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INTERPOLATION AND RANDOM INTERPOLATION IN DE BRANGES-ROVNYAK SPACES

ANDREAS HARTMANN AND GIUSEPPE LAMBERTI

ABSTRACT. The aim of this paper is to characterize universal and multiplier interpolating sequences for de Branges-Rovnyak spaces $\mathcal{H}(b)$ where the defining function b is a general non-extreme rational function. Our results carry over to recently introduced higher order local Dirichlet spaces and thus generalize previously known results in classical local Dirichlet spaces. In this setting, we also investigate random interpolating sequences with prescribed radii, providing a 0–1 law.

1. INTRODUCTION

Let \mathcal{H} be a reproducing kernel Hilbert space on a domain $\Omega \subset \mathbb{C}^d$ and let k be its reproducing kernel. Using the notation $k_\lambda(z) = k(\lambda, z)$, $\lambda, z \in \Omega$, we say that a sequence Λ on Ω is (*universal*) *interpolating* for \mathcal{H} (or \mathcal{H} -interpolating) if the operator

$$R_\Lambda: \mathcal{H} \longrightarrow \ell^2$$

$$f \mapsto \left(\frac{f(\lambda_n)}{\|k_{\lambda_n}\|_{\mathcal{H}}} \right)_{n \in \mathbb{N}}$$

is bounded and surjective. We need to introduce another notion of interpolation. For this purpose we recall the *multiplier algebra* of \mathcal{H} as

$$\mathcal{M}(\mathcal{H}) := \{\varphi \in \text{Hol}(\Omega) : \varphi f \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}.$$

We mention, among other properties of this algebra, that every function that belongs to $\mathcal{M}(\mathcal{H})$ must be bounded on Ω . We say that a sequence $\Lambda \subset \Omega$ is *multiplier interpolating* for \mathcal{H} if for every sequence $(a_n)_{n \in \mathbb{N}} \in \ell^\infty$ there exists $\varphi \in \mathcal{M}(\mathcal{H})$ such that $\varphi(\lambda_n) = a_n$. In the literature these sequences are also called interpolating for $\mathcal{M}(\mathcal{H})$ or $\mathcal{M}(\mathcal{H})$ -interpolating.

In general, every multiplier interpolating sequence is also a universal interpolating sequence (see [28, Chapter 2, Proposition 1]), although the converse is not always true (e.g. the Bergman space). However the reverse implication does hold for any reproducing kernel Hilbert space with the Nevanlinna-Pick property. For a recent account on this theory, we refer the reader to [1].

We now proceed to describing more in details the setting of our work. Consider the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and its topological boundary $\mathbb{T} := \partial\mathbb{D}$. The *pseudohyperbolic distance* ρ is defined as

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right| \quad z, w \in \mathbb{D}.$$

An example of a space where the two notions of interpolating sequence introduced before are equivalent is the Hardy space $H^2(\mathbb{D})$, the space of holomorphic function f on \mathbb{D} such that

$$\|f\|_{H^2}^2 := \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty,$$

where dm is the normalized Lebesgue measure on \mathbb{T} . It is not difficult to see that the multiplier algebra of H^2 is precisely H^∞ , the space of bounded holomorphic functions on \mathbb{D} . In [7], Carleson proved that a sequence Λ is interpolating for H^∞ if and only if it satisfies the so-called *Carleson condition*, i.e.

$$\inf_{n \in \mathbb{N}} \prod_{k \neq n} \rho(\lambda_n, \lambda_k) > 0. \quad (\text{C})$$

Later on Shapiro and Shields [30] proved that the same condition characterizes also interpolating sequences for H^2 (and more generally in H^p).

We now introduce the space of holomorphic functions that is the focus of our study. Consider $b \in H^\infty(\mathbb{D})$ such that $\|b\|_{H^\infty} \leq 1$. The *de Branges-Rovnyak space* $\mathcal{H}(b)$ is defined as the reproducing kernel Hilbert space of holomorphic functions on \mathbb{D} with the reproducing kernel given by

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z} \quad z, w \in \mathbb{D}.$$

These spaces were first introduced by de Branges and Rovnyak [16]. Since then, they have been well-studied and are regarded as fundamental within the theory of holomorphic functions, playing a significant role in various areas of function theory and operator theory. For a recent and thorough treatment of $\mathcal{H}(b)$ spaces we refer the reader to [20].

We recall that whenever $\|b\|_{H^\infty} < 1$, the space $\mathcal{H}(b)$ actually coincides with H^2 provided with an equivalent norm. For this reason we focus our attention to $\mathcal{H}(b)$ spaces where the function b has norm equal to 1. In addition we require that b is a rational function, and a *non-extreme* point of the unit ball of H^∞ , the latter meaning that

$$\int_{\mathbb{T}} \log(1 - |b(\zeta)|) dm(\zeta) > -\infty.$$

When b is non-extreme, then there exists a unique outer function $a \in H^\infty$, $\|a\|_{H^\infty} = 1$, such that $a(0) > 0$ and

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

We refer to a as the *Pythagorean mate* of b and we call (a, b) a *pair*. By the Fejér-Riesz theorem, whenever b is rational, then a is also rational.

Our main result strongly relies on the following description of this class of spaces. This was proved in [14, Lemma 4.3] (see also [5] for another proof).

Theorem 1.1. *Let (a, b) a rational pair and ζ_1, \dots, ζ_l the zeroes of a on \mathbb{T} with multiplicity m_1, \dots, m_l . Then*

$$\mathcal{H}(b) = \left(\prod_{j=1}^l (z - \zeta_j)^{m_j} \right) H^2 \oplus P_{N-1}, \quad (1)$$

where $N = \sum_j m_j$ and P_{N-1} is the space of polynomials of degree at most $N - 1$.

Let $M(\bar{a}) = T_{\bar{a}}H^2$ be the range space of the Toeplitz operator $T_{\bar{a}}$, then, using [20, Theorems 23.2 and 28.7], it can be shown that $\mathcal{H}(b) = M(\bar{a})$ as sets if and only if (a, b) forms a *corona pair*, which means that

$$\inf_{z \in \mathbb{D}} |a(z)|^2 + |b(z)|^2 > 0.$$

This condition is automatic when b is rational non inner so that we can assume $\mathcal{H}(b) = M(\bar{a})$. By standard results in functional analysis, the corresponding norms are equivalent. In [18], the authors demonstrated that the decomposition (1) is orthogonal in the metric of $M(\bar{a})$.

Interpolating sequences, both multiplier and universal, for de Branges-Rovnyak spaces have previously been studied for certain specific choices of b . The objective of this paper is to extend those results. To be more precise, in [29] the author provides a complete characterization of multiplier interpolating sequences for $\mathcal{H}(b)$ when this space can be viewed as a *local Dirichlet space* \mathcal{D}_μ (for a precise definition, we refer the reader to Section 2). Later, it was shown in [8] that, under these conditions, multiplier and universal interpolating sequences coincide with sequences that satisfies a separation and a Carleson measure conditions. It is worth noting that the space \mathcal{D}_μ has the complete Nevanlinna-Pick property, while, more generally, determining when a de Branges-Rovnyak space has this property is a more delicate question (see [12]).

The first main result of this paper is the following theorem.

