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## ► To cite this version:

Ryoto Hakamata, Masayoshi Yoshikawa, Toshihiko Ogawa. Raison d'être of mathematical works and epistemological responsibility in inquiry: the case of the Fibonacci sequence under various moduli. Fifth conference of the International Network for Didactic Research in University Mathematics, Centre de Recerca Matemàtica [CRM], Jun 2024, Barcelona, Spain. <hal-04944234>

**HAL Id: hal-04944234**

**<https://hal.science/hal-04944234v1>**

Submitted on 13 Feb 2025

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# **Raison d'être of mathematical works and epistemological responsibility in inquiry: the case of the Fibonacci sequence under various moduli**

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*This study explores raison d'être of mathematical works and epistemological responsibility of students in university mathematics education, by analyzing the case of inquiry into the Fibonacci sequence under various moduli. Employing the anthropological theory of the didactic, it examines how the students construct the milieu through the inquiry. As a result, the mathematical works were learned in a manner different from the commonly disseminated teaching style. However, while the students were given some responsibility for constructing the milieu, the teacher often determined the necessary mathematical works, indicating challenges in fully delegating epistemological responsibility to students. The findings suggest the possibilities for inquiry-based approaches in university mathematics education.*

*Keywords: Teachers' and students' practices at university level, transition to, across and from university mathematics, anthropological theory of the didactic, didactic paradigm, milieu construction.*

## **INTRODUCTION: THEORETICAL BACKGROUND**

Research on university mathematics education has paid significant attention to the problem of transition, namely, discontinuity between secondary-level and tertiary-level mathematics, particularly in the context of teacher education (e.g., Gueudet et al., 2016). Our study also addresses this issue based on the *anthropological theory of the didactic* (ATD). The reason for adopting ATD is that, as will be detailed below, it helps to identify the research problem in university mathematics education related to the transition, and provides approaches to address the issue. Over the past decade, research within ATD has used the concept of the *didactic paradigm* to clarify the problematic (e.g., Bosch et al., 2018). These studies have referred to two typical didactic paradigms: the *paradigm of visiting works*, which has been widely disseminated, and its counterpart, the *paradigm of questioning the world*.

The core issue noted with the paradigm of visiting works is the disappearance of the *raison d'être* of the objects (to be) learned and the reduction of the *epistemological responsibility* of students. In this paradigm, a curriculum is composed of a sequence of mathematical works constrained by various factors. Students learn such works as if visiting monuments, treating them as valuable in themselves. As a result, these works have lost their function within the original body of knowledge. ATD refers to this as the disappearance of the *raison d'être* of the works (cf. Chevallard, 2015). Besides this, students are considered to be taking on epistemological responsibility when they are deciding, on their own initiative, how to construct the necessary environment to solve

problems or answer questions. In the traditional paradigm, students merely follow teacher's guidance in visiting works, and thus have little epistemological responsibility.

The paradigm of questioning the world has been proposed as an alternative. In this paradigm, learning principally occurs through *inquiry*. ATD defines inquiry in a broad sense, as a process in which students, with teachers' support, generate their own answers to an initial question. These questions and answers involve not just a single simple question and its corresponding answer, but rather a chain of derivative questions and partial answers. In an inquiry process, students may study works from books or the Internet (called *media* in ATD), or obtain data through experiments and simulations. ATD refers to a set of derivative questions, partial answers, works, and data as (didactic) *milieu*. Students continue to construct their milieu and, at the same time, interact with it to generate the final answer. Such a process is conceptualized as the *study and research path* (SRP). This concept emphasizes the dialectical development of the two activities of study and research in inquiry.

Milieu is a key concept for considering *raisons d'être* of works and the epistemological responsibility of students. When construction of the milieu is undertaken with the purpose of "answering the question," the works learned in the inquiry can be said to have a *raison d'être*. Furthermore, when students are responsible for selecting and deciding how to construct the milieu, such as choosing which media to reference or which existing answers to apply, they are considered to be assuming the epistemological responsibility for the final answer. Therefore, delegating the responsibility for milieu construction to students within inquiry appears to be a promising approach to addressing the issue raised by the paradigm of visiting works.

