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On the zero capillarity limit for the Euler-Korteweg system

Corentin Audiard *, Marc-Antoine Vassenet[†]

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Abstract

We study the Euler-Korteweg equations with a weak capillarity tensor. It formally converges to the Euler equations in the zero capillarity limit. Our aim is two-fold : first we prove rigorously this limit in \mathbb{R}^d , $d \geq 1$, and obtain a more precise BKW expansion of the solution, second we initiate the study of the problem on the half space. In this case we obtain a priori estimates for the solutions that degenerate as the capillary coefficient converges to zero, and we explain this degeneracy with the construction of a (formal) BKW expansion that exhibits boundary layers.

The results on the full space extend and improve a classical result of Grenier (1998) on the semi-classical limit of nonlinear Schrödinger equations.

The analysis on the half space is restricted to the case of quantum fluids with irrotational velocity.

Résumé

1 Introduction

The Euler-Korteweg system is a modification of the compressible Euler equations that adds a capillary tensor in the momentum equation

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla g(\rho_\varepsilon) = \varepsilon^2 \nabla \left(K(\rho_\varepsilon) \Delta \rho_\varepsilon + \frac{1}{2} K'(\rho_\varepsilon) |\nabla \rho_\varepsilon|^2 \right), & (x, t) \in \Omega \times [0, T] \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (1.1)$$

The term $\varepsilon^2 K$ is the capillary coefficient. We are interested in the study of the limit $\varepsilon \rightarrow 0$, where we recover formally the usual Euler equations.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = 0, & (x, t) \in \Omega \times [0, T] \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (1.2)$$

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We consider solutions of the form $\rho = \rho_\infty + r$, with $(r, u) \in C([0, T], H^{n+1} \times H^n)$, n large, ρ_∞ is a constant such that $g'(\rho_\infty) > 0$, and $u \in C([0, T], H^n)$. Their (local) existence for fixed ε is known since the work of Benzoni, Danchin and Descombes [5].

Given (ρ_0, u_0) smooth, we study the convergence of smooth solutions of (1.1) to the solution of (1.2). We consider two geometric settings : $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^{+*} := \mathbb{R}_+^d$. Precise statements are given later, but our results for these two cases are significantly different and can be broadly summarized as follows :

1. In the full space case, we prove the existence of a time interval independent of ε on which the solutions of (1.1) converge to the solution of the Euler equations (“approximate solution”), with explicit rate of convergence. Moreover, thanks to BKW analysis, we obtain a higher order expansion of the approximate solution with arbitrarily high order of convergence.
2. In the half space case, with boundary condition $\rho|_{x_d=0} = 1$, $u_d|_{x_d=0} = 0$, we obtain a priori estimates of the solution that degenerate as $\varepsilon \rightarrow 0$. This feature is explained by the construction of an approximate solution which features terms varying rapidly in a “boundary layer” of size ε near $x_d = 0$, this explains the divergence of the higher order H^n norms of the solution as $\varepsilon \rightarrow 0$.

A discussion on other choices of boundary conditions that can be found in the litterature, and the associated BKW expansion, is provided at the end of the article, section 4.5.

Link with the Schrödinger equation There is an abundant litterature on the analysis of perturbations of hyperbolic problems, the problem studied here has most striking similarities with the semi-classical limit for the nonlinear Schrödinger equation

$$i\varepsilon \partial_t \psi_\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi_\varepsilon = g(|\psi_\varepsilon|^2) \psi. \quad (1.3)$$

Indeed the Madelung transform $\psi_\varepsilon = \sqrt{\rho_\varepsilon} e^{i\varphi_\varepsilon/\varepsilon}$ allows to formally reformulate (1.3) as the so-called quantum Euler system

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla g(\rho_\varepsilon) = \frac{\varepsilon^2}{4} \nabla \left(\frac{\Delta \rho_\varepsilon}{\rho_\varepsilon} - \frac{|\nabla \rho_\varepsilon|^2}{2\rho_\varepsilon^2} \right), \end{cases} \quad (1.4)$$

we recognize (1.1) with $K(\rho) = 1/(4\rho)$.

The Schrödinger equation on the full space The rigorous analysis of the semi-classical limit for (1.3) was initiated by Gérard [11], who proved the convergence to the Euler system in periodic, analytic settings. This was later extended to the Sobolev framework by Grenier [14] thanks to a change of variable (different from the Madelung transform) which allowed to reformulate (1.3) as a symmetrizable hyperbolic system with a dispersive perturbation *which commutes with the symmetrizer*. His main result is¹ :

¹The exact statement in [14] is slightly different, for the convenience of the reader we rephrase it in a way which is simpler for comparison in our settings.

Theorem 1.1 (Grenier '98). *Let ψ_ε solution of (1.3) with, for some $J \in \mathbb{N}$, $\psi_\varepsilon|_{t=0} = a_0(x, \varepsilon)e^{i\varphi_0(x, \varepsilon)/\varepsilon}$, $a_0 = \sum_{j=0}^J \varepsilon^j a_0^j(x) + \varepsilon^J r_\varepsilon^J(x)$, $\varphi_0 = \sum_{j=0}^J \varepsilon^j \varphi_0^j + \varepsilon^J \delta_\varepsilon^J$. Assume*

$$f' > 0, \lim_{\varepsilon \rightarrow 0} \|(r^J, \delta^J)\|_{H^s(\mathbb{R}^d)} = 0 \text{ for some } s > 2N + 2 + d/2.$$

Then there exists $T > 0$ such that ψ_ε has the form $\psi_\varepsilon = a_\varepsilon e^{i\varphi_\varepsilon/\varepsilon}$ on $[0, T] \times \mathbb{R}^d$, and there exists functions $a := \sum_{j=0}^J \varepsilon^j a^j$ complex valued, $\varphi = \sum_{j=0}^J \varepsilon^j \varphi^j$ defined on $[0, T] \times \mathbb{R}^d$ given by the BKW method such that

$$\|(a_\varepsilon - a, \varphi_\varepsilon - \varphi)\|_{L^\infty([0, T], H^{s-2J-2-d/2})} = o(\varepsilon^J).$$

This fundamental result received several extensions : addition of a subquadratic potential (Carles [9]), solutions that do not cancel at infinity (Alazard and Carles [2]), a degenerate non-linearity with $f'(0) = 0$ (Alazard-Carles [1] with some technical limitations on the regularity, later lifted by Chiron and Rousset [10]). In all the results mentioned, the fluid formulation (1.4) is never used for the proof of convergence. Rather the authors work either directly on the Schrödinger equation, or on the equations satisfied by $a_\varepsilon, \varphi_\varepsilon$, where a_ε is *complex valued*. This is a key feature since it allows to work on equations that have a better structure (less nonlinear, more skew-symmetric).

The Schrödinger equation on a domain This case is significantly more involved since the boundary conditions of (1.3) are in general not compatible with those of the limit system (1.2). The construction of approximate solution through a BKW expansion then requires to add corrector terms that are rapidly varying, see section 4.2 for details. When the spatial domain is the exterior of a smooth compact set in dimension 2, Lin and Zhang [18] proved the convergence of the fluid variables of (1.3) with $g(\rho) = \rho - 1$ (Gross-Pitaevskii equation), $\partial_n \psi_\varepsilon|_{\partial\Omega} = 0$ (Neumann boundary condition) to the solution of the Euler equation (1.2):

$$(\rho_\varepsilon, u_\varepsilon) := (|\psi_\varepsilon|^2, \varepsilon \operatorname{Im}(\overline{\psi_\varepsilon} \nabla \psi_\varepsilon)) \rightarrow (\rho, \rho u), \text{ in } L^\infty([0, T], L^2 \times L_{\text{loc}}^1).$$

The proof is fundamentally different from the argument of Grenier as it merely uses a modulated energy

$$H_\varepsilon(\psi_\varepsilon, \rho, u) = \int_\Omega |(\varepsilon \nabla - iu)\psi_\varepsilon|^2 + (|\psi_\varepsilon|^2 - \rho)^2 dx,$$

it does not extend to higher order of convergence or smoother functional settings.

When the domain is the half space $\mathbb{R}^2 \times \mathbb{R}^{+*}$, arbitrarily precise approximate solutions $(\rho_{\text{app}}, \varphi_{\text{app}})$ and high order of convergence were obtained by Chiron and Rousset [10] thanks to difficult energy estimates on the error $e^{-i\varphi_{\text{app}}/\varepsilon} \psi_\varepsilon - \sqrt{\rho_{\text{app}}}$. In particular, the skew symmetric nature of the linearized operator

$$i\frac{\varepsilon}{2}\Delta + u_{\text{app}} \cdot \nabla + \frac{\operatorname{div}(u_{\text{app}})}{2}$$

played a key role.

For Dirichlet boundary conditions $\psi_\varepsilon|_{x_d=0} = 0$, the analysis is even more difficult as the amplitude of the boundary layer terms is $O(1)$ (instead of $O(\varepsilon)$ for Neumann conditions), Gui and Zhang managed to push further the analysis from [10] to obtain results similar to the Neumann case, with the restriction that data are small (but with smallness independent of ε).