Theorem 1.2. *Let b be a non-extreme and rational function of the unit ball of H^∞ and a its Pythagorean mate. Let ζ_1, \dots, ζ_l be the zeroes of a on \mathbb{T} with multiplicity m_1, \dots, m_l and Λ be a sequence on \mathbb{D} . The following are equivalent.*

- (i) Λ is $\mathcal{M}(\mathcal{H}(b))$ -interpolating.
- (ii) Λ is $\mathcal{H}(b)$ -interpolating.
- (iii) Λ satisfies the Carleson condition and

$$\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta_j - \lambda_n|^{2m_j}} < \infty \quad \text{for every } 1 \leq j \leq l. \quad (2)$$

We now introduce the random setting we are interested in. If not differently specified, by the term *random sequence* we mean a sequence of the following kind. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence of radii such that $0 \leq r_n < 1$ for every n and consider a family $(\vartheta_n)_{n \in \mathbb{N}}$ of independent random variables uniformly distributed in $[0, 2\pi]$. Define

$$\Lambda(\omega) = (\lambda_n(\omega))_{n \in \mathbb{N}} = (r_n e^{i\vartheta_n(\omega)})_{n \in \mathbb{N}}.$$

Such random sequences, sometimes mentioned in the literature as Steinhaus sequences, have already been studied in the last years. We mention the pioneering works of Cochran [13] and Rudowicz [26] leading to a 0–1 law for interpolating sequences for the Hardy space and the work of Bogdan [6] where this model was used to better understand random zero sets in Dirichlet spaces. It is worth mentioning more recent works, such as [10], which deals with interpolation in Dirichlet spaces, [9, 15], on interpolation and related properties for spaces of holomorphic functions in several variables, and [21], which deals with random interpolation in the Nevanlinna class.

In order to state our result, we need to introduce a partition into dyadic annuli of the unit disk and a counting function. Consider for $k \geq 0$,

$$A_k = \left\{ z \in \mathbb{D} : 1 - 2^k \leq |z| < 1 - 2^{-(k+1)} \right\},$$

$$N_k = \#(A_k \cap \Lambda).$$

Note that N_k is actually a number and not a random variable, since it depends only on the sequence of the *a priori* fixed radii.

We prove the following theorem.

Theorem 1.3. *Let Λ be a random sequence and b a non-extreme, rational function. Consider the decomposition of $\mathcal{H}(b)$ as described in (1) and define*

$$M = \max(\{m_j : 1 \leq j \leq l\}).$$

Then

$$\mathbb{P}(\Lambda \text{ is interpolating for } \mathcal{H}(b)) = \begin{cases} 1 & \text{if } \sum_{k \in \mathbb{N}} N_k 2^{-\frac{k}{2M}} < \infty \\ 0 & \text{if otherwise} \end{cases}$$

where with the term *interpolating* we mean both *multiplier* and *universal interpolating*.

In view of Kolmogorov's 0-1 law the above statements translate into if and only if statements. The reader should also notice that the above condition only involves the highest multiplicity m_j but not the number of zeros ζ_k .

2. PRELIMINARIES

2.1. de Branges-Rovnyak spaces and local Dirichlet spaces. We start this section describing the existing connection between de Branges-Rovnyak and local Dirichlet spaces, that was already mentioned in the introduction. Let μ be a finite positive Borel measure on \mathbb{T} and let $P[\mu]$ its Poisson extension. The associated *generalized Dirichlet space* \mathcal{D}_μ consists of those $f \in H^2$ such that

$$D_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty.$$

In particular whenever $\mu = \delta_\zeta$ for some $\zeta \in \mathbb{T}$, the space is usually called *local Dirichlet space*. In this situation it is immediate that

$$D_{\delta_\zeta}(f) = \int_{\mathbb{T}} \left| \frac{f(\omega) - f(\zeta)}{\omega - \zeta} \right|^2 dm(\omega).$$

We will write \mathcal{D}_ζ instead of $\mathcal{D}_{\delta_\zeta}$ and D_ζ instead of D_{δ_ζ} in the rest of the paper to simplify the notation.

The spaces \mathcal{D}_μ were first introduced by Richter [24]. Later on it was shown in [25] that \mathcal{D}_ζ can be viewed as de Branges-Rovnyak space for some b with equivalence of norm. This result was strengthened by Sarason [27], who showed how to choose b in order to have $\mathcal{D}_\zeta = \mathcal{H}(b)$ with equality of norms. The converse of this result was then proved in [11]. Finally in [14], the authors considered the problem of when $\mathcal{D}_\mu = \mathcal{H}(b)$ without necessarily equality of norms. We report here one of the main result of their paper.

Theorem 2.1 ([14, Theorem 4.1]). *Let (a, b) be a rational pair and let μ be a finite positive measure on \mathbb{T} . Then $\mathcal{D}_\mu = \mathcal{H}(b)$ if and only if:*

- (i) *the zeroes of a on \mathbb{T} are all simple;*
- (ii) *the support of μ is exactly equal to this set of zeros.*

In [22] the authors introduce the *local Dirichlet spaces of higher order* \mathcal{D}_ζ^N , with $N \in \mathbb{N}$. We say that $f \in \mathcal{D}_\zeta^N$ if $f \in H^2$ and

$$D_\zeta^N(f) := \int_{\mathbb{T}} \left| \frac{f(\omega) - T_{N-1}(f, \zeta)(\omega)}{(\omega - \zeta)^N} \right|^2 dm(\omega) < \infty,$$

where $T_{N-1}(f, \zeta)$ is the $(N-1)$ -th order Taylor polynomial of f at ζ ,

$$T_{N-1}(f, \zeta)(\omega) = \sum_{j=0}^{N-1} \frac{f^{(j)}(\zeta)}{j!} (\omega - \zeta)^j.$$

The norm is then defined to be

$$\|f\|_{\mathcal{D}_\zeta^N}^2 := \|f\|_{H^2}^2 + D_\zeta^N(f).$$

In the same paper the authors also proved that these spaces can be seen as de Branges-Rovnyak spaces. The theorem is the following.

Theorem 2.2 ([22, Theorem 1.1]). *Let b be a non-extreme point of the unit ball of H^∞ with $b(0) = 0$, and let $N \in \mathbb{N}$. Then the following are equivalent:*

- (i) *b is a rational function of degree N such that the mate a has a single zero of multiplicity N at a point $\zeta \in \mathbb{T}$;*
- (ii) *there is a $\zeta \in \mathbb{T}$ and a polynomial \tilde{p} of degree $< N$ with $\tilde{p}(\zeta) \neq 0$ such that $\|f\|_{\mathcal{H}(b)}^2 = \|f\|_{H^2}^2 + D_\zeta^N(\tilde{p}f)$.*

If this conditions holds, then there are polynomials p and q of degree $\leq N$ such that $b = \frac{p}{q}$, $a = \frac{(z-\zeta)^N}{q}$, $\tilde{p}(z) = z^N \overline{p(1/\bar{z})}$, for $z \in \mathbb{D}$, and $|q(\omega)|^2 = |p(\omega)|^2 + |\omega - \zeta|^{2N}$ for all $\omega \in \mathbb{T}$. Furthermore, $\mathcal{H}(b) = \mathcal{D}_\zeta^N$ with equivalence of norms.

We also mention some recent related results on the equality $D_\mu = H(b)$ in [23].

To complete the preliminaries needed on de Branges-Rovnyak spaces, we recall the following description of the multiplier algebra of $\mathcal{H}(b)$ spaces, proved in [19, Proposition 2.6, Proposition 3.1].