### **Research question**

Our research targets *book-club-style* mathematics seminars that are typical in Japanese university mathematics education. In this format, a small group of students reads a book together with a teacher. Typically, the book to be read is preselected by the teacher at the time the seminar is designed, and it is customary to read the book sequentially from the first page. Textbooks in mathematics are logically organized in a sequence where many definitions and propositions are introduced because they are needed for proving theorems that appear later. This implies that the students often find the *raison d'être* of these works to be unclear and have low epistemological responsibility towards the constructed knowledge.

This paper discusses a case where a teacher, who had traditionally used this style, redesigned the seminar as an SRP to overcome these issues. The SRP was implemented at a Japanese national teacher training university for two years. It formed part of a seminar for third-year undergraduate students specializing in mathematics education. This SRP was initially planned within the area of elementary number theory, and the topic chosen was the Fibonacci sequence under various moduli. The initial question of the inquiry was formulated as follows: *What mathematical properties hold for the Fibonacci sequence under various moduli?* The aim of our study is to describe and

analyse, from the perspective of milieu construction, the *raison d'être* of the works and the students' epistemological responsibilities in the SRP. This serves to demonstrate the potential and possibilities of incorporating inquiry into university mathematics education. From this, the following research question emerges:

RQ: In the context of a university mathematics seminar focused on the Fibonacci sequence under various moduli, how and for what reasons is a milieu constructed, and how is the responsibility for its construction distributed among members in the SRP?

## **METHODOLOGY**

### **Didactic engineering**

This research adopts the methodology of *didactic engineering* (Artigue, 2020). In the process of didactic engineering, an *a priori analysis* describes how the SRP can develop from the initial question. It is common to represent the development of the SRP using a tree structure known as a *Q-A map* (Winsløw et al., 2013). In addition to this, in this paper, we also consider mathematical works that can become a component of the milieu. A priori analysis allows us to predict the possible development of an SRP prior to its implementation, enabling us to contemplate what occurred and what did not in the actual SRP, as well as its characteristics. Then, *in vivo* analysis is conducted. The development of the implemented SRP is described as the actualized Q-A map. This analysis is conducted continuously over the long-term implementation period of the SRP. A posteriori analysis involves the consideration of the characteristic phenomena. In this study, a posteriori analysis can be seen as equivalent to the "Discussion" section in a general research paper. By comparing the prediction made in the a priori analysis with the reality of the implemented SRP, it is possible to highlight the characteristics of the results. Furthermore, the a posteriori analysis includes the perspective of *ecological analysis* in ATD. This involves examining the factors that led to such characteristics, i.e., the conditions and constraints.

### **Participants and procedures for data collection and analysis**

This paper focuses on a two-year SRP conducted during a mathematics seminar for two third-year undergraduate students at a national teacher training university. At the end of the second year, the students presented about what they had studied to their supervisor and another professor in the department. The analysis covers a total of 30 periods of teaching in the first year, 13 periods in the spring semester and 17 periods in the autumn semester (each period of 90 minutes).

One of the two students aspired to be a primary school teacher, while the other aspired to be a high school mathematics teacher. Before starting this SRP, the students had taken two algebra courses that covered mathematical knowledge such as properties of integers (e.g., Euclidean Algorithm, prime factorization), modular arithmetic (e.g., residue classes, the Chinese Remainder Theorem), and matrices. However, they did not understand them well enough to apply their knowledge to this inquiry. The teacher (the

second author) is a mathematics researcher specializing in algebraic combinatorics. In his previous teaching experience, he conducted seminars by book-club-style. This was the first time he conducted a seminar in the form of an inquiry.