The general case of the Euler-Korteweg system There are several reasons to study the Euler-Korteweg system. It includes the physically relevant Schrödinger equation, but more importantly it has also been widely considered with other capillarity coefficients. For example, in the framework of weak solutions, with techics similar to the modulated energy estimates, Bresch, Gisclon, and Lacroix-Violet [8] studied the case $K(\rho)$ proportional to ρ^s , $s \in \mathbb{R}$, Giesselmann, Lattanzio and Tzavaras [12] considered constant and general positive capillarity coefficient K . A conditional convergence result of weak solutions of the Euler-Korteweg system (1.1) to the Euler equations (1.2) was deduced from these methods by Giesselmann and Tzavaras [13], under ad hoc regularity assumptions on the solutions, and special algebraic relations for K and g . To our knowledge the existence of global weak solutions for the Euler-Korteweg system with general capillarity is still an open problem. Other relevant capillary coefficients such as $1 + \kappa/(\rho - 1)$, $\kappa > 0$ appear in the framework of quasi-linear Schrödinger equations (e.g. [17]).

When the spatial domain is an open set different from \mathbb{R}^d , another motivation to consider the fluid formulation of the Schrödinger equation is the study of the boundary value problem where one prescribes on the boundary the physical quantities $\rho|_{\partial\Omega}$ and $u \cdot n|_{\partial\Omega}$. This is considered as the physically relevant boundary conditions for quantum fluids, see [19] section 2.1. Indeed since we have $\rho = |\psi|^2$, $u = \text{Im}(\bar{\psi}\nabla\psi)/|\psi|^2$, the boundary conditions on the original Schrödinger variable are highly nonlinear, and make the analysis quite difficult. The analysis on the half space is the subject of section 4.

Finally, another important point is that the analysis of the semi-classical limit for the Schrödinger equation is restricted to the case of irrotational velocity fields : $\text{curl}(u) = 0$. This limitation is lifted here by working directly on the fluid formulation, to the price of more technical energy estimates.

It should be noted that the convergence of solutions of (1.1) to solutions of other models (Burgers, KdV, Kadomtsev-Petviashvili) in the long wave regimes was studied by Benzoni and Chiron [6], the analysis relied notably on an improvement of the energy estimates introduced in [5], quite similar to proposition 3.2.

Main results When the spatial domain is \mathbb{R}^d , we obtain the existence of arbitrarily precise approximate solution, and their convergence as $\varepsilon \rightarrow 0$ to the exact solution :

Proposition 1.2 (Existence of an approximate solution). *Let $\Omega = \mathbb{R}^d$, $N \in \mathbb{N}$, $n > d/2 + 3 + [3N/2]$, $[\cdot]$ the integer part, $\rho_\infty > 0$, data $(r_0^k, u_0^k) \in H^{n_k}$, $0 \leq k \leq N$, with $n_k = n - [3k/2]$, $0 \leq k \leq N$, and such that $g' \circ (\rho_\infty + r_0^0) \geq c > 0$.*

There exists $\varepsilon_0 > 0$ and $T > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists an approximate solution $(\rho^{app} - \rho_\infty, u^{app}) := \sum_0^N \varepsilon^k (r^k, u^k)$, (r_k, u_k) given by the BKW expansion, solution of

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = e_1, \\ \partial_t u + u \cdot \nabla u + \nabla g = \varepsilon^2 \nabla (K \Delta \rho + \frac{1}{2} K' |\nabla \rho|^2) + e_2, \\ (\rho - \rho_\infty, u)|_{t=0} = \sum_{k=0}^N \varepsilon^k (r_0^k, u_0^k), \end{cases}$$

with

$$\|(e_1, e_2)\|_{C_T(H^{n_{N-1}} \times H^{n_{N-3}})} = O(\varepsilon^{N+1}), \quad (1.5)$$

$$\inf_{(x,t) \in \mathbb{R}^d \times [0,T]} \min(g'(\rho^{app}), \rho^{app}) \geq \alpha/2. \quad (1.6)$$

Theorem 1.3 (Convergence of the approximate solution). *Consider $(\rho_\varepsilon, u_\varepsilon)$ solution of the Euler-Korteweg system*

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla g = \varepsilon^2 \nabla (K \Delta \rho_\varepsilon + \frac{1}{2} \tilde{K}' |\nabla \rho_\varepsilon|^2), \\ (\rho_\varepsilon - \rho_\infty, u_\varepsilon)|_{t=0} = \sum_{k=0}^N \varepsilon^k (r_0^k, u_0^k), \end{cases}$$

with for any $0 \leq k \leq N$, $(r_0^k, u_0^k) \in H^{n-[3k/2]}$, $n > d/2 + 4 + [3N/2]$.

Let $(\rho^{app}, u^{app}) \in C([0, T], (\rho_\infty + H^{n+1-[3N/2]}) \times H^{n-[3N/2]})$ given by Proposition 1.2. For ε small enough, and $n - [3N/2] - 3$ even, the exact solution $(\rho_\varepsilon, u_\varepsilon)$ exists on $[0, T]$ and satisfies

$$\|(\rho_\varepsilon - \rho^{app}, u_\varepsilon - u^{app})\|_{L^\infty([0, T], H^{n-[3N/2]-3})} = O(\varepsilon^{N+1}).$$

Remark The restriction “ $n - [3N/2] - 3$ even” is purely technical and related to our choice of energy for simplicity of the proof. The restriction can be lifted by using a bit of pseudo-differential calculus as in [5], and replaced by the sharper condition $n > d/2 + 4 + [3N/2]$ with n real rather than an integer.

When the spatial domain is \mathbb{R}_+^d , our results are not as complete : even the derivation of energy estimates requires to work in the special case $K(\rho) = 1/\rho$. Arbitrarily precise approximate solutions in the sense of Proposition 1.2 exist, but the convergence to the exact solution is still open. For consistency in section 4, we construct the approximate solution in the special case $K = 1/\rho$, but this part can be easily generalized to general K . On the other hand irrotationality is an important simplification. For technical simplicity, we do not track the precise regularity assumptions in this case (instead we work with smooth functions) and we restrict the analysis to an irrotational velocity. In order to take into account the fast variation of the solution near the boundary, we introduce the notation $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^{+*}$. The use of a capital letter for a function F generically means that it writes as $F(x, t) = \tilde{F}(x', x_d/\varepsilon, t)$.

As usual for boundary value problems, the smoothness of the data is not enough to ensure the smoothness of the solution, we refer to section 4 for a description of the additional compatibility conditions, and 2 for the functional settings.

Proposition 1.4 (Approximate solution as a two scale expansion). *Assume $K(\rho) = 1/\rho$.*

Let $N \in \mathbb{N}$, $(r_0^k, u_0^k)_{0 \leq k \leq N} \in H^\infty(\mathbb{R}_+^d)$ satisfying the compatibility conditions, and $(u_0^k)_{0 \leq k \leq N}$ irrotational.

There exists $\varepsilon > 0$ and $T > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists an approximate solution $(\rho^{app} - \rho_\infty, u^{app}) = \sum_0^N \varepsilon^k (r^k + R^k, u^k + U^k)$, where $R^k(x, t) = \tilde{R}^k(x', x_d/\varepsilon, t)$, $U^k = \nabla \Phi^k$, $\Phi^k(x, t) = \tilde{\Phi}^k(x', x_d/\varepsilon, t)$, and $\Phi^0 = \Phi^1 = 0$, solution of

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = e_1 + E_1, \\ \partial_t u + u \cdot \nabla u + \nabla g = \varepsilon^2 \nabla (K \Delta \rho + \frac{1}{2} \tilde{K}' |\nabla \rho|^2) + e_2 + E_2, \\ (\rho - a, u)|_{t=0} = \sum_{k=0}^N \varepsilon^k (r_0^k, u_0^k), \end{cases}$$

with for any $n > 0$, $\|(e_1, e_2)\|_{C_T H^n} = O(\varepsilon^{N+1})$, and $E_1 = \tilde{E}_1(x', x_d/\varepsilon)$ (respectively E_2) is $O(\varepsilon^{N+1})$ in \mathcal{E}_T , respectively $O(\varepsilon^N)$.

In the rest of the paper, we shall assume $\rho_\infty = 1$. This can always be done with the change of unknown $\rho = \rho_\infty \tilde{\rho}$, since it preserves the assumption $g'(1) > 0$.

Plan of the paper Section 2 is devoted to basic notations and reminder on Sobolev spaces. Section 3 is focused on the proof of theorem 1.3 : we first prove uniform energy estimates which imply that the solution of the Euler-Korteweg system (1.1) remains smooth on a time interval independent of ε , then we prove a general “drift estimate” on the difference between an exact and an approximate solution. The construction of an approximate solution by BKW expansion is described in section 3.3, the convergence of the approximate solution to the exact solution is then a direct consequence of the general “drift estimate”.