Theorem 2.3. *Let (a, b) a rational pair. Then $\mathcal{M}(\mathcal{H}(b)) = H^\infty \cap \mathcal{H}(b)$.*

It is worth pointing out that the aforementioned theorem was established for the space $M(\bar{a})$; however, as already mentioned, in our context this space coincides with $\mathcal{H}(b)$. In general, $H^\infty \cap \mathcal{H}(b)$ does not need to be an algebra (see [19] for references).

We will need another result concerning the derivatives of Blaschke products. This was proved in [2, 3], but we report here the version presented in [20, Theorem 21.8].

Theorem 2.4. *Let $(\lambda_n)_{n \in \mathbb{N}}$ be a Blaschke sequence in \mathbb{D} , and let B_Λ be the corresponding Blaschke product. Assume that, for an integer $N \geq 0$ and a point $\zeta \in \mathbb{T}$, we have*

$$\sum_{n=0}^{\infty} \frac{1 - |\lambda_n|}{|\zeta - \lambda_n|^{N+1}} \leq A.$$

Then the following hold.

- (i) *For each $0 \leq j \leq N$, both limits*

$$B_\Lambda^{(j)}(\zeta) := \lim_{r \rightarrow 1^-} B_\Lambda^{(j)}(r\zeta) \quad \text{and} \quad \lim_{r \rightarrow 1^+} B_\Lambda^{(j)}(r\zeta)$$

exist and are equal.

(ii) There is a constant $C = C(N, A)$ such that the estimation

$$|B_\Lambda^{(j)}(r\zeta)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

From the proof presented in [20, Theorem 21.8], it can also be deduced that, under the conditions of the theorem, the successive derivatives up to N of the partial Blaschke products $B_n = \prod_{k=1}^n \frac{|\lambda_k|}{\lambda_k} \frac{\lambda_k - z}{1 - \overline{\lambda_k} z}$ converge at ζ to the respective successive derivatives of $B(\zeta)$.

2.2. Probability background. On the probabilistic side, we need the following classical theorem (see for instance [4]).

Theorem 2.5 (Kolmogorov's three series theorem). *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables and for any $c > 0$ let be $Y_n := X_n \chi_{\{|X_n| \leq c\}}$. Consider the series*

- (i) $\sum_n \mathbb{P}(|X_n| > c)$
- (ii) $\sum_n \mathbb{E}[Y_n]$
- (iii) $\sum_n \mathbb{V}[Y_n]$.

In order that $\sum_n X_n$ converges almost surely it is necessary that the three series converges for all positive c and sufficient that they converge for some positive c .

3. INTERPOLATING SEQUENCES: PROOF OF THEOREM 1.2

Referring to Theorem 1.2, since we already know that (i) \implies (ii), the plan of the proof is to establish (ii) \implies (iii) \implies (i). We start proving (ii) \implies (iii). In this direction, we mention the following lemma, that holds for any choice of the function b . It was first proved in [17], see also [20, Theorem 31.16].

Lemma 3.1. *If Λ is interpolating for $\mathcal{H}(b)$ then Λ satisfies the Carleson condition.*

The following lemma, on the other hand, clearly relies on the description of the $\mathcal{H}(b)$ spaces provided in (1).

Lemma 3.2. *If Λ is interpolating for $\mathcal{H}(b)$, then Λ satisfies (2).*

Proof. Observe first that since Λ is interpolating for $\mathcal{H}(b)$, the restriction operator R_Λ is bounded. Moreover, since b is non extreme, $\mathcal{H}(b)$ contains the polynomials and in particular the constant function 1. Hence

$$R_\Lambda 1 = \left(\frac{1}{\|k_{\lambda_n}\|_{\mathcal{H}}} \right)_{n \in \mathbb{N}} \in \ell^2.$$

In particular, any idempotent sequence $(e_n)_{n \in \mathbb{N}}$, $e_n \in \{0, 1\}$, can be interpolated by a function in $\mathcal{H}(b)$.

Suppose now that (2) does not hold for some r , $1 \leq r \leq l$. Define

$$u_n := \frac{1 - |\lambda_n|^2}{|\zeta_r - \lambda_n|^{2m_r}}.$$

Then there exists a subsequence $(\lambda_{n_k})_{k \in \mathbb{N}}$ of Λ such that $\lambda_{n_k} \rightarrow \zeta_r$ as $k \rightarrow \infty$. We can divide this subsequence into $(\lambda_{n_{k,1}})_{k \in \mathbb{N}}$ and $(\lambda_{n_{k,2}})_{k \in \mathbb{N}}$ such that both $u_{n_{k,1}}$ and $u_{n_{k,2}}$ are not in ℓ^1 . Since Λ is $\mathcal{H}(b)$ -interpolating, there exists $f \in \mathcal{H}(b)$ such that

$$\begin{aligned} f(\lambda_{n_{k,1}}) &= 1 \\ f(\lambda_{n_{k,2}}) &= 0. \end{aligned}$$

From Theorem 1.1 we know that

$$f(z) = \prod_{j=1}^l (z - \zeta_j)^{m_j} g(z) + \sum_{j=0}^{m_r-1} c_j (z - \zeta_r)^j = \prod_{j=1}^l (z - \zeta_j)^{m_j} g(z) + p_r(z)$$

with $g \in H^2$. We claim that there exist $C > 0$ and $k_0 \in \mathbb{N}$ such that either one of

$$\begin{aligned} |1 - p_r(\lambda_{n_k,1})| &= \left| 1 - \sum_{j=0}^{m_r-1} c_j (\lambda_{n_k,1} - \zeta_r)^j \right| \geq C \quad \forall k > k_0, \quad \text{or} \\ |p_r(\lambda_{n_k,2})| &= \left| \sum_{j=0}^{m_r-1} c_j (\lambda_{n_k,2} - \zeta_r)^j \right| \geq C \quad \forall k > k_0 \end{aligned}$$

holds. Suppose $c_0 = 0$. Since $\lambda_{n_k,1} \rightarrow \zeta_r$, there exists $k_1 \in \mathbb{N}$ such that $|p_r(\lambda_{n_k,1})| \leq 1/2$, for every $k > k_1$, which yields

$$\left| 1 - \sum_{j=0}^{m_r-1} c_j (\lambda_{n_k,1} - \zeta_r)^j \right| \geq \frac{1}{2} \quad \forall k > k_1.$$

If $c_0 \neq 0$ then, again because $\lambda_{n_k,2} \rightarrow \zeta_r$, there exists $k_2 \in \mathbb{N}$ such that

$$\left| \sum_{j=1}^{m_r-1} c_j (\lambda_{n_k,2} - \zeta_r)^j \right| \leq \frac{|c_0|}{2} \quad \forall k > k_2,$$

which yields $|p_r(\lambda_{n_k,2})| \geq |c_0|/2$ for every $k > k_2$. If we take $C = \min(|c_0|/2, 1/2)$ and $k_0 = \max(k_1, k_2)$ we have the statement. Since both $u_{n_k,1}$ and $u_{n_k,2}$ are not in ℓ^1 , we have that

$$\sum_{k>k_0} u_{n_k,1} = \infty \quad \sum_{k>k_0} u_{n_k,2} = \infty.$$

From Lemma 3.1 we know that Λ satisfies (C). This means that Λ is H^2 -interpolating. Now if $c_0 = 0$, then setting $N = m_1 + \dots + m_l$,

