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INFINITE-WISE INTERACTIONS: MEAN-FIELD AND GRAPH LIMITS FOR MULTIPLE-WISE DISTINGUISHABLE AGENT SYSTEMS

THIERRY PAUL, STEFANO ROSSI, AND EMMANUEL TRÉLAT

ABSTRACT. We consider N distinguishable agents with m -body interactions encoded by a multiple-wise kernel. We derive the mesoscopic mean-field limit $N \rightarrow \infty$ and a macroscopic (graph-limit) equation for deterministic label configurations. Our main novelty is the infinite-wise regime $m \rightarrow \infty$: without assuming convergence of the kernels, we prove (along subsequences) existence and well-posedness of limiting mesoscopic and macroscopic equations by combining the Hewitt–Savage theorem with an Arzelà–Ascoli compactness argument, and we obtain uniform-in- m quantitative estimates under a natural $1/m$ -Lipschitz scaling. We illustrate the framework on higher-order Hegselmann–Krause and Kuramoto models and on complex contagion dynamics.

CONTENTS

1. Introduction	1
1.1. Contributions	3
1.2. Plan of the paper	4
1.3. Notation	5
2. Setting and main results	5
2.1. Main assumption	5
2.2. Multiple-wise limit	6
2.3. Infinite-wise limit	12
3. Proof of Theorem 2.1: Validation of the mesoscopic equation	17
4. Proof of Theorem 2.4: Validation of the macroscopic equation	20
5. Proof of Theorem 2.9: Infinite-wise limit	23
References	26

1. INTRODUCTION

Many mathematical models — ranging from mechanical systems in physics, to agent-based models in biophysics, and those arising in opinion dynamics — commonly assume that interactions between agents are pairwise. In this article, we investigate more general models involving what we entitle *multiple-wise* interactions.

Specifically, given $N \in \mathbb{N}^* = \{1, 2, \dots\}$ and $m \in \{1, \dots, N\}$, we consider a system of N agents described by the variables $\{\xi_i^N(t)\}_{i=1}^N \subset \mathbb{R}^d$ (for some $d \in \mathbb{N}^*$) communicating via multiple-wise interactions of order m , of the type

$$\dot{\xi}_i^N(t) = \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N G_{i, j_1, \dots, j_m}^N(t, \xi_i^N(t), \xi_{j_1}^N(t), \dots, \xi_{j_m}^N(t)), \quad i \in \{1, \dots, N\}, \quad (1.1)$$

where $G_{i, j_1, \dots, j_m}^N : \mathbb{R}_t \times \mathbb{R}_\xi^{d(m+1)} \rightarrow \mathbb{R}^d$ represents the *multiple-wise* interaction between the agent i and the agents j_1, \dots, j_m , and where the indices may possibly coincide.

Models incorporating multiple-wise interactions — of which system (1.1) is a mathematical generalization — have recently gained attention in a wide variety of fields, among which we mention

consensus dynamics [33], synchronization of oscillators [29, 30], epidemic spreading and social contagions (see [5, 8], also for a general overview of the field). In all the cases, these models provide new deep insights into the emergent phenomena related to complex systems.

Even in the case $m = 1$, corresponding to pairwise interactions, system (1.1) describes many significant situations ranging from classical physics to flocking dynamics (see [39, 14, 15, 32, 4]), vehicular flows (see [6]) and opinion dynamics (see [23, 1, 2]) that have been thoroughly investigated in recent years. Particularly in the opinion dynamic framework (see e.g. [34]) — where the value $\xi_i^N(t)$ represents the opinion of agent i — group behaviors often cannot be fully explained by pairwise interactions alone, since opinions are frequently governed by simultaneous multiple-wise interactions ($m > 1$) of the type described above.

This paper focuses on the rigorous justification of suitable partial differential equations (PDEs) used to statistically describe systems as in (1.1), in the large N and m limits. For sufficiently large N , the statistical properties of such systems do not depend on their exact size, allowing reduced descriptions via PDEs. In order to rigorously establish this, we analyze three distinct asymptotic regimes: the mesoscopic limit, the macroscopic limit, and an infinite-wise limit, where m diverges after N .

In the pairwise case $m = 1$, numerous works in the mathematical literature have rigorously investigated the passage across different scales — microscopic, mesoscopic, and macroscopic — for mechanical systems, flocking models, animal collective behavior, synchronization of oscillators etc. (just to mention a few, see [17, 35, 12, 13]). Typically, the models considered assume that the interactions do not depend explicitly on the labels i and j of the particles and this implies that the particles are indistinguishable, i.e. each permutation of the particle system is again a solution of (1.1) with the same vector field.

More recently, however, models involving distinguishable particles have been studied, i.e. models in which the interaction depends explicitly on the labels of the agents. In this context different approaches have been proposed. Some works make use of graphon theory to derive the appropriate limiting equations across scales (see [31, 26, 3]). In others, such as the earlier work [36] (see also [7] for a similar point of view in the theory of mean-field games), the dependence on agents' labels is directly embedded into the physical interaction among the agent variables, see (2.2).

In this work, we extend the analysis to the multiple-wise case ($m > 1$) with distinguishable particles, exploring whether limit mesoscopic and macroscopic equations can be derived as $m \rightarrow +\infty$ (infinite-wise limit), preserving the key characteristics of the interaction type. We mention that multiple-wise interactions have been recently studied in [3] for systems with a different structure which prevent the derivation of the infinite-wise limit equations obtained in this work (we refer to (ii) in Remark 2.10 for a comparison). Moreover, a related mean-field framework for hypergraph limit objects—based on directed hypergraph measures and leading to well-posed Vlasov equations that can be approximated by empirical measures—was developed in [27].

A central assumption in our paper is the introduction of a reference space Ω for the labels $i \in \{1, \dots, N\}$, so that the dependence of the system on the agent labels $\{1, \dots, N\}$ is directly embedded into the physical interaction between agent variables through points $\{x_i\}_{i=1}^N$ in Ω (see (2.2)).

The different scales are described as follows: In the mesoscopic limit, agent labels $x \in \Omega$ and variables $\xi \in \mathbb{R}^d$ are treated as independent quantities, jointly distributed according to a statistical distribution on the agent's label-variable phase space. By studying suitable marginals of this distribution function (see equation (2.7)), we show that, as $N \rightarrow +\infty$, the discrete dynamics is well approximated by an m -dependent multiple-wise interacting mesoscopic equation.

Differently, in the macroscopic limit, variables are considered as functions of agent labels, which are now distributed according to a probability density. We derive the corresponding m -dependent integro-differential equation for these functions by passing from the mesoscopic to the macroscopic level for monokinetic solutions (see (2.12)). Additionally, we show how to bypass the mesoscopic

level and rigorously pass directly from the microscopic to the macroscopic level using the concept of graph limit. The underlying idea is to study the convergence via a limit of Riemann sums, which allows one to recover the macroscopic interaction operator as a continuum limit of the finite system.

These mesoscopic and macroscopic systems are derived from the microscopic one at fixed m , through quantitative estimates that make the multiple-wise m -dependence explicit.

The main novelty of the work is the study of the infinite-wise limit where $m \rightarrow +\infty$. Here, we rigorously derive limiting differential equations at the mesoscopic and macroscopic level (infinite-wise equations, see (2.29) and (2.31)), respectively for the distribution function on the one-agent phase-space and for the macroscopic state function, in the limit when the mesoscopic and macroscopic systems interact with infinitely many other agents. These new equations retain the averaged characteristics of the interactions through an asymptotic vector field and they are obtained, via the Hewitt–Savage Theorem, by passing to the limit through subsequences in the solutions of the corresponding mesoscopic and macroscopic equations with finite interaction order. For this reason, in general, the limit infinite-wise equations are not explicitly characterized (see the discussion in 2.3.3).

By combining the three asymptotic procedures, we obtain, as a main novelty of this work (see part (i), part (ii) and part (iii) of Theorem 2.11), convergence results that rigorously connect the dynamics of N agents interacting via m -body interactions, to the limiting equations satisfied by the distribution function and the macroscopic state function as both N and m tend to infinity. In particular, we prove the following result, here stated in the particular case of the reference space $\Omega = (0, 1]$ (see Theorem 2.11 for the general case).

Theorem 1.1. *Let $\{\xi_i^N(t)\}_{i=1}^N$ be the state variables defined by the m -order multiple-wise agent system in (1.1) and assume that, for $i \in \{1, \dots, N\}$, $\xi_i^N(0) = y_0(\frac{i}{N})$ with $y_0 : (0, 1] \rightarrow \mathbb{R}^d$ a Lipschitz continuous initial datum. For any $m \in \{1, \dots, N\}$, we assume that there exists a regular function $G^{(m)} : \mathbb{R}_t \times ((0, 1] \times \mathbb{R}_\xi^d)^{m+1} \rightarrow \mathbb{R}^d$ such that, for every N , any $t \in \mathbb{R}$, $(\xi, \xi_1, \dots, \xi_m) \in \mathbb{R}^{d(m+1)}$ and any $\{i, j_1, \dots, j_m\} \subset \{1, \dots, N\}$,*

$$G_{i, j_1, \dots, j_m}^N(t, \xi, \xi_1, \dots, \xi_m) = G^{(m)}\left(t, \frac{i}{N}, \xi, \frac{j_1}{N}, \xi_1, \dots, \frac{j_m}{N}, \xi_m\right),$$

where G_{i, j_1, \dots, j_m}^N is defined as in (1.1). Then, for a given $T > 0$ and under suitable assumptions on the interaction functions $G^{(m)}$ (see assumptions A and C), it holds that, up to a subsequence in m ,

$$\lim_{\substack{N, m \rightarrow +\infty \\ m \leq N}} \sup_{t \in [0, T]} \max_{i=1, \dots, N} \left| \xi_i^N(t) - y^\infty\left(t, \frac{i}{N}\right) \right| = 0,$$

where $y^\infty : [0, T] \times (0, 1] \rightarrow \mathbb{R}^d$ is solution of a suitable limit macroscopic equation with initial datum y_0 (see (2.31) in Theorem 2.9).

The proof of Theorem 1.1 follows as a corollary of the general result stated in Theorem 2.11 and will be presented in Section 2.3, after Theorem 2.11.

1.1. Contributions. The main results of this paper can be summarized as follows:

- **Mesoscopic mean-field limit at fixed m :** For distinguishable agents with m -wise interactions, we justify the mesoscopic equation (2.5) by quantitative Dobrushin estimates and propagation of chaos (Theorem 2.1).
- **Macroscopic (graph-limit) model at fixed m :** We derive the macroscopic equation (2.13) for monokinetic solutions and provide a direct micro-to-macro graph-limit estimate with explicit dependence on m (Theorem 2.4).
- **Infinite-wise limit $m \rightarrow \infty$:** Under Assumption C, we construct a limit interaction operator \mathcal{G}^∞ and prove well-posedness and convergence (possibly along subsequences) of the mesoscopic and macroscopic dynamics towards the infinite-wise equations (2.29) and (2.31)

(Theorem 2.9), using symmetrization together with the Hewitt–Savage theorem and Arzelà–Ascoli compactness.

- **Uniform-in- m bounds and combined limits:** As anticipated and discussed in Theorem 1.1, a key new feature of our analysis is the derivation of quantitative estimates uniform in m (notably (5.9) and (5.10)), which yield joint $(N, m) \rightarrow \infty$ convergence statements connecting the N -agent dynamics with m -body interactions to the limiting equations satisfied by the distribution function and the macroscopic state function in the infinite-wise limit (see (2.33), (2.34), and (2.35) in Theorem 2.11).
- **Applications and examples:** We discuss how the theoretical framework developed can be applied to three concrete examples where the limit infinite-wise equations can be explicitly characterized. These are multiple-wise extensions of the Hegselmann–Krause model for opinion dynamics, the Kuramoto model for oscillator synchronization, and of a complex contagion model used in sociology and biology. Moreover, we include an example where \mathcal{G}^∞ is not explicitly computable and is obtained only via compactness (Section 2.3.3).

For the reader’s convenience, Figure 1.1 summarizes the chain of limits connecting the microscopic dynamics (2.3)–(2.7) to the mesoscopic equation (2.5) (mean-field/propagation of chaos, see [17, 38]), to the macroscopic graph-limit equation (2.13) (see [31, 36]), and to the infinite-wise limit (2.29)–(2.31) established in Section 2.3.

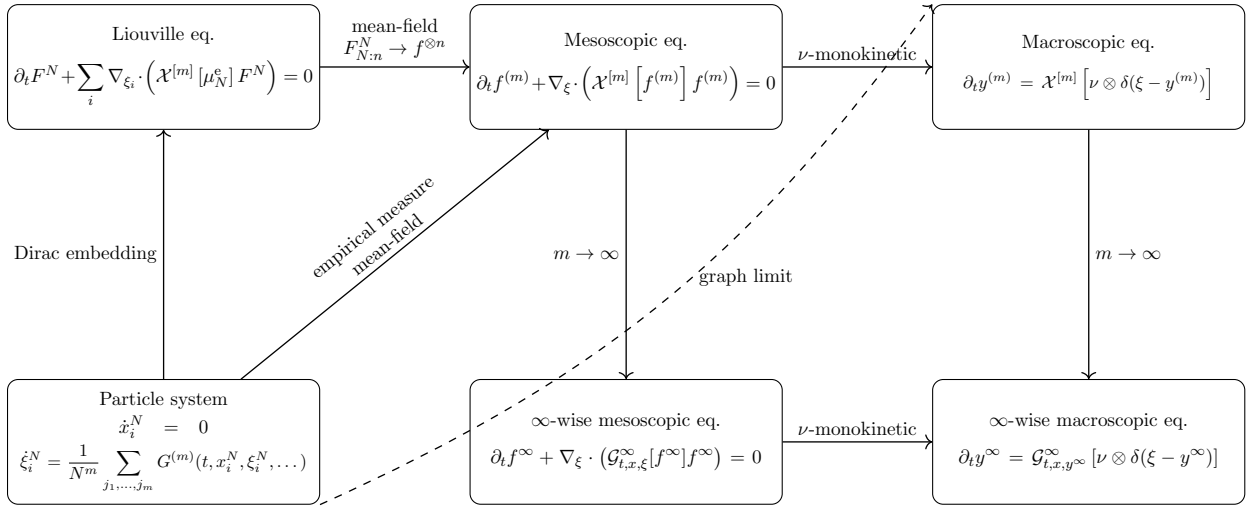


FIGURE 1. Scale limits and main equations.

1.2. Plan of the paper. The paper is organized as follows. The main assumptions and theorems are presented in Section 2, together with the discussion of some examples and remarks.

We first state the main assumption on the interaction functions of system (1.1) in Subsection 2.1. Then, in Subsection 2.2, we introduce the equations that we aim to derive rigorously for a fixed m , and we state the main results concerning the mesoscopic and macroscopic limits for fixed m , given by Theorems 2.1 and 2.4. We also include three examples of multiple-wise models, namely the higher-order extensions of the Hegselmann–Krause, Kuramoto and complex contagion models.