In section 4, we initiate the analysis of the boundary value problem on a half space for (1.1) with boundary conditions $u|_{x_d=0} \cdot e_d = 0$, $\rho|_{x_d=0} = 1$. We first prove non optimal energy estimates on the solution that degenerate² as $\varepsilon \rightarrow 0$. Then (section 4.2 and after), as a possible explanation for the blow up of high H^s norms in the limit $\varepsilon \rightarrow 0$, we construct a BKW expansion with boundary layer terms that are smooth functions of x_d/ε . In concluding remarks (section 4.5), we compare the effect on the boundary layers of other choices of boundary conditions.

2 Notations, functional settings

We denote $A \lesssim B$ when there exists a constant C such that $A \leq CB$. The possible dependance of C with respect to some parameters will always be clear in whenever the notation is used.

Differential calculus A multi-index is generically denoted $\alpha \in \mathbb{N}^d$, its order is $|\alpha| = \sum_1^d \alpha_k$, the derivative of order α is $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

The gradient of a vector field is the matrix

$$\nabla u = (\partial_i u_j)_{1 \leq i, j \leq d}.$$

Irrotational and solenoidal vector fields We denote \mathbb{Q} , respectively \mathbb{P} , the projector on irrotational, respectively solenoidal, vector fields :

$$\mathbb{Q} = \Delta^{-1} \nabla \operatorname{div}, \quad \mathbb{P} = \mathbb{I} - \mathbb{Q}.$$

They are continuous self-adjoint projectors on $L^2(\mathbb{R}^d)$. We underline that $\Delta \mathbb{Q}$ and $\Delta \mathbb{P}$ are differential operators, in particular for f a smooth function and any $n \geq 1$, the commutator $[\Delta^n \mathbb{P}, f]$ is a differential operator of order $2n - 1$, while if f is not smooth we can use the mild estimates to bound $\|[\Delta^n \mathbb{P}, f]g\|_2 \lesssim \|f\|_{H^{2n}} \|g\|_{W^{1,\infty}} + \|f\|_{W^{1,\infty}} \|g\|_{H^{2n-1}}$.

The following simple identities will be often used without mention :

$$\mathbb{Q} \nabla = \nabla, \quad \mathbb{P} \nabla = 0, \quad \operatorname{div} = \operatorname{div} \mathbb{Q}, \quad \operatorname{div} \mathbb{P} = 0.$$

Functional spaces For $s \in \mathbb{R}$ (though we will only use s nonnegative integer), the H^s spaces are defined as

$$H^s(\mathbb{R}^d) = \{f \in L^2 : \int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty\}.$$

²We point out that even for $\varepsilon = 1$, this is a new result.

Of course, when s is an integer, they coincide with the set of L^2 functions that have distributional derivatives in L^2 up to order s . Due to their even higher simplicity, we shall in particular use the H^{2n} spaces for n integer, that are equivalently defined as

$$H^{2n}(\mathbb{R}^d) = \{f \in L^2 : \Delta^n f \in L^2\},$$

with equivalent norm $\|f\|_2 + \|\Delta^n f\|_2$.

The space \mathcal{E}_T is the set of functions smooth and exponentially decaying in the x_d variable:

$$\mathcal{E}_T = \{F \in H^\infty(\mathbb{R}_+^d \times [0, T]), \exists \gamma > 0 : \forall j \in \mathbb{N}, e^{\gamma x_d} \partial_d^j F \in L^\infty(\mathbb{R}_{x_d}^+, H^\infty(\mathbb{R}^{d-1} \times [0, T]))\}.$$

There is no natural norm on \mathcal{E}_T , so in proposition 1.4 by $E_\varepsilon = O(\varepsilon^N)$, we mean that there exists γ independent of ε such that for any $n \in \mathbb{N}$,

$$\sum_{j \leq n} \|e^{\gamma x_d} \partial_d^j E_\varepsilon\|_{L^\infty(\mathbb{R}_{x_d}^+, H^n(\mathbb{R}^{d-1} \times [0, T]))} = O(\varepsilon^N).$$

We recall a few standard properties of Sobolev spaces (see e.g. [4]):

Proposition 2.1. *Sobolev embedding* : For $0 < s < d/2$, $H^s \hookrightarrow L^p$, $p = \frac{2d}{d-2s}$.

For $s > d/2$, $s \notin d/2 + \mathbb{N}$, $H^s \hookrightarrow C_b^{s-d/2}$.

Gagliardo-Nirenberg type estimates :

$$\forall |\alpha| + |\beta| \leq k \in \mathbb{N}, \|D^\alpha f D^\beta g\|_2 \lesssim \|f\|_\infty \|g\|_{H^k} + \|f\|_{H^k} \|g\|_\infty.$$

Composition rules : for F smooth on some interval I , $F(0) = 0$, $u \in H^n \cap L^\infty$, $n \in \mathbb{N}$, $\overline{\text{Im}(u)} \subset I$.

$$\|F(u)\|_{H^n} \leq C(\|u\|_\infty) \|u\|_{H^n}.$$

A simple consequence of proposition 2.1 is that for $|\alpha| + |\beta| = n + 1$, $n > d/2 + 1$, $1 \leq \min(|\alpha|, |\beta|)$,

$$\|\partial^\alpha f \partial^\beta g\|_2 \lesssim \|f\|_{W^{1,\infty}} \|g\|_{H^n} + \|f\|_{H^n} \|g\|_{W^{1,\infty}} \lesssim \|f\|_{H^n} \|g\|_{H^n}.$$

In particular, we will frequently use the mild estimate : for $n > d/2 + 1$,

$$\forall \alpha \in \mathbb{N}^d, |\alpha| = n : \partial^\alpha (f \nabla g) = f(\partial^\alpha \nabla g) + R, \|R\|_2 \lesssim \|f\|_{W^{1,\infty}} \|g\|_{H^n} + \|f\|_{H^n} \|g\|_{W^{1,\infty}}. \quad (2.1)$$

3 Analysis on the whole space

3.1 Energy estimates and the time of existence

We remind that thanks to the change of variable $\rho \rightarrow \rho_\infty \rho$ we assume $\rho_\infty = 1$. Energy estimates for the Euler-Korteweg system have been derived in numerous settings, including the case with a small parameter [6]. We include here for completeness a self contained proof more in the spirit of [3] that does not use pseudo-differential calculus.

It relies on the following reformulation (due to Frédéric Coquel) : set $w = \varepsilon \sqrt{K/\rho} \nabla \rho = \nabla l$ where $l(\rho) = \varepsilon \int \sqrt{K/\rho} d\rho$, $l(1) = 0$, so that $\nabla l = w$. Then

$$\partial_t l + u \cdot w + \varepsilon \sqrt{\rho K} \operatorname{div} u = 0.$$

Set $a = \sqrt{\rho K}$, $z = u + iw$, after some computations

$$\partial_t z + u \cdot \nabla z + i \nabla z \cdot w + g'(\rho) \nabla \rho + i \varepsilon \nabla(\operatorname{adiv} z) = 0. \quad (3.1)$$

The hierarchy of modified energies is the following

$$\forall n \in \mathbb{N}, E_n(\rho, u) = \frac{1}{2} \int a^{2n} \rho |\mathbb{Q} \Delta^n z|^2 + |\mathbb{P}(\theta_n \Delta^n z)|^2 + g' a^{2n} |\Delta^n(\rho - 1)|^2 dx,$$

where θ_n is a function of ρ such that $\frac{1}{2}((\theta_n)^2)' = \frac{\varphi_n}{a} \sqrt{K/\rho}$, that we will choose positive on a suitable interval. Heuristically, the first two terms in E_n control $(\nabla \rho, u)$ in H^{2n} , but this control degenerates as $\varepsilon \rightarrow 0$, which is why we incorporate the third lower order term.

The weights $a^{2n} \rho$ be guessed from the case $n = 0$, where the conserved energy is $\int \rho |z|^2 / 2 + G(\rho) dx$, and at first order $G(\rho) = g'(\rho)(\rho - 1)^2 / 2$. For $n > 0$, one adds the weight a^n for commutation with the differential operator $\nabla(\operatorname{adiv} \cdot)$.