$$\begin{aligned} \infty &> \sum_{k \in \mathbb{N}} (1 - |\lambda_{n_k,1}|^2) |g(\lambda_{n_k,1})|^2 \\ &= \sum_{k \in \mathbb{N}} (1 - |\lambda_{n_k,1}|^2) \frac{|1 - p_r(\lambda_{n_k,1})|^2}{\prod_{j=1}^l |\lambda_{n_k,1} - \zeta_j|^{2m_j}} \\ &\geq \frac{C^2}{2^{2(N-m_r)}} \sum_{k>k_0} \frac{1 - |\lambda_{n_k,1}|^2}{|\lambda_{n_k,1} - \zeta_r|^{2m_r}}, \end{aligned}$$

where the last inequality follows from $|\lambda_{n_k,1} - \zeta_j| \leq 2$ and from the previous claim. This gives a contradiction. While if $c_0 \neq 0$, then

$$\begin{aligned} \infty &> \sum_{k \in \mathbb{N}} (1 - |\lambda_{n_k,2}|^2) |g(\lambda_{n_k,2})|^2 \\ &= \sum_{k \in \mathbb{N}} (1 - |\lambda_{n_k,2}|^2) \frac{|p_r(\lambda_{n_k,2})|^2}{\prod_{j=1}^l |\lambda_{n_k,2} - \zeta_j|^{2m_j}} \\ &\geq \frac{C^2}{2^{2(N-m_r)}} \sum_{k>k_0} \frac{1 - |\lambda_{n_k,2}|^2}{|\lambda_{n_k,2} - \zeta_r|^{2m_r}}, \end{aligned}$$

which gives again a contradiction. □

We now complete the proof of Theorem 1.2, i.e. deducing the multiplier interpolation result from the Carleson condition and (2). As it turns out, given a bounded sequence the interpolating function of this sequence involves the Blaschke product B_Λ of Λ . Our next aim will thus be to prove that (2) implies $B_\Lambda \in \mathcal{H}(b)$, see Lemma 3.6. In order to keep the computations more tractable we first establish the result for the case $a(z) = a_1(z) = \frac{(z-1)^N}{2^N}$ and denote by b_1 its Pythagorean mate. In this scenario, (2) in Theorem 1.2(iii) is equivalent to

$$\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2N}} < \infty. \quad (3)$$

We introduce the following functions:

$$G_N(z) := \frac{\varphi_n(z) - T_{N-1}(\varphi_n, 1)(z)}{(z-1)^N},$$

$$H_N(z) := \frac{\varphi_n(z)f(z) - T_{N-1}(\varphi_n f, 1)(z)}{(z-1)^N},$$

where $\varphi_n(z) = \frac{z - \lambda_n}{1 - \overline{\lambda_n}z}$ is the elementary Blaschke factor (Möbius transform) at λ_n .

Lemma 3.3. *Let φ_n be the Blaschke factor at λ_n . Then*

$$D_1^N(\varphi_n) = \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2N}} |\lambda_n|^{2(N-1)}.$$

Proof. A standard inductive argument gives

$$\varphi_n^{(j)}(z) = j! \overline{\lambda_n}^{j-1} \frac{1 - |\lambda_n|^2}{(1 - \overline{\lambda_n}z)^{j+1}}, \quad j \geq 1.$$

In particular we have

$$\varphi_n^{(j)}(1) = j! \overline{\lambda_n}^{j-1} \frac{1 - |\lambda_n|^2}{(1 - \overline{\lambda_n})^{j+1}}.$$

We now proceed to computing $D_1^N(\varphi_n)$. We note that $D_1^N(\varphi_n) = \|G_N\|_{H^2}^2$ and we claim that

$$G_N(z) = \frac{1 - |\lambda_n|^2}{(1 - \overline{\lambda_n})^N} \overline{\lambda_n}^{N-1} k_{\lambda_n}(z),$$

where $k_{\lambda_n}(z)$ is the Szëgo kernel at λ_n . We proceed by induction. From [29, Lemma 2.1] (or by direct computation) we know that the claim is true for $N = 1$. Now suppose the claim is

true for N . Then

$$\begin{aligned}
G_{N+1}(z) &= \frac{\varphi_n(z) - \sum_{j=0}^{N-1} \frac{\varphi_n^{(j)}(1)}{j!} (z-1)^j - \frac{\varphi_n^{(N)}(1)}{N!} (z-1)^N}{(z-1)^{N+1}} \\
&= \frac{\varphi_n(z) - \sum_{j=0}^{N-1} \frac{\varphi_n^{(j)}(1)}{j!} (z-1)^j - \frac{\overline{\lambda_n}^{N-1} (1-|\lambda_n|^2)}{(1-\overline{\lambda_n})^{N+1}} (z-1)^N}{(z-1)^{N+1}} \\
&= \frac{G_N(z)}{z-1} - \frac{\overline{\lambda_n}^{N-1} (1-|\lambda_n|^2)}{(1-\overline{\lambda_n})^{N+1} (z-1)} \\
&= \frac{1}{z-1} \left(G_N(z) - \frac{\overline{\lambda_n}^{N-1} (1-|\lambda_n|^2)}{(1-\overline{\lambda_n})^{N+1}} \right) \\
&= \frac{1}{z-1} \left(\frac{1-|\lambda_n|^2}{(1-\overline{\lambda_n})^N} \overline{\lambda_n}^{N-1} k_{\lambda_n}(z) - \frac{\overline{\lambda_n}^{N-1} (1-|\lambda_n|^2)}{(1-\overline{\lambda_n})^{N+1}} \right) \\
&= \frac{1}{z-1} \frac{\overline{\lambda_n}^{N-1} (1-|\lambda_n|^2)}{(1-\overline{\lambda_n})^{N+1}} \left(\frac{1-\overline{\lambda_n}}{1-\overline{\lambda_n}z} - 1 \right) \\
&= \frac{1-|\lambda_n|^2}{(1-\overline{\lambda_n})^{N+1}} \overline{\lambda_n}^N k_{\lambda_n}(z).
\end{aligned}$$

We can now easily compute $D_1^N(\varphi_n)$:

$$D_1^N(\varphi_n) = \|G_N\|_{H^2}^2 = \frac{1-|\lambda_n|^2}{|1-\lambda_n|^{2N}} |\lambda_n|^{2(N-1)},$$

which ends the proof. \square

Lemma 3.4. *Let φ_n be the Blaschke factor at λ_n . Then there exists $C > 0$ such that for every $f \in \mathcal{H}(b_1)$ we have*

$$D_1^N(\varphi_n f) \leq D_1^N(f) + C \max\left\{ |f^{(j)}(1)/j!|^2 : j = 0, \dots, N-1 \right\} \frac{1-|\lambda_n|^2}{|1-\lambda_n|^{2N}}.$$

Proof. We first claim that the function H_N , introduced before Lemma 3.3, is equal to

$$\begin{aligned}
H_N(z) &= \sum_{j=0}^{N-1} \frac{f^{(j)}(1)}{j!} \frac{\varphi_n(z) - T_{(N-1)-j}(\varphi_n, 1)(z)}{(z-1)^{N-j}} + \varphi_n(z) \frac{f(z) - T_{N-1}(f, 1)(z)}{(z-1)^N} \\
&= \sum_{j=0}^{N-1} \frac{f^{(j)}(1)}{j!} G_{N-j}(z) + \varphi_n(z) \frac{f(z) - T_{N-1}(f, 1)(z)}{(z-1)^N}.
\end{aligned}$$

