering a synthetic and holistic view on mathematics, and to emphasize relations with problems posed at school. This also concerns their mutual motivation and in particular addressing the rationale of school mathematics content beyond references that are currently emphasised as being immediately relevant to everyday life.

At Klein's time, modern mathematics had not yet taken off, but Klein was aware that a process of conceptual rewriting of mathematics was taking place in the natural historical development of the field, and that the fruits of this process should make it possible to modernise and invigorate mathematics teaching at all levels:

The normal process of development [...] of a science is the following: higher and more complicated parts become gradually more elementary, due to the increase in the capacity to understand the concepts and to the simplification of their exposition. It constitutes the

task of the school to verify, in view of the requirements of general education, whether the introduction of elementarised concepts into the syllabus is necessary or not. (Klein and Schimmack 1907, p. 90)

This discourse resonates with the didactical benefits of mathematical structuralism later put in the fore by Bourbaki in the Manifesto *The Architecture of Mathematics*. Concepts and structures are key ingredients of an integrated perspective on mathematics, and generality may clarify and simplify the exposition. We do not claim that Klein was structuralist in his pedagogy; on the contrary, he paid great attention to balance logic and intuition. Our perspective, in a context where mathematical structuralism has massively impacted university mathematics curricula, is to examine the extent to which the teaching and learning of abstract mathematical structures at university can contribute to the implementation of Klein's vision or depart from it.

We focus in this paper on the case of the relationship between real analysis, as it is taught at the secondary-tertiary transition, and abstract analysis (metric spaces, topology, Banach spaces, Hilbert spaces,...). In previous work (Hausberger and Hochmuth, 2023), we gave historical anchor points of the emergence of such a realm of structures that generalised real analysis and applied a model, initially developed in the framework of the Anthropological Theory of the Didactic (ATD) for abstract algebra, to account for the transition from real to abstract analysis throughout the Bachelor and Master degree programmes. We used this model to study excerpts of a textbook (Bridges, 1998), used at the Bachelor-Master transition and chosen for its didactic project: to make visible how the concepts and theorems of abstract analysis enlighten real analysis, which is first recapitulated in view of its generalisation. The main point was to detect the continuities and ruptures that might be observed in the shift towards abstraction that accompanied the rise of mathematical structures in analysis, or in our ATD terms the development of structuralist praxeologies. Moreover, the model puts in the fore a dialectic of objects and structures (see theoretical framework), in other words a dialectic of the particular and the general or of the concrete and the abstract that characterise structuralist thinking. The vitality of this dialectic was observed in relation to the issue of motivating abstract concepts.

This paper is a continuation of the previous work, but with a slightly different focus: we now consider relationships with school mathematics and teacher training, in the spirit of Klein. As a case study, we analyse from the viewpoint of structuralist praxeologies the tasks assigned by Bridges in his textbook around the topological notion of connectedness in metric spaces and in relation to real analysis. For example, Bridges generalises the intermediate value theorem (IVT) which is used as a main motivation for connectedness. Our main research question is the following: on the basis of this textbook and in the light of our analysis tools, can we defend teaching the notion of connectedness to teacher students as linking real analysis and abstract analysis in the spirit of Klein? We begin by presenting our theoretical framework, and then go on to analyse selected extracts from the textbook using these tools.

Finally, we draw conclusions in relation to the problem posed by Klein, and conclude by outlining a few extensions we envisage to this research.

THEORETICAL FRAMEWORK

According to ATD (Chevallard & Bosch, 2020), every human activity consists in the coordination of a praxis and a logos, hence the key notion of a praxeology, represented by a quadruple $[T/\tau/\theta/\Theta]$. Its practical-technical block (or know-how) consists of a type of tasks T together with a corresponding technique τ (useful to carry out the tasks $t \in T$). The technological-theoretical block (or know-why) comprises the technology θ , a discourse on the technique, and the theory Θ , the ultimate level of justification. We continue by presenting the tools developed specifically in ATD to analyse praxeologies based on mathematical structuralism. The reader may wish to consult our previous paper (Hausberger and Hochmuth, 2023) for connections with other works offering a praxeological modelling of mathematical practices in the analysis track at university.