The following lemma is elementary, a proof can be found in [3], end of appendix A:

Lemma 3.1. *Assume there exists some $\alpha > 0$ such that $\operatorname{Im}(\rho) \subset I \subset [\alpha, 1/\alpha[$, $\|\rho\|_{W^{1,\infty}} \leq \alpha^{-1}$ and $\forall s \in I$, $g'(s) \geq \alpha > 0$. Choose θ_n such that $\theta_n|_I \geq \alpha$, then there exists $c_\alpha, C_\alpha > 0$ such that*

$$c_\alpha (\|z\|_{H^{2n}}^2 + \|\rho - 1\|_{H^{2n}})^2 \leq E_0 + E_n \leq C_\alpha (\|z\|_{H^{2n}}^2 + \|\rho - 1\|_{H^{2n}})^2,$$

In what follows, we denote $E_n(t) = E_n(\rho(t), u(t))$, (ρ, u) solution of the Euler-Korteweg system. For conciseness in the computations, we define

$$\varphi_n(\rho) := a^{2n} \rho, \quad \psi_n(\rho) := a^{2n} g'(\rho). \quad (3.2)$$

Proposition 3.2. *Let $(\rho_0 - 1, u_0) \in H^{2n+1} \times H^{2n}$, $2n > d/2 + 1$, $(\rho - 1, u) \in C([0, T], H^{2n+1} \times H^{2n})$ the unique local solution of (1.1).*

Assume that for some $T' \leq T$, $\alpha > 0$, $\rho(\mathbb{R}^d \times [0, T']) \subset I \subset [\alpha, \infty[$, such that $g'|_I \geq \alpha$, then we have for $t \in [0, T']$, $k \leq n$

$$\frac{dE_k}{dt} \leq C(\|u\|_\infty, \|\rho\|_{W^{1,\infty}}, \alpha) (\|z\|_{H^{2k}} + \|\rho - 1\|_{H^{2k}})^2 (\|u\|_{W^{1,\infty}} + \|\rho\|_{W^{2,\infty}}). \quad (3.3)$$

In particular the following holds :

1. *If $\operatorname{Im}(\rho_0) \subset I' \subset [2\alpha, \infty[$, such that on I' , $g' \geq 2\alpha$, then the solution exists on some time interval $[0, T_1]$, T_1 independent of ε , $(\rho - 1, u) \in C_{T_1}(H^{2n} \times H^{2n})$ with bounds independent of ε , and there exists an interval $I_1 \subset [\alpha, \infty[$ on which $g' \geq \alpha$ such that $\rho([0, T_1] \times \mathbb{R}^d) \subset I_1$, .*
2. *In the limit $\|(z_0, \rho_0 - 1)\|_{H^{2n} \times H^{2n}} \rightarrow 0$, there exists $c > 0$ such that the time of existence is bounded from below by $c / \|(z_0, \rho_0 - 1)\|_{H^{2n} \times H^{2n}}$.*

Proof. If the estimate (3.3) is true, the other points follow from the usual bootstrap argument (combined with lemma 3.1), so we focus on (3.3). Note also that the existence result of Benzoni-Danchin-Descombes implies the smoothness of the solutions, hence up to a standard approximation argument we can assume that the solution (ρ, u) is as smooth as needed for the computations.

In the following computations, R denotes generically a term which is controlled by the right hand side in (3.3). We shall use very often (2.1) without mention, e.g. to replace $\mathbb{Q} \Delta^n (u \cdot \nabla z)$ by $u \cdot \nabla(\mathbb{Q} \Delta^n z)$ plus terms that can be absorbed in R . Let us differentiate the three terms in E_n .

Computation of $D_1 := \frac{d}{2dt} \int_{\mathbb{R}^d} a^{2n} \rho |\mathbb{Q}\Delta^n z|^2 dx$

$$\begin{aligned} D_1 &= \operatorname{Re} \int_{\mathbb{R}^d} -\varphi_n \Delta^n \mathbb{Q} (u \cdot \nabla z + i \nabla z \cdot w + i \varepsilon \nabla (a \operatorname{div} z) + \nabla g) \Delta^n \mathbb{Q} \bar{z} dx \\ &= \operatorname{Im} \int_{\mathbb{R}^d} \underbrace{\varphi_n (\nabla (\Delta^n z) \cdot w + \varepsilon \mathbb{Q} \Delta^n \nabla (a \operatorname{div} z)) \cdot \mathbb{Q} \Delta^n \bar{z}}_{I_1} \underbrace{- \varphi_n g' (\mathbb{Q} \Delta^n \nabla \rho) \cdot \mathbb{Q} \Delta^n u}_{I_2} dx \\ &\quad - \operatorname{Re} \int_{\mathbb{R}^d} \varphi_n (u \cdot \nabla (\mathbb{Q} \Delta^n z)) \cdot \mathbb{Q} \Delta^n \bar{z} dx + R. \end{aligned}$$

We have $\operatorname{Re}((u \cdot \nabla \mathbb{Q} \Delta^n z) \cdot \mathbb{Q} \Delta^n \bar{z}) = u \cdot \nabla |Q \Delta^n z|^2$, hence with an integration by parts we include this term in R . In order to bound I_1 , we first point out that the factor ε in $\varepsilon \mathbb{Q} \Delta^n \nabla (a \operatorname{div} z)$ is essential, indeed it implies thanks to (2.1)

$$\varepsilon \mathbb{Q} \Delta^n \nabla (a \operatorname{div} z) = \nabla (a \Delta^n \operatorname{div} z + 2n \nabla a \cdot \Delta^{n-1} \nabla \operatorname{div} z) + \mathcal{R}, \quad \|\mathcal{R}\|_2 \leq C(\alpha) \|z\|_{W^{1,\infty}} \|z\|_{H^{2n}},$$

We now bound I_1 :

$$\begin{aligned} I_1 &= \operatorname{Im} \int_{\mathbb{R}^d} \varphi_n (\nabla \Delta^n z \cdot w + \varepsilon \nabla (a \Delta^n \operatorname{div} z + 2n \nabla a \cdot \nabla \Delta^{n-1} \operatorname{div} z)) \mathbb{Q} \Delta^n \bar{z} dx + R \\ &= \operatorname{Im} \int_{\mathbb{R}^d} \varphi_n (\nabla \Delta^n z \cdot w) \cdot \mathbb{Q} \Delta^n \bar{z} - \varepsilon (a \Delta^n \operatorname{div} z + 2n \nabla a \cdot \mathbb{Q} \Delta^n z) \operatorname{div}(\varphi_n \mathbb{Q} \Delta^n \bar{z}) dx + R \\ &= \operatorname{Im} \int_{\mathbb{R}^d} -\varphi_n (\Delta^n z \cdot w) \operatorname{div} \mathbb{Q} \Delta^n \bar{z} + \varepsilon \operatorname{div} \mathbb{Q} \Delta^n z (-a \nabla \varphi_n + 2n \varphi_n \nabla a) \cdot \mathbb{Q} \Delta^n \bar{z} dx. \end{aligned}$$

We use $-a \varphi_n' + 2n \varphi_n a' = -a(a^{2n} + 2n \rho a^{2n-1} a') + 2n \rho a^{2n} a' = -\varphi_n \sqrt{K/\rho}$ to obtain

$$\begin{aligned} I_1 &= \operatorname{Im} \int_{\mathbb{R}^d} -\varphi_n (\Delta^n z \cdot w) \operatorname{div} \mathbb{Q} \Delta^n \bar{z} - \varphi_n (\operatorname{div} \mathbb{Q} \Delta^n z) (w \cdot \mathbb{Q} \Delta^n \bar{z}) dx + R \\ &= \operatorname{Im} \int_{\mathbb{R}^d} -\varphi_n (\mathbb{P} \Delta^n z \cdot w) \operatorname{div}(\mathbb{Q} \Delta^n \bar{z}) dx + R. \end{aligned}$$

This contains a loss of derivatives, which will be cancelled later thanks to the derivative of the solenoidal term in the energy.

Compensation of I_2 Note that without further computation the term I_2 is already without loss of derivatives, but with a loss in ε . Using $\operatorname{div}(\rho u) = \rho \operatorname{div}(Qu) + u \cdot \nabla \rho$ we find

$$\begin{aligned} I_2 + \frac{d}{2dt} \int_{\mathbb{R}^d} \psi_n |\Delta^n (\rho - 1)|^2 dx &= \int -\varphi_n g' \Delta^n \nabla \rho \cdot \mathbb{Q} \Delta^n u - \psi_n \Delta^n \rho \Delta^n (u \cdot \nabla \rho + \rho \operatorname{div} Qu) dx \\ &= \int -\varphi_n g' \Delta^n \nabla \rho \cdot \mathbb{Q} \Delta^n u + \psi_n \nabla \Delta^n \rho \cdot (\mathbb{Q} \Delta^n u) \rho \\ &\quad - \psi_n u \cdot \nabla \Delta^n \rho \Delta^n \rho dx + R \\ &= \int (-\varphi_n g' + \psi_n \rho) \Delta^n \nabla \rho \cdot \mathbb{Q} \Delta^n u + \operatorname{div}(\psi_n u) \frac{|\Delta^n \rho|^2}{2} dx + R \\ &= R, \end{aligned}$$

indeed thanks to definition (3.2), we have $-\varphi_n g' + \psi_n \rho = 0$.