In Subsection 2.3, we present the main novelty of this work, given by Theorem 2.9, which establishes the infinite-wise limit as $m \rightarrow \infty$ for the mesoscopic and macroscopic equations derived in Theorems 2.1 and 2.4. In this subsection, we also apply the infinite-wise limit to the three examples introduced above (Hegselmann–Krause, Kuramoto and complex contagion models) and finally state the main Theorem 2.11 linking the N -agent model (1.1) to the solutions of the infinite-wise mesoscopic and macroscopic equations.

The remaining Sections 3, 4, and 5 are devoted to the proofs of the main Theorems 2.1, 2.4, and 2.9.

1.3. Notation. Given a metric space (Ω, d_Ω) and $p \geq 1$, we denote by $\mathcal{P}_p(\Omega)$ the set of probability measures on Ω with finite p -th order moment, i.e.

$$\int_{\Omega} d_\Omega(x, x_0)^p d\mu(x) < +\infty, \quad \text{for any } \mu \in \mathcal{P}_p(\Omega) \quad \text{and} \quad x_0 \in \Omega.$$

In order to quantify the difference between two probability measures $\mu, \nu \in \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d)$, we introduce the Wasserstein distance

$$\mathcal{W}_p(\mu, \nu) := \inf_{\Pi \in \mathcal{C}(\mu, \nu)} \left(\int_{(\Omega_x \times \mathbb{R}_\xi^d)^2} (d_\Omega(x^1, x^2) + |\xi^1 - \xi^2|)^p d\Pi(x^1, \xi^1, x^2, \xi^2) \right)^{\frac{1}{p}},$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings between μ and ν : i.e. probability measures on the product space $(\Omega_x \times \mathbb{R}_\xi^d)^2$ with first and second marginal respectively μ and ν .

We also compare two probability measures $\mu_k, \nu_k \in \mathcal{P}_p((\Omega_x \times \mathbb{R}_\xi^d)^k)$: In this case, denoting

$$Z_k^i := (x_1^i, \xi_1^i, \dots, x_k^i, \xi_k^i) \in (\Omega_x \times \mathbb{R}_\xi^d)^k \quad (1.2)$$

for $i = 1, 2$, we define the q -modified Wasserstein distance

$$\mathcal{W}_p^{[q]}(\mu_k, \nu_k) := \inf_{\Pi \in \mathcal{C}(\mu_k, \nu_k)} \left(\int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2k}} \left(d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}(Z_k^1, Z_k^2) \right)^p d\Pi(Z_k^1, Z_k^2) \right)^{\frac{1}{p}},$$

where, given $q \geq 1$, we have

$$d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}(Z_k^1, Z_k^2) := \left\| (d_{\Omega \times \mathbb{R}^d}(z_1^1, z_1^2), \dots, d_{\Omega \times \mathbb{R}^d}(z_k^1, z_k^2)) \right\|_{\ell^q}, \quad d_{\Omega \times \mathbb{R}^d}(z_i^1, z_i^2) := d_\Omega(x_i^1, x_i^2) + |\xi_i^1 - \xi_i^2|, \quad (1.3)$$

with $z_j^i = (x_j^i, \xi_j^i) \in \Omega_x \times \mathbb{R}_\xi^d$ for $i = 1, 2$ and $j \in \{1, \dots, k\}$. We also use the notation

$$X_k^i := (x_1^i, \dots, x_k^i) \in \Omega_x^k, \quad \Xi_k^i := (\xi_1^i, \dots, \xi_k^i) \in \mathbb{R}_\xi^{dk}, \quad \text{for } i = 1, 2.$$

Moreover, if (X, μ) is a probability space, Y a measurable space and $\phi : X \rightarrow Y$ a measurable function, we define the push-forward $\phi_*\mu \in \mathcal{P}(Y)$ of μ along ϕ by

$$\phi_*\mu(A) := \mu(\phi^{-1}(A)), \quad A \subset Y \quad \text{measurable.}$$

Finally, we denote by $\mathbf{1}_A$ the characteristic function of the set A , defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.

2. SETTING AND MAIN RESULTS

In this section, we introduce the main assumption underlying our analysis of system (1.1). We then rigorously state the main results of the paper.

2.1. Main assumption. In the following, for any $m \in \{1, \dots, N\}$, we assume that there exist a complete metric space (Ω, d_Ω) and a *continuous* function

$$G^{(m)} : \mathbb{R}_t \times (\Omega_x \times \mathbb{R}_\xi^d)^{m+1} \rightarrow \mathbb{R}^d \quad (2.1)$$

which is globally Lipschitz on $\Omega_x^{m+1} \times \mathbb{R}_\xi^{d(m+1)}$, uniformly on every compact subset of \mathbb{R}_t and such that, for every N , there exist $x_1, \dots, x_N \in \Omega$ such that

$$G^{(m)}(t, x_i, \xi, x_{j_1}, \xi_1, \dots, x_{j_m}, \xi_m) = G_{i, j_1, \dots, j_m}^N(t, \xi, \xi_1, \dots, \xi_m), \quad (2.2)$$

for any $t \in \mathbb{R}$, $(\xi, \xi_1, \dots, \xi_m) \in \mathbb{R}^{d(m+1)}$ and any $\{i, j_1, \dots, j_m\} \subset \{1, \dots, N\}$ and where G_{i, j_1, \dots, j_m}^N is defined as in (1.1).

This assumption is to be understood as the existence of a function defined on the extended phase space $(\Omega \times \mathbb{R}^d)^{m+1}$, providing a continuous interpolation of the interaction's dependence on the discrete labels $\{1, \dots, N\}$. Such an assumption was already employed in the previous work [36] by the first and third authors (see also the recent work [7] for a similar point of view).

Thanks to (2.2), we can equivalently define the multi-agent system (1.1) as

$$\begin{aligned} \dot{x}_i^N(t) &= 0 \\ \dot{\xi}_i^N(t) &= \frac{1}{Nm} \sum_{j_1, \dots, j_m=1}^N G^{(m)}(t, x_i^N, \xi_i^N(t), x_{j_1}^N, \xi_{j_1}^N(t), \dots, x_{j_m}^N, \xi_{j_m}^N(t)), \quad i \in \{1, \dots, N\}. \end{aligned} \quad (2.3)$$

In this way, the indices $\{1, \dots, N\}$ are represented by stationary labels $\{x_i^N\}_{i=1}^N$ related to the variables $\{\xi_i^N\}_{i=1}^N$. Notice that the new system $\{x_i^N, \xi_i^N(t)\}_{i=1}^N$ is exchangeable, in the sense that any permutation of the set of indices yields a solution of the same problem with the permuted initial data.

2.2. Multiple-wise limit. We present the main results of this work concerning the mesoscopic and macroscopic limits with multiple-wise order m fixed. The results are divided into two parts: the first concerns the passage to the mesoscopic level, while the second to the macroscopic level. Finally, we discuss three examples of multiple-wise models.

2.2.1. Mesoscopic description. We start by discussing the mesoscopic equation. Given $t \in \mathbb{R}$, we introduce the empirical measure at time t associated with system (2.3); this is given by

$$\mu_N^e(t, dx, d\xi) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}(x) \otimes \delta_{\xi_i^N(t)}(\xi) \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d). \quad (2.4)$$

It is straightforward to verify that $\mu_N^e(t)$ weakly solves the following mesoscopic equation of Vlasov-type:

$$\partial_t f(t, x, \xi) + \nabla_\xi \cdot \left(\mathcal{X}^{[m]}[f(t)](t, x, \xi) f(t, x, \xi) \right) = 0, \quad (2.5)$$

where $\mathcal{X}^{[m]}[\mu_N]$ is the multiple-wise mean-field force of order m given by

$$\mathcal{X}^{[m]}[\mu_N](t, x, \xi) := \int_{(\Omega_x \times \mathbb{R}_\xi^d)^m} G^{(m)}(t, x, \xi, x_1, \xi_1, \dots, x_m, \xi_m) d\mu_N^{\otimes m}(x_1, \xi_1, \dots, x_m, \xi_m). \quad (2.6)$$

In the following, we justify the use of equation (2.5) by proving the existence and uniqueness of its weak measure-valued solutions and showing the following validity result: If $\mu_N^e(0)$ is converging to a given probability distribution $f_0 \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d)$ as $N \rightarrow +\infty$, the same holds for a time $t > 0$, i.e. $\mu_N^e(t)$ converges to the unique solution $f(t) \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d)$ to (2.5) with initial datum f_0 . This can be proved quantitatively by obtaining a Dobrushin-type estimate (as referred by the seminal paper [17]).

A different but related point view is that of propagation of chaos (see, e.g., [38]): Given $f(0) \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d)$, assume that at the initial time the agents are independently and identically distributed with law $F^N(0) := f(0)^{\otimes N} \in \mathcal{P}((\Omega_x \times \mathbb{R}_\xi^d)^N)$. For a time $t > 0$, the statistical distribution of the agents is given by $F^N(t) \in \mathcal{P}((\Omega_x \times \mathbb{R}_\xi^d)^N)$ weak solution of the following Liouville equation

$$\partial_t F^N(t, Z_N) + \sum_{i=1}^N \frac{1}{Nm} \sum_{j_1, \dots, j_m=1}^N \nabla_{\xi_i} \cdot \left(G^{(m)}(t, x_i, \xi_i, x_{j_1}, \xi_{j_1}, \dots, x_{j_m}, \xi_{j_m}) F^N(t, Z_N) \right) = 0, \quad (2.7)$$

where, for $k \in \{1, \dots, N\}$, $Z_k := (x_1, \xi_1, \dots, x_k, \xi_k)$. Defining

$$F^N(t)_{N:k}(Z_k) := \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{N-k}} F^N(t)(Z_N) dx_{k+1} d\xi_{k+1} \dots dx_N d\xi_N,$$

we expect that, for N large, the behavior of the small group of k particles will become nearly independent, so that at time t and as $N \rightarrow +\infty$, each particle will follow the same statistical distribution $f(t)$ given by the solution of (2.5) with initial datum $f(0)$: i.e., in a suitable topology, $F^N(t)_{N:k}$ will converge to $f^{\otimes k}(t)$.

Both the Dobrushin-type estimate and propagation of chaos will be proved in the following Theorem 2.1. Before stating it, we list some technical assumptions concerning the regularity of the interaction function $G^{(m)}$ introduced in (2.2).

Assumptions on $G^{(m)}$.

A: There exists $M > 0$ independent of m such that

$$\left\| G^{(m)} \right\|_{L^\infty(\mathbb{R}_t \times (\Omega_x \times \mathbb{R}_\xi^d)^{m+1})} \leq M; \quad (2.8)$$

B: There exist $q \in [1, +\infty]$ and a constant $\text{Lip}_{x,\xi}^{[q]}(G) \geq 0$ such that $G^{(m)}$ is globally Lipschitz continuous on $(\Omega_x \times \mathbb{R}_\xi^d)^{m+1}$ with respect to the ℓ^q distance $d_{(\Omega \times \mathbb{R}^d)^{m+1}}^{[q]}$ defined in (1.3), i.e.

$$\left| G^{(m)}(Z_{m+1}^1) - G^{(m)}(Z_{m+1}^2) \right| \leq \text{Lip}_{x,\xi}^{[q]}(G) d_{(\Omega \times \mathbb{R}^d)^{m+1}}^{[q]}(Z_{m+1}^1, Z_{m+1}^2).$$

By the equivalence of norms in finite dimensional normed spaces, if $G^{(m)}$ is globally Lipschitz respect to the ℓ^q norm, the same holds for any other $q' \in [1, +\infty]$, but with a different Lipschitz constant $\text{Lip}_{x,\xi}^{[q']}(G)$. Notice that, since the interaction $G^{(m)}$ is assumed to be globally Lipschitz, the classical Cauchy-Lipschitz theory ensures that the solutions to (2.3) are globally defined in time.

Theorem 2.1 (Validation of the mesoscopic equation). *Given $m \in \mathbb{N}^*$, let $G^{(m)}$ be as in (2.1) satisfying assumptions A, and B for some $q \in [1, +\infty]$. The following results hold:*

(i) *(Existence and uniqueness). Given $p \geq 1$ and $f_0 \in \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d)$, there exists a unique global solution $f \in \mathcal{C}([0, +\infty); \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d))$ to the mesoscopic equation (2.5) with initial datum f_0 . Equivalently, f is the unique solution of the equation:*

$$f(t) = \varphi_{f_0}(t)_* f_0, \quad t \in \mathbb{R} \quad (2.9)$$

where $\varphi_{f_0}(t) : \Omega_x \times \mathbb{R}_\xi^d \rightarrow \Omega_x \times \mathbb{R}_\xi^d$ is the Vlasov flow defined by $\varphi_{f_0}(t)(x_0, \xi_0) := (x_0, \xi(t))$ where

$$\begin{cases} \dot{x}(t) = 0 \\ \dot{\xi}(t) = \mathcal{X}^{[m]}[f(t)](t, x(t), \xi(t)) \\ x(0) = x_0, \xi(0) = \xi_0, \quad (x_0, \xi_0) \in \Omega_x \times \mathbb{R}_\xi^d, \end{cases}$$

with $\mathcal{X}^{[m]}[f(t)]$ as in (2.6).

(ii) *(Dobrushin-type estimate). Given $f_0 \in \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d)$, let $f(t) \in \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d)$ be the weak solution of the mesoscopic equation (2.5) with initial datum f_0 and $\mu_N^e(t)$ be the empirical measure defined in (2.4) with initial datum $\mu_N^e(0)$. Then, for $m \in \{1, \dots, N\}$ and $p \geq 1$, it holds that*

$$\mathcal{W}_p(\mu_N^e(t), f(t)) \leq \exp\left(\text{Lip}_{x,\xi}^{[q]}(G)(1+m)t\right) \mathcal{W}_p(\mu_N^e(0), f(0)). \quad (2.10)$$

(iii) *(Propagation of chaos). Let $p \geq 1$ and assume that $\Omega_x \subset \mathbb{R}^d$. For $k \in \{1, \dots, N\}$, let $F^N(t)_{N:k}$ be the k -marginal of the solution to the Liouville equation (2.7) with initial datum $F_0^N(Z_N) = \prod_{i=1}^N f_0(x_i, \xi_i)$, with $f_0 \in \mathcal{P}_z(\Omega_x \times \mathbb{R}_\xi^d)$ such that $z > p$. Let $f(t) \in \mathcal{P}_z(\Omega_x \times \mathbb{R}_\xi^d)$ be*

the unique solution to (2.5) with initial datum f_0 , then,

$$\begin{aligned} \mathcal{W}_p^{[q]}(F^N(t)_{N:k}, f(t)^{\otimes k}) &\leq Ck^{\frac{1}{q}} \exp\left(\text{Lip}_{x,\xi}^{[q]}(G)(2+m)t\right) \mathcal{M}_z(f(0))^{1/z} \\ &\times \begin{cases} N^{-\frac{1}{2p}} + N^{-(z-p)/(pz)} & \text{if } p > d \text{ and } z \neq 2p \\ N^{-\frac{1}{2p}}(\log(1+N))^{1/p} + N^{-(z-p)/(pz)} & \text{if } p = d \text{ and } z \neq 2p \\ N^{-\frac{1}{2d}} + N^{-(z-p)/(pz)} & \text{if } p \in (0, d) \text{ and } z \neq 2d/(2d-p), \end{cases} \end{aligned} \quad (2.11)$$

where $\mathcal{M}_z(f) := \int |x|^z df(x)$.