The starting point is the consideration of mathematical structuralism as a method, which consists of reasoning in terms of classes of objects, relations between these classes and stability properties for operations on structures (Hausberger, 2018). The general view of structures thus allows particular properties of objects to be demonstrated by making them appear as consequences of more general facts (theorems about structures). Dually, generalisations are put to the test of objects, hence a *dialectical relationship between objects and structures*. In praxeological terms, the structuralist method consists in the passage from a praxeology $\mathbf{P} = [T/?/?/\Theta_{\text{particular}}]$ where it is unclear which technique to apply, to a structuralist praxeology $\mathbf{P}_s = [T^s/\tau/\theta/\Theta_{\text{structure}}]$ where, modulo generalisation of the type of tasks (T^s), the theory of a given type of structure guides the mathematician in solving the problem. It is important to point out that this transition (called type 1) leads to a structuralist praxeology whose rationale is related to its ability to solve concrete problems (related to \mathbf{P}) by a gain in technology permitted by the insight of structures.

Moreover, Hausberger (2018) distinguishes two structuralist levels of praxeologies: at level 1, structures act as a vocabulary and appear mainly through definitions (e.g. a task of type T “show that a given function between given metric spaces is bounded” is solved by checking that the definition of boundedness is satisfied); at level 2, the technique mobilises general results about structures (e.g. any continuous function from a compact space into a metric space is bounded). In the process of developing the level 2 contextualised (since the function and metric spaces are given) structuralist praxeology, an abstract task is assigned (prove the theorem) to establish its technology.

The latter task is of type T^s “show that any function between metric spaces that fulfils given conceptual properties is bounded”. At this stage, it remains more or less an isolated task. But in the teaching and learning of structures, the stage is reached when praxeologies based on such abstract types of tasks (that concern abstractly defined

classes of objects, e.g. generic functions between generic metric spaces) are developed. In this context, the key structuralist insight (for the preceding example) that compactness is preserved by continuous mappings comes in the fore together with other connections between the various concepts that are involved (continuity, closedness, boundedness, compactness). This is called the type 2 transition, situated between the Bachelor and Master degree programs. It is important to note that the new purely abstract praxis II' shall be anchored on reasoning with concepts that take their origin in the logos of previously developed structuralist praxeologies denoted P_s . This connection between the two types of transition is vital for a sound (properly motivated) development of abstract analysis.

One shall note that generic objects such as generic real functions already appear in early analysis courses in the context of abstract tasks of what we called pre-structuralist praxeologies (Laukert et al., 2023), but the properties of functions and their domain/co-domain (\mathbf{R} or \mathbf{R}^n) that play a role are not fully elucidated in terms of structures (topological concepts, metric spaces and functional analysis), hence the terminology pre-structuralist. In fact, analysis mixes different kinds of structures and some results in real analysis that closely intertwine different structuralist aspects may be hard to extend to natural structuralist statements or may lead to different general statements that capture some aspects of the initial problem while abstracting other aspects. Real analysis certainly constitutes a body of praxeologies that cannot be reduced to its structuralist dimensions developed and revealed through the abstract analysis.

To conclude this theoretical framework, let us emphasize key features of structuralist praxeologies that relate to the perspective of Klein. Our main research hypothesis is that smooth transitions of type 1 (and 2, to some extent) with a vitality of the dialectic of objects and structures allows to meet the vision of Klein since: i) structuralist concepts become tools to solve concrete problems ii) structures unify different branches of mathematics (e.g. geometry and analysis, by bringing geometrical insights into analysis through topology) iii) the conceptual perspective brings a new foundation to real analysis in terms of more general principles, which increases the understanding of the reasons why theorems hold true iv) the type 1 transition connects university to school mathematics. On the opposite, discontinuities in the type 1 transition hinders the realisation of Klein's perspective.

ANALYSIS OF THE TEXTBOOK EXCERPTS ON CONNECTEDNESS

The subchapter starts with a definition of connectedness: a metric space is called connected if it is not the union of two disjoint nonempty open subsets (p. 158). Since the family of open subsets gives the topology of the metric space, connectedness is a topological property. Connectedness formalizes the idea that a metric space, or a subspace of it, "cannot be split into smaller, separated parts". Since closed subsets are precisely the complementary sets of the open subsets, connectedness can analogously be characterized by the non-existence of two disjoint nonempty closed subsets whose

union gives the whole space (3.4.1 (ii)). The point of view that connectedness is a topological notion is further strengthened by presenting a characterization using continuous functions (i.e. functions whose preimage of an open set is an open set): a metric space is connected if and only if there is no continuous mapping from the space onto $\{0,1\}$, the typical disconnected subset (Exercise 3.4.2.2). It is also a first step in the direction of a generalisation of the intermediate value theorem (IVT) as it connects the concepts of connectedness and continuity of mappings.