In this SRP, the students worked on their inquiry basically in pairs. The teacher intervened in their inquiry as needed, for example, by providing questions or presenting the media needed for their inquiry. Moreover, although they worked on inquiry primarily in the seminars, the teacher often assigned homework as needed, and they worked on inquiry individually outside the seminars.

This SRP was initially planned within the area of elementary number theory and the topic chosen was the Fibonacci sequence under various moduli. The initial question  $Q_0$ , was “What mathematical properties hold for the Fibonacci sequence under various moduli?” This SRP was conducted over a total of 20 periods, spanning the latter part of the spring semester (3 periods) and the entire autumn semester (17 periods). During the first 10 periods of the spring semester, a preliminary investigation into the mathematical properties of the Fibonacci sequence was conducted.

Data were collected by recording the dialogs of the students and the teacher. And, the first and third authors also observed the seminars and collected memoranda of the seminar observations, photographs of the blackboard, and students’ notebooks. Based on these data, we describe how the inquiry was actually addressed.

### **A PRIORI ANALYSIS: A POSSIBLE DEVELOPMENT OF THE SRP**

At the start of the inquiry, a typical question derived from  $Q_0$  could be  $Q_1$ : “Is the Fibonacci sequence modulo any integer  $n \geq 2$  always periodic?” Concrete data from the Fibonacci sequences modulo some integers obtained through manual calculations, spreadsheet software, or programming will play an important role. It is expected that concepts related to the pigeonhole principle and elementary number theory such as congruences, will be used in response to  $Q_1$ . By learning and applying such mathematical works as needed, a positive response can be obtained; that is, the answer  $A_1$ : “It is always periodic.”

Subsequently, more detailed properties related to periodicity can be explored. For instance, considering the Fibonacci sequence modulo  $n$  and denoting the length of its period as  $\pi(n)$ , a question such as “Is it true that  $n \neq 2 \Rightarrow \pi(n)$  is even?” ( $Q_{2.1}$ ) can be formulated. This question can be considered as a specific example of a broader question,  $Q_2$ , which asks about the patterns in the lengths of the periods. Further questions, like “Does  $\pi(p)$  have any particular properties or patterns for primes  $p$ ?” ( $Q_{2.2}$ ), “Is there a relationship between  $\pi(p^k)$  and  $\pi(p)$ ?” ( $Q_{2.3}$ ), or “Is there a relationship between  $\pi(p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l})$  and  $\pi(p_i^{e_i})$ ?” can be derived.

The answers to the questions  $Q_{2.1}$  to  $Q_{2.4}$  can be summarized as follows. First,  $Q_{2.1}$ , formulated as a conjecture, is positively resolved using mathematical induction ( $A_{2.1}$ ).  $Q_{2.2}$  focuses on the periods of the Fibonacci sequences modulo primes. For this question, properties like “for a prime  $p$  in the form  $5p \pm 1$ ,  $\pi(p) | p - 1$ ” or “for a