To summarize,

$$\frac{d}{2dt} \int_{\mathbb{R}^d} \varphi_n |\mathbb{Q} \Delta^n z|^2 + \psi_n |\Delta^n (\rho - 1)|^2 dx = \operatorname{Im} \int_{\mathbb{R}^d} -\varphi_n (\mathbb{P} \Delta^n z \cdot w) \operatorname{div}(\mathbb{Q} \Delta^n \bar{z}) dx + R. \quad (3.4)$$

Computation of $D_2 = \frac{d}{2dt} \int_{\mathbb{R}^d} |\mathbb{P}(\theta_n \Delta^n z)|^2 dx$. We use $\mathbb{P}\nabla = 0$,

$$\begin{aligned} D_2 &= -\operatorname{Re} \int_{\mathbb{R}^d} \mathbb{P}(\theta_n \Delta^n (u \cdot \nabla z + i \nabla z \cdot w + i \varepsilon \nabla(a \operatorname{div} z) + \nabla g)) \cdot \mathbb{P}(\theta_n \Delta^n \bar{z}) dx \\ &= -\operatorname{Re} \int_{\mathbb{R}^d} \left(u \cdot \nabla(\mathbb{P} \theta_n \Delta^n z) + i \varepsilon \theta_n \nabla(2n \nabla a \cdot \Delta^n \mathbb{Q}z + a \Delta^n \operatorname{div} \mathbb{Q}z) \right. \\ &\quad \left. + \theta_n g' \Delta^n \nabla \rho \right) \cdot \mathbb{P}(\theta_n \Delta^n \bar{z}) dx. \end{aligned}$$

With the usual integration by parts, we find

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^d} (u \cdot \nabla(\theta_n \Delta^n z)) \cdot \mathbb{P} \theta_n \Delta^n z dx &= R, \\ \int_{\mathbb{R}^d} (\theta_n \nabla \Delta^n z \cdot w) \cdot \mathbb{P}(\theta_n \Delta^n z) dx &= - \int_{\mathbb{R}^d} (\theta_n \Delta^n z \cdot w) \operatorname{div} \mathbb{P}(\theta_n \Delta^n z) dx + R = R. \end{aligned}$$

We deduce

$$\begin{aligned} D_2 &= \operatorname{Im} \int_{\mathbb{R}^d} \varepsilon \theta_n \nabla(2n \nabla a \cdot \Delta^n \mathbb{Q}z + a \Delta^n \operatorname{div} \mathbb{Q}z) \cdot \mathbb{P}(\theta_n \Delta^n \bar{z}) dx + R \\ &= - \int_{\mathbb{R}^d} \varepsilon (2n \nabla a \cdot \Delta^n \mathbb{Q}z + a \Delta^n \operatorname{div} \mathbb{Q}z) \nabla \theta_n \cdot \mathbb{P}(\theta_n \Delta^n \bar{z}) dx + R \\ &= - \int_{\mathbb{R}^d} \varepsilon (a \operatorname{div} \mathbb{Q} \Delta^n z) \theta_n' \nabla \rho \cdot \mathbb{P}(\theta_n \Delta^n \bar{z}) dx + R. \end{aligned}$$

We can replace $\mathbb{P} \theta_n \Delta^n z$ by $\theta_n \mathbb{P} \Delta^n z$ up to terms of order $2n - 1$, which are then absorbed in R with an integration by parts, this leads to

$$D_2 = - \int_{\mathbb{R}^d} \varepsilon (a \operatorname{div} \mathbb{Q} \Delta^n z) \theta_n \theta_n' \nabla \rho \cdot \mathbb{P}(\Delta^n \bar{z}) dx + R. \quad (3.5)$$

Conclusion Using (3.4) and (3.5) we obtain

$$\begin{aligned} \frac{dE_n}{dt} &= \operatorname{Im} \int_{\mathbb{R}^d} -\varphi_n (\mathbb{P} \Delta^n z \cdot w) \operatorname{div}(\mathbb{Q} \Delta^n \bar{z}) + a \sqrt{\frac{\rho}{K}} \theta_n \theta_n' (\mathbb{P} \Delta^n z \cdot w) \operatorname{div} \mathbb{Q} \Delta^n \bar{z} dx + R, \\ &= R, \end{aligned}$$

indeed the definition of θ_n ensures $\theta_n \theta_n' a \sqrt{\rho/K} - \varphi_n = 0$. \square

Remark 1. Note that in the limit $\varepsilon \rightarrow 0$, we recover the usual time of existence for quasi-linear hyperbolic equations, with blow up criterion on $\|(\rho - 1, u)\|_{W^{1,\infty}}$.

3.2 Difference estimates

Consider a smooth approximate solution (ρ_1, u_1) , ρ_1 bounded away from 0, satisfying

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}(\rho_1 u_1) = e_1, \\ \partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla g(\rho_1) = \varepsilon^2 \nabla (K(\rho_1) \Delta \rho_1 + \frac{1}{2} K'(\rho_1) |\nabla \rho_1|^2) + e_2, \end{cases} \quad (3.6)$$

(e_1, e_2) some functions assumed to be smooth, say H^∞ .

Set generically f_1 for a function evaluated at ρ_1, u_1 , in particular as previously $l_1 = l(\rho_1)$

$$\begin{cases} \partial_t l_1 + u_1 \cdot w_1 + \varepsilon a_1 \operatorname{div} u_1 = \varepsilon \sqrt{\frac{K_1}{\rho_1}} e_1, \\ \partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla g(\rho_1) = \varepsilon \nabla (a_1 \operatorname{div} w_1) + e_2, \end{cases} \quad (3.7)$$

Set also generically $\tilde{f} = f - f_1$, in particular $\tilde{u} = u - u_1$, $\tilde{r} = \rho - \rho_1$, $\tilde{w} = w - w_1$. The equation on z_1 is

$$\partial_t z_1 + u_1 \cdot \nabla z_1 + i \nabla z_1 \cdot w_1 + g'_1 \nabla \rho_1 + i \varepsilon \nabla (a_1 \operatorname{div} z_1) = e_3. \quad (3.8)$$

where $e_3 = e_2 + i \varepsilon \nabla \left(\sqrt{\frac{K_1}{\rho_1}} e_1 \right)$ the difference equations on \tilde{r}, \tilde{z} are

$$\begin{cases} \partial_t \tilde{r} + \operatorname{div}(\rho \tilde{u} + \tilde{r} u_1) & = -e_1, \\ \partial_t \tilde{z} + u \cdot \nabla \tilde{z} + i \nabla \tilde{z} \cdot w + g' \nabla \tilde{r} + i \varepsilon \nabla (a \operatorname{div} \tilde{z}) & = -\tilde{u} \cdot \nabla z_1 - i \nabla z_1 \cdot \tilde{w} - \tilde{g}' \nabla r_1 \\ & + i \varepsilon \nabla (\tilde{a} \operatorname{div} z_1) - e_3. \end{cases} \quad (3.9)$$

In the same spirit as the previous section, we define the energies

$$\tilde{E}_n = \frac{1}{2} \int \rho a^{2n} |\mathbb{Q} \Delta^n \tilde{z}|^2 + |\mathbb{P}(\theta_n \Delta^n \tilde{z})|^2 + a^{2n} g'(\rho) |\Delta^n \tilde{r}|^2 dx.$$

The analog of lemma 3.1 is true :

Lemma 3.3. *Assume there exists some $\alpha > 0$ such that $\operatorname{Im}(\rho) \subset I \subset [\alpha, 1/\alpha[$, $\|\rho\|_{W^{1,\infty}} \leq \alpha$ and $\forall a \in I$, $g'(a) \geq \alpha > 0$. Choose θ_n such that $\theta_n|_I \geq \alpha$, then there exists $c_\alpha, C_\alpha > 0$ such that*

$$c_\alpha (\|\tilde{z}\|_{H^{2n}}^2 + \|\tilde{r}\|_{H^{2n}})^2 \leq E_0 + E_n \leq C_\alpha (\|\tilde{z}\|_{H^{2n}}^2 + \|\tilde{r}\|_{H^{2n}})^2,$$

Proposition 3.4. *Let $2n > d/2 + 1$, $(\rho - 1, u) \in C_T H(H^{2n+1} \times H^{2n})$ given by the first point of proposition 3.2, (ρ_1, u_1) an approximate solution in $C_T(H^{2n+3} \times H^{2n+2})$, $\rho_1 \geq \alpha$, then for $k \leq n$*

$$\begin{aligned} \frac{d\tilde{E}_n}{dt} &\leq C(\|\tilde{z}, z_1, \varepsilon z_1\|_{H^{2n} \times H^{2n+1} \times H^{2n+2}} + \|\tilde{r}\|_{H^{2n}} + \|r_1\|_{H^{2n}}, \alpha) (\|\tilde{z}\|_{H^{2n}} + \|\tilde{r}\|_{H^{2n}})^2 \\ &\quad + \|e_2\|_{H^{2n}}^2 + \|(e_1, \varepsilon \nabla e_1)\|_{(H^{2n})^2}^2. \end{aligned} \quad (3.10)$$

Proof. This is a rather straightforward modification of the proof of estimate (3.3). For conciseness we only sketch the computations for the irrotational case, $\mathbb{Q}z = z$: performing similar computations as for energy estimates, we obtain, with R a term that is controlled by the right hand side of (3.10):

$$\begin{aligned} \frac{d\tilde{E}_n}{dt} &= R - \operatorname{Re} \int \varphi_n \Delta^n \left(\tilde{u} \cdot \nabla z_1 + i \nabla z_1 \cdot \tilde{w} + \tilde{g}' \nabla r_1 + i \varepsilon \nabla (\tilde{a} \operatorname{div} z_1) - e_3 \right) \overline{\Delta^n \tilde{z}} \\ &\quad - \int a^{2n} \rho g' \nabla \Delta^n \tilde{r} \cdot \Delta^n \tilde{u} - \int a^{2n} g' \Delta^n \tilde{r} \Delta^n (\operatorname{div}(\rho \tilde{u} + \tilde{r} u_1) - e_1). \end{aligned}$$

A first observation is that all terms of the first line can easily be absorbed in R , for example $\operatorname{Re} \int i\varepsilon\varphi_n \Delta^n \nabla(\tilde{a} \operatorname{div} z_1) \cdot \tilde{z}$ is roughly bounded by $C\|\tilde{r}\|_{L^\infty} \|\varepsilon z_1\|_{H^{2n+2}} \|z\|_{H^{2n}}$.