To prove this we proceed by induction. As shown in [29, Lemma 2.1] the claim is true for $N = 1$. We suppose it is true for N . Then

$$\begin{aligned}
H_{N+1}(z) &= \frac{\varphi_n(z)f(z) - T_{N-1}(\varphi_n f, 1)(z)}{(z-1)^{N+1}} - \frac{(\varphi_n f)^{(N)}(1)}{N!(z-1)} \\
&= \frac{H_N(z)}{z-1} - \sum_{j=0}^N \binom{N}{j} \frac{\varphi_n^{(j)}(1) f^{(N-j)}(1)}{N!(z-1)},
\end{aligned}$$

and using the formula for H_N ,

$$\begin{aligned} &= \sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n(z) - \sum_{k=0}^{N-1-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j! (z-1)^{N+1-j}} \\ &\quad + \varphi_n(z) \frac{f(z) - \sum_{k=0}^{N-1} \frac{f^{(k)}(1)}{k!} (z-1)^k}{(z-1)^{N+1}} - \sum_{j=0}^N \frac{\varphi_n^{(j)}(1) f^{(N-j)}(1)}{(N-j)! j! (z-1)}. \end{aligned}$$

Let us rewrite the first sum:

$$\begin{aligned} &\sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n(z) - \sum_{k=0}^{N-1-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j! (z-1)^{N+1-j}} \\ &= \sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n(z) - \sum_{k=0}^{N-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j! (z-1)^{N+1-j}} + \sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n^{(N-j)}(1) (z-1)^{N-j}}{j! (N-j)! (z-1)^{N+1-j}} \\ &= \sum_{j=0}^N f^{(j)}(1) \frac{\varphi_n(z) - \sum_{k=0}^{N-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j! (z-1)^{N+1-j}} - f^{(N)}(1) \frac{\varphi_n(z) - \varphi_n(1)}{N! (z-1)} \\ &\quad + \sum_{j=0}^{N-1} f^{(j)}(1) \frac{\varphi_n^{(N-j)}(1)}{j! (N-j)! (z-1)} \end{aligned}$$

We deduce

$$\begin{aligned} H_{N+1}(z) &= \sum_{j=0}^N f^{(j)}(1) \frac{\varphi_n(z) - \sum_{k=0}^{N-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j! (z-1)^{N+1-j}} - f^{(N)}(1) \frac{\varphi_n(z) - \varphi_n(1)}{N! (z-1)} \\ &\quad + \sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n^{(N-j)}(1)}{(N-j)! j! (z-1)} - \sum_{j=0}^N \frac{\varphi_n^{(j)}(1) f^{(N-j)}(1)}{(N-j)! j! (z-1)} + \varphi_n(z) \frac{f(z) - \sum_{k=0}^{N-1} \frac{f^{(k)}(1)}{k!} (z-1)^k}{(z-1)^{N+1}} \end{aligned}$$

Consider the three middle terms:

$$\begin{aligned} &-f^{(N)}(1) \frac{\varphi_n(z) - \varphi_n(1)}{N! (z-1)} + \sum_{j=0}^{N-1} \frac{f^{(j)}(1) \varphi_n^{(N-j)}(1)}{(N-j)! j! (z-1)} - \sum_{j=0}^N \frac{\varphi_n^{(j)}(1) f^{(N-j)}(1)}{(N-j)! j! (z-1)} \\ &= -f^{(N)}(1) \frac{\varphi_n(z)}{N! (z-1)} + \sum_{j=0}^N \frac{f^{(j)}(1) \varphi_n^{(N-j)}(1)}{(N-j)! j! (z-1)} - \sum_{j=0}^N \frac{\varphi_n^{(j)}(1) f^{(N-j)}(1)}{(N-j)! j! (z-1)} \\ &= -f^{(N)}(1) \frac{\varphi_n(z)}{N! (z-1)} \end{aligned}$$

Finally

$$\begin{aligned}
H_{N+1}(z) &= \sum_{j=0}^N f^{(j)}(1) \frac{\varphi_n(z) - \sum_{k=0}^{N-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j!(z-1)^{N+1-j}} - f^{(N)}(1) \frac{\varphi_n(z)}{N!(z-1)} \\
&\quad + \varphi_n(z) \frac{f(z) - \sum_{k=0}^{N-1} \frac{f^{(k)}(1)}{k!} (z-1)^k}{(z-1)^{N+1}} \\
&= \sum_{j=0}^N f^{(j)}(1) \frac{\varphi_n(z) - \sum_{k=0}^{N-j} \frac{\varphi_n^{(k)}(1)}{k!} (z-1)^k}{j!(z-1)^{N+1-j}} + \varphi_n(z) \frac{f(z) - \sum_{k=0}^N \frac{f^{(k)}(1)}{k!} (z-1)^k}{(z-1)^{N+1}} \\
&= \sum_{j=0}^N \frac{f^{(j)}(1)}{j!} G_{N+1-j}(z) + \varphi_n(z) \frac{f(z) - T_N(f, 1)(z)}{(z-1)^{N+1}}.
\end{aligned}$$

Note that G_N is a scalar multiple of the reproducing kernel k_{λ_n} and is thus orthogonal in H^2 to the shift invariant subspace $\varphi_n H^2$. Note also that φ_n , as a Blaschke factor, is inner and hence an isometric multiplier on H^2 . Therefore from Lemma 3.3 we obtain

$$\begin{aligned}
D_1^N(\varphi_n f) &= \|H_N\|_{H^2}^2 \\
&= \left| \sum_{j=0}^{N-1} \frac{f^{(j)}(1)}{j!} \frac{1 - |\lambda_n|^2}{(1 - \overline{\lambda_n})^{N-j}} \overline{\lambda_n}^{-N-1-j} \right|^2 \frac{1}{1 - |\lambda_n|^2} + D_1^N(f) \\
&= \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2N}} \left| \sum_{j=0}^{N-1} \frac{f^{(j)}(1)}{j!} (1 - \overline{\lambda_n})^j \overline{\lambda_n}^{-N-1-j} \right|^2 + D_1^N(f) \\
&\leq C \max\left\{ |f^{(j)}(1)/j!|^2 : j = 0, \dots, N-1 \right\} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2N}} + D_1^N(f).
\end{aligned}$$

□

By an easy induction, the following corollary is an immediate consequence of Lemma 3.4 and Theorem 2.4 (which relates somehow the sum (3) to the derivatives of the Blaschke product at ζ).

Corollary 3.5. *Suppose that (3) holds and consider*

$$B_n(z) = \prod_{k=1}^n \frac{|\lambda_k|}{\lambda_k} \varphi_k(z).$$

Then there exists $C > 0$ which depends on N and on the sum (3) such that

$$D_1^N(B_n) \leq C \sum_{k=1}^n \frac{1 - |\lambda_k|^2}{|1 - \lambda_k|^{2N}}.$$

Lemma 3.6. *If (3) holds, then $B_\Lambda \in \mathcal{H}(b_1)$ and $|B_\Lambda(1)| = 1$.*

Observe that the above conclusion trivially holds for every subproduct $B_{\Lambda'}$ with $\Lambda' \subset \Lambda$.