Immediately afterwards the following question is posed: what are precisely the connected sets in \mathbf{R} ? This question indicates the type of task T_1 “determine connected subsets of a given metric space”. The proof for \mathbf{R} (Proposition 3.4.3) uses the intermediate value property (IVP) of an interval I , which we see as a pre-structuralist characterisation: I contains the segment $[c,d]$ for any two points $c < d$ in I . Intervals have been defined comprehensively as subsets of the form $|a,b|$ (open, closed or semi-open) and characterized by this property (Proposition 1.3.3) in the chapter on real analysis. The supremum axiom of \mathbf{R} is used, but not enlightened in structuralist terms (concept of completeness). The embryo of a praxeology that is being developed remains of structuralist level 1, since it uses mainly the definition of connectedness (and that of open/closed subsets).

Every result on connectedness, contextualised to \mathbf{R} , can thus be reduced to the case of intervals. This leads to question what the structure-oriented extension of connectedness in metric spaces may bring to real analysis. The answer remains unclear at this stage. The IVP involves the partial order in \mathbf{R} while connectedness applies to topological spaces in general, but the benefit of generality remains to be seen.

The following Exercise 3.4.4 establishes the type of tasks T_2 “prove that a subset of a metric space is connected”. In the assigned tasks, subsets are given abstractly by union/intersection of nonempty closed subspaces. The aim is obviously to prove a structuralist theorem to feed a level 2 contextualised structuralist praxeology based on the same type of tasks, but no example of such application is given. The next Proposition 3.4.5, posing the stability of connectedness under taking adherence points (S to \bar{S}), as well as Proposition 3.4.6 and exercises 3.4.7 1-2 fit with T_2 : again, the considered subsets are defined abstractly by other properties (e.g. $\bar{S} \cap T \neq \emptyset$ is non empty and S, T connected). So far, the course focuses on structuralist stability properties of connectedness. With regard to \mathbf{R} , however, the results are without specific gain: e.g., intervals obviously remain intervals by adding adherence points.

New notions are introduced and considered through exercises 3.4.7.3-6: chain connectedness, connected component, total disconnectedness (the connected component of a point is the singleton), local connectedness. There is again a task of type T_2 , resulting in the theorem that a compact, chain connected set is connected. Moreover, further properties of connected components are established, but without any contextualisation, except the following: the notions of connected components and

local connectedness are applied in Exercise 3.4.7.7 to give a conceptual proof of the description of open subsets of \mathbf{R} as unions of open intervals. This characterisation has already been proved in Proposition 1.3.6 of the real analysis chapter. Here, too, we cannot see any particular gain from the abstract treatment in metric spaces (except aesthetics to the eye of a structuralist mathematician, which is not a didactical criteria).

Then two tasks of type T_3 “prove that a subset of a metric space is totally disconnected” are assigned. In the case of \mathbf{R} , any countable subset like \mathbf{Q} turns out to be disconnected (first task), but also the set of irrational numbers (second task), which are known to be uncountable. One may note that the type of tasks T_2 may have been assigned, instead of T_3 , to both the rational and irrational numbers (viewed as metric spaces for the distance inherited from \mathbf{R}), in order to introduce the notion of totally disconnectedness from a bottom-up perspective.

The stability of connectedness under continuous mappings is expressed by Proposition 3.4.8. That the range of a continuous mapping from a connected metric space is connected is an immediate consequence of the topological definitions of connectedness and continuity. The consequence for continuous mappings from a connected metric space to \mathbf{R} is then addressed as the generalised IVT (figure 1) and is an important consequence of this structuralist principle: theorem 3.4.9 thus elucidates the structure of the domain of the function so that the IVP on the range holds, but the codomain remains \mathbf{R} without further structuralist insights. Also, the role played by completeness of \mathbf{R} remains somehow implicit in this result, as it was in the observation that intervals are the connected subsets in \mathbf{R} . Finally, this new theorem is not applied to more general contexts than that of the IVT itself (real numbers), where as a contextualised level 2 structuralist praxeology based on T_4 “prove that the range of a mapping has the IVP” could have been developed to assign a practical rationale to connectedness and the generalised IVT altogether.

A very important consequence of Proposition (3.4.8) is the following generalised *Intermediate Value Theorem*.

(3.4.9) Theorem. *Let f be a continuous mapping of a connected metric space X into \mathbf{R} , and a, b points of $f(X)$ such that $a < b$. Then for each $y \in (a, b)$ there exists $x \in X$ such that $f(x) = y$.*

Proof. By Propositions (3.4.8) and (3.4.3), $f(X)$ is an interval. The result follows immediately. \square

Figure 1: the generalised intermediate value theorem

Then a couple of exercises that draw general consequences from the continuity of the distance in metric spaces are posed (e.g. 3.4.10.1 about unbounded connected metric spaces), without hinting at the scope of these results and without further enlightening the real analysis context.