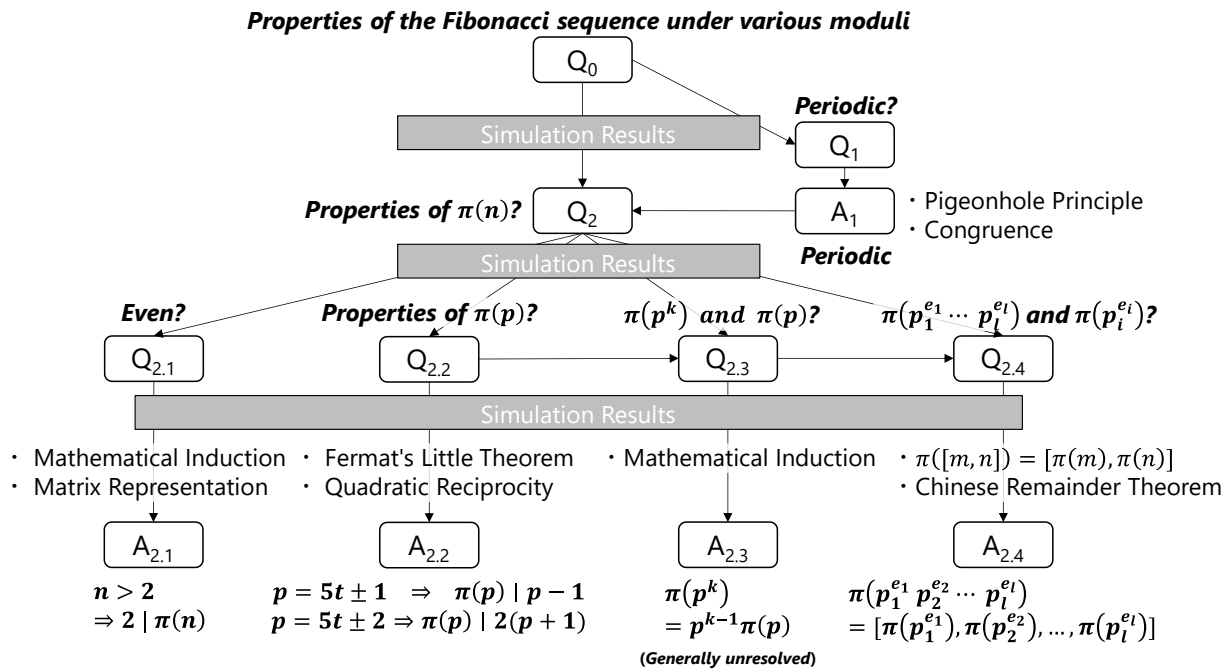
prime  $p$  in the form  $5p \pm 2$ ,  $\pi(p) | 2(p + 1)$ ” are possible answers ( $A_{2.2}$ ). Uncovering these properties likely requires some familiarity with elementary number theory. This is because one must not only focus on primes, but also classify these primes according to the residues modulo 5 and pay attention to the divisibility by the length of the period  $\pi(p)$ . These properties can be proven using Fermat’s Little Theorem and the Law of Quadratic Reciprocity. While tackling  $Q_{2.2}$  may appear challenging due to the need for some works in elementary number theory, the inquiry from  $Q_{2.2}$  to  $A_{2.2}$  becomes plausible considering references to media. Students can engage with some webpages, applying their works to their own milieu, advancing their inquiry while constructing knowledge.

While  $Q_{2.3}$  may derive from  $Q_{2.2}$ , typically, it would be deduced by focusing on periods for prime powers such as  $\pi(2^k)$  or  $\pi(3^k)$ , based on the generated data. It is anticipated that  $Q_{2.3}$  will spontaneously be engaged with ease. This is because the students are expected to be familiar with focusing on prime powers. As an answer to this question,  $A_{2.3}$ , it can be considered that  $\pi(p^k) = p^{k-1}\pi(p)$ . In fact, this conjecture remains an unsolved problem, and no general proof has been presented yet. However, this does not mean that students cannot tackle it in their inquiry; it is conceivable that they could construct proofs for some specific prime numbers.

Question  $Q_{2.4}$ , based on the uniqueness of prime factorization, is a natural question to consider in the inquiry, and the exploration of such questions should be encouraged. An answer  $A_{2.4}$  to this question could be expressed as follows:  $\pi(p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}) = [\pi(p_1^{e_1}), \pi(p_2^{e_2}), \dots, \pi(p_l^{e_l})]$ , where the square brackets mean the least common multiple of several numbers. In fact, more generally, for any two natural numbers  $m$  and  $n$ ,  $\pi([m, n]) = [\pi(m), \pi(n)]$  holds. This is something that students can discover through interactions with the milieu.

The proof of this property requires works of elementary number theory, such as the Chinese Remainder Theorem. The students under consideration in this paper had previously taken lectures on elementary number theory. Therefore, it is expected that they can advance their inquiry by applying known works as one of the elements within the milieu. However, there may be situations where known works cannot be effectively applied, and in such cases, it is anticipated that some form of media reference, such as learning through Wikipedia’s “Chinese Remainder Theorem” article or a textbook on elementary number theory, may occur.

