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An experimentation to address Klein's discontinuity in the case of the integral

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In this article, we present an experiment based on the modeling of the second Klein discontinuity developed by Carl Winsløw within the framework of the anthropological theory of the Didactic. This experiment was conducted in the institutional context of secondary teacher training in France, involving a population of students holding a bachelor's degree in mathematics (L3). The case study focuses on the connections between the integral taught in high school in France, introduced as the area under the curve, and its relationship with the Riemann integral and measure theory taught at the university.

Keywords: Teaching and learning of analysis and calculus; Transition to, across and from university mathematics; Klein second discontinuity; Integral.

INTRODUCTION

The problem of the second discontinuity was formalized as early as 1908 by Felix Klein. This second discontinuity occurs when a student leaves university to become a secondary school mathematics teacher, while the first discontinuity happens upon entering university. To address this issue, Klein set out to present mathematics in a series of books based on three principles: emphasizing connections between mathematical domains, demonstrating how academic mathematics relates to school mathematics, and highlighting the links between mathematics and real-world applications. These three principles constitute his so-called Plan B for mathematics education.

The second discontinuity appears to persist today (Wasserman et al., 2018), and students still struggle to perceive the connections between university-level mathematics and the mathematics to be taught in secondary education. Recent empirical results (Hoth et al., 2020) illustrate that the transfer of knowledge from academic to school mathematics is not automatic.

What mathematical knowledge is useful for a future teacher? What types of connections need to be developed and strengthened between university-level mathematics and school mathematics in training programs to promote the professional development of teachers? These are ongoing debates that currently make the second discontinuity of Klein a vibrant question in mathematics education research and a significant challenge for the teaching and training profession. In this direction, new tools have recently been introduced (Winsløw and Grønbaek, 2014; Winsløw 2020) by addressing the issue of Klein's second discontinuity with the Anthropological Theory

of Didactics (ATD; Chevallard and Bosch, 2020). Winsløw employs the concept of an individual's relation to an object of knowledge within an institution, using the Anthropological Theory of the Didactic. He distinguishes between high school (*HS*) and university (*U*) institutions, as well as three different institutional positions: high school student (*s*), university student (σ), and high school teacher (*t*). An object of knowledge (in the case of this article, taking the integral as an example), which exists across both institutions, is denoted as "*o*" in high school and as " ω " when it pertains to a theory of integration (Riemann or Lebesgue, linked to the general theory of measure) taught at the university. Winsløw (2014) then proposes the following modeling of discontinuities:

$$R_{HS}(s, o) \rightarrow R_U(\sigma, \omega) \rightarrow R_{HS}(t, o)$$

where where $R_U(\sigma, \omega)$ reads “the relation of a university student sigma to the object of knowledge omega within the institution “University”. Klein's response to the transfer problem consists of establishing a connection $R_U^*(\sigma, \omega)$ weaving connections between *o* and ω in the light of the change in position indicated by the last arrow. In a subsequent modeling, Winsløw (2020) denotes $R_U(\sigma, o \cup \omega)$ this new integrator relationship, and it is this notation that we will retain. How can we construct a sequence in teacher training to enable the development of this new relationship? This article introduces such a sequence, experimented within in the MEEF master's program¹ in France in 2020.

THEORETICAL FRAMEWORK

This research is anchored in the Anthropological Theory of the Didactic (ATD). Firstly, it offers a language for modeling Klein's double discontinuity, as explained in the introduction. Secondly, the theory of praxeologies plays a crucial role in bridging the gap between High School (HS) and University (U) knowledge. ATD emphasizes the relative nature of knowledge objects (*o*) in relation to the institutions (I) that develop, standardize, and transmit them, as well as the individuals (*x*) subjected to these institutions (Chevallard & Bosch, 2020). Therefore, ATD focuses more on the generic positions (*p*) individuals occupy, such as teacher (*t*) or student (*s*), rather than the individuals themselves. The study aims to examining the institutional relations ($R_I(p, o)$) of individuals within the institution (*I*), in their respective positions (*p*), with regards to the knowledge object (*o*). The arrow diagram presented by Winsløw and Grønbaek (2014) in the introduction summarizes the various institutions, institutional positions, and knowledge objects involved in Klein's double discontinuity. Our research primarily addresses institutions where mathematics is taught.