Proposition 3.4.11 connects to the notion of compactness and uniform convergence: whenever X is connected, the uniform continuity of every continuous real valued function on X is equivalent to the compactness of X . We see this result as emblematic

of the transition of type 2. Nevertheless, the uniform continuity of functions defined on compact, i.e. bounded and closed, intervals of \mathbf{R} , is a well-known result in real analysis. That bounded and closed are not only sufficient but also necessary is usually justified by counter-examples considering continuous functions on unbounded as well as on bounded and open intervals. The proposition enlightens the role of the concept of compactness in a more general context considering real-valued functions on connected metric spaces. It in particular shows in which sense compactness is necessary for uniform continuity to hold and establishes the real analysis result on a more general ground. Dually, the real analysis background serves as an anchor point for the rise towards abstraction, hence serves the transition. Unfortunately, these connections remain implicit in the textbook, they are not discussed.

In the end of the chapter, the notion of path-connectedness is considered in metric spaces. It generalises the idea that a subset does not consist of isolated separate parts in the sense that two points can be joined by a path lying wholly in the subset. Proposition 3.4.12 establishes that path-connectedness implies connectedness. Path-connectedness is a notion which is considered in multivariable real analysis contexts to also generalise the concept of interval in \mathbf{R} . Another, perhaps more straightforward possibility, would be convexity, where the choice of paths is restricted to straight lines. Convexity is stronger than path-connectedness. That convexity is not discussed as alternative can be interpreted as a symptom for the focus of the author on topological perspectives in his presentation and a lack of meta-discourse on the *raison d'être* of generalised notions in a view to Real Analysis needs. Regarding \mathbf{R}^n , Proposition 3.4.13 establishes that connected open subsets of \mathbf{R}^n are path-connected, i.e. connectedness and path-connectedness are equivalent for open sets in \mathbf{R}^n . The obvious question about the case of closed connected subsets of \mathbf{R}^n is not raised (a negative answer). Thus, with a view on a real analysis context, the difference between connectedness and path-connectedness remains weakly clarified.

Altogether, the praxeology based on the type of tasks T_5 “Prove that a subset of a metric space is path-connected (or not)” is only weakly developed. Obviously, neither T_2 nor T_5 are relevant in \mathbf{R} , since the connected subsets in \mathbf{R} are the intervals and intervals are trivially path-connected. Are the types of tasks T_2 and T_5 relevant when contextualised to \mathbf{R}^n i.e. in the multivariable real analysis context? Exercises 3.4.16.1 contains the only example of subsets in \mathbf{R}^n whose path-connectedness is questioned.

The set $B = \{(x, y) \in \mathbf{R}^2 : 0 < x \leq 1, y = \sin \frac{\pi}{x}\}$, i.e. the graph of a function defined on a semi-open interval, and the set $A = \{(0, y) \in \mathbf{R}^2 : -1 \leq y \leq 1\}$, i.e. the set of adherence points of the graph, are considered. Then $A \cup B$ is not path-connected. We may infer that, in the structuralist perspective of the author, this task of type T_5 is assigned in order to present a counter-example concerning path-connectedness of closed connected subsets and stability of path-connectedness under taking adherence points.

At this point, we asked ourselves why the following relationship to real analysis, which also appears relevant for school mathematics, was not established: at school,

the continuity of a function is typically explained by the fact that the graph of the function can be drawn “without lifting the pen”. This descriptive property can be formally interpreted as the path-connectedness of the graph. This is supported by the observation/theorem that a real-valued function defined on \mathbf{R} is continuous if and only if its graph is path-connected in \mathbf{R}^2 .

Finally exercises 3.4.16 2-3 question the stability of path-connectedness under union and intersection, which holds true under some extra conditions. Again, these are structuralist aspects whose practical usefulness is not underlined through contextualised level 2 structuralist praxeologies.

DISCUSSION

To summarise, we observed that the definition of the concepts and their embedding in the theory of metric spaces, as well as resulting new concept-based proofs, appear to be poorly motivated, despite the relationship with the intermediate value theorem (IVT). A contributing factor is that the scope of structuralist praxeologies on the topic of connectedness is scarcely or not developed at all and the types of tasks remain fairly limited in number and diversity. In particular, level 2 contextualised structuralist praxeologies are missing. As a result, a *raison d'être* cannot come to life. In addition, the role played by other properties of real numbers (such as completeness or the ordered field structure) is not discussed in relation to the IVT. Only connectedness is elucidated. This is another reason why the results in the chapter under consideration tend to remain isolated. Large parts focus on theoretical development (the type 2 transition), which for the reasons mentioned has little explicit connections to the type 1 transition. These observations, formulated in praxeological terms with a view to structuralist aspects and their transitions, have consequences on Klein's project that we will now underline.