Remark 2.2. We collect here a few additional observations related to the previous theorem:

- (i) Theorem 2.1 can be readily extended to the case of a locally Lipschitz vector field G by analyzing the time of existence of the N -particle system and controlling the temporal evolution of the supports of the measures (see [36] for the case $m = 1$). This extension is particularly relevant for second-order systems where interactions depend on the agents' velocities.
- (ii) In order for the estimate in the distance $\mathcal{W}_p^{[q]}$ to hold in part (iii) of Theorem 2.1, we require that the initial data f_0 possesses finite moments of order $z > p$. This assumption is necessary to apply a law-of-large-numbers-type estimate for the expected value of the Wasserstein distance between the empirical measure and its law. Here we rely on the estimates by Fournier and Guillin [19]. If f_0 has sufficiently high-order finite moments (for instance, if f_0 has compact support, or if $z > 2p$ when $p \geq d$, or $z > \frac{2dp}{2d-p}$ when $p \in (0, d)$), then the powers of N in (2.11) depending on z are of lower order and can be neglected.
- (iii) If assumption **B** is replaced by the stronger assumption **C**, the estimates (2.10) and (2.11) can be strengthened, yielding uniform bounds with respect to the multiple-wise parameter m (see (5.9)).

2.2.2. Macroscopic description. We now introduce the macroscopic level of description. Let us consider again the mesoscopic equation (2.5). By taking a monokinetic weak solution to (2.5) of the form

$$f_{\text{mon}}(t, x, \xi) := \nu(x) \otimes \delta_{y(t,x)}(\xi) \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d), \quad (2.12)$$

with $\nu \in \mathcal{P}(\Omega_x)$ and $y(t, x)$ a measurable function of class \mathcal{C}^1 with respect to time, it is straightforward to verify that $y(t, x)$ solves the macroscopic equation

$$\partial_t y(t, x) = \int_{\Omega^m} G^{(m)}(t, x, y(t, x), x_1, y(t, x_1), \dots, x_m, y(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m). \quad (2.13)$$

In the following Theorem 2.4, after proving the existence and uniqueness of solutions to (2.13), we show how to pass directly from the N -particle dynamics defined in (1.1) to the macroscopic equation (2.13) through the concept of graph limit.

To describe this limit procedure, we first give the following definition.

Definition 2.3. Let $\nu \in \mathcal{P}(\Omega)$. We say that $(\mathcal{A}^N, \mathcal{X}_N)_{N \in \mathbb{N}^*}$ is a family of tagged partitions of Ω associated with ν if $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$ is a N -tuple of disjoint subsets $\Omega_i^N \subset \Omega$ such that

$$\Omega = \bigcup_{i=1}^N \Omega_i^N, \quad \text{with } \nu(\Omega_i^N) = \frac{1}{N} \quad \text{and } \text{diam}_\Omega(\Omega_i^N) \leq \frac{C_\Omega}{Nr}, \quad \forall i \in \{1, \dots, N\}, \quad (2.14)$$

for some $C_\Omega > 0$ and $r > 0$ not depending on N and $\mathcal{X}^N = (x_1^N, \dots, x_N^N)$ is a N -tuple of points $x_i^N \in \Omega_i^N$.

Families of tagged partitions always exist when Ω is a compact n -dimensional smooth manifold having a boundary or not and ν is a Lebesgue measure on Ω with $r = 1/n$, see, e.g., [20].

Given a family of tagged partitions related to (Ω, ν) , we consider the following discrete function

$$y^{N,(m)}(t, x) := \sum_{i=1}^N \xi_i^N(t) \mathbf{1}_{\Omega_i^N}(x), \quad \forall (t, x) \in \mathbb{R} \times \Omega, \quad (2.15)$$

where $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$ is solution of the particle system (1.1) with multiple-wise interactions of order m and given initial data $\xi_i^N(0)$ for every $i \in \{1, \dots, N\}$.

The graph limit procedure consists in proving the quantitative convergence of the discrete function $y^{N,(m)}$ in (2.15) towards the solution $y^{(m)}$ of the limit macroscopic equation (2.13) with multiple-wise order m .

Theorem 2.4 (Validation of the macroscopic equation). *Given $m \in \mathbb{N}^*$, let $G^{(m)}$ be as in (2.1) satisfying assumptions A, and B for some $q \in [1, +\infty]$. The following results hold:*

- (i) (Monokinetic measures) *Given $T > 0$, let $\nu \in \mathcal{P}(\Omega_x)$ and $y(t, \cdot) \in \mathcal{C}^1([0, T]; L^\infty(\Omega_x; \mathbb{R}^d))$. Then*

$$f_{\text{mon}}(t, x, \xi) := \nu(x) \otimes \delta_{y(t,x)}(\xi) \in \mathcal{P}(\Omega_x \times \mathbb{R}_\xi^d)$$

is a (weak) solution of the Vlasov-type equation (2.5) if and only if $y(t, x)$ is solution of the macroscopic equation (2.13), ν_x -almost everywhere on $[0, T]$.

- (ii) (Existence and uniqueness) *Given $\nu \in \mathcal{P}(\Omega_x)$ and $y_0 \in L^\infty(\Omega_x; \mathbb{R}^d)$, there exists a unique global solution $y^{(m)}(t, \cdot) \in L^\infty(\Omega_x; \mathbb{R}^d)$ which is continuously differentiable in time of the macroscopic equation (2.13) such that $y(0) = y_0$.*

- (iii) (Graph limit: From micro to macro) *Let Ω be a compact, complete metric space, and given $\nu \in \mathcal{P}(\Omega)$, assume there exists a family of tagged partitions of Ω associated with ν as in (2.14). Given $T > 0$, consider the unique solution $t \mapsto y^{(m)}(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$ on the interval $[0, T]$ of the macroscopic equation (2.13) such that $y^{(m)}(0, \cdot) = y_0(\cdot)$ where y_0 a Lipschitz continuous initial datum. Consider, for any $N \in \mathbb{N}^*$ and $m \in \{1, \dots, N\}$, the unique solution $t \mapsto (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$ on $[0, T]$ of the particle system (1.1) such that $\xi_i^N(0) = y_0(x_i^N)$ for every $i \in \{1, \dots, N\}$. Then*

$$\left\| y^{N,(m)}(t, \cdot) - y^{(m)}(t, \cdot) \right\|_{L^\infty(\Omega)} \leq m^{\frac{1}{q}} \frac{C_\Omega}{N^r} (1 + \text{Lip}_x(y_0)) e^{t \text{Lip}_{x,\xi}^{[q]}(G)(2+m\frac{1}{q})}, \quad (2.16)$$

where $y^{N,(m)}(t, \cdot)$ is defined in (2.15). In particular, since $y^{N,(m)}(t, \cdot)$ is piecewise constant on each Ω_i^N , for any $i \in \{1, \dots, N\}$ we have

$$\left| \xi_i^N(t) - y^{(m)}(t, x) \right| \leq m^{\frac{1}{q}} \frac{C_\Omega}{N^r} (1 + \text{Lip}_x(y_0)) e^{t \text{Lip}_{x,\xi}^{[q]}(G)(2+m\frac{1}{q})} \quad \text{for any } x \in \Omega_i^N. \quad (2.17)$$

Remark 2.5. *We conclude with two additional observations concerning the results of Theorem 2.4:*

- (i) *Although the solutions of the mesoscopic Vlasov equation (2.5) with measure-valued data exist globally in time, as ensured by part (i) of Theorem 2.1, the preservation of the monokinetic structure is guaranteed only locally in time, up to some $T > 0$. In particular, the statement of part (i) of Theorem 2.4 holds as long as no blow-up of the monokinetic property occurs.*
- (ii) *As in Remark 2.2, if assumption B is replaced by the stronger assumption C, the estimates (2.17) and (2.16) can be refined to yield uniform bounds with respect to the multiple-wise parameter m (see (5.10)).*

2.2.3. Examples. We now consider three examples of multiple-wise systems represented by (1.1). These are multiple-wise extension of the Hegselmann–Krause, Kuramoto and complex contagion models.

Example 2.6 (Multiple-wise Hegselmann–Krause model). *Given $N \in \mathbb{N}^*$ agents with opinions $\{\xi_i(t)\}_{i=1}^N \subset \mathbb{R}^d$ and $m \in \{1, \dots, N\}$, we consider the following generalization of the Hegselmann–Krause model [23] given by*

$$\dot{\xi}_i(t) = \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N w_{i, j_1, \dots, j_m}^N \left(\frac{\sum_{h=1}^m \xi_{j_h}(t)}{m} - \xi_i(t) \right), \quad i \in \{1, \dots, N\},$$

where w_{i, j_1, \dots, j_m}^N is the m -th order multiple-wise interaction weight between agent i and the agents j_1, \dots, j_m . In this model the opinion of the agent i is updated by considering a weighted sum of the interactions with all the average opinions of other m agents. Notice that in general w_{i, j_1, \dots, j_m}^N is not symmetric. A simplified version of this model has been discussed in [33] and in [3] with a different normalization.

We assume there exists a continuous function $w^{(m)} : \Omega^{m+1} \rightarrow \mathbb{R}^+$ such that for any $i, j_1, \dots, j_m \in \{1, \dots, N\}$ there exist $x_i, x_{j_1}, \dots, x_{j_m} \in \Omega_x$ so that

$$w_{i, j_1, \dots, j_m}^N = w^{(m)}(x_i, x_{j_1}, \dots, x_{j_m}).$$

In this case, using notation (2.2) for the interaction function, we get that

$$G^{(m)}(x, \xi, x_1, \xi_1, \dots, x_m, \xi_m) = w^{(m)}(x, x_1, \dots, x_m) \left(\frac{1}{m} \sum_{h=1}^m \xi_h - \xi \right),$$

for $\xi, \xi_1, \dots, \xi_m \in D_\xi$ a domain in \mathbb{R}^d . We make the following assumptions on the $w^{(m)} : \Omega_x^{m+1} \rightarrow \mathbb{R}^+$:

- (1) There exists $C > 0$ such that $\|w^{(m)}\|_{L^\infty(\Omega^{m+1})} \leq C$ for every $m \in \mathbb{N}^*$.
- (2) For each $x, x_1, \dots, x_m, x'_1, \dots, x'_m \in \Omega$, there exists $C > 0$ such that

$$\left| w^{(m)}(x, x_1^1, \dots, x_m^1) - w^{(m)}(x, x_1^2, \dots, x_m^2) \right| \leq C d_{\Omega^m}^{[q]}(X_m^1, X_m^2).$$

For m fixed, the mesoscopic and macroscopic equations rigorously justified in the limit $N \rightarrow \infty$ by parts (ii), (iii) of Theorem 2.1 and by part (iii) of Theorem 2.4 are respectively

$$\partial_t f(t, x, \xi) + \nabla_\xi \cdot \left(\int_{(\Omega \times \mathbb{R}^d)^m} w^{(m)}(x, x_1, \dots, x_m) \left(\frac{1}{m} \sum_{i=1}^m \xi_i - \xi \right) df^{\otimes m}(t, Z_m) f(t, x, \xi) \right) = 0, \quad (2.18)$$

and

$$\partial_t y(t, x) = \int_{\Omega^m} w^{(m)}(x, x_1, \dots, x_m) \left(\frac{1}{m} \sum_{i=1}^m y(t, x_i) - y(t, x) \right) d\nu^{\otimes m}(x_1, \dots, x_m). \quad (2.19)$$

Example 2.7 (Multiple-wise Kuramoto model). *Given $N \in \mathbb{N}^*$ oscillators with phases $\{\xi_i(t)\}_{i=1}^N \subset [0, 2\pi)$, we consider the following extension of the classical Kuramoto model [28] given by*

$$\dot{\xi}_i(t) = \omega_i + \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N a_{i, j_1, \dots, j_m}^N \sin \left(\frac{1}{m} \sum_{h=1}^m \xi_{j_h}(t) - \xi_i(t) \right), \quad i \in \{1, \dots, N\},$$

where $\omega_i \in [0, 2\pi)$ are the natural frequencies of each oscillator and the phases $\xi_i(t)$ are interacting by a sum of multiple-wise interactions among groups of m oscillators weighted by an interaction weight a_{i, j_1, \dots, j_m}^N . This type of model has been discussed in [8, Section 4.2.1, (48)], [29, II, (1)] and [5] and also in [3] with a different normalization.

We proceed as in the previous example. In this case, we assume there exist continuous functions $\omega : \Omega \rightarrow \mathbb{R}^+$ and $a^{(m)} : \Omega^{m+1} \rightarrow \mathbb{R}^+$ such that for any $i, j_1, \dots, j_m \in \mathbb{N}^*$ there exist $x_i, x_{j_1}, \dots, x_{j_m} \in \Omega$ so that

$$\omega_i = \omega(x_i) \quad \text{and} \quad a_{i, j_1, \dots, j_m}^N = a^{(m)}(x_i, x_{j_1}, \dots, x_{j_m}).$$

With the notations in (2.2), we have

$$G^{(m)}(x, \xi, x_1, \xi_1, \dots, x_m, \xi_m) = \omega(x) + a^{(m)}(x, x_1, \dots, x_m) \sin\left(\frac{1}{m} \sum_{h=1}^m \xi_h - \xi\right).$$

To apply the mesoscopic and macroscopic limits, we make the following assumptions on the $a^{(m)} : \Omega_x^{m+1} \rightarrow \mathbb{R}^+$:

- (1) There exists $C > 0$ such that $\|a^{(m)}\|_{L^\infty(\Omega^{m+1})} \leq C$ for every $m \in \mathbb{N}^*$.
- (2) For each $x, x_1, \dots, x_m, x'_1, \dots, x'_m \in \Omega$, there exists $C > 0$ such that

$$\left| a^{(m)}(x, x_1^1, \dots, x_m^1) - a^{(m)}(x, x_1^2, \dots, x_m^2) \right| \leq C d_{\Omega^m}^{[q]}(X_m^1, X_m^2).$$

For m fixed, by parts (ii) and (iii) of Theorem 2.1 and part (iii) of Theorem 2.4, the mesoscopic and macroscopic equations rigorously justified in the limit are respectively

$$\begin{aligned} & \partial_t f(t, x, \xi) + \omega(x) \partial_\xi f(t, x, \xi) \\ & + \partial_\xi \left(\int_{(\Omega \times [0, 2\pi])^m} a^{(m)}(x, x_1, \dots, x_m) \sin\left(\frac{1}{m} \sum_{i=1}^m \xi_i - \xi\right) df^{\otimes m}(t, Z_m) f(t, x, \xi) \right) = 0, \end{aligned} \quad (2.20)$$

and

$$\partial_t y(t, x) = \omega(x) + \int_{\Omega^m} a^{(m)}(x, x_1, \dots, x_m) \sin\left(\frac{1}{m} \sum_{i=1}^m y(t, x_i) - y(t, x)\right) d\nu^{\otimes m}(x_1, \dots, x_m). \quad (2.21)$$

Example 2.8 (Multiple-wise smooth complex contagion model). For $N \in \mathbb{N}^*$ and $m \in \{1, \dots, N\}$, we consider a smooth version of complex contagion given by the multi-agent system

$$\dot{\xi}_i(t) = \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N \alpha_i \left(\frac{1}{m} \sum_{h=1}^m \kappa_{i, j_h} \sigma(\xi_{j_h}(t) - \theta_i) \right)^2 (1 - \xi_i(t)) - \beta_i \xi_i(t),$$

for $i \in \{1, \dots, N\}$, where the state variable $\xi \in [0, 1] \subset \mathbb{R}$ stands for an “adoption level”, and the dynamics is driven by group-reinforcement modeled by a \mathcal{C}^1 sigmoid $\sigma : \mathbb{R} \rightarrow (0, 1)$ (e.g. logistic) with Lipschitz constant L_σ , quantifying the response to the gap $\xi_j - \theta_i$ (level ξ_j exceeds personal threshold θ_i) and by influence weights $\kappa_{i, j} \geq 0$ and heterogeneous parameters $\alpha_i, \beta_i \geq 0$, and $\theta_i \in (0, 1)$. Note that if $\xi_i(0) \in [0, 1]$, then $\xi_i(t) \in [0, 1]$ for all $t \geq 0$.