The above is an a priori analysis of the expected inquiry process. When summarized as a Q-A map, it appears as shown in Figure 1. Each question is labelled briefly to indicate its content, and the typical components of the milieu that will be constructed during the inquiry process are also included. In particular, the data obtained as a result of simulations are expected to have an interactive role, contributing to various questions and answers, rather than serving a one-time purpose.



**Figure 1: The Q-A map for the possible process of the inquiry**

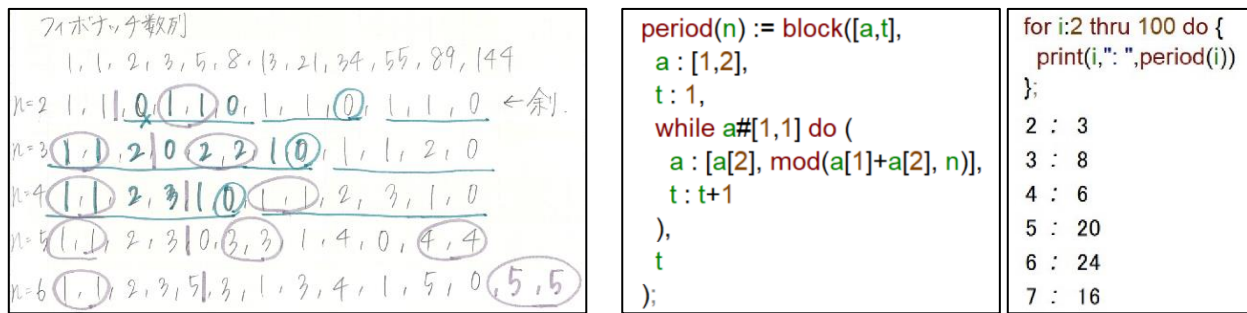
In the following sections, we will describe the realised SRP and analyse its characteristics through a comparison with the results of the a priori analysis.

## DESCRIPTION OF THE SRP IMPLEMENTED

### Phase 1: Emergence of Questions Related to Q<sub>2.4</sub> and Q<sub>2.1</sub>

Initially, the teacher introduced the Fibonacci sequence under some moduli and presented the question Q<sub>0</sub>: “What mathematical properties hold for the Fibonacci sequence under various moduli?” Over four periods, mathematically formulated conjectures were gradually developed.

The students first manually calculated and recorded the Fibonacci sequences modulo some integers (the left side of Figure 2). They quickly noticed their periodicity. This observation led to the conjecture Q<sub>1</sub>, related to periodicity. The teacher engaged the students in discussions about the structure of the recurrence relations and the pigeonhole principle, sharing ideas for their proofs (A<sub>1</sub>). The students then attempted to use spreadsheet software (Excel) for calculating the Fibonacci sequences modulo more integers, aiming to gather more data. However, they realized that with large numbers, the expression of the calculation result became insufficient in spreadsheet software. Consequently, the teacher suggested using the computer algebra system Maxima for simulation. Subsequently, they calculated and outputted the lengths of the periods for integers from 2 to approximately 100, and investigated their characteristics. The right side of Figure 2 shows a part of the code and output from the program actually used.



**Figure 2: The Fibonacci sequences modulo some integers manually calculated (left side) and the length of the periods calculated by Maxima (right side).**

An agreement was reached to explore the following two conjectures, which the students derived from their simulation results:  $\alpha) n|m \Rightarrow \pi(n)|\pi(m)$ ,  $\beta) n \geq 3 \Rightarrow 2|\pi(n)$ . Attempts were made to search for related information on the internet, but no relevant Japanese literature was found. Searching with terms like “Fibonacci period modulo” led to Marc Renault’s website, and from there, his master’s thesis on the Fibonacci sequence under various moduli was discovered and downloaded (Renault, 1996). This thesis contained sections related to the above conjectures. The teacher introduced a theorem related to Conjecture  $\alpha$ , formalized as  $\pi([m, n]) = [\pi(m), \pi(n)]$ . However, it was not introduced as an already established theorem. The teacher made students create some specific examples to help them realize that this general claim seems plausible. Following this, the students took on this claim as a conjecture and set out to prove it. We will refer this conjecture as  $\alpha'$ .