For the second line, using $\Delta^n \tilde{r} \nabla \Delta^n \tilde{r} = \frac{1}{2} \nabla(\Delta^n \tilde{r})^2$ and integration by parts, we find

$$\begin{aligned} \int a^{2n} \rho g' \nabla \Delta^n \tilde{r} \cdot \Delta^n \tilde{u} + a^{2n} g' \Delta^n \tilde{r} \Delta^n \operatorname{div}(\rho \tilde{u} + \tilde{r} u_1) &= R + \int a^{2n} \rho g' \nabla \Delta^n \tilde{r} \cdot \tilde{u} \Delta^n \rho \\ &\quad + a^{2n} g' \Delta^n \tilde{r} (u_1 \cdot \nabla \Delta^n \tilde{r} + (\Delta^n \operatorname{div} u_1) \tilde{r}) \\ &= R + \int a^{2n} \rho g' \nabla \Delta^n \tilde{r} \cdot \tilde{u} \Delta^n r_1, \end{aligned}$$

once again this last term is taken care of with an integration by part. \square

3.3 BKW analysis and convergence

This part can be done exactly as in previous works on the Schrödinger equation (see for example [10] section 3.2, or [14]), so we only recall the basic facts.

Write formally $r = \rho - 1$, $\rho = 1 + r^0 + \sum_1^\infty \varepsilon^k r^k$, $u = \sum_0^\infty \varepsilon^k u^k$, and plug this ansatz in (1.1). We obtain at rank 0 and 1

$$\begin{cases} \partial_t \rho^0 + \operatorname{div}(\rho^0 u^0) = 0, & \partial_t r^1 + \operatorname{div}(r^1 u^0 + \rho^0 u_1) = 0, \\ \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla(g(\rho^0)) = 0, & \partial_t u^1 + u^1 \cdot \nabla u^0 + u^0 \cdot \nabla u^1 + \nabla(g'(\rho^0) r^1) = 0, \end{cases} \quad (3.11)$$

and generically the equation of rank k is the linearization of the equation at order 0 with some source terms depending on the lower order terms $(r^j, u^j)_{0 \leq j \leq k-1}$

$$\begin{cases} \partial_t r^k + \operatorname{div}(r^k u^0 + \rho^0 u^k) = f_1^k, \\ \partial_t u^k + u^0 \cdot \nabla u^k + u^k \cdot \nabla u^0 + \nabla(g'(\rho^0) r^k) = f_2^k. \end{cases} \quad (3.12)$$

More precisely, $f_1^k = -\operatorname{div} \left(\sum_{0 < j, l, j+l=k} r^j u^l \right)$ involves derivatives of order at most 1 of terms

$(r^j, u^j)_{j \leq k-1}$, while f_2^k involves similar terms and derivatives up to order 3 of terms $(r^j)_{0 \leq j \leq k-2}$. It is less easy to write, but the main term for counting loss of derivatives is clearly $K(\rho^0) \nabla \Delta r^{k-2}$.

Unsurprisingly, the system of rank 0 is the Euler equations, that are well known to be symmetrizable (with symmetrizer $\operatorname{diag}(g'(\rho^0)/\rho^0, 1 \dots, 1)$), higher order equations are the linearization of the Euler equations near (ρ^0, u^0) , with forcing terms. The following result of well-posedness for symmetrizable hyperbolic systems is standard ([7], [4] theorem 4.15):

Theorem 3.5. *For initial data $(\rho_0^0 - 1, u_0^0) \in H^n$, $n > d/2 + 1$, $\operatorname{Im}(\rho^0) \subset I \subset [2\alpha, \infty[$, with $g'|_I \geq 2\alpha > 0$. There exists a time $T(\rho_0^0, u_0^0)$ such that system (3.11) has a unique solution in $\cap_{j=0}^n C^j([0, T], H^{n-j})$, with $\inf_{[0, T] \times \mathbb{R}^d} \min(\rho^0, g'(\rho^0)) \geq \alpha$.*

For any $k \geq 1$, data $(r_0^k, u_0^k) \in H^n(\mathbb{R}^d)$, $n > p > d/2 + 1$ and forcing terms $(f_1^k, f_2^k) \in H^p([0, T] \times \mathbb{R}^d)$, the system (3.12) has a unique solution in $\cap_{j=0}^p C^j([0, T], H^{p-j})$, with T the time of existence of (ρ^0, u^0) , it satisfies

$$\|(r^k, u^k)\|_{\cap_{j=0}^p C^j([0, T], H^{p-j})} \leq C(\|(r_0^0, u_0^0)\|_{H^p}, \alpha) \left(\|(r_0^k, u_0^k)\|_{H^n} + \|(f_1^k, f_2^k)\|_{\cap_{j=0}^p C^j H^{p-j}} \right).$$

As a consequence of this result and composition/product rules in Sobolev spaces, we may now state a precise version of proposition 1.2:

Corollary 3.6. *Let $N \in \mathbb{N}$. For $n > d/2 + 1 + [3N/2]$, $[\cdot]$ the integer part, data $(r_0^k, u_0^k) \in H^{n_k}$, $0 \leq k \leq N$, with $n_k = n - [3k/2]$, $0 \leq k \leq N$, there exists solutions of the systems (3.11),(3.12) up to order N , with $(r^k, u^k) \in \cap_{j=0}^{n_k} C^j([0, T], H^{n_k-j})$.*

If moreover $n > d/2 + 3 + [3N/2]$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the function $(\rho^{app}, u^{app}) := (\rho^0, u^0) + \sum_1^N \varepsilon^k (r^k, u^k)$ is an approximate solution of the Euler-Korteweg system as in (3.6) with

$$\|(e_1, e_2)\|_{C_T(H^{n_N-1} \times H^{n_N-3})} = O(\varepsilon^{N+1}), \quad (3.13)$$

$$\inf_{(x,t) \in \mathbb{R}^d \times [0, T]} \min(g'(\rho^{app}), \rho^{app}) \geq \alpha/2. \quad (3.14)$$

Proof. The proof is an immediate application of theorem 3.5 and composition rules in Sobolev spaces, so we only underline two points. First it is necessary to choose ε small enough in order to ensure inequality (3.14). Second, the numerology : as pointed out previously, (f_1^k, f_2^k) are functions that contain first order derivatives of (r^{k-1}, u^{k-1}) and third order derivatives of r^{k-2} . Hence $(r^0, u^0) \in \cap_{j=0}^n C_T^j H^{n-j}$, $(r^1, u^1) \in \cap_{j=0}^{n-1} C_T^j H^{n-j-1}$, then as f_2^k contains third order derivatives of ρ^0 , we have $(r^2, u^2) \in \cap_{j=0}^{n-3} C_T^j H^{n-j-3}$, the proof is ended by induction.

For the estimate of (e_1, e_2) , it suffices to observe that the worst terms in e_1 are derivatives of order one of (ρ^N, u^N) , while the worst term in e_2 is a derivative of third order of ρ^N . \square

Remark 2. The result is a bit better for $N = 0$, as (ρ^0, u^0) is an approximate solution of order 2 if $(r_0^1, u_0^1) = 0$. This is in particular the case if the initial data is simply of the form (ρ_0, u_0) .

Putting together corollary 3.6 and proposition 3.4, we can now prove the main result.