This example is built on the classical idea of complex contagion, where adoption of a change of behavior requires reinforcement from multiple contacts (see, e.g., [22, 18, 11, 10]). More recently, higher-order contagion rules have been formalized on simplicial complexes and hypergraphs [25, 16, 37], revealing that reinforcement at group level can produce nonlinear phenomena such as discontinuous transitions, multistability, and mesoscopic localization.

We assume there exists a compact metric space Ω and continuous functions

$$\alpha : \Omega \rightarrow [0, \alpha_{\max}], \quad \beta : \Omega \rightarrow [0, \beta_{\max}], \quad \theta : \Omega \rightarrow (0, 1), \quad \text{and} \quad \kappa : \Omega \times \Omega \rightarrow [0, \kappa_{\max}],$$

for $\alpha_{\max}, \beta_{\max}, \kappa_{\max} \geq 0$, so that, for any $i, j \in \{1, \dots, N\}$ there exist $x_i, x_j \in \Omega$ for which

$$\alpha_i = \alpha(x_i), \quad \beta_i = \beta(x_i), \quad \theta_i = \theta(x_i), \quad \kappa_{i, j} = \kappa(x_i, x_j),$$

with influence kernel κ assumed to be L_κ -Lipschitz in both variables.

We now define the m -wise interaction mapping

$$G^{(m)}(x, \xi; x_1, \xi_1, \dots, x_m, \xi_m) := \alpha(x) \left(\frac{1}{m} \sum_{h=1}^m \kappa(x, x_h) \sigma(\xi_h - \theta(x)) \right)^2 (1 - \xi) - \beta(x) \xi.$$

For m fixed, by parts (ii) and (iii) of Theorem 2.1, the mesoscopic equation rigorously justified in the limit is

$$\partial_t f(t, x, \xi) + \partial_\xi \left(\mathcal{X}^{[m]}[f(t)](x, \xi) f(t, x, \xi) \right) = 0, \quad (2.22)$$

where, given $\mu \in \mathcal{P}(\Omega \times [0, 1])$, we introduced the mesoscopic interaction term

$$\mathcal{X}^{[m]}[\mu](x, \xi) := \alpha(x) Q^{[m]}[\mu](x) (1 - \xi) - \beta(x) \xi, \quad (2.23)$$

and the squared empirical mean

$$Q^{[m]}[\mu](x) := \int_{(\Omega \times [0, 1])^m} \left(\frac{1}{m} \sum_{j=1}^m Z(x; x_j, \xi_j) \right)^2 d\mu^{\otimes m}(Z_m), \quad Z(x; x', \xi') := \kappa(x, x') \sigma(\xi' - \theta(x)). \quad (2.24)$$

For monokinetic data $f(t, x, \xi) = \nu \otimes \delta_{y(t, x)}(\xi)$ with $\nu \in \mathcal{P}(\Omega)$, by part (iii) of Theorem 2.4, we also obtain the macroscopic equation

$$\partial_t y(t, x) = \alpha(x) Q^{[m]}[\nu \otimes \delta_{y(t, x)}](x) (1 - y(t, x)) - \beta(x) y(t, x). \quad (2.25)$$

2.3. Infinite-wise limit. As a main novelty of this work, we study a new situation where the order m of the multiple-wise interaction in the mesoscopic and macroscopic equations (2.5) and (2.13) tends to infinity. To do this, the following stronger assumption on the interaction will be assumed:

C: $G^{(m)}$ is globally Lipschitz continuous in the first (x, ξ) variables with a constant $\text{Lip}(G)$ independent of m , i.e.

$$\left| G^{(m)}(t, x, \xi, \cdot) - G^{(m)}(t, x', \xi', \cdot) \right| \leq \text{Lip}(G) (|x - x'| + |\xi - \xi'|), \quad (2.26)$$

for $x, x' \in \Omega_x$ and $\xi, \xi' \in \mathbb{R}_\xi^d$. Moreover we assume the following stronger Lipschitz conditions

$$\begin{aligned} \left| G^{(m)}(\cdot, x_1^1, \xi_1^1, \dots, x_m^1, \xi_m^1) - G^{(m)}(\cdot, x_1^2, \xi_1^2, \dots, x_m^2, \xi_m^2) \right| \\ \leq \frac{\text{Lip}(G)}{m} \sum_{i=1}^m \left(d_\Omega(x_i^1, x_i^2) + |\xi_i^1 - \xi_i^2| \right), \end{aligned} \quad (2.27)$$

with $(x_1^i, \xi_1^i, \dots, x_m^i, \xi_m^i) \in (\Omega_x \times \mathbb{R}_\xi^d)^m$ for $i = 1, 2$.

2.3.1. Main results. In the following Theorem 2.9, we identify a limit vector field \mathcal{G}^∞ defined on the space of probability measures which can be viewed as a limit of the symmetrization of the interaction $G^{(m)}$ as m goes to infinity and we obtain limit equations (at the mesoscopic and macroscopic levels) which retains the average features of the interaction and of the distribution of the labels through the vector field \mathcal{G}^∞ (see (2.29) and (2.31) in parts (i) and (ii) of Theorem 2.9 and the examples discussed in Section 2.3.2).

Theorem 2.9 (Infinite-wise limit). *Given $m \in \mathbb{N}^*$, let $G^{(m)}$ be as in (2.1) satisfying assumptions A and C.*

(i) (Infinite-wise mesoscopic limit) *Let Ω_x be compact and $D \subset \mathbb{R}_\xi^d$ a closed and bounded domain of the Euclidean space. Given $f_0 \in \mathcal{P}(\Omega_x \times D_\xi)$, there exists $f^\infty \in \mathcal{C}([0, T]; \mathcal{P}(\Omega_x \times D_\xi))$ and a map*

$$\mathcal{G}^\infty : [0, T] \times (\Omega_x \times D_\xi) \times \mathcal{P}(\Omega_x \times D_\xi) \rightarrow \mathbb{R}^d \quad (2.28)$$

Lipschitz continuous in all its variables, such that f^∞ is the unique weak solution of

$$\partial_t f^\infty(t, x, \xi) + \nabla_\xi \cdot (\mathcal{G}_{t, x, \xi}^\infty[f^\infty] f^\infty(t, x, \xi)) = 0 \quad (2.29)$$

with initial datum f_0 , and, up to a subsequence,

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \mathcal{W}_1 \left(f^{(m)}(t), f^\infty(t) \right) = 0, \quad (2.30)$$

where $f^{(m)} \in \mathcal{C}([0, T], \mathcal{P}(\Omega_x \times D_\xi))$ is the unique solution to the mesoscopic equation (2.5) with multiple-wise order m with initial datum f_0 .

- (ii) (Infinite-wise macroscopic limit) Let Ω_x be compact and consider $\nu \in \mathcal{P}(\Omega_x)$ and a Lipschitz datum $y_0 \in L^\infty(\Omega_x; \mathbb{R}^d)$. Then, there exists a unique solution $y^\infty \in \mathcal{C}^1([0, T]; L^\infty(\Omega_x; \mathbb{R}^d))$ with initial datum y_0 to the equation

$$\partial_t y^\infty(t, x) = \mathcal{G}_{t, x, y^\infty(t, x)}^\infty [\nu \otimes \delta(\xi - y^\infty(t, \cdot))], \quad (2.31)$$

where \mathcal{G}^∞ is given in (2.28) and, up to a subsequence,

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \|y^{(m)}(t, \cdot) - y^\infty(t, \cdot)\|_{L^\infty(\Omega_x)} = 0, \quad (2.32)$$

where $y^{(m)} \in \mathcal{C}^1([0, T]; L^\infty(\Omega_x; \mathbb{R}^d))$ is the unique solution to the macroscopic equation (2.13) with multiple-wise order m .

Remark 2.10. We collect here two further comments concerning Theorem 2.9:

- (i) In parts (i) and (ii) of Theorem 2.9, the convergence results are stated along subsequences. If, in addition, the family $\{\mathcal{X}^{[m]}\}_{m \in \mathbb{N}}$ in (2.6) forms a Cauchy sequence with respect to the uniform-Lipschitz topology on $[0, T] \times (\Omega \times D) \times \mathcal{P}(\Omega \times D)$, then

$$\mathcal{G}_{t, x, \xi}^\infty[\mu] = \lim_{m \rightarrow \infty} \mathcal{X}^{[m]}[\mu](t, x, \xi)$$

is uniquely determined, and the convergences in (2.30) and (2.32) hold for the entire sequence.

- (ii) We now compare our results with those obtained in [3], which studies a different class of multiple-wise interaction systems. The model and the regularity assumptions in [3] differ substantially from those considered here. In [3], the authors investigate non-exchangeable systems defined on hypergraphs with unbounded rank, deriving a deterministic limit via a bi-coupling distance that simultaneously encodes interactions of infinitely many orders. Consequently, the limit in [3] as $m \rightarrow \infty$ arises from replacing a finite sum with an infinite series under a suitable summability condition. In contrast, our sequential $m \rightarrow \infty$ limit at fixed order leads to a different limiting object: our \mathcal{G}^∞ , defined in (2.28), emerges as the limit of symmetric kernels depending on the empirical measure, rather than as an infinite series of rank contributions. Furthermore, the regularity assumptions differ: while our approach requires a Lipschitz constant of order $1/m$ (see (2.27)), the assumptions in [3, Equations (1.7)–(1.9)] involve a sequence of Lipschitz constants $\{L_m\}$ satisfying the summability condition $\sum_m mL_m < \infty$.

We end this section by presenting the main theorem which directly connect the microscopic dynamics with the infinite-interaction limit of the mesoscopic and macroscopic equations (2.5) and (2.13). This is obtained by combining the result about the infinite-wise limit of Theorem 2.9 with the mesoscopic and macroscopic limits with fixed m given in Theorems 2.1 and 2.4.

Theorem 2.11. Let $G^{(m)}$ be as in (2.1) satisfying assumptions A and C. The following results hold:

- (i) (From micro to infinite-wise meso through Liouville) Let Ω_x be compact and $D \subset \mathbb{R}_\xi^d$ a closed and bounded domain. Given $f_0 \in \mathcal{P}(\Omega_x \times D_\xi)$, let $F^{N, (m)}(t)_{N:1}$ be the first marginal of the solution to the Liouville equation (2.7) with multiple-wise order m and initial datum $f_0^{\otimes N}$. Then,

$$\lim_{\substack{N, m \rightarrow +\infty \\ m \leq N}} \sup_{t \in [0, T]} \mathcal{W}_1(F^{N, (m)}(t)_{N:1}, f^\infty(t)) = 0, \quad (2.33)$$

where f^∞ is the unique weak solution of (2.29) with initial datum f_0 .

- (ii) (From micro to infinite-wise macro through Liouville) Let $\Omega_x \subset \mathbb{R}^d$ be compact and, given $\nu \in \mathcal{P}(\Omega_x)$ and $y_0 \in L^\infty(\Omega_x)$ Lipschitz, let $F^{N, (m)}(t)_{N:1}$ be the first marginal of the solution

to the Liouville equation (2.7) with multiple-wise order m and monokinetic initial datum

$$F_{\text{mon}}^N(0, X_N, \Xi_N) := \prod_{i=1}^N \nu(x_i) \otimes \delta_{y_0(x_i)}(\xi_i).$$

Moreover, let us denote by $Y_{N:1}^{(m)}(t, x) := \int \xi F^{N,(m)}(t)_{N:1}(t, x, \xi) d\xi$ its first moment in ξ . Then, for any $\phi \in \mathcal{C}(\Omega_x)$,

$$\lim_{\substack{N, m \rightarrow +\infty \\ m \leq N}} \sup_{t \in [0, T]} \int_{\Omega_x} \phi(x) (Y_{N:1}^m(t, dx) - y^\infty(t, x) \nu(dx)) = 0, \quad (2.34)$$

where y^∞ is the unique solution of (2.31) with initial datum y_0 .

(iii) (From micro to infinite-wise macro through the graph limit) Let Ω be a compact complete metric space and, given $\nu \in \mathcal{P}(\Omega)$, assume there exists a family of tagged partitions of Ω associated with ν as in (2.14). Given $T > 0$, consider the unique solution $t \mapsto y^\infty(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$ on $[0, T]$ of the infinite-wise equation (2.31) such that $y^\infty(0, \cdot) = y_0(\cdot)$ with y_0 a Lipschitz continuous initial datum. Consider, for any $N \in \mathbb{N}^*$, the unique solution $t \mapsto (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$ on $[0, T]$ of the particle system (1.1) such that $\xi_i^N(0) = y_0(x_i^N)$ for every $i \in \{1, \dots, N\}$ and set $y^{N,(m)}(t, x)$ as in (2.15).

Then, the following convergence in strong norm holds up to subsequence:

$$\lim_{\substack{N, m \rightarrow +\infty \\ m \leq N}} \|y^{N,(m)}(t, \cdot) - y^\infty(t, \cdot)\|_{L^\infty(\Omega)} = 0. \quad (2.35)$$

In particular, being by definition $y^{N,(m)}(\cdot, t)$ piecewise constant on each Ω_i^N , we have

$$\lim_{\substack{N, m \rightarrow +\infty \\ m \leq N}} |\xi_i^N(t) - y^\infty(t, x)| = 0 \quad \text{for any } x \in \Omega_i^N, \quad i \in \{1, \dots, N\}. \quad (2.36)$$

As a special case of Theorem 2.11-(iii), we can deduce Theorem 1.1, which was stated in the Introduction.