At this point,  $\alpha'$  and  $\beta$  concerning  $Q_{2.4}$  and  $Q_{2.1}$  respectively were conjectured. The subsequent inquiry would proceed while reading the master’s thesis as the main media.

### Phase 2: Inquiry into $Q_{2.4}$ and $Q_{2.1}$

The proof of  $\pi([m, n]) = [\pi(m), \pi(n)]$  was constructed while reading and understanding the thesis. In the seminar, the students explained the proof based on the thesis, and the teacher asked questions and provided clarifications. The following fact is notable at this stage: rather than reading the master’s thesis from beginning to end sequentially, the students and the teacher often read the thesis retrospectively while searching for mathematical works necessary for the proof of Conjecture  $\alpha'$ . The mathematical works, such as other theorems mentioned in the thesis and the Chinese Remainder Theorem, were not pre-learned, but became elements of the milieu as needed. Through this process, the students found that the equation  $\pi(p_1^{e_1} p_2^{e_2} \dots p_l^{e_l}) = [\pi(p_1^{e_1}), \pi(p_2^{e_2}), \dots, \pi(p_l^{e_l})]$  holds. This equation was what we had presented as the answer  $A_{2.4}$  to  $Q_{2.4}$  in the a priori analysis. That is to say, in the actual SRP, contrary to our anticipated sequence,  $Q_{2.4}$  was resolved before other questions (such as  $Q_{2.1}$ ).

During the process of constructing the proof of Conjecture  $\alpha'$ , it was discovered that the proposition formulated in Conjecture  $\beta$ , “ $n \geq 3 \Rightarrow 2|\pi(n)$ ”, was also proved as a theorem within the same thesis. Following this discovery, the proof of  $\beta$  was constructed.

### Phase 3: Inquiry into Q<sub>2.3</sub>

The final phase is Q<sub>2.3</sub>, namely, the inquiry into the equation  $\pi(p^k) = p^{k-1}\pi(p)$ . As mentioned above, this is an unsolved problem, and a general proof has not yet been discovered. The teacher developed activities by starting with having the students construct specific examples, eventually formalizing the conjecture. Subsequently, a proof was constructed for the case of  $p = 2$ . Since the proof was included in the same thesis, students followed that proof. It is noteworthy that, even here, the way of tracing back the works necessary for the proof continued. For example, properties of the Fibonacci sequence such as  $F_{2n} = F_n(F_{n-1} + F_{n+1})$ ,  $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$ , etc., were learned and utilized in the proof for the case of  $p = 2$ .

### DISCUSSION: A POSTERIORI ANALYSIS

#### Construction of the milieu with *raison d'être* of the mathematical works

One of the characteristics of the SRP is the way in which the milieu was constructed, particularly the order of it, which differs from that of a typical seminar-style mathematics lesson. The SRP may superficially appear to be similar to a mathematics seminar in a book-club format, because one particular medium—the master's thesis in our case—held a privileged position and the students read it with the support of their teacher. However, in this SRP, the mathematical works were frequently employed in a sequence opposite to that in the traditional style. The students read the master's thesis by going back to the necessary sections and learned the works as needed. A typical example of this characteristic is the properties of the Fibonacci sequence used in Phase 3. These properties could be presented as valuable in their own right. However, in this SRP, they were learned and utilized because they were necessary for the inquiry. This indicates that the mathematical works learned had a clear function in answering the questions of the inquiry, that is, they had the *raison d'être*.