Proof of theorem 1.3. We apply proposition 3.4 on the difference $(\tilde{r}, \tilde{u}) = (\rho - \rho^{app}, u - u^{app})$, and for $0 \leq n \leq (n_N - 3)/2$. Thanks to the bounds on the approximate solution, we obtain

$$\begin{aligned} \|\tilde{z}(t)\|_{H^{n_N-3}}^2 + \|\tilde{r}\|_{H^{n_N-3}}^2 &\leq \int_0^t C(\|\tilde{z}\|_{H^{n_N-3}} + \|\tilde{r}\|_{H^{n_N-3}}, \alpha) (\|\tilde{z}\|_{H^{n_N-3}}^2 + \|\tilde{r}\|_{H^{n_N-3}}^2) (s) ds \\ &\quad + O(\varepsilon^{N+1}). \end{aligned}$$

Gronwall's lemma ensures that, as long as $\|\tilde{z}\|_{H^{n_N-3}} + \|\tilde{r}\|_{H^{n_N-3}} = O(1)$ and $g'(\rho), \rho$ are bounded away from 0, we have

$$\|\tilde{z}(t)\|_{H^{n_N-3}}^2 + \|\tilde{r}(t)\|_{H^{n_N-3}}^2 \leq C(\alpha)\varepsilon^{N+1}. \quad (3.15)$$

For ε small enough a standard bootstrap argument ensures that on $[0, T]$ the solution exists with $g'(\rho), \rho$ bounded away from 0 and (3.15) holds. In particular, we have $\|\rho - \rho^{app}\|_{C_T H^{n_N-3}} = O(\varepsilon^{N+1})$, and $\|\tilde{u}\|_{C_T H^{n_N-3}} = \|\text{Re}(\tilde{z})\|_{C_T H^{n_N-3}} = O(\varepsilon^{N+1})$. \square

4 Analysis on the half space

The case of the half space is more intricate. Even for $\varepsilon = 1$, there are no well-posedness results for the boundary value problem for the Euler-Korteweg system. Our aim in this section is to

initiate the analysis of the problem, by first deriving a priori estimates, and then performing a formal BKW expansion of the -hypothetical solution- to give an intuition of the effect of a boundary in the limit $\varepsilon \rightarrow 0$. We restrict the analysis to the case of the quantum Euler equation, that is $K(\rho) = 1/\rho$.

A reminder on compatibility conditions For boundary value problems, this is most easily done in general abstract setting : consider a problem of the form

$$\begin{cases} \partial_t U = F(U), \\ CU|_{x_d=0} = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where F is a smooth function of U and its space derivatives. C is a constant rectangular matrix (for our problem, $U = (\rho, u^t)^t$, $C = (e_1, e_{d+1})^t$). Obviously, by continuity we have

$$0 = CU|_{x_d=0, t=0} = CU_0|_{x_d=0},$$

this is the compatibility condition of order 0. By differentiation in time of $CU|_{x_d=0} = 0$ and use of the pde, we obtain the compatibility condition of order 1: $0 = C\partial_t U|_{x_d=0, t=0} = CF(U_0)|_{x_d=0}$, the sequence of higher order compatibility condition is obtained by iteration of the differentiation in time and use of $\partial_t U = F$.

In our settings, where F and U_0 depend on ε , there is a further manipulation : sorting by powers of ε , for each compatibility condition of a fixed order we obtain a hierarchy of conditions, for example if $U_0 = \sum \varepsilon^k U_0^k$, the compatibility condition of order 0 implies for any k , $CU_0^k|_{x_d=0} = 0$, the hierarchy of compatibility conditions of order 1 is then obtained by Taylor expansion of the relation $CF(\sum \varepsilon^k U_0^k) = 0$, and so on.

There exists non trivial data that satisfy the compatibility conditions at all orders, for example $(r_0, u_0) \in C_c^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{+*})$.

4.1 A priori estimates on the half space

In this section we derive a priori estimates for irrotational solutions of the Euler-Korteweg system in the half space in the special case of quantum hydrodynamics $K = 1/\rho$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \varepsilon^2 \nabla \left(\frac{\Delta \rho}{\rho} - \frac{|\nabla \rho|^2}{2\rho^2} \right), \quad (x', z) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, \quad t \geq 0. \\ (\rho, u)|_{t=0} = (\rho_0, u), \\ (\rho, u_3)|_{z=0} = (1, 0). \end{cases} \quad (4.1)$$

This leads to a major simplification, indeed the main order term for the reformulated system on $z = u + i\varepsilon \nabla \rho / \rho$ becomes linear :

$$\partial_t z + u \cdot \nabla z + i \nabla z \cdot w + \nabla g + i\varepsilon \Delta z = 0, \quad \text{with } w = \frac{\varepsilon \nabla \rho}{\rho}. \quad (4.2)$$

Nonetheless, the analysis of the boundary value problem is quite intricate : when carrying the energy method as in the full space, boundary terms coming from integration by parts must be

tracked, moreover the problem becomes characteristic in the limit $\varepsilon \rightarrow 0$, this causes the energy estimates to be non uniform in ε . Let $x = (x', x_d)$. Regularity in the tangential variables (t, x') is handled differently from the regularity in the normal variable x_d , accordingly we introduce the following functionals (abusively written as norms) : for z defined on $[0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}^+$,

$$\begin{aligned} \|z(t)\|_{X^n} &= \sum_{2\alpha_0 + \sum_1^d \alpha_k \leq n} \|\partial^\alpha z(t)\|_{L^2}, \\ \|z(t)\|_{X_{\text{tan}}^n} &= \sum_{2\alpha_0 + \sum_1^{d-1} \alpha_k \leq n} \|\partial^\alpha z(t)\|_{L^2}. \end{aligned}$$

In the same spirit as the full space, we define the following energies : for any tangential multi-index $\alpha = (\alpha_0, \dots, \alpha_{d-1}) \in \mathbb{N}^d$,

$$E_\alpha(t) = \frac{1}{2} \int_{\mathbb{R}_+^d} \rho |\partial^\alpha z|^2 + g'(\rho) |\partial^\alpha \rho|^2 dx.$$

It will be used numerous times without mention that due to the boundary conditions, for any tangential derivative

$$\partial^\alpha \rho|_{x_d=0} = 0, \quad \partial^\alpha u_d|_{x_d=0} = 0.$$

Our main result here is :

Lemma 4.1. *If (ρ, u) is a smooth, bounded away from 0, solution of (4.1), then for $n \in \mathbb{N}$, $2n > d/2 + 1$,*

$$\sum_{2\alpha_0 + \alpha_1 + \dots + \alpha_{d-1} \leq 2n} \frac{d}{dt} E_\alpha(t) \leq C(\|\nabla z\|_\infty + \|\nabla \rho\|_\infty) \|z\|_{X^{2n}}^2 \quad (4.3)$$

with $C = C(\|\rho\|_\infty + \|\rho\|_\infty^{-1}, \|\nabla z\|_\infty + 1)$ a continuous function. Moreover for $0 \leq j \leq n$ there exists a continuous function F_j

$$\|\partial_d^{2j} z\|_{X_{\text{tan}}^{2(n-j)}} \lesssim \frac{F_j(\|u\|_{X_{\text{tan}}^{2n}})}{\varepsilon^{4n}} \quad (4.4)$$

Remark 3. Independently of the limit $\varepsilon \rightarrow 0$, the estimates are the first step toward a well-posedness result similar to the one from [5] in the full space case. We expect that standard existence methods from the field of quasi-linear hyperbolic boundary value problems (e.g. [7] chapter 11: existence for the linearized system with a duality argument, then an iteration scheme) can be tracked to our settings, since the higher order dispersive part is linear, but a detailed proof is beyond the aim of this section.

Proof. We recall the reformulated equations (4.2) :

$$\partial_t z + u \cdot \nabla z + i \nabla z \cdot w + \nabla g + i \varepsilon \nabla \operatorname{div}(z) = 0.$$

We perform the same computations as for Proposition 3.2, but we have to check the cancellation of boundary terms. As a warm up, we prove the conservation of energy

$$\frac{d}{dt} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \rho |z|^2 + G(\rho) dx = 0, \text{ where } G' = g.$$

Indeed, denoting $\mathbf{n} = -\mathbf{e}_d$ the outward normal

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \frac{\rho |z|^2}{2} + G(\rho) dx &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \frac{-\operatorname{div}(\rho u) |z|^2}{2} \\ &\quad + \operatorname{Re} \left(\rho (-u \cdot \nabla z - i \nabla z \cdot w - \nabla g - i \varepsilon \nabla \operatorname{div}(z)) \bar{z} \right) \\ &\quad - g(\rho) \operatorname{div}(\rho u) dx \\ &= \int_{\mathbb{R}^{d-1}} - \left(\frac{|z|^2}{2} + g \right) \rho u \cdot \mathbf{n} + \operatorname{Im}(\rho \operatorname{div}(z) \bar{z} \cdot \mathbf{n}) dx' \\ &\quad - \operatorname{Im} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \rho (\nabla z \cdot w) \cdot \bar{z} - \varepsilon \operatorname{div}(z) \nabla \rho \cdot \bar{z} dx. \end{aligned}$$