Proof. It is well known that $B_n \rightarrow B_\Lambda$ in H^2 as $n \rightarrow \infty$. This also implies that some subsequence of (B_n) converges for almost every ζ to B_Λ , say (B_{n_k}) . Define

$$\psi_k(\zeta) = \left| \frac{B_{n_k}(\zeta) - T_{N-1}(B_{n_k}, 1)(\zeta)}{(\zeta - 1)^N} \right|, \quad \zeta \in \mathbb{T} \setminus \{1\}.$$

By the preceding observation, $B_{n_k}(\zeta) \rightarrow B_\Lambda(\zeta)$ a.e. Moreover, in view of Theorem 2.4 and the remark thereafter, the Taylor polynomial $T_{N-1}(B_{n_k}, 1)(\zeta)$ converges at every ζ to $T_{N-1}(B_\Lambda, 1)(\zeta)$. Hence

$$\psi_k(\zeta) \rightarrow \psi(\zeta) = \left| \frac{B_\Lambda(\zeta) - T_{N-1}(B_\Lambda, 1)(\zeta)}{(\zeta - 1)^N} \right|, \quad \text{a.e. } \zeta \in \mathbb{T} \text{ when } k \rightarrow +\infty.$$

Then we can use the above corollary and Fatou's lemma to obtain

$$\begin{aligned} D_1^N(B_\Lambda) &= \int_{\mathbb{T}} \left| \frac{B_\Lambda(\zeta) - T_{N-1}(B_\Lambda, 1)(\zeta)}{(\zeta - 1)^N} \right|^2 dm(\zeta) = \int_{\mathbb{T}} |\psi(\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} \lim_{k \rightarrow \infty} |\psi_k(\zeta)|^2 dm(\zeta) \\ &\leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{T}} |\psi_k(\zeta)|^2 dm(\zeta) \\ &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{T}} \left| \frac{B_{n_k}(\zeta) - T_{N-1}(B_{n_k}, 1)(\zeta)}{(\zeta - 1)^N} \right|^2 dm(\zeta) \\ &\leq \sup_k D_1^N(B_{n_k}) \\ &\leq C \sum_{j=1}^{\infty} \frac{1 - |\lambda_j|}{|1 - \lambda_j|^{2N}}, \end{aligned}$$

which proves the first statement.

The second statement is already known from Frostman's theorem (see for instance [3, Theorem 1]). \square

Before showing the multiplier interpolation result, let us make the following useful observation saying that it is always possible to add a finite number of points to a (multiplier-) interpolating sequence (once the equivalence of interpolation and multiplier interpolation established it is clear that the result is true also in $\mathcal{H}(b)$). It is of course enough to show that we can add one point.

Lemma 3.7. *Let $\Lambda = \{\lambda_n\}_{n \geq 1}$ be $\mathcal{M}(\mathcal{H}(b))$ -interpolating, and $\lambda_0 \in \mathbb{D} \setminus \Lambda$. Then $\Lambda \cup \{\lambda\} = \{\lambda_n\}_{n \geq 0}$ is $\mathcal{M}(\mathcal{H}(b))$ -interpolating.*

Proof. Let $(v_n)_{n \geq 0} \in \ell^\infty$. Since Λ is interpolating there is $f_0 \in \mathcal{M}(\mathcal{H}(b))$ such that $f_0(\lambda_n) = v_n$, $n \geq 1$. Now, by the previous results, multiplier interpolation implies (2) and thus $B_\Lambda \in \mathcal{M}(\mathcal{H}(b))$. Setting now

$$f = f_0 + \frac{(z-1)^N B_\Lambda}{(\lambda_0-1)^N B_\Lambda(\lambda_0)} (v_0 - f_0(\lambda_0)) \quad (4)$$

we obtain obviously a function $f \in \mathcal{M}(\mathcal{H}(b))$ with $f(\lambda_n) = v_n$, $n \geq 0$. \square

The reader might wonder why we have added the factor $\frac{(z-1)^N}{(\lambda_0-1)^N}$ in the formula (4) (the proof would work without this factor). Still, we will need later that when $f_0 \in (z-1)^N H^2 \cap H^\infty$, then f will be also in this intersection. This means that for the (multiplier) interpolation we do not really need the polynomial part.

Lemma 3.8. *If Λ satisfies (3) and (C), then Λ is interpolating for $\mathcal{M}(\mathcal{H}(b_1))$. Moreover the interpolating function can be chosen in $aH^2 \cap H^\infty$.*

Recall that $\mathcal{H}(b_1) = aH^2 + P_{N-1}$, where $a(z) = (z-1)^N$ and $\mathcal{M}(\mathcal{H}(b_1)) = \mathcal{H}(b_1) \cap H^\infty$.

Proof. Let $(v_n)_{n \in \mathbb{N}} \in \ell^\infty$. We have to construct an interpolating function $f \in \mathcal{M}(\mathcal{H}(b))$: $f(\lambda_n) = v_n$, $n \in \mathbb{N}$. Observe first that from Lemma 3.6, we know that $B_\Lambda \in \mathcal{H}(b_1)$, i.e.

$$B_\Lambda(z) = (z-1)^N g(z) + \sum_{j=0}^{N-1} c_j (z-1)^j = (z-1)^N g(z) + p_N(z), \quad (5)$$

where $g \in H^2$. Note in particular that $|c_0| = |B_\Lambda(1)| = 1$, which gives us that $c_0 \neq 0$.

Let $\mathcal{Z}(p_N)$ be the zero set of p_N and set $\{\zeta_j\}_{j=1}^J = \mathcal{Z}(p_N) \cap \mathbb{T}$. Observe that $0 \leq j \leq N-1$ since p_N is not the zero polynomial. In view of $p_N(1) \neq 0$, we also have $|\zeta_j - 1| \geq \delta$ for some $\delta > 0$ and every j . Hence the set

$$\Lambda_1 := \{\lambda_{n,1}\} := \bigcup_{j=1}^J D(\zeta_j, \delta/2) \cap \Lambda$$

is far from 1. Since Λ_1 satisfies (C) and $|\lambda_{n,1} - 1| \geq \delta/2$, there exists $h_1 \in H^\infty$ such that

$$h_1(\lambda_{n,1}) = \frac{v_{n,1}}{(\lambda_{n,1} - 1)^N}.$$

Then define $h(z) = (z-1)^N h_1(z) \in \mathcal{H}(b_1) \cap H^\infty$ which solves the interpolation problem $h(\lambda_{n,1}) = v_{n,1}$.

The remaining sequence $\Lambda \setminus \Lambda_1$ might contain zeros of p_N , so let us first consider $\Lambda_2 = \Lambda \setminus (\Lambda_1 \cup \mathcal{Z}(p_N))$ the elements of which we will denote by $\lambda_{n,2}$. Then for every $n \in \mathbb{N}$, we have $|p_N(\lambda_{n,2})| \geq \varepsilon$ for some $\varepsilon > 0$ (Λ_2 does not contain nor accumulate to any zero of p_N). Note also that $|B_{\Lambda_1}(\lambda_{n,2})| \geq |B_{\Lambda \setminus \lambda_{n,2}}(\lambda_{n,2})| \geq \eta > 0$ by the Carleson condition. Hence the sequence

$$u_n := -\frac{v_{n,2} - h(\lambda_{n,2})}{B_{\Lambda_1}(\lambda_{n,2})p_N(\lambda_{n,2})}$$

is uniformly bounded, so that there exists $f_1 \in H^\infty$ with $f_1(\lambda_{n,2}) = u_n$. Then, in view of (5), we have

$$f(z) = (B_\Lambda(z) - p_N(z))f_1(z) \in aH^2 \cap H^\infty \subset \mathcal{H}(b) \cap H^\infty,$$

and by construction

$$f(\lambda_{n,2}) = \frac{v_{n,2} - h(\lambda_{n,2})}{B_{\Lambda_1}(\lambda_{n,2})}.$$

We can now define $F = B_{\Lambda_1}f + h$. Since $f, h \in aH^2 \cap H^\infty$ and $B_{\Lambda_1} \in H^\infty$, we get $F \in aH^2 \cap H^\infty \subset \mathcal{H}(b_1) \cap H^\infty$, and by construction $F(\lambda_n) = v_n$ for $\lambda_n \in \Lambda_3 := \Lambda_1 \cup \Lambda_2$.