Praxeologies form the core of ATD, emphasizing the analysis of human activities. A praxeology (*P*) comprises both a praxis Π and a discourse Λ on that praxis. ATD

¹ MEEF : Métier de l'Enseignement, de l'Education et de la Formation. The MEEF master's program prepares students for careers in education and teaching.

suggests that the relationship $(R_I(p, o))$ arises from praxeologies in which the object (o) is involved, operating at various levels within the praxeology: the type of task (T), the technique (τ) employed to solve tasks of this type (t), the technology (θ) supporting the technique, or the theory (Θ) providing the ultimate basis for the praxis. This set of praxeologies can be described in the form of a structured model that is called a reference praxeological model (RPM; Florensa et al., 2015). RPMs are reconstructions of the knowledge to be taught, obtained by considering different levels of the didactic transposition (via historical epistemology, official programs, textbooks, and teaching materials).

METHODOLOGY

The experimented sequence is intended for students undergoing teacher training as part of the MEEF master's program at the University of Montpellier, who hold a bachelor's degree in mathematics from the University of Montpellier (other profiles are enrolled in this master's program but are not subjects of the study). To conduct this study, we chose a subject of study: the integral. We hypothesize that students' praxeological equipment regarding integration after their bachelor's degree corresponds to what is expected by the university institution, in particular that they have studied the Riemann integral and the Lebesgue integral with measure theory. Thus, the design of our experimentation is based on the description of $R_U(\sigma, \omega)$ using a RPM. We thus constructed a dominant praxeological model, which will be our RPM for teaching integration at the University of Montpellier. Our modeling reveals two sectors: the first sector is related to the Riemann integral and contains five local mathematical organizations (integrability, properties of the integral, primitives, integration calculations, Riemann and Darboux sums). The second sector pertains to measure theory and the Lebesgue integral, revealing four regional mathematical organizations around the general theory of measure, the general theory of integration, image and product measures, and finally L^p spaces. On the other hand, since $R_{HS}(t, o)$ contains $R_{HS}(s, o)$ we have also constructed a RPM for the integral at the high school level. The Winsløw's model highlights the need to create a new relationship $R_U(\sigma, o \cup \omega)$ to facilitate the transfer of advanced knowledge, where " o " designates the high school integral, and ω designates the university integral. From our RPM, we will then formalize a mathematical organization which, in our view, realizes the Klein plan for integration in the sense that it will mobilize elements of logic and praxis from praxeological models related to high school integration, Riemann, and Lebesgue. To carry out the Klein plan and strengthen the connections between high school mathematics on integration and university mathematics on integration, we based our construction on the proof of a manual of the fundamental theorem of analysis (Figure 1.), as it is required in high school. The task corresponding to the RPM related to the high school integral is t_{FTC} "Demonstrate that if f a non negative, increasing and continuous function on $[a; b]$, with $a < b$, then the function $\phi: x \mapsto \int_a^x f(t) dt$ is differentiable and the derivative function is f ". The technique employed (see below), as described in Figure 1, involves the intuitive notion of area and some of its properties:

The area under the curve of a continuous (even just increasing) and positive function admits an area; the area is additive; the area of a segment is 0. Our objective is therefore to construct a logic that allows justifying, within the rigorous standard of the university, the various steps of the proof of this theorem. Our epistemological investigation has identified the Jordan measure theory of quareable sets (see below) as underlying the theory of areas. Thus, our project is to use this transitional element, the Jordan measure of squarable sets, to highlight the connections between the different theories. In the following section, we present the various tasks proposed to students aimed at reconstructing this logic, and then, based on a priori analysis, the praxeologies that are targeted. We will thus observe the development of two types of praxeologies: the first type, denoted as P^* , represents a praxeology stemming from the dominant praxeological model of the University institution but whose engineering work has modified certain components in order to establish connections between σ and ω (which will subsequently appear in the study process). The second type, which we denote as P^\sim , corresponds to praxeologies originating from the High School institution but which, during the study process, are enriched by praxeological elements related to ω .