Proof of Theorem 1.1. This result follows directly as a corollary of (2.36) in Theorem 2.11-(iii) by taking $\Omega = (0, 1]$ with the Euclidean distance as the space of labels, and considering the tagged partition

$$\Omega_i^N = \left(\frac{i-1}{N}, \frac{i}{N} \right], \quad x_i^N = \frac{i}{N}, \quad \text{for } i \in \{1, \dots, N\},$$

together with $\nu = dx$, where dx denotes the Lebesgue measure. \square

2.3.2. Examples. We will apply to the models introduced in Example 2.6 and Example 2.7 the results stated in Theorem 2.9, getting an explicit form for the asymptotic vector field \mathcal{G}^∞ presented in (2.29) and (2.31). Although the Hegselmann–Krause model is linear in the opinion variables and the use of our techniques is therefore unnecessary in this case, we believe it is still instructive to consider it.

Example 2.12 (Infinite-wise Hegselmann–Krause model). *We want to get an explicit expression for the infinite-wise limit of these mesoscopic and macroscopic equations introduced in (2.18) and (2.19). In this case, we make the following on the $w^{(m)} : \Omega_x^{m+1} \rightarrow \mathbb{R}^+$:*

- (1) *There exists $C > 0$ such that $\|w^{(m)}\|_{L^\infty(\Omega^{m+1})} \leq C$ for every $m \in \mathbb{N}^*$.*
- (2) *For each $x, x_1, \dots, x_m, x'_1, \dots, x'_m \in \Omega$, there exists $C > 0$ such that*

$$\left| w^{(m)}(x, x_1, \dots, x_m) - w^{(m)}(x, x'_1, \dots, x'_m) \right| \leq C \mathcal{W}_1 \left(\frac{1}{m} \sum_{k=1}^m \delta_{x_k}, \frac{1}{m} \sum_{h=1}^m \delta_{x'_h} \right).$$

For the mesoscopic equation, by part (i) of Theorem 2.9, we know there exists a map

$$\mathcal{G}^\infty : [0, T] \times (\Omega_x \times D_\xi) \times \mathcal{P}(\Omega_x \times D_\xi) \rightarrow \mathbb{R}^d$$

Lipschitz continuous in all its variables, such that the infinite-wise limit equation is given by

$$\partial_t f^\infty(t, x, \xi) + \nabla_\xi \cdot (\mathcal{G}_{t,x,\xi}^\infty[f^\infty(t)]f^\infty(t, x, \xi)) = 0.$$

In this case, we can go further by explicitly characterizing the asymptotic vector field (2.28). Indeed, as in the proof of part (ii) of Theorem 2.9, for any $x \in \Omega$ we can assume w.l.o.g. that $w^{(m)}(x, \cdot)$ is a sequence of symmetric functions on Ω^m . Since it is uniformly bounded and Lipschitz continuous in the sense stated in C, there exists $w^\infty(x, \cdot) \in \mathcal{C}^0(\mathcal{P}(\Omega))$ such that, up to a subsequence,

$$\lim_{m \rightarrow +\infty} \sup_{(x_1, \dots, x_m) \in \Omega^m} \left| w^{(m)}(x, x_1, \dots, x_m) - w^\infty\left(x, \frac{1}{m} \sum_{i=1}^m \delta_{x_i}\right) \right| = 0.$$

Noting that, for every m ,

$$\frac{1}{m} \sum_{h=1}^m \xi_h = \int_{\Omega \times \mathbb{R}^d_\xi} \xi d[\mu_{Z_m}^e], \quad \text{and} \quad \frac{1}{m} \sum_{i=1}^m \delta_{x_i} = \int_{\mathbb{R}^d_\xi} d\mu_{Z_m}^e$$

where $\mu_{Z_m}^e = \frac{1}{m} \sum_{h=1}^m \delta_{x_h} \otimes \delta_{\xi_h}$, by part (ii) of Proposition 5.1 and by triangle inequality we have:

$$\begin{aligned} & \int_{(\Omega \times \mathbb{R}^d)^m} w^{(m)}(t, x, x_1, \dots, x_m) \left(\frac{1}{m} \sum_{i=1}^m \xi_i - \xi \right) d[f^\infty]^{\otimes m}(t, Z_m) \\ & \quad \rightarrow w^\infty\left(x, \int_{\mathbb{R}^d_\xi} f^\infty(t, x, \xi) d\xi\right) \left(\int_{\Omega \times \mathbb{R}^d} \xi f^\infty(t, x, \xi) dx d\xi - \xi \right). \end{aligned} \quad (2.37)$$

Finally, we get the infinite-wise mesoscopic limit of the higher order Hegselmann–Krause model:

$$\partial_t f^\infty(t, x, \xi) + \nabla_\xi \cdot \left(\left[w^\infty(x, \rho^\infty(t)) \left(\int_{\Omega \times \mathbb{R}^d_\xi} \xi df^\infty(t) - \xi \right) \right] f^\infty \right) = 0,$$

with $\rho^\infty(t, x) = \int_{\mathbb{R}^d} f^\infty(t, x, \xi) d\xi$.

For the macroscopic equation, by part (ii) of Theorem 2.9 we know that the limit infinite-wise macroscopic equation is given by

$$\partial_t y^\infty(t, x) = \mathcal{G}_{x, y^\infty(t)}^\infty[\nu \otimes \delta(\xi - y^\infty(t, x))],$$

where

$$\mathcal{G}_{x,\xi}^\infty[f] = w^\infty\left(x, \int_{\mathbb{R}^d_\xi} df(x', \xi')\right) \left(\int_{\Omega \times \mathbb{R}^d_\xi} \xi' df(x', \xi') - \xi \right), \quad \forall f \in \mathcal{P}(\Omega \times \mathbb{R}^d)$$

is the limit function obtained in (2.37). It follows that the infinite-wise limit of the higher order Hegselmann–Krause model has the form

$$\partial_t y^\infty(t, x) = w^\infty(x, \nu) \left(\int_{\Omega} y^\infty(t, x') d\nu(x') - y^\infty(t, x) \right).$$

Example 2.13 (Infinite-wise Kuramoto model). *We now want to get an explicit expression for the infinite-wise limit of the mesoscopic and macroscopic equations for the Kuramoto model introduced in (2.20) and (2.21). We assume that:*

- (1) There exists $C > 0$ such that $\|a^{(m)}\|_{L^\infty(\Omega^{m+1})} \leq C$ for every $m \in \mathbb{N}^*$.
- (2) For each $x, x_1, \dots, x_m, x'_1, \dots, x'_m \in \Omega$, there exists $C > 0$ such that

$$\left| a^{(m)}(x, x_1, \dots, x_m) - a^{(m)}(x, x'_1, \dots, x'_m) \right| \leq CW_1 \left(\frac{1}{m} \sum_{k=1}^m \delta_{x_k}, \frac{1}{m} \sum_{h=1}^m \delta_{x'_h} \right).$$

Proceeding as in the previous example, we get the infinite-wise mesoscopic and macroscopic equations for the higher order Kuramoto model given by

$$\begin{aligned} \partial_t f^\infty(t, x, \xi) + \omega(x) \partial_\xi f^\infty(t, x, \xi) \\ + \partial_\xi \left(a^\infty(x, \rho^\infty(t)) \sin \left(\int_{\Omega \times \mathbb{R}^d} \xi' df^\infty(t, x', \xi') - \xi \right) f^\infty(t, x, \xi) \right) = 0, \end{aligned}$$

with $\rho^\infty(t, x) = \int_{\mathbb{R}^d} f^\infty(t, x, \xi) d\xi$, and

$$\partial_t y^\infty(t, x) = \omega(x) - a^\infty(x, \nu) \sin \left(y^\infty(t, x) - \int_{\Omega} y^\infty(t, x') d\nu(x') \right),$$

where $a^\infty(x, \cdot) \in \mathcal{C}^0(\mathcal{P}(\Omega))$ is the limit of the symmetric functions $a^{(m)}(x, \cdot)$ considered.

Example 2.14 (Infinite-wise smooth complex contagion model). *In this last example, we get an explicit expression for the infinite-wise limit of the mesoscopic and macroscopic equations for the smooth complex contagion model introduced in (2.22) and (2.25).*

We recall the assumptions in Example 2.8 and the notation for the reinforcement functional $Q^{[m]}$ in (2.24). Since $G^{(m)}(\cdot, Z_m)$ is globally Lipschitz on $\Omega \times [0, 1]$ with a constant depending only on $\alpha_{\max}, \beta_{\max}, \kappa_{\max}, L_\kappa, L_\sigma$ independent of m , and $\mu \mapsto Q^{[m]}[\mu]$ is Lipschitz for the \mathcal{W}_1 distance on $\mathcal{P}(\Omega \times [0, 1])$, with a Lipschitz constant independent of m , we can apply again the procedure in Theorem 2.9 to obtain the mesoscopic and macroscopic limits.

However in this case we notice that, by expanding the square,

$$Q^{[m]}[\mu](x) = \frac{1}{m} M_2[\mu](x) + \frac{m-1}{m} M_1[\mu](x)^2, \quad x \in \Omega, \quad \mu \in \mathcal{P}(\Omega \times [0, 1])$$

where

$$M_1[\mu](x) := \int Z(x; x', \xi') d\mu(x', \xi'), \quad M_2[\mu](x) := \int Z(x; x', \xi')^2 d\mu(x', \xi'),$$

with $Z(x; x', \xi')$ defined in (2.24). Therefore, defining $Q^\infty[\mu](x) := M_1[\mu](x)^2$, we get the convergence rate

$$|Q^{[m]}[\mu] - Q^\infty[\mu]| \leq \frac{1}{m} |M_2[\mu] - M_1[\mu]^2| \leq \frac{\kappa_{\max}^2}{m}, \quad (2.38)$$

since $Z(x, x', \xi') \in [0, \kappa_{\max}]$. Hence, we have that both $Q^{[m]}$ and $\mathcal{X}^{[m]}$ defined in (2.23) are Cauchy sequences respectively in $\Omega \times \mathcal{P}(\Omega \times [0, 1])$ and $\Omega \times [0, 1] \times \mathcal{P}(\Omega \times [0, 1])$ (compare this with (i) in Remark 2.10).

Finally, we get the infinite-wise mesoscopic equation

$$\partial_t f^\infty(t, x, \xi) + \partial_\xi ([\alpha(x) Q^\infty[f^\infty(t)](x) (1 - \xi) - \beta(x) \xi] f^\infty(t, x, \xi)) = 0,$$

and the macroscopic equation

$$\partial_t y^\infty(t, x) = \alpha(x) Q^\infty[\nu \otimes \delta_{y^\infty(t, x)}](1 - y^\infty(t, x)) - \beta(x) y^\infty(t, x).$$

Remark 2.15. We collect here some observations related to the Examples 2.12, 2.13 and 2.14:

(i) In the case of the complex contagion model in Example 2.14, one can obtain an explicit rate of convergence of $f^{(m)}$ to f^∞ by using (2.38), namely, for $t \in [0, T]$,

$$\mathcal{W}_1(f^{(m)}(t), f^\infty(t)) \lesssim \frac{\exp(CT)}{m}.$$

(ii) Consider the multiple-wise and infinite-wise Kuramoto models in Examples 2.7 and 2.13, and assume for simplicity that the weights $a^{(m)}, a^\infty$ are constant and equal to $K > 0$. When $m = 1$, the mesoscopic pairwise interaction term is

$$\mathcal{X}^{[1]}[\mu](\xi) = K \int_{\Omega \times [0, 2\pi]} \sin(\xi' - \xi) d\mu(x', \xi'), \quad (2.39)$$

which differs from the infinite-wise mesoscopic term

$$\mathcal{X}^\infty[\mu](\xi) = K \sin \left(\int_{\Omega \times [0, 2\pi]} \xi' d\mu(x', \xi') - \xi \right). \quad (2.40)$$

Equation (2.39) describes a μ -average of the pairwise interactions between all states, whereas equation (2.40) represents an interaction with the μ -average of the states. The same conclusions apply to the corresponding macroscopic equation, as well as to the Hegselmann–Krause model with non-constant weights and to the complex contagion model.

2.3.3. *An example where the limit \mathcal{G}^∞ is not explicit.* Consider the generalization of the complex contagion model introduced in Example 2.8, defined by

$$G^{(m)}(x, \xi; z_1, \dots, z_m) = \alpha(x) F_m \left(x; \frac{1}{m} \sum_{\ell=1}^m \delta_{z_\ell} \right) (1 - \xi) - \beta(x) \xi, \quad z_\ell = (x_\ell, \xi_\ell),$$

where $\{F_m\}_{m \geq 1}$ is a sequence of maps $F_m : \Omega \times \mathcal{P}(\Omega \times D) \rightarrow [0, 1]$ satisfying

$$|F_m(x; \mu) - F_m(x'; \nu)| \leq L_F (d_\Omega(x, x') + \mathcal{W}_1(\mu, \nu)) \quad \text{for all } x, x' \in \Omega, \mu, \nu \in \mathcal{P}(\Omega \times D).$$

With the notation of Example 2.8, we have

$$\mathcal{X}^{[m]}[\mu](x, \xi) = \alpha(x) S_m(x; \mu) (1 - \xi) - \beta(x) \xi,$$

where $S_m(x; \mu) := \int_{(\Omega \times D)^m} F_m(x, \eta_{Z_m}^{(m)}) d\mu^{\otimes m}(Z_m)$ and $\eta_{Z_m}^{(m)} = \frac{1}{m} \sum_{\ell=1}^m \delta_{z_\ell}$.

Under the assumptions of Example 2.8, the family $\{\mathcal{X}^{[m]}\}_{m \geq 1}$ is uniformly bounded and equi-Lipschitz on the compact domain $\Omega \times [0, 1] \times \mathcal{P}(\Omega \times D)$ endowed with the metric $d_\Omega + |\cdot| + \mathcal{W}_1$. By the Arzelà–Ascoli theorem, there exists a subsequence (m_k) and a continuous limit map

$$\mathcal{G}^\infty : \Omega \times [0, 1] \times \mathcal{P}(\Omega \times D) \rightarrow \mathbb{R}$$

such that

$$\mathcal{X}^{[m_k]} \longrightarrow \mathcal{G}^\infty \quad \text{uniformly on } \Omega \times [0, 1] \times \mathcal{P}(\Omega \times D).$$

In particular, no convergence of the sequence $\{F_m\}$ is assumed, and different subsequences may yield different cluster points. The limit is therefore not explicit.

3. PROOF OF THEOREM 2.1: VALIDATION OF THE MESOSCOPIC EQUATION

Proof of part (i) of Theorem 2.1: Existence and uniqueness of weak solutions.

The existence and uniqueness of the solution follow by a classical contraction argument by defining, for $T > 0$, the map

$$\mathcal{F} : \mathcal{C}([0, T]; \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d)) \rightarrow \mathcal{C}([0, T]; \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d))$$

such that $\mathcal{F}(\nu)(t) := \varphi_{\nu(0)}(t) * \mu_0$, for $\nu \in \mathcal{C}([0, T]; \mathcal{P}_p(\Omega_x \times \mathbb{R}_\xi^d))$. The contractivity and global existence follow directly from the assumption that $G^{(m)}$ is globally Lipschitz. \square

Proof of part (ii) of Theorem 2.1: Dobrushin-type estimate.