As a result, Λ_3 is $\mathcal{M}(\mathcal{H}(b_1))$ -interpolating, and the interpolating function F is in $aH^2 \cap H^\infty$.

Finally, from Lemma 3.7, since $\Lambda_0 := \mathcal{Z}(p_N) \cap \Lambda$ is finite, it follows that $\Lambda = \Lambda_3 \cup \Lambda_0$ is $\mathcal{M}(\mathcal{H}(b_1))$ -interpolating. And moreover, since $F \in aH^2 \cap H^\infty$, formula (4) shows that the function interpolating on Λ is also in $aH^2 \cap H^\infty$. \square

With the previous lemma we have completed the proof of Theorem 1.2 for the case $a(z) = a_1(z) = \frac{(z-1)^N}{2^N}$ and b_1 as its Pythagorean mate. Furthermore, we observe that everything established up to this point can be readily extended to $a(z) = a_\zeta(z) = \frac{(z-\zeta)^N}{2^N}$ and its Pythagorean mate b_ζ . To complete the proof for the general case, we need one final lemma.

Lemma 3.9. *A sequence Λ is interpolating for $\mathcal{M}(\mathcal{H}(b))$ if and only if it is interpolating for every $\mathcal{M}(\mathcal{H}(b_{\zeta_j}))$, $1 \leq j \leq l$.*

Proof. This proof is a variation of [29, Theorem 4.1].

It is immediate that $\mathcal{H}(b) \subset \bigcap_j \mathcal{H}(b_{\zeta_j})$. Indeed, if $f \in \mathcal{H}(b)$, then for every $1 \leq j \leq l$,

$$f = ag + p_N = \prod_{k=1}^l (z - \zeta_k)^{m_k} g + p_N = (z - \zeta_j)^{m_j} \left(\prod_{k \neq j} (z - \zeta_k)^{m_k} \right) g + p_N,$$

where $g \in H^2$ and $p_N \in P_{N-1}$. Now expanding p_N as a Taylor series at ζ_j , we get $p_N = \sum_{i=0}^{m_j-1} \frac{p_N^{(i)}(\zeta_j)}{i!} (z - \zeta_j)^i + (z - \zeta_j)^{m_j} q$ for some polynomial q , which yields the required inclusion. Hence $\mathcal{M}(\mathcal{H}(b)) \subset \mathcal{M}(\mathcal{H}(b_{\zeta_j}))$ for every j , and every interpolating sequence for $\mathcal{M}(\mathcal{H}(b))$ is interpolating for every $\mathcal{M}(\mathcal{H}(b_{\zeta_j}))$.

For the converse, suppose that Λ is interpolating for every $\mathcal{M}(\mathcal{H}(b_{\zeta_j}))$.

Fix a bounded sequence $(v_n)_{n \in \mathbb{N}}$. Since Λ is interpolating for $\mathcal{M}(\mathcal{H}(b_{\zeta_1}))$, from Lemma 3.8, we know that we can interpolate (v_n) by a function of the form $F_1 = (z - \zeta_1)^{m_1} f_1(z) \in \mathcal{M}(\mathcal{H}(b_{\zeta_1}))$ (no polynomial part).

Now since the constant sequence (1) can be interpolated by functions in any $\mathcal{M}(\mathcal{H}(b_{\zeta_j}))$, $2 \leq j \leq l$, there are f_j such that $F_j = (z - \zeta_j)^{m_j} f_j(z) \in \mathcal{M}(\mathcal{H}(b_{\zeta_j}))$, $2 \leq j \leq l$, and $F_j(\lambda_n) = 1$ for every $2 \leq j \leq l$ and $n \in \mathbb{N}$. Then the function $F = \prod_j F_j$ belongs to $\mathcal{M}(\mathcal{H}(b))$ and interpolates (a_n) . \square

From the previous lemmas it follows that $(iii) \implies (i)$, which complete the proof of Theorem 1.2.

4. RANDOM INTERPOLATION: PROOF OF THEOREM 1.3

Lemma 4.1. *Let Λ be a random sequence, $\zeta \in \mathbb{T}$ and $M \in \mathbb{N}$. Then*

$$\mathbb{P} \left(\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|^{2M}} < \infty \right) = \begin{cases} 1 & \text{if } \sum_{k \in \mathbb{N}} N_k 2^{-\frac{k}{2M}} < \infty \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since the random sequences we are considering are invariant by rotations, we can assume without loss of generality that $\zeta = 1$.

Let us introduce

$$X_n = \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2M}}.$$

By construction, X_n are independent. We start noting that

$$\begin{aligned} \mathbb{P}(X_n > 1) &= \mathbb{P} \left((1 - |\lambda_n|^2)^{1/M} > 1 + |\lambda_n|^2 - 2|\lambda_n| \cos(\vartheta_n) \right) \\ &= \mathbb{P} \left(\cos(\vartheta_n) > \frac{1 + |\lambda_n|^2 - (1 - |\lambda_n|^2)^{1/M}}{2|\lambda_n|} \right). \end{aligned}$$

Define

$$u_n := \frac{1 + |\lambda_n|^2 - (1 - |\lambda_n|^2)^{1/M}}{2|\lambda_n|},$$

then we obtain

$$\begin{aligned}\mathbb{P}(X_n > 1) &= \mathbb{P}(0 < \vartheta_n < \arccos(u_n)) + \mathbb{P}(2\pi - \arccos(u_n) < \vartheta_n < 2\pi) \\ &= \frac{\arccos(u_n)}{\pi}.\end{aligned}$$

For $n \rightarrow \infty$ we have that $u_n \rightarrow 1$ which implies $\arccos(u_n) \sim C(1 - |u_n|^2)^{1/2}$. A computation yields

$$1 - |u_n|^2 \simeq (1 - |\lambda_n|^2)^{1/M},$$

which finally implies

$$\mathbb{P}(X_n > 1) \simeq (1 - |\lambda_n|^2)^{1/2M}.$$

Since

$$\sum_{n \in \mathbb{N}} (1 - |\lambda_n|^2)^{1/2M} \simeq \sum_{k \in \mathbb{N}} N_k 2^{-\frac{k}{2M}},$$

we obtain that if $\sum_k N_k 2^{-\frac{k}{2M}} = \infty$ then $\sum_{n \in \mathbb{N}} \mathbb{P}(X_n > 1) = +\infty$. By Theorem 2.5 (or directly by the Borel-Cantelli lemma), we conclude that $\mathbb{P}(\sum_n X_n < \infty) = 0$, which yields the second part of the claim.