The starting point of the contribution was the question of the educational benefits to teach student teachers connectedness in metric spaces. Criteria for an answer emerged from connections between Klein's project and the point of view of structuralist praxeologies, mainly the idea of smooth structuralist transitions as developed in the theoretical framework. From our analysis of a typical textbook, our position is divided. We begin by the drawbacks, which relate to inadequate didactic means to achieve the type 1 transition on the topic of connectedness.

- a) Basically, the concept of connectedness remains weakly motivated by considerations on the real numbers. Starting from intervals and their properties, other possibilities to generalise their features could have been reflected in the transition to \mathbf{R}^n , such as convexity or path-connectedness. To decide between these different alternatives, the question of the stability of properties under a continuous mapping could have been asked, in the structuralist spirit. In particular, this structuralist behaviour is the key point to tackle the following issue: which alternative leads to a generalisation of the intermediate value theorem?

- b) The textbook primarily presents conceptual proofs whose potential to tackle interesting questions in the perspective of Klein's project is still doubtful. The general concepts appear to be motivated by the goal of generalisation itself, but are hardly anchored in real analysis issues (other than the IVT). For example: Exercise 3.4.7.7 (determination of the open sets of the real line) mobilises a conceptual proof of a connection already established in real analysis, which may be interesting for pure mathematicians and may be satisfying in terms of aesthetic value, but the added practical value for student teachers remains at least unclear and is actually not worked out.
- c) With regard to the concept of connectedness, the text to a certain extent constitutes something that remains stuck between two worlds: on the one hand, a (relatively) concrete world of real analysis based on real numbers as school objects and already considered in terms of axioms, and on the other hand a (relatively) abstract world of metric spaces. A dialectic between concrete and abstract (in our framework, objects and structures) is thus not brought to life.

Nevertheless, the text provides starting points for a depiction of connectedness that could contribute to Klein's agenda. What are such starting points?

- a) Connectedness in \mathbf{Q} could have been investigated: are the connected sets there also the intervals? Unlike the textbook, the definition of a totally disconnected subset would have arisen as a concluding step of this investigation and not as a starting point.
- b) Similarly, path-connectedness may have been related to the process of formalisation of the intuitive idea that the curve of a continuous function can be drawn without lifting the pen: continuous real functions are characterised by the path-connectedness of their graph. Exercise 3.4.16.1 of the textbook further elaborates on this idea to construct a counter-example to structuralist assertions without making explicit this connection with the intuitive notion of graph from school mathematics.
- c) The question we formulated in point a) of the drawbacks could have been investigated to motivate the notion of connectedness. The idea that a missing point in the interior of an interval decomposes it into two disjoint closed subsets may serve as an argument to introduce the definition, among other arguments. On a meta level, the following issue needs to be addressed to implement didactical aspects of this investigation: what degree of generalisation beyond metric spaces would be adequate to foster the development of helpful structuralist insights among teacher students? In particular, to what extent should general topology be developed?

Although the textbook attempted to elaborate abstract analysis from real analysis in a bottom-up perspective, the overarching viewpoint of structures is dominating the presentation. In our opinion, fulfilling Klein's project would require a different kind of textbook that more successfully implements a dialectical point of view between real and abstract analysis, in other words between objects and structures.

GENERAL CONCLUSION AND OUTLOOK

The notion of structuralist praxeology in ATD, with its structuralist levels, its two transitions and the objects-structures dialectic, has made it possible to study Klein's didactic problematic by virtue of the continuities and ruptures that it highlights in the passage from real analysis to abstract analysis. Both the research on structuralist praxeologies and that on Klein's pedagogical programme are inscribed in a questioning of transitions (Klein's double discontinuity for the former and transitions within university for the latter).

The use of the notion of structuralist praxeology is not limited to the analysis of what already exists, but also makes it possible to engage in didactic design with the aim of developing structuralist praxeologies useful to future teachers. In this endeavour, particular attention should be paid to the objects-structures dialectic and the process of questioning objects (e.g. continuous real functions and their properties) in the light of structures (Hausberger, 2019) may be envisaged in the didactic perspective of Questioning the World (Chevallard & Bosch, 2020) in ATD. This is one of the main avenues for the development of this work, as our analysis has shown punctual weaknesses of a textbook while also identifying avenues for the implementation of Klein's ideas, around the notion of connectedness as an emblematic case.

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