Let $\Pi_0 \in \mathcal{C}(\mu_N^e(0), f(0)) \in \mathcal{P}_p((\Omega_x \times \mathbb{R}_\xi^d)^2)$ be an optimal coupling between $\mu_N^e(0)$ and $f(0)$ and let $\varphi_{\mu_N^e(0)}(t)$ be the Vlasov flow at time t generated by the particle system in (1.1) and $\varphi_{f_0}(t)$ the Vlasov flow generated by the solutions to the mesoscopic equation (2.5). Then the probability measure on $\mathcal{P}_p((\Omega_x \times \mathbb{R}_\xi^d)^2)$ given by

$$\Pi_t(x^1, \xi^1, x^2, \xi^2) := \left(\varphi_{\mu_N^e(0)}(t) \otimes \varphi_{f_0}(t) \right)_* \Pi_0(x^1, \xi^1, x^2, \xi^2),$$

is a coupling between $\mu_N^e(t)$ and $f(t)$, solutions to the mesoscopic Vlasov-type equation (2.5) with initial data respectively $\mu_N^e(0)$ and $f(0)$. Indeed notice that the marginals of Π_t on the first (x^1, ξ^1) and

second (x^2, ξ^2) couple of variables are respectively $\mu_N^e(t)$ and $f(t)$, since $\mu_N^e(t) = (\varphi_{\mu_N^e(0)}(t))_* \mu_N^e(0)$ and $f(t) = (\varphi_{f_0}(t))_* f_0$ (see also [21, Lemma 3.2]).

By definition of Wasserstein distance, we have that

$$\mathcal{W}_p(\mu_N^e(t), f(t)) \leq \left(\int_{(\Omega \times \mathbb{R}^d)^2} (\mathrm{d}_\Omega(x^1, x^2) + |\xi^1 - \xi^2|)^p d\Pi_t(x^1, \xi^1, x^2, \xi^2) \right)^{\frac{1}{p}} =: D(t)$$

and, by definition of push-forward,

$$D(t)^p = \int_{(\Omega \times \mathbb{R}^d)^2} (\mathrm{d}_\Omega(x^1, x^2) + |\xi^1(t; x^1, \xi^1) - \xi^2(t; x^2, \xi^2)|)^p d\Pi_0(x^1, \xi^1, x^2, \xi^2),$$

where

$$\xi^1(t; x^1, \xi^1) := \pi_\xi(\varphi_{\mu_N^e(0)}(t)(x^1, \xi^1)), \quad \xi^2(t; x^2, \xi^2) := \pi_\xi(\varphi_{f(0)}(t)(x^2, \xi^2))$$

and $\pi_\xi : \Omega_x \times \mathbb{R}_\xi^d \rightarrow \mathbb{R}_\xi^d$ is the projection on the second component. We have

$$\begin{aligned} |\xi^1(t; x^1, \xi^1) - \xi^2(t; x^2, \xi^2)| &\leq |\xi^1 - \xi^2| \\ &+ \int_0^t \underbrace{\left| \mathcal{X}^{[m]}[\mu_N^e(\tau)](\tau, x^1, \xi^1(\tau; x^1, \xi^1)) - \mathcal{X}^{[m]}[f(\tau)](\tau, x^1, \xi^1(\tau; x^1, \xi^1)) \right|}_{=: \mathcal{T}_1} d\tau \\ &+ \int_0^t \underbrace{\left| \mathcal{X}^{[m]}[f(\tau)](\tau, x^1, \xi^1(\tau; x^1, \xi^1)) - \mathcal{X}^{[m]}[f(\tau)](\tau, x^2, \xi^2(\tau; x^2, \xi^2)) \right|}_{=: \mathcal{T}_2} d\tau. \end{aligned}$$

For $q \in [1, +\infty]$, by using the regularity assumption (2) on $G^{(m)}$, we get

$$\begin{aligned} \mathcal{T}_1 &\leq \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2m}} \left| G^{(m)}(\tau, x^1, \xi^1(\tau; x^1, \xi^1), Z_m) - G^{(m)}(\tau, x^1, \xi^1(\tau; x^1, \xi^1), Z'_m) \right| d\Pi_\tau^{(m)}(Z_m, Z'_m) \\ &\leq \mathrm{Lip}_{x, \xi}^{[q]}(G) \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2m}} \mathrm{d}_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[q]}(Z_m, Z'_m) d\Pi_\tau^{(m)}(Z_m, Z'_m), \end{aligned}$$

for any coupling $\Pi_\tau^{(m)} \in \mathcal{P}_p((\Omega_x \times \mathbb{R}_\xi^d)^{2m})$ between $(\mu_N^e)^{\otimes m}(\tau)$ and $f^{\otimes m}(\tau)$ and where

$$Z_m = (x_1, \xi_1, \dots, x_m, \xi_m) \in (\Omega_x \times \mathbb{R}_\xi^d)^m, \quad Z'_m = (x'_1, \xi'_1, \dots, x'_m, \xi'_m) \in (\Omega_x \times \mathbb{R}_\xi^d)^m.$$

The second term can be bounded thanks to the regularity assumption (2) on $G^{(m)}$:

$$\begin{aligned} \mathcal{T}_2 &\leq |\mathcal{X}^{[m]}[f(\tau)](\tau, x^1, \xi^1(\tau; x^1, \xi^1)) - \mathcal{X}^{[m]}[f(\tau)](\tau, x^2, \xi^2(\tau; x^2, \xi^2))| \\ &\leq \mathrm{Lip}_{x, \xi}^{[q]}(G) (\mathrm{d}_\Omega(x^1, x^2) + |\xi^1(\tau; x^1, \xi^1) - \xi^2(\tau; x^2, \xi^2)|). \end{aligned}$$

By triangle inequality on the p -norm and the Minkowski's integral inequality, we arrive at

$$\begin{aligned} D(t) &\leq \mathcal{W}_p(\mu_N^e(0), f(0)) + \mathrm{Lip}_{x, \xi}^{[q]}(G) \int_0^t D(\tau) d\tau \\ &\quad + \mathrm{Lip}_{x, \xi}^{[q]}(G) \int_0^t \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2m}} \mathrm{d}_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[q]}(Z_m, Z'_m) d\Pi_\tau^{(m)}(Z_m, Z'_m) d\tau, \end{aligned}$$

where we also used the optimality of Π_0 . By choosing

$$\Pi_\tau^{(m)}(Z_m, Z'_m) := \Pi_\tau^{\otimes m}(Z_m, Z'_m)$$

and using the inequality

$$\mathrm{d}_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[q]}(Z_m, Z'_m) \leq \mathrm{d}_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[1]}(Z_m, Z'_m),$$

we get

$$\begin{aligned} \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2m}} d_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[q]}(Z_m, Z'_m) d\Pi_\tau^{(m)}(Z_m, Z'_m) &\leq \int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2m}} d_{(\Omega_x \times \mathbb{R}_\xi^d)^m}^{[1]}(Z_m, Z'_m) d\Pi_\tau^{\otimes m}(Z_m, Z'_m) \\ &= m \int_{\Omega_x \times \mathbb{R}_\xi^d} d_{\Omega_x \times \mathbb{R}_\xi^d}(z^1, z^2) d\Pi_\tau(z^1, z^2) \leq mD(\tau). \end{aligned}$$

We conclude that

$$D(t) \leq \mathcal{W}_p(\mu_N^e(0), f(0)) + \text{Lip}_{x,\xi}^{[q]}(G)(1+m) \int_0^t D(\tau) d\tau,$$

and we obtain (2.10) by the Grönwall's lemma. \square

Proof of part (iii) of Theorem 2.1: Propagation of chaos.

We start constructing a symmetric coupling $\Pi_t \in \mathcal{C}(F^N(t), f(t)^{\otimes N}) \subset \mathcal{P}_p((\Omega_x \times \mathbb{R}_\xi^d)^{2N})$ in the following way: At the initial time we consider

$$\Pi_0(Z_N^1, Z_N^2) := \delta_{(Z_N^2)}(Z_N^1) \otimes f(0)^{\otimes N}(Z_N^2),$$

where $Z_N^i \in (\Omega_x \times \mathbb{R}_\xi^d)^N$ for $i = 1, 2$ are defined in (1.2). We then introduce

$$\Pi_t(Z_N^1, Z_N^2) := (\Phi^N(t) \otimes \varphi_{f_0}(t)^{\otimes N})_* \Pi_0(Z_N^1, Z_N^2),$$

where $\Phi^N(t)$ is the particle flow at time t generated by the particle system in (1.1), while $\varphi_{f_0}(t)$ is the Vlasov flow given in (2.9). Notice that Π_t is a coupling between $F^N(t)$ and $f^{\otimes N}(t)$ since $F^N(t) = \Phi^N(t)_* F^N(0)$ and $f^{\otimes N}(t) = [\varphi_{f_0}^{\otimes N}(t)]_* f_0^{\otimes N}$.

Hence, since $(\Pi_t)_{N:k} \in \mathcal{C}(F^N(t)_{N:k}, f^{\otimes k}(t))$, we get

$$\begin{aligned} \mathcal{W}_p^{[q]}(F^N(t)_{N:k}, f(t)^{\otimes k}) &\leq \left(\int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2k}} \left(d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}(Z_k^1, Z_k^2) \right)^p d(\Pi_t)_{N:k}(Z_k^1, Z_k^2) \right)^{\frac{1}{p}} \\ &= \left(\int_{(\Omega_x \times \mathbb{R}_\xi^d)^{2N}} \left(d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}(Z_k^1, Z_k^2) \right)^p d\Pi_t(Z_N^1, Z_N^2) \right)^{\frac{1}{p}} \\ &= \left(\int \|\Xi_k^1(t; Z_N^2) - \Xi_k^2(t; Z_N^2)\|_{\ell^q}^p df(0)^{\otimes N}(Z_N^2) \right)^{\frac{1}{p}}, \end{aligned} \quad (3.1)$$

where $\Xi_k^1(t; Z_N^2) := \pi_\xi^{\otimes k}(\Phi^N(t)(Z_N^2))$ and $\Xi_k^2(t; Z_N^2) := \pi_\xi^{\otimes k}(\varphi_{f_0}(t)^{\otimes N}(Z_N^2))$ with $\pi_\xi : \Omega_x \times \mathbb{R}_\xi^d \rightarrow \mathbb{R}_\xi^d$ the projection on the ξ variable.

Let $\mu_N^e(t)$ be the empirical measure supported on the particle system at time t with initial data Z_N^2 . Similarly to the proof of part (ii) of Theorem 2.1, we have

$$\begin{aligned} \|\Xi_k^1(t) - \Xi_k^2(t)\|_{\ell^q} &\leq \text{Lip}_{x,\xi}^{[q]}(G) \int_0^t \|\Xi_k^1(\tau) - \Xi_k^2(\tau)\|_{\ell^q} d\tau + k^{\frac{1}{q}} \text{Lip}_{x,\xi}^{[q]}(G) \int_0^t \mathcal{W}_1^{[q]}(\mu_N^e(\tau)^{\otimes m}, f(\tau)^{\otimes m}) d\tau \\ &\leq \text{Lip}_{x,\xi}^{[q]}(G) \int_0^t \|\Xi_k^1(\tau) - \Xi_k^2(\tau)\|_{\ell^q} d\tau + k^{\frac{1}{q}} m \text{Lip}_{x,\xi}^{[q]}(G) \int_0^t \mathcal{W}_1(\mu_N^e(\tau), f(\tau)) d\tau, \end{aligned}$$

where we used that $\mathcal{W}_1^{[q]}(\mu_N^e(\tau)^{\otimes m}, f(\tau)^{\otimes m}) \leq m \mathcal{W}_1(\mu_N^e(\tau), f(\tau))$ (see e.g. [36, Lemma 14 (Appendix A.1.4)]). By the Dobrushin's estimate obtained in (2.10) and the Grönwall's lemma, we get

$$\|\Xi_k^1(t) - \Xi_k^2(t)\|_{\ell^q} \leq \exp\left(\text{Lip}_{x,\xi}^{[q]}(G)t\right) k^{\frac{1}{q}} \frac{m}{m+1} \left[\exp\left(\text{Lip}_{x,\xi}^{[q]}(G)(1+m)t\right) - 1 \right] \mathcal{W}_1(\mu_N^e(0), f(0)).$$

Since $\mathcal{W}_1 \leq \mathcal{W}_p$, by (3.1) we obtain

$$\mathcal{W}_p^{[q]}(F^N(t)_{N:k}, f(t)^{\otimes k}) \leq k^{\frac{1}{q}} \exp\left(\text{Lip}_{x,\xi}^{[q]}(G)(2+m)t\right) \left(\int \mathcal{W}_p(\mu_N^e(0), f(0))^p df_0\right)^{\frac{1}{p}}.$$

We now use a quantitative version of the law of large numbers given by the Fournier and Guillin's bound (see [19]). Assuming that $\Omega_x \subset \mathbb{R}^d$, for $z > 0$ and $p \geq 1$, we get

$$\int \mathcal{W}_p(\mu_N^e(0), f(0))^p df_0 \leq C \mathcal{M}_z(f(0))^{p/z} \begin{cases} N^{-\frac{1}{2}} + N^{-(z-p)/z} & \text{if } p > d \text{ and } z \neq 2p \\ N^{-\frac{1}{2}} \log(1+N) + N^{-(z-p)/z} & \text{if } p = d \text{ and } z \neq 2p \\ N^{-\frac{p}{2d}} + N^{-(z-p)/z} & \text{if } p \in (0, d) \text{ and } z \neq 2d/(2d-p), \end{cases}$$

where $\mathcal{M}_z(f) = \int |x|^z df(x)$. \square

4. PROOF OF THEOREM 2.4: VALIDATION OF THE MACROSCOPIC EQUATION

Proof of part (i) and (ii) of Theorem 2.4: Monokineticity and existence of solutions.