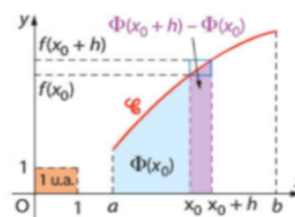
PRESENTATION OF THE SEQUENCE

In our praxeological study (Planchon, 2022), we have identified a task, denoted as t_{FTC} , that is found in different institutions (HS and U).

In the institution “high school”, the technique to be implemented consists of recognizing that, for $x_0 \in [a; b]$ and $h > 0$, then $\phi(x_0 + h) - \phi(x_0)$ represents the area under the curve, between the lines $x = x_0$ and $x = x_0 + h$. One can then bound this area with the area of two rectangles and conclude. The proof, required for high

Theorem : Let f be a continuous and positive function on an interval $[a; b]$. Then the function $\phi : x \mapsto \int_a^x f(t)dt$ is differentiable on $[a; b]$ and $\phi' = f$.

Proof : We consider the case where f is increasing on $I = [a; b]$. x_0 and h are two numbers : $x_0 \in I$, $h \neq 0$ and $x_0 + h \in I$.



- If $h > 0$, since f is increasing, $f(x_0) \leq f(x_0 + h)$. $\phi(x_0 + h) - \phi(x_0)$ represents the area under the curve C over $[x_0; x_0 + h]$. We bound this area by the areas of two rectangles with the same width and heights $f(x_0)$ and $f(x_0 + h)$:

$$h \times f(x_0) \leq \phi(x_0 + h) - \phi(x_0) \leq h \times f(x_0 + h).$$

$$\text{Hence, } f(x_0) \leq \frac{\phi(x_0 + h) - \phi(x_0)}{h} \leq f(x_0 + h).$$

- If $h < 0$, we can similarly show that : $f(x_0 + h) \leq \frac{\phi(x_0 + h) - \phi(x_0)}{h} \leq f(x_0)$.
- Now, as f is continuous at x_0 , we have $\lim_{h \rightarrow 0} f(x_0 + h) = f(x_0)$. Therefore, by the squeeze theorem, $\lim_{h \rightarrow 0} \frac{\phi(x_0 + h) - \phi(x_0)}{h} = f(x_0)$. Thus, ϕ is differentiable at x_0 and $\phi'(x_0) = f(x_0)$.

This result holds for all x_0 in $[a; b]$, so ϕ is differentiable on $[a; b]$ and $\phi' = f$.

Figure 1. Proof of the FTC (translated from textbook “Transmath”, (Bonneval et al., 2012))

school students, is written below (Figure 1) and is found in various textbooks at this level. The technique is justified here by the definition of the derivative, but also by various properties related to the concept of area: the area of a rectangle, the growth of the area, themselves justified by the intuitive notion of area, as stated in the official curriculum. Although the concept of area is related to the concept of measure, our RPM does not mention an explicit link between measure and the concept of area in the tasks proposed to students in the undergraduate program. In the University institution, this task denoted as t_U is also present, but the technique to be implemented is different: for $x_0 \in [a; b]$ and $h > 0$, we have:

$$|\phi(x_0 + h) - \phi(x_0) - hf(x_0)| = \int_{x_0}^{x_0+h} |f(x) - f(x_0)| dx$$

The continuity of f ensures that, then, ϕ is differentiable in x_0 and $\phi'(x_0) = f(x_0)$. This technique is justified by the definition of the derivative, but also by the various properties of the Riemann integral, themselves justified by the theory of the Riemann integral (Planchon, 2022). Later, in the third year of university, we encounter the same task within the framework of Lebesgue integral theory and measure theory, but the technique is limited to noting that if a positive function f is continuous, then it is Riemann integrable, and thus we reduce it to the case of the Riemann integral. Thus, the praxeological equipment of students at the end of the third year at the mathematic university includes the praxeologies related to the type of task of which t_{FTC} is an instantiation. In the perspective of generating a new relationship, we started by describing a theory, a new logos Λ_M^* , that justifies the technique implemented in high school for the execution of t_{FTC} . Daubelcour (1998), Douady (1987), Perrin-Glorian (1999) emphasized the Jordan measure for the measurement of areas. The theory that will form the basis of our mathematical organization is therefore the measure of measurable sets, as presented by Lebesgue (1975). This measure μ enjoys various properties that are mobilized in the proof of the fundamental theorem of analysis at the high school level: it is simply additive², invariant under isometry³, the measure of the unit square is 1, the measure of a segment is 0, and the measure is increasing.