Let us now consider the case when $\sum_k N_k 2^{-\frac{k}{2M}} < +\infty$. The sum condition translates immediately to $\sum_{n \in \mathbb{N}} \mathbb{P}(X_n > 1) < +\infty$, which already yields condition (i) of that theorem. In order to check the conditions (ii) and (iii), let us introduce

$$Y_n = \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2M}} \chi_{\{X_n \leq 1\}}.$$

We proceed computing $\mathbb{E}[Y_n]$. Since $\vartheta \mapsto |1 - \lambda_n e^{i\vartheta}|$ is even, we get

$$\begin{aligned}\mathbb{E}[Y_n] &= \frac{1}{2\pi} \int_{\arccos(u_n)}^{2\pi - \arccos(u_n)} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n e^{i\vartheta}|^{2M}} d\vartheta = \frac{1}{\pi} \int_{\arccos(u_n)}^{\pi} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n e^{i\vartheta}|^{2M}} d\vartheta \\ &= \frac{1}{\pi} \int_{\arccos(u_n)}^{\pi/2} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n e^{i\vartheta}|^{2M}} d\vartheta + \frac{1}{\pi} \int_{\pi/2}^{\pi} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n e^{i\vartheta}|^{2M}} d\vartheta.\end{aligned}\tag{6}$$

Let us discuss the first integral. Observe that $2|1 - r e^{i\vartheta}|^2 \geq |1 - e^{i\vartheta}|^2 = 2(1 - \cos(\vartheta))$ for $r \in (0, 1)$ and $\vartheta \in [0, \pi/2]$. Hence

$$\begin{aligned}\frac{1}{\pi} \int_{\arccos(u_n)}^{\pi/2} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n e^{i\vartheta}|^{2M}} d\vartheta &\leq \frac{1}{\pi} \int_{\arccos(u_n)}^{\pi/2} \frac{1 - |\lambda_n|^2}{(1 - \cos(\vartheta))^M} d\vartheta \\ &= \frac{2(1 - |\lambda_n|^2)}{2^M \pi} \int_{c_n}^{1/2} \frac{1}{t^{2M} \sqrt{1 - t^2}} dt,\end{aligned}$$

where in the last equality we have used the change of variable $t = \sin(\vartheta/2)$ and $c_n = \sin(\frac{\arccos(u_n)}{2})$. If $t \in [c_n, 1/2]$, then the function $t \mapsto \sqrt{1 - t^2}$ is bounded from above and below. This yields

$$\begin{aligned}\frac{2(1 - |\lambda_n|^2)}{2^M \pi} \int_{c_n}^{1/2} \frac{1}{t^{2M} \sqrt{1 - t^2}} dt &\simeq \frac{2(1 - |\lambda_n|^2)}{2^M \pi} \int_{c_n}^{1/2} \frac{1}{t^{2M}} dt \\ &= \frac{2(1 - |\lambda_n|^2)}{2^M \pi (1 - 2M)} (2^{2M-1} - \frac{1}{c_n^{2M-1}}).\end{aligned}$$

Since, as already shown, $\arccos(u_n) \sim C_1(1 - |\lambda_n|^2)^{1/2}$ for $n \rightarrow \infty$, we have that $c_n \simeq C_2(1 - |\lambda_n|^2)^{1/2M}$. Hence

$$\begin{aligned} \frac{2(1 - |\lambda_n|^2)}{2^M \pi} \int_{c_n}^{1/2} \frac{1}{t^{2M}} \frac{dt}{\sqrt{1-t^2}} &\simeq \frac{2(1 - |\lambda_n|^2)}{2^M \pi (2M-1)} \left(\frac{1}{(1 - |\lambda_n|^2)^{(2M-1)/(2M)} - 2^{2M-1}} \right) \\ &\simeq (1 - |\lambda_n|^2)^{1/(2M)} \end{aligned}$$

Then it is immediate that the first integral is asymptotic to $C(1 - |\lambda_n|^2)^{1/2M}$ as $n \rightarrow \infty$. Since the denominator of the second integral in (6) is comparable to a constant, this second integral behaves like $(1 - |\lambda_n|^2)$, which is summable by the Blaschke condition. We thus obtain condition (ii) of the theorem:

$$\sum_{n \in \mathbb{N}} \mathbb{E}[Y_n] \simeq \sum_{n \in \mathbb{N}} \left((1 - |\lambda_n|^2)^{1/2M} + (1 - |\lambda_n|^2) \right) \simeq \sum_{k \in \mathbb{N}} N_k 2^{-\frac{k}{2M}} < +\infty.$$

In order to check condition (iii), it remains to estimate $\mathbb{V}[Y_n]$. Note that by our previous discussions, whenever $\vartheta \in [\arccos(u_n), 2\pi - \arccos(u_n)]$ then

$$\frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^{2M}} = \frac{1 - |\lambda_n|^2}{|1 - \lambda_n| e^{i\vartheta}|^{2M}} \leq 1.$$

With this in mind, we obtain

$$\begin{aligned} \mathbb{E}[Y_n^2] &= \frac{1}{2\pi} \int_{\arccos(u_n)}^{2\pi - \arccos(u_n)} \left(\frac{1 - |\lambda_n|^2}{|1 - \lambda_n| e^{i\vartheta}|^{2M}} \right)^2 d\vartheta \\ &\leq \frac{1}{2\pi} \int_{\arccos(u_n)}^{2\pi - \arccos(u_n)} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n| e^{i\vartheta}|^{2M}} d\vartheta \\ &= \mathbb{E}[Y_n], \end{aligned}$$

which yields

$$\mathbb{V}[Y_n] \leq \mathbb{E}[Y_n](1 - \mathbb{E}[Y_n]) \simeq \mathbb{E}[Y_n].$$

Consequently

$$\sum \mathbb{V}[Y_n] \leq \sum \mathbb{E}[Y_n] < +\infty,$$

which is condition (iii). By Theorem 2.5 we conclude that

$$\sum_{n \in \mathbb{N}} X_n = \sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|^{2M}} < \infty \quad \text{almost surely.}$$

□

Proof of Theorem 1.3. Suppose first that $\sum_k N_k 2^{-\frac{k}{2M}} < \infty$. From [13, Theorem 2] and [26, Theorem p.160] we know that a random sequence Λ satisfies the Carleson condition almost surely if and only if

$$\sum_{k \in \mathbb{N}} N_k^2 2^{-k} < \infty.$$

Since $M \geq 1$, then $(N_k 2^{-k/2})_{k \in \mathbb{N}} \in \ell^1$ which implies that $(N_k 2^{-k/2})_{k \in \mathbb{N}} \in \ell^2$, so the random sequence Λ almost surely satisfies the Carleson condition.

Moreover, Lemma 4.1 implies that for arbitrary $\zeta \in \mathbb{T}$

$$\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|^{2M}} < \infty \quad \text{almost surely.}$$

By hypothesis, $m_j \leq M$, $1 \leq j \leq l$ so that

$$\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta_j - \lambda_n|^{2m_j}} < \infty \quad \text{almost surely for } 1 \leq j \leq l.$$

We conclude with Theorem 1.2, that Λ is almost surely interpolating (multiplier or universal).

If $\sum N_k 2^{-\frac{k}{2M}} = \infty$, then, applying Lemma 4.1 once more, there exists r , $1 \leq r \leq l$ such that

$$\sum_{n \in \mathbb{N}} \frac{1 - |\lambda_n|^2}{|\zeta_r - \lambda_n|^{2m_r}} = \infty \quad \text{almost surely.}$$

This means that (2) does almost surely not hold and so we get

$$\mathbb{P}(\Lambda \text{ is interpolating for } \mathcal{H}(b)) = 0.$$

□

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