The proof of part (i) follows by testing $f_{\text{mon}}(t, x, \xi) = \nu(x) \otimes \delta_{y(t,x)}(\xi)$ against a test function and noticing that

$$\mathcal{X}^{[m]}[f_{\text{mon}}(t)](t, x, \xi) = \int_{\Omega^m} G^{(m)}(t, x, \xi, x_1, y(t, x_1), \dots, x_m, y(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m).$$

The existence and uniqueness of global solutions to (2.13) follows again by a contraction argument. Indeed, given $y_0 \in L^\infty(\Omega, \mathbb{R}^d)$, we define the operator

$$\mathcal{F} : L^\infty\left([0, +\infty); L^\infty(\Omega_x; \mathbb{R}_\xi^d)\right) \rightarrow L^\infty\left([0, +\infty); L^\infty(\Omega_x; \mathbb{R}_\xi^d)\right)$$

such that

$$\mathcal{F}(\psi)(t, x) := y_0(x) + \int_0^t \int_{\Omega^m} G^{(m)}(s, x, \psi(s, x), x_1, \psi(s, x_1), \dots, x_m, \psi(s, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) ds.$$

By the Lipschitz assumption on $G^{(m)}$ given in B, it follows that, for any $C_m > 0$,

$$\sup_{t \in [0, +\infty)} e^{-2C_m t} \|\mathcal{F}(\psi_1)(t) - \mathcal{F}(\psi_2)(t)\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq \frac{\text{Lip}_{x,\xi}^{[q]}(G^{(m)})(m+1)}{2C_m} \sup_{t \in [0, +\infty)} e^{-2C_m t} \|\psi_1 - \psi_2\|_{L^\infty(\Omega, \mathbb{R}^d)}. \quad (4.1)$$

Choosing $C_m := \text{Lip}_{x,\xi}^{[q]}(G^{(m)})(m+1)$, we obtain that \mathcal{F} is a contraction in $L^\infty\left([0, +\infty); L^\infty(\Omega_x; \mathbb{R}_\xi^d)\right)$ with respect to the weighted metric defined in (4.1).

Hence, there exists a unique $y^{(m)} \in L^\infty\left([0, +\infty); L^\infty(\Omega_x; \mathbb{R}_\xi^d)\right)$ such that

$$y^{(m)}(t, x) = y_0 + \int_0^t \int_{\Omega^m} G^{(m)}(s, x, y^{(m)}(s, x), x_1, y^{(m)}(s, x_1), \dots, x_m, y^{(m)}(s, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) ds. \quad (4.2)$$

By the integral representation in (4.2) and the regularity of $G^{(m)}$, it follows that $y^{(m)}(\cdot, x)$ is also continuously differentiable in time and that (2.13) holds. \square

Proof of part (iii) of Theorem 2.4: From micro to macro through the graph limit.

We start by proving two technical lemmas.

Lemma 4.1. (*Continuity in the label variable*)

Let $y^{(m)}(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$ be solution to the macroscopic equation (2.13) with initial data y_0 and let $x, x' \in \Omega$ be arbitrary. We have

$$|y^{(m)}(t, x) - y^{(m)}(t, x')| + d_\Omega(x, x') \leq e^{\text{Lip}_{x,\xi}^{[q]}(G)t} (|y_0(x) - y_0(x')| + d_\Omega(x, x')) \quad \forall t \geq 0. \quad (4.3)$$

Proof of Lemma 4.1. By definition, we have

$$\begin{aligned} & \partial_t y^{(m)}(t, x) - \partial_t y^{(m)}(t, x') \\ &= \int_{\Omega^m} \left[G^{(m)}(t, x, y^{(m)}(t, x), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) \right. \\ & \quad \left. - G^{(m)}(t, x', y^{(m)}(t, x'), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) \right] d\nu^{\otimes m}(x_1, \dots, x_m) \end{aligned}$$

and using the Lipschitz assumption on $G^{(m)}$, we obtain

$$\left| \partial_t(y^{(m)}(t, x) - y^{(m)}(t, x')) \right| \leq \text{Lip}_{x, \xi}^{[q]}(G) \left(|y^{(m)}(t, x) - y^{(m)}(t, x')| + d_\Omega(x, x') \right)$$

and (4.3) follows by the Grönwall's lemma. \square

Lemma 4.2. (*Approximation by Riemann sums*)

Let Ω be a compact complete metric space and, given $\nu \in \mathcal{P}(\Omega)$, assume there exists a family of tagged partitions $(\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N), \mathcal{X}_N = (x_1^N, \dots, x_N^N))$ of Ω associated with ν as in (2.14). Moreover, let $y^{(m)}(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$ be solution to the macroscopic equation (2.13) with a Lipschitz continuous initial data y_0 . Given $N \in \mathbb{N}^*$, we have

$$\begin{aligned} & \left| \int_{\Omega^m} G^{(m)}(x, y^{(m)}(t, x), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) \right. \\ & \quad \left. - \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N G^{(m)}(x, y^{(m)}(t, x), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \right| \\ & \leq m^{\frac{1}{q}} \text{Lip}_{x, \xi}^{[q]}(G) \frac{C_\Omega}{N^r} e^{t \text{Lip}_{x, \xi}^{[q]}(G)} (\text{Lip}_x(y_0) + 1). \end{aligned} \quad (4.4)$$

Proof. Writing

$$\int_{\Omega^m} G^{(m)}(\cdot) d\nu^{\otimes m} = \sum_{j_1, \dots, j_m=1}^N \int_{\Omega_{j_1}^N \times \dots \times \Omega_{j_m}^N} G^{(m)}(\cdot) d\nu^{\otimes m}$$

and using that $\nu(\Omega_i^N) = \frac{1}{N}$ and $\text{diam}_\Omega(\Omega_i^N) \leq \frac{C_\Omega}{N^r}$ (see (2.14)), we have

$$\begin{aligned} \epsilon_x^N(t) &:= \left| \int_{\Omega^m} G^{(m)}(t, x, y^{(m)}(t, x), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) \right. \\ & \quad \left. - \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N G^{(m)}(t, x, y^{(m)}(t, x), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \right| \\ & \leq \sum_{j_1, \dots, j_m=1}^N \int_{\Omega_{j_1}^N \times \dots \times \Omega_{j_m}^N} \left| G^{(m)}(t, x, y^{(m)}(t, x), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) \right. \\ & \quad \left. - G^{(m)}(t, x, y^{(m)}(t, x), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \right| d\nu^{\otimes m}(X_m) \\ & \leq \text{Lip}^{[q]}(G) \sum_{j_1, \dots, j_m=1}^N \int_{\Omega_{j_1}^N \times \dots \times \Omega_{j_m}^N} d_{\Omega \times \mathbb{R}^d}^{[q]}((\mathcal{X}_m, \mathcal{Y}^{(m)}(t, \mathcal{X}_m)), (X_m, Y^{(m)}(t, X_m))) d\nu^{\otimes m}(X_m), \end{aligned}$$

where $\mathcal{X}_m := (x_{j_1}^N, \dots, x_{j_m}^N)$, $\mathcal{Y}^{(m)}(t, \mathcal{X}_m) := (y^{(m)}(t, x_{j_1}^N), \dots, y^{(m)}(t, x_{j_m}^N))$ and $Y^{(m)}(t, X_m) := (y^{(m)}(t, x_1), \dots, y^{(m)}(t, x_m))$.

Notice that, by (4.3) in Lemma 4.1, we get

$$d_{\Omega \times \mathbb{R}^d}^{[q]}((\mathcal{X}_m, \mathcal{Y}^{(m)}), (X_m, Y^{(m)})) \leq \left(\sum_{i=1}^m (e^{\text{Lip}_{x,\xi}^{[q]}(G)t} (\text{Lip}_x(y_0) + 1) d_{\Omega}(x_{j_i}^N, x_i))^q \right)^{\frac{1}{q}}.$$

Since $x_i, x_{j_i}^N \in \Omega_{j_i}^N$, we get that $d_{\Omega}(x_{j_i}^N, x_i) \leq \text{diam}_{\Omega}(\Omega_{j_i}^N) \leq C_{\Omega}/N^r$, it follows that

$$\epsilon_x^N(t) \leq m^{\frac{1}{q}} \text{Lip}^{[q]}(G) \frac{C_{\Omega}}{N^r} e^{\text{Lip}_{x,\xi}^{[q]}(G)t} (\text{Lip}_x(y_0) + 1)$$

which gives (4.4). \square

We are now ready to prove part (iii) of Theorem 2.4. We set $r_i^N(t) := y^{(m)}(t, x_i^N) - \xi_i^N(t)$, for $i = 1, \dots, N$. By definition, we have

$$\begin{aligned} \dot{r}_i^N(t) &= \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N \left(G^{(m)}(t, x_i^N, y^{(m)}(t, x_i^N), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \right. \\ &\quad \left. - G^{(m)}(t, x_i^N, \xi_i^N(t), x_{j_1}^N, \xi_{j_1}^N(t), \dots, x_{j_m}^N, \xi_{j_m}^N(t)) \right) + \epsilon_i^N(t) \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \epsilon_i^N(t) &= \int_{\Omega^m} G^{(m)}(t, x_i^N, y^{(m)}(t, x_i^N), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) \\ &\quad - \frac{1}{N^m} \sum_{j_1, \dots, j_m=1}^N G^{(m)}(t, x_i^N, y^{(m)}(t, x_i^N), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \end{aligned}$$

with $r_i^N(0) = 0$, for every $i \in \{1, \dots, N\}$. On the one part, we have

$$\begin{aligned} &\left| G^{(m)}(t, x_i^N, y^{(m)}(t, x_i^N), x_{j_1}^N, y^{(m)}(t, x_{j_1}^N), \dots, x_{j_m}^N, y^{(m)}(t, x_{j_m}^N)) \right. \\ &\quad \left. - G^{(m)}(t, x_i^N, \xi_i^N(t), x_{j_1}^N, \xi_{j_1}^N(t), \dots, x_{j_m}^N, \xi_{j_m}^N(t)) \right| \leq \text{Lip}_{x,\xi}^{[q]}(G) (m+1)^{\frac{1}{q}} \|R^N(t)\|_{\ell^\infty}, \end{aligned} \quad (4.6)$$

where $R^N(t) = (r_1^N(t), \dots, r_N^N(t))$. On the other part, using (4.4) in Lemma 4.2, we have

$$|\epsilon_i^N(t)| \leq m^{\frac{1}{q}} \frac{C_{\Omega}}{N^r} e^{t \text{Lip}_{x,\xi}^{[q]}(G)} (\text{Lip}_x(y_0) + 1). \quad (4.7)$$

Finally, we infer from (4.5), (4.6) and (4.7) that

$$\frac{d}{dt} \|R^N(t)\|_{\ell^\infty} \leq \|\dot{R}^N(t)\|_{\ell^\infty} \leq \text{Lip}_{x,\xi}^{[q]}(G) (m+1)^{\frac{1}{q}} \|R^N(t)\|_{\ell^\infty} + m^{\frac{1}{q}} \frac{C_{\Omega}}{N^r} e^{t \text{Lip}_{x,\xi}^{[q]}(G)} (\text{Lip}(y_0) + 1)$$

from which we obtain (2.17).

Let us establish (2.16). For every $x \in \Omega$ there exists $i \in \{1, \dots, N\}$ such that $x \in \Omega_i^N$, and thus $d_{\Omega}(x, x_i^N) \leq \text{diam}_{\Omega}(\Omega_i^N) \leq \frac{C_{\Omega}}{N^r}$ (by (2.14)). It follows that

$$\|y^{(m)}(t, x) - y^{(m)}(t, x_i^N)\| \leq \text{Lip}_x(y^{(m)}(t, \cdot)) d_{\Omega}(x, x_i^N) \leq \frac{C_{\Omega}}{N^r} e^{t \text{Lip}_{x,\xi}^{[q]}(G)} (\text{Lip}_x(y(0, \cdot)) + 1)$$

and, noting that $y^N(t, x) = \xi_i^N(t)$, (2.16) follows by the triangular inequality, using (2.17). \square

5. PROOF OF THEOREM 2.9: INFINITE-WISE LIMIT

In order to prove Theorem 2.9, we need some results about the limit of symmetric functions of many variables, recalled in the following proposition.

Proposition 5.1. *Given (Ω, d_Ω) a compact metric space and $m \in \mathbb{N}^*$, let $F^{(m)} : \Omega^m \rightarrow \mathbb{R}^d$ a symmetric function of the m variables, i.e.*

$$F^{(m)}(x_1, \dots, x_m) = F^{(m)}(x_{\sigma(1)}, \dots, x_{\sigma(m)}), \quad x_1, \dots, x_m \in \Omega$$

where $\sigma \in \mathfrak{S}_m$, the set of permutations of $\{1, \dots, m\}$. The following results hold:

(i) Assume that

- there exists $C_1 > 0$ independent of m such that $\|F^{(m)}\|_{L^\infty(\Omega^m)} \leq C_1$;
- there exists $C_2 > 0$ independent of m such that

$$|F^{(m)}(X_m) - F^{(m)}(Y_m)| \leq C_2 \mathcal{W}_1(\mu_{X_m}^e, \mu_{Y_m}^e), \quad (5.1)$$

for any $X_m = (x_1, \dots, x_m), Y_m = (y_1, \dots, y_m) \in \Omega^m$ and where $\mu_{X_m}^e = \frac{1}{m} \sum_{j=1}^m \delta_{x_j}$; then, there exists a continuous map $F^\infty : \mathcal{P}(\Omega) \rightarrow \mathbb{R}^d$ and a subsequence $m_k \rightarrow +\infty$ as $k \rightarrow +\infty$ such that

$$\lim_{k \rightarrow +\infty} \sup_{X_{m_k} \in \Omega^{m_k}} |F^{(m_k)}(X_{m_k}) - F^\infty(\mu_{X_{m_k}}^e)| = 0. \quad (5.2)$$

Moreover F^∞ is Lipschitz continuous¹ with the same constant C_2 as in (5.1),

$$|F^\infty(\mu) - F^\infty(\nu)| \leq C_2 \mathcal{W}_1(\mu, \nu), \quad \text{for any } \mu, \nu \in \mathcal{P}(\Omega).$$

(ii) (Hewitt–Savage Theorem) Given $\nu \in \mathcal{P}(\Omega)$, there exist a subsequence $m_k \rightarrow +\infty$ as $k \rightarrow +\infty$ and a continuous map $F^\infty : \mathcal{P}(\Omega) \rightarrow \mathbb{R}^d$ given by (5.2) such that

$$\lim_{k \rightarrow +\infty} \int_{\Omega^{m_k}} F^{(m_k)}(x_1, \dots, x_{m_k}) d\nu^{\otimes m_k}(x_1, \dots, x_{m_k}) = F^\infty(\nu).$$

Proof of Proposition 5.1. The statements in points (i) and (ii) coincide with those in [9, Theorem 2.1, Theorem 5.14] (see also [24] for point (ii)), where the corresponding proofs can be found. Since the only difference with respect to [9] lies in the Lipschitz regularity of the limit function in (i), we will focus below on establishing this additional regularity.

In particular, assuming w.l.o.g. $d = 1$, the main point consists in introducing the extension

$$\mathbf{F}^{(m)}(\mu) := \inf_{X_m \in \Omega_m} \{F^{(m)}(X_m) + C_2 \mathcal{W}_1(\mu_{X_m}^e, \mu)\},$$

of $F^{(m)}$ to the space $\mathcal{P}(\Omega_x)$. The existence of the continuous map $F^\infty : \mathcal{P}(\Omega) \rightarrow \mathbb{R}^d$ follows by showing that the sequence of functions $\{\mathbf{F}^{(m)}\}$ is equi-continuous and equi-bounded, and applying the Arzelà-Ascoli theorem.