In the following, we provide an explanation of the document given to the students, particularly outlining the various tasks they will be required to complete. The first part of the guide introduces a concept, called area measure, defined axiomatically as follows: *We assume that there exists a subset Q of the set of subsets of \mathbb{R}^2 containing points, segments, interiors of polygons, stable under finite intersection and union. An area measure is a function μ defined on Q with values in \mathbb{R}_+ , simply additive, invariant under isometry, and such that the measure of the square $[0; 1[\times [0; 1[$ is 1.* In this first part, we do not provide more details about what the set is. Three tasks are then assigned:

² If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

³ If s is an isometry, then $\mu(s(A)) = \mu(A)$

t_1 : Show that the area measure is a diffuse measure⁴ ;

t_2 : Determine the area measure of a rectangle in terms of its dimensions with justification;

t_3 : Drawing on the area measure, rewrite the proof of the Fundamental Theorem of Analysis extracted from the textbook with the rigor standard of the university (see Figure 1).

The choice of tasks t_1 and t_2 is based on our RPM related to the Lebesgue integral at the university. Indeed, with the aim of designing tasks that generate the new integrator relationship, $R_U(\sigma, \sigma \cup \omega)$, we wanted to enable students to make connections with their previous knowledge, which had been seen in measure theory. In particular, we chose to use the term "diffuse" in the description of task t_1 to explicitly refer to a task already encountered by students (showing that a translation-invariant measure on \mathbb{R} is diffuse).

Regarding task t_1 , two techniques can be employed: either a proof by contradiction or a direct proof. In both cases, the growth of the measure is a key point and must be proven by the students. The techniques are theoretically known to the students. Therefore, the task involves adapting proofs already studied in the third year of the undergraduate program to the specific context of the theory of areas.

For task t_2 , the goal is to distinguish the set of measures of rectangles. It is possible to limit ourselves to rectangles parallel to the axes (due to invariance under isometry). When the measures of the sides of the rectangles are integers, it suffices to mobilize the additivity of the area. This is also the case for rational dimensions. For the case where the measures of the sides are positive real numbers, which are non-rational, it is necessary to mobilize the density of \mathbb{Q} in \mathbb{R} and again mobilize the growth of μ . The praxeological elements to be mobilized here theoretically form part of the praxeological equipment of students who have studied in a mathematics undergraduate program. We model these two tasks as instantiations of two types of tasks from the RPM relative to measure theory at the university (demonstrating a property of a measure, determining the measure of a set for a given measure), but with a logo block modified, as it is not from measure theory.

Finally, for task t_3 , the goal is to formalize a proof from the textbook. The choice to introduce a school textbook, through the proof of the fundamental theorem of analysis, highlights the relevance of the work done previously to analyze material from the school environment. The task to be performed here can be considered as a professional task, which distinguishes it from tasks t_1 and t_2 . Here, the task corresponds to a task type originating from the RPM related to high school integration. But here, the technique to be implemented requires an adapted logo block, which contains the elements of Λ_M^* . But a question must emerge: Is the area bounded by a continuous

⁴ the measure of points is 0

function curve an element of Q ? The study of this issue is the subject of the second part of the activity, developed below.

The second part of the activity introduces the definition proposed by Lebesgue of measurable sets in the plane: For any natural number i , we call a level i grid the grid Q_i in the Euclidean plane \mathbb{R}^2 , referred to an orthonormal coordinate system, whose vertices are the points with coordinates $(\frac{a}{10^i}, \frac{b}{10^i})$, where a and b are integers. A closed square surface (in the sense of the usual topology of the plane) of level i is a set of the form $\{(x, y) \in \mathbb{R}^2, \frac{a}{10^i} \leq x \leq \frac{b}{10^i}, \frac{a}{10^i} \leq y \leq \frac{b}{10^i}\}$. If S is a bounded subset of the plane (i.e., contained in a square surface), we consider the set of closed square surfaces of level i contained in S and denote by s_i their union and by n_i their number. Then $s_i \subset S$. Similarly, we consider the closed square surfaces of level i that intersect S , denote by S_i their union, and by N_i their number, so that $S \subset S_i$. Finally, we define $u_i = \frac{n_i}{100^i}$ and $v_i = \frac{N_i}{100^i}$. We say that S is *quarrable* when $\lim_{i \rightarrow +\infty} u_i = \lim_{i \rightarrow +\infty} v_i$. The common limit is then denoted by $\mu(S)$ and called the area of the surface S (in the sense of Lebesgue).