The way in which the milieu was constructed in this case is different from the knowledge development that Barquero et al., (2013) described as an *applicationism* phenomenon. Under the epistemology of applicationism, mathematically fundamental works should be learned before being applied to extra-mathematical situations. Although the authors used this term in the context of applied mathematics, we can expand its scope to encompass all of mathematics. Namely, applicationism can be understood as the view that any work should be learned before being applied in some situations. It can be said that many logically-organized mathematics textbooks are based on this epistemology, and thus, the problematic phenomena caused by applicationism, namely, the dilution of *raison d'être* of mathematical works, can fully occur within an inner-mathematical context as well. The above suggests that redesigning university mathematics courses through SRPs will be a promising approach to addressing the issue of transition, even within the context of pure mathematics.

## **Construction of the milieu under the epistemological responsibility of students**

Another characteristic of this SRP is that a part of the responsibility for constructing the milieu was devolved to the students. A typical instance was the teacher asking students at each juncture of the inquiry, “What kind of question would you like to consider next?” The two questions  $Q_{2.1}$  and  $Q_{2.4}$ , formulated in Phase 1, were based on several conjectures proposed by the students. Furthermore, at the end of Phase 2, a similar question was asked, leading to the proof of a theorem on prime powers. The way in which questions appeared and the order in which they were tackled were different from what was shown in the a priori analysis, which is an indication that some responsibility was devolved to the students. On the other hand, there was not much autonomous reference to media by the students. This suggests that another part of the responsibility for constructing the milieu was monopolized by the teacher. Specifically, this refers to the responsibility of determining at certain points in the inquiry which mathematical works might be necessary and/or which might not be useful. The teacher occasionally instructed, “Try reading this part of the thesis”, and the students followed these instructions to learn the works or follow the proofs. Then, the students did not seek out additional information that might be useful for the inquiry beyond the section.

This implies that there was little occurrence of *adidactisation* (Chevallard & Strømskag, 2022). When the milieu is described as adidactic for students, it means that the students are *not* trying to figure out the teacher’s didactic intentions behind the milieu. As this definition shows, adidactisation is crucial in delegating epistemological responsibility of inquiry to students. The students, in the SRP, largely relied on the teacher to decide where to find the elements of the milieu. In other words, the milieu was constructed through the students’ assumption that “it must be necessary for our inquiry because the teacher has presented it at this moment.” In this sense, the milieu was not adidactized for them. Therefore, in terms of media usage, much of the epistemological responsibility lay with the teacher, not the students.

One of the constraints on the adidactisation would be the existing epistemology that the students have spontaneously formed through their mathematics learning under the paradigm of visiting works. Under this paradigm, media to be referenced are predetermined, and furthermore, the timing of their referencing is also planned in advance. Such learning experiences are likely to lead students to the view regarding knowledge construction that “media and milieu are presented by the teacher when needed.” To update such an epistemology, long-term new mathematical experiences across both secondary and higher levels of mathematics are necessary.

## **CONCLUSION**

This study examined how the milieu was constructed and how the responsibility for its construction was distributed among members in a university mathematics seminar framed as an SRP. The analysis revealed two key characteristics of the SRP. First, the mathematical works were learned and utilized in a sequence that differed from the typical seminar in a book-club style. This suggests that the SRP can help address the

issue of the disappearance of the *raison d'être* of mathematical works, which is a common problem in traditional university mathematics education. Second, while the students were given some responsibility to formulate questions and direct the inquiry, their agency in constructing the milieu was somewhat constrained. This indicates that fully delegating the epistemological responsibility to students is challenging, likely due to their existing epistemological views shaped by past experiences in the paradigm of visiting works. The findings suggest that incorporating inquiry-based approaches like the SRP into university mathematics education holds promise for addressing the transition issue. However, more research is needed to further understand how to foster students' autonomy in constructing the milieu and taking on full epistemological responsibility within such inquiry processes. Longitudinal studies investigating how students' mathematical epistemologies evolve through experiencing diverse inquiry-oriented activities across secondary and tertiary levels would be particularly valuable.

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