It remains to verify the Lipschitz continuity of the limit function F^∞ which can be proven by observing that $\mathbf{F}^{(m)}$ is equi-Lipschitz continuous. Indeed, given $\varepsilon > 0$, $\mu, \nu \in \mathcal{P}(\Omega_x)$ and taking $X_m \in \Omega_m$ such that

$$|\mathbf{F}^{(m)}(\nu) - \mathbf{F}^{(m)}(X_m) - C_2 \mathcal{W}_1(\mu_{X_m}^e, \nu)| \leq \varepsilon$$

we get

$$\mathbf{F}^{(m)}(\mu) \leq \mathbf{F}^{(m)}(X_m) + C_2 \mathcal{W}_1(\mu_{X_m}^e, \mu) \leq \mathbf{F}^{(m)}(\nu) + C_2 \mathcal{W}_1(\mu, \nu) + \varepsilon.$$

The thesis follows by the fact that the Lipschitz continuity is preserved in the limit by the Arzelà-Ascoli theorem. \square

¹We thank Pierre-Louis Lions, who mentioned to Thierry Paul the possibility of obtaining improved regularity for the limit F^∞ by imposing stronger assumptions on the sequence of symmetric functions.

Proof of part (i) of Theorem 2.9: Infinite-wise mesoscopic limit.

For a fixed $m \in \mathbb{N}$ and given $T > 0$, we consider the unique solution $f^{(m)} \in \mathcal{C}([0, T], \mathcal{P}_p(\Omega_x \times D_\xi))$ (for every $p \geq 1$) to the mesoscopic equation

$$\partial_t f^{(m)}(t, x, \xi) + \nabla_\xi \cdot \left(\mathcal{X}^{[m]}[f^{(m)}(t)](t, x, \xi) f^{(m)}(t, x, \xi) \right) = 0, \quad (5.3)$$

with initial datum $f_0 \in \mathcal{P}(\Omega_x \times \mathbb{R}^d)$. Let $\{f^{(m)}\}_{m=1}^\infty \subset \mathcal{C}([0, T], \mathcal{P}_p(\Omega_x \times D_\xi))$ be the sequence of probability measures, solutions to (5.3) for $m \in \mathbb{N}$ in the time interval $[0, T]$. By (2.8), since $\|\mathcal{X}^{[m]}\|_\infty \leq M$ uniformly in m , the map $t \mapsto f^{(m)}(t)$ is M -Lipschitz in \mathcal{W}_1 . Since $\mathcal{P}(\Omega \times D)$ is compact, the family is equi-continuous and relatively compact in $\mathcal{C}([0, T]; \mathcal{P}(\Omega \times D))$. Hence, there exists of a subsequence $\{f^{(m_k)}\}_{k \in \mathbb{N}}$ that converges weakly to a given $f^\infty \in \mathcal{C}([0, T], \mathcal{P}_p(\Omega_x \times D_\xi))$ uniformly in $[0, T]$. Thanks to assumption C, we have

$$\begin{aligned} \sup_{x, \xi} \left| \mathcal{X}^{[m]}[f^{(m)}](t, x, \xi) - \mathcal{X}^{[m]}[f^\infty](t, x, \xi) \right| &\leq \frac{\text{Lip}(G)}{m} \mathcal{W}_1^{[1]}([f^{(m)}]^{\otimes m}, [f^\infty]^{\otimes m}) \\ &\leq \text{Lip}(G) \mathcal{W}_1(f^{(m)}, f^\infty), \end{aligned} \quad (5.4)$$

where we used that $\mathcal{W}_1^{[1]}([f^{(m)}]^{\otimes m}, [f^\infty]^{\otimes m}) \leq m \mathcal{W}_1(f^{(m)}, f^\infty)$ (see e.g. [36, Lemma 14 (Appendix A.1.4)]). Notice that

$$\begin{aligned} &\int_{\Omega_x^m \times D_\xi^m} G^{(m)}(t, x, \xi, x_1, \xi_1, \dots, x_m, \xi_m) d(f^\infty)^{\otimes m}(x_1, \xi_1, \dots, x_m, \xi_m) \\ &= \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \int_{\Omega_x^m \times D_\xi^m} G^{(m)}(t, x, \xi, x_{\sigma(1)}, \xi_{\sigma(1)}, \dots, x_{\sigma(m)}, \xi_{\sigma(m)}) d(f^\infty)^{\otimes m}(x_1, \xi_1, \dots, x_m, \xi_m). \end{aligned}$$

Hence, defining

$$\mathcal{G}_{t, x, \xi}^{(m)}(x_1, \xi_1, \dots, x_m, \xi_m) := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} G^{(m)}(t, x, \xi, x_{\sigma(1)}, \xi_{\sigma(1)}, \dots, x_{\sigma(m)}, \xi_{\sigma(m)}),$$

we have that $\mathcal{G}_{t, x, \xi}^{(m)}$ is symmetric in the variables $(x_i, \xi_i)_{i=1}^m \in \Omega_x \times D_\xi$. Moreover, by assumptions A and C, $\mathcal{G}_{t, x, \xi}^{(m)}$ satisfies the hypotheses of part (i) of Proposition 5.1.

Hence there exist a continuous function $\mathcal{G}_{t, x, \xi}^\infty : \mathcal{P}(\Omega \times D) \rightarrow \mathbb{R}^d$ and a subsequence $\{m_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow +\infty} \sup_{Z_{m_k}} \left| \mathcal{G}_{t, x, \xi}^{(m_k)}(Z_{m_k}) - \mathcal{G}_{t, x, \xi}^\infty[\mu_{Z_{m_k}}^\varepsilon] \right| = 0. \quad (5.5)$$

The convergence holds uniformly in $t, x, \xi \in [0, T] \times \Omega \times D$ and this implies also that the limit map $\mathcal{G}_{t, x, \xi}^\infty[\cdot]$ is Lipschitz continuous with constant $\text{Lip}(G)$ in all its dependencies. By the triangle inequality on $\mathcal{G}^{(m)}$ and by part (ii) of Proposition 5.1, we get

$$\lim_{k \rightarrow +\infty} \int_{\Omega^{m_k} \times \mathbb{R}^{dm_k}} \mathcal{G}_{t, x, \xi}^{(m_k)}(x_1, \xi_1, \dots, x_{m_k}, \xi_{m_k}) d(f^\infty)^{\otimes m_k}(x_1, \xi_1, \dots, x_{m_k}, \xi_{m_k}) = \mathcal{G}_{t, x, \xi}^\infty[f^\infty]. \quad (5.6)$$

Taking $\varphi \in \mathcal{C}_b^1(\Omega_x \times D_\xi)$, we have

$$\begin{aligned} \partial_t \langle f^{(m)}, \varphi \rangle &= \left\langle \mathcal{X}^{(m)}[f^{(m)}] f^{(m)}, \nabla_\xi \varphi \right\rangle = \left\langle \left(\mathcal{X}^{(m)}[f^{(m)}] - \mathcal{X}^{(m)}[f^\infty] \right) f^{(m)}, \nabla_\xi \varphi \right\rangle \\ &\quad + \left\langle \left(\mathcal{X}^{(m)}[f^\infty] - \mathcal{G}^\infty[f^\infty] \right) f^{(m)}, \nabla_\xi \varphi \right\rangle + \left\langle \mathcal{G}^\infty[f^\infty] f^{(m)}, \nabla_\xi \varphi \right\rangle. \end{aligned}$$

The first term on the r.h.s. is going to zero thanks to (5.4), the second term thanks to (5.6) and the uniform convergence in the parameters x, ξ . The third term is converging to the limit equation, since \mathcal{G}^∞ is regular in the parameters.

We conclude the proof by observing that the existence and uniqueness of the solution to (2.29) follow from a fixed-point argument, analogous to the one used in the proof of part (i) of Theorem 2.1 and part (i) of Theorem 2.4, thanks to the Lipschitz continuity of the limit map $\mathcal{G}_{t,x,\xi}^\infty[\cdot]$ in all its variables. \square

Remark 5.2. *It is worth noting that the Arzelà–Ascoli theorem could be directly applied to the vector field $\mathcal{X}^{[m]}[\mu](t, x, \xi)$, which is equi-bounded and equi-Lipschitz on $[0, T] \times (\Omega \times D) \times \mathcal{P}(\Omega \times D)$, providing the convergence, up to a subsequence, to a limiting vector field $\mathcal{X}_{t,x,\xi}^\infty[\mu]$. With the argument presented above, we have moreover shown that $\mathcal{X}_{t,x,\xi}^\infty[\mu] = \mathcal{G}_{t,x,\xi}^\infty[\mu]$, where \mathcal{G}^∞ denotes the limit, up to a subsequence, of the symmetrization of G^m introduced in (5.5).*

Proof of part (ii) of Theorem 2.9: Infinite-wise macroscopic limit.

We now study the limit of the macroscopic equation (2.13) as the order m of the multiple-wise interaction goes to infinity. Given Ω_x compact, let us write again the macroscopic equation

$$\partial_t y^{(m)}(t, x) = \int_{\Omega_x^m} G^{(m)}(t, x, y^{(m)}(t, x), x_1, y^{(m)}(t, x_1), \dots, x_m, y^{(m)}(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m), \quad (5.7)$$

for $\nu \in \mathcal{P}(\Omega_x)$ and $m \in \mathbb{N}^*$.

Given $T > 0$, by part (ii) of Theorem 2.4, we know there exists a unique solution $y^{(m)} \in \mathcal{C}^1([0, T]; L^\infty(\Omega_x; \mathbb{R}^d))$ to (5.7) with initial datum $y_0 \in L^\infty(\Omega_x; \mathbb{R}^d)$. If moreover y_0 is Lipschitz in Ω_x , thanks to the assumptions A and C, the sequence of solutions $\{y^{(m)}\}_{m=1}^\infty$ is uniformly bounded and Lipschitz equi-continuous. Indeed, by integrating in time (5.7), we get

$$\sup_{t \in [0, T]} \|y^{(m)}(t, \cdot)\|_{L^\infty(\Omega_x)} \leq \|y_0\|_{L^\infty(\Omega_x)} + T \|G^{(m)}\|_{L^\infty} \leq \|y_0\|_{L^\infty(\Omega_x)} + TM,$$

where M is defined in (2.8); moreover, by the Lipschitz regularity of y_0 and using the Grönwall’s lemma,

$$\left| y^{(m)}(t_1, x_1) - y^{(m)}(t_2, x_2) \right| \leq M|t_1 - t_2| + e^{\text{Lip}(G)T} (\text{Lip}(y_0) + T\text{Lip}(G)) d_\Omega(x_1, x_2), \quad (5.8)$$

where $\text{Lip}(G)$ is the constant defined in (2.26). By the Arzelà–Ascoli Theorem, we deduce there exists $y^\infty \in \mathcal{C}([0, T], \mathcal{C}(\Omega_x; \mathbb{R}^d))$ Lipschitz-continuous with the same constants obtained in (5.8) such that, up to a subsequence, $y^{(m)}$ converges to y^∞ in L^∞ .

We now want to show that y^∞ satisfies (2.31). Notice that, since the measure $\nu^{\otimes m}$ is tensorized,

$$\begin{aligned} & \int_{\Omega_x^m} G^{(m)}(t, x, y^\infty(t, x), x_1, y^\infty(t, x_1), \dots, x_m, y^\infty(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) \\ &= \int_{\Omega_x^m \times \mathbb{R}^{dm}} G^{(m)}(t, x, y^\infty(t, x), x_1, \xi_1, \dots, x_m, \xi_m) \prod_{i=1}^m \nu(dx_i) \otimes \delta(\xi_i - y^\infty(t, x_i)), \end{aligned}$$

and that, we can assume without loss of generality, that $G^{(m)}$ is symmetric with respect to the variables $\{x_i, \xi_i\}_{i=1}^m$. By (5.6) in the proof of part (i) of Theorem 2.9 and by part (ii) of Proposition 5.1, we get that

$$\begin{aligned} & \lim_{m \rightarrow +\infty} \int_{\Omega_x^m} G^{(m)}(t, x, y^\infty(t, x), x_1, y^\infty(t, x_1), \dots, x_m, y^\infty(t, x_m)) d\nu^{\otimes m}(x_1, \dots, x_m) \\ &= \mathcal{G}_{t,x,y^\infty(t,x)}^\infty[\nu \otimes \delta(\xi - y^\infty(t, \cdot))]. \end{aligned}$$

\square

Proof of parts (i), (ii) and (iii) of Theorem 2.11: From micro to the infinite-wise mesoscopic and macroscopic equations. The proof follows as a corollary from the previous points. We provide the details of cases (ii) and (iii), as they are less straightforward.

Concerning part (ii), given $\phi \in \mathcal{C}(\Omega_x)$, by triangle inequality we have

$$\left| \int_{\Omega_x} \phi(x) (Y_{N:1}^m(t, dx) - y^\infty(t, x)\nu(dx)) \right| \leq \underbrace{\left| \int_{\Omega_x} \phi(x) (Y_{N:1}^m(t, dx) - y^{(m)}(t, x)\nu(dx)) \right|}_{=:T_1} + \underbrace{\left| \int_{\Omega_x} \phi(x) (y^{(m)}(t)\nu(dx) - y^\infty(t, x)\nu(dx)) \right|}_{=:T_2}.$$

By the characterization of weak convergence through the Wasserstein distance \mathcal{W}_1 , we get

$$\begin{aligned} T_1 &= \left| \int_{\Omega_x \times \mathbb{R}^d} \phi(x) \xi \left(dF^{N,(m)}(t)_{N:1}(x, \xi) - \delta(\xi - y^{(m)}(t, x)) \otimes d\nu(x) \right) \right| \\ &\leq \|\phi\|_{L^\infty} \sup_{t \in [0, T]} \mathcal{W}_1(F^{N,(m)}(t)_{N:1}, \delta(\xi - y^{(m)}(t, x))\nu(dx)), \end{aligned}$$

while, since Ω_x is compact,

$$T_2 \leq \|\phi\|_{L^\infty} \sup_{t \in [0, T]} \|y^{(m)}(t, \cdot) - y^\infty(t, \cdot)\|_{L^\infty(\Omega)}.$$

The conclusion follows by the results in (2.11) and (2.32), by noticing that thanks to assumption C, inequality (2.11) can be improved obtaining a uniform estimate in the multiple-wise parameter m given by

$$\mathcal{W}_1(F^{N,(m)}(t)_{N:1}, f^{(m)}(t)) \leq C \exp\left(2t\text{Lip}(G)\right) \times \begin{cases} N^{-\frac{1}{2d}} & \text{if } d > 1 \\ N^{-\frac{1}{2}} \log(1 + N) & \text{if } d = 1. \end{cases} \quad (5.9)$$

Similarly, concerning part (iii) of Theorem 2.9, the conclusion follows by the results in (2.16) and (2.32), by noticing that thanks to assumption C, inequality (2.16) can be improved obtaining a uniform estimate in the multiple-wise parameter m given by

$$\|y^{N,(m)}(t, \cdot) - y^{(m)}(t, \cdot)\|_{L^\infty(\Omega)} \leq 2\text{Lip}(G) \frac{C_\Omega}{N^r} (1 + \text{Lip}_x(y_0)) e^{3t\text{Lip}(G)}. \quad (5.10)$$

□

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