Three tasks are then assigned:

t_4 : Demonstrate that the semi-open unit square, $C = [0; 1[\times [0; 1[$, is quarrable and has a Lebesgue area equal to 1, and that the function μ also satisfies the additivity axiom.

t_5 : What are the properties of μ that are mobilized in the reasoning presented in Figure 2? Relate these properties to the axioms or to general propositions that, if necessary, can be considered as new axioms.

In his treatise (around 250 BC), Archimede demonstrates that the disk has the same area as the right-angled triangle whose legs are respectively the radius and the circumference of this circle. Here is an interpretation of his approach with a contemporary reformulation, particularly incorporating the concept of limit and the properties of trigonometric functions:

Consider a circle with a radius of 1, and let p be the length of its semi-perimeter. For any non-zero integer n , denote P_n and Q_n as the regular polygons with n sides inscribed and circumscribed to the circle. The polygon P_n is the union of n isometric triangles at OAB , and the polygon Q_n is the union of n isometric triangles at OGF .

The Lebesgue areas of these polygons are $n \cos(p_n) \sin(p_n)$ and $n \tan(p_n)$. The difference between the areas of Q_n and P_n tends toward 0, so the disk is quarrable, and its area is given by

$$\lim_{n \rightarrow +\infty} \mu(Q_n) = p_n.$$

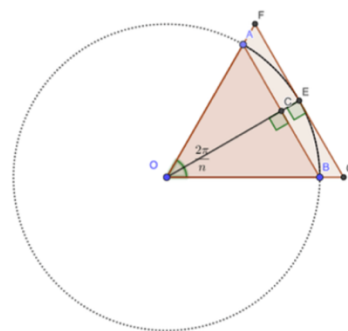


Figure 2. Area of the disk

t_6 : Show that the area bounded by the curve of an increasing function is measurable.

Task t_4 is divided into two sub-tasks: sub-task $t_{4,1}$ is « demonstrate that the square $[0; 1[\times [0; 1[$ is quarrable with a Lebesgue area equal to 1, i.e., that the normalization axiom is satisfied », and sub-task $t_{4,2}$ is « demonstrate that μ satisfies the additivity axiom ». For sub-task $t_{4,1}$, taking the notations from the statement, we have $n_i =$

$100^i - (10^i + 10^i - 1)$ and $N_i = n_i + (10^i + 10^i - 1) + (10^i + 10^i + 1)$, and then $u_i = 1 - \left(\frac{2}{10^i} - \frac{1}{100^i}\right)$ and $v_i = u_i + \frac{4}{100^i}$. The sequences (u_i) and (v_i) are indeed sequences that converge to 1. Finally, the expected formalization for sub-task $t_{4,2}$ consists of introducing, at a fixed i , the number of level- i squares that are included in A , in B , and in $A \cup B$, as well as the number of level- i squares that intersect A , in B , and $A \cup B$. Denoting a_i, b_i, d_i as the number of squares in A, B, D and A_i, B_i, D_i as the number that intersect A, B, D , we have $a_i + b_i \leq d_i \leq D_i \leq A_i + B_i$. Finally, it is found that the sequences $\left(\frac{a_i}{100^i}\right)$ and $\left(\frac{D_i}{100^i}\right)$ are adjacent, so they converge to the same limit. This limit is the limit of $\frac{a_i}{100^i} + \frac{b_i}{100^i}$, so $\mu(A \cup B) = \mu(A) + \mu(B)$. The completion of this task involves elements of praxeologies in analysis related to the concept of the limit of a sequence, especially on adjacent sequences. Again, we model these task t_4 as instantiation of a type of task from our RPM relatives to the measure theory at the University (showing that an application is a measure). The task t_5 was constructed based on a singular task found in the RPM at the high school level, which can be modeled as « find an approximation of π based on the area of the disk ». This task involves, once again, formalizing reasoning that can be found in school textbooks. It is of the same type as task t_3 , it means a task of a type encountered in the RPM related to high school integration, but with the logo block enriched by mathematical elements from Λ_M^* . From the study of this task, the logo is enriched by the following proposition: A surface S is quarrable if and only if there exist two sequences of polygons (P_n) and (Q_n) such that, for every n , $P_n \subset S \subset Q_n$ and $\lim_{n \rightarrow +\infty} \mu(Q_n) - \mu(P_n) = 0$. This proposition can be used in the completion of the last task t_6 , which is to demonstrate that the set bounded by the curve of an increasing and positive function, and then the one bounded by the curve of a positive continuous function, is a quarrable set.

For this, students are asked to consider, when f is an increasing function on $[a; b]$, the set $\Omega = \{(x; y) \in \mathbb{R}^2, a \leq x \leq b, 0 \leq y \leq f(x)\}$.

To complete this task, it is necessary to adapt the technique used in the task “show that an increasing function on an interval is Riemann-integrable over that interval”, which is a task encountered by students in the second year of their undergraduate studies. Thus, the technique to be implemented is: for every n in \mathbb{N}^* consider for $k \in \{0, \dots, n - 1\}$, the rectangles:

$$r_k = \{(x, y) \in \mathbb{R}^2, a + k \frac{b-a}{n} \leq a + (k+1) \frac{b-a}{n}, 0 \leq y \leq f\left(a + k \frac{b-a}{n}\right)\} \text{ and}$$

$$R_k = \{(x, y) \in \mathbb{R}^2, a + k \frac{b-a}{n} \leq a + (k+1) \frac{b-a}{n}, 0 \leq y \leq f\left(a + (k+1) \frac{b-a}{n}\right)\}$$

By defining $P_n = \bigcup_{k=0}^{n-1} r_k$ and $Q_n = \bigcup_{k=0}^{n-1} R_k$ as described, we indeed have $P_n \subset \Omega \subset Q_n$

(due to the growth of f), and $\mu(Q_n) - \mu(P_n) = \frac{(b-a)(f(b)-f(a))}{n}$ tends to 0. Therefore,

Ω is quarrable. Here, we encounter a task that belongs to a type typically found in the domain related to measure theory (demonstrating measurability of a set), with once again a modified logo block.

CONCLUSION AND RESULTS

In this article, we have presented our experimentation as a response to Klein's problem for the integral, within the institutional context of secondary teacher training in France. The sequence presented here aims to approach secondary school mathematical concepts with the perspective of undergraduate knowledge. The development of the sequence was based on RPM related to the integral at the high school and university levels. These models were complemented by a mathematical organization constructed from our epistemological study. In the six tasks proposed here, tasks t_1, t_2, t_4, t_6 are derived from praxeologies whose praxis originates from the dominant praxeological model related to the integral at the university level (this means that the task proposed is of a type encountered at the university). The technique to be implemented requires confronting the technique developed at the university, adapting it to the specific situation, which necessitates questioning the discourse of the praxeology. The new type of praxeology we are modelling here is called P^* . The completion of these tasks, which thus mobilize university-level mathematics, aims to provide students with theoretical elements to justify, with the rigor of the university, the proof of the fundamental theorem of analysis. This work of justification is the focus of tasks t_3 and t_5 . Here, the task originates from a type encountered in the high school institution. The new type of praxeology we are modelling here is called P^\sim . This formalization brings out the notion of Kleinian praxeologies (Planchon, 2022), the development of which generates the new relation. Thus, an important result of this study is to produce a proposal that responds to Winslow's formalization of generating a new relation. The analysis of student activity can highlight the links we aim to construct and may ultimately make it possible to detect the obstacles to the development of the targeted praxeologies.

The methodology described here can also be used, by mobilizing objects other than the integral, to construct learning situations that encourage the development of Kleinian praxeologies, of the P^* and the P^\sim type.

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