The first integral cancels, indeed $u \cdot \mathbf{n} = -u_d = 0$, and

$$-\operatorname{Im}(\rho \operatorname{div}(z) \bar{z} \cdot \mathbf{n}) = \rho \operatorname{div}(w) u_d - \rho \operatorname{div}(u) w_d = -\rho \operatorname{div}(u) w_d,$$

and we have from the equation of mass conservation on the boundary:

$$0 = (\partial_t \rho + \operatorname{div}(\rho u))|_{x_d=0} = \operatorname{div}(u) + u \cdot \nabla \rho = \operatorname{div}(u).$$

To cancel the second integral, we use $\rho w = \varepsilon \nabla \rho$, and the boundary conditions $u_d|_{x_d=0} = 0$, $w_i|_{x_d=0} = 0$, $1 \leq i \leq d-1$:

$$\begin{aligned} \operatorname{Im} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \rho (\nabla z \cdot w) \cdot \bar{z} - \varepsilon \operatorname{div}(z) \nabla \rho \cdot \bar{z} dx &= \operatorname{Im} \varepsilon \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \partial_i z_j \partial_j \rho \bar{z}_i - \partial_j \rho \bar{z}_j \partial_i z_i dx \\ &= -\operatorname{Im} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} \rho (z_j w_j \partial_i \bar{z}_i + \bar{z}_j w_j \partial_i z_i) dx \\ &\quad - \operatorname{Im} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^+} z_j (\partial_i \partial_j \rho) \bar{z}_i dx \\ &\quad - \operatorname{Im} \int_{\mathbb{R}^{d-1}} z \cdot w \bar{z}_d dx' \\ &= \int_{\mathbb{R}^{d-1}} -(u \cdot w) w_d + |w|^2 u_d dx' = 0. \end{aligned}$$

The higher order estimates are similar : if $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}}$ contains only tangential derivatives, we denote $f_\alpha := \partial^\alpha f$ and we have

$$\partial_t z_\alpha + u \cdot \nabla z_\alpha + i(\nabla z_\alpha) \cdot w + \partial^\alpha \nabla g + i \varepsilon \nabla(\operatorname{div} z_\alpha) = \mathcal{C},$$

where \mathcal{C} is a quadratic commutator term that contains derivatives of order at most $|\alpha|$ of z . We differentiate E_α , and denote R a generic term which has a L^2 bound of the form $C(\|\rho\|_\infty +$

$\|\rho^{-1}\|_\infty)(\|\nabla z\|_\infty + \|\nabla \rho\|_\infty)\|z(t)\|_{H^{2n}}^2$, as in the statement of the lemma. Thanks to Gagliardo-Nirenberg type inequality (2.1) we find

$$\begin{aligned} \frac{d}{dt}E_\alpha(t) &= \int_{\mathbb{R}_+^d} \varepsilon \operatorname{Im}((\nabla z_\alpha \cdot \nabla \rho)\bar{z}_\alpha - \operatorname{div}(z_\alpha)\nabla \rho \cdot \bar{z}_\alpha) - \rho(\partial^\alpha \nabla g) \cdot u_\alpha - g' \rho_\alpha \partial^\alpha \operatorname{div}(\rho u) dx + R \\ &\quad + \varepsilon \operatorname{Im} \int_{\partial \mathbb{R}_+^d} \rho \operatorname{div}(z_\alpha)\bar{z}_\alpha \cdot e_d dx'. \end{aligned}$$

As for the conservation of energy, we have $\operatorname{Im}(\operatorname{div}(z_\alpha)\bar{z}_\alpha \cdot e_d) = -\operatorname{div}(u_\alpha)w_{\alpha,d} = 0$, so after integration by parts

$$\begin{aligned} \frac{d}{dt}E_\alpha(t) &= \varepsilon \int_{\partial \mathbb{R}_+^d} u_\alpha \cdot \nabla \rho w_{\alpha,d} dx \\ &\quad - \int_{\mathbb{R}_+^d} \rho g' \nabla \rho_\alpha \cdot u_\alpha + g' \rho_\alpha \operatorname{div}(\rho u_\alpha + u \rho_\alpha) dx + R \\ &= \int_{\partial \mathbb{R}_+^d} g' \rho_\alpha \rho u_{\alpha,d} + \int_{\mathbb{R}_+^d} g' u \cdot \frac{\nabla(\rho_\alpha)^2}{2} dx + R \\ &= - \int_{\partial \mathbb{R}_+^d} g' u_d \frac{(\rho_\alpha)^2}{2} dx' + R = R. \end{aligned}$$

This is (4.3).

Now to control normal derivatives, we shall use the equation to prove inductively

$$\forall j \leq n, \quad \|\partial_d^{2j} z\|_{X_{\tan}^{2(n-j)}} \lesssim \frac{F(\|u\|_{X_{\tan}^{2n}})}{\varepsilon^{4j}}, \quad (4.5)$$

where F is a generic smooth function that cancels at 0.

Denote $\Delta' = \sum_1^{d-1} \partial_i^2$ the tangential laplacian, we use noncharacteristicity :

$$\partial_d^2 z = -\Delta' z + i \frac{\partial_t z}{\varepsilon} + \frac{i}{\varepsilon} (u \cdot \nabla z + i \nabla z \cdot w + \nabla g). \quad (4.6)$$

To bound $\|\partial_d^2 z\|_{X_{\tan}^{2n-2}}$ we crudely bound $\|-\Delta' z + i \partial_t z / \varepsilon\|_{X_{\tan}^{2n-2}} \lesssim \|z(t)\|_{X_{\tan}^{2n}} / \varepsilon$. The nonlinear terms are estimated with Gagliardo-Nirenberg type inequalities, for conciseness we focus on the worst term $u_d \partial_d \partial^\alpha z$, with ∂^α a tangential derivative of order $2n-2$. We use the following interpolation inequality

$$\forall k \in \mathbb{N}^*, \quad \|f'\|_{L^2(\mathbb{R}^+)} \lesssim \|f\|_{L^2(\mathbb{R}^+)}^{1-1/k} \|f^{(k)}\|_{L^2(\mathbb{R}^+)}^{1/k}. \quad (4.7)$$

The inequality is easy when the domain is \mathbb{R} instead of \mathbb{R}^+ , it is deduced from this case by using extension operators. Applying this to $u_d \partial_d z$ we find for some $C, C_1 > 0$ and fixed (x', t) :

$$\begin{aligned} \frac{1}{\varepsilon} \|u_d \partial_d \partial^\alpha z(x', \cdot, t)\|_{L^2(\mathbb{R}_+^d)} &\leq \|u_d\|_\infty \|\partial^\alpha \partial_d z\|_{L^2} \leq \frac{C}{\varepsilon} (\|z\|_{L^2} \|\partial_d z\|_{L^2})^{1/2} \|z\|_{L^2}^{1/2} \|\partial^\alpha \partial_d^2 z\|_{L^2(\mathbb{R}^+)}^{1/2} \\ &\leq \frac{C_1}{\varepsilon} \|z\|_{L^2}^{5/4} \|\partial_d^2 z\|_{X_{\tan}^{2n-2}}^{3/4} \\ &\leq \frac{C_1^4}{4\varepsilon^4} \|z\|_{L^2}^5 + \frac{3}{4} \|\partial_d^2 z\|_{X_{\tan}^{2n-2}}. \end{aligned}$$

Thanks to Sobolev's embedding, $\| \|z\|_{L^2(\mathbb{R}^+)}^5 \|z\|_{L^2(\mathbb{R}^{d-1})} \lesssim \|z\|_{X_{\text{tan}}^{2n}}^5$.

We may now proceed to the induction : assume (4.5) is true for $1 \leq j \leq k-1$. To estimate $\partial^\alpha \partial_d^{2k} z$, ∂^α a tangential derivative of order $2n-2k$, we use equation (4.6) and we focus on the estimate of $\|u_d \partial_d^{2k-1} \partial^\alpha z / \varepsilon\|$, ∂^α tangential of order $2n-2k$:

$$\frac{1}{\varepsilon} \|u_d \partial_d^{2k-1} \partial^\alpha z\|_{L^2(\mathbb{R}^+)} \lesssim \frac{1}{\varepsilon} \|z\|_{L^2}^{1/2} \|\partial_d z\|_{L^2}^{1/2} \|\partial_d^{2k} \partial^\alpha z\|_2^{1/2} \|\partial_d^{2k-2} \partial^\alpha z\|_2^{1/2},$$

we deduce for any $C > 0$

$$\frac{1}{\varepsilon} \|u_d \partial_d^{2k-1} \partial^\alpha z\|_{L^2(\mathbb{R}_+^d)} \leq \frac{C'}{\varepsilon^2} \|z\|_{L^\infty(\mathbb{R}^{d-1}, L^2(\mathbb{R}^+))} \|\partial_d z\|_{L^\infty L^2} \|\partial_d^{2k-2} z\|_{X_{\text{tan}}^{2n-2k}(\mathbb{R}_+^d)} + \frac{\|\partial_d^{2k} z\|_{X_{\text{tan}}^{2n-2k}}}{C}.$$

Note that from Sobolev's embedding and interpolation

$$\|\partial_d z\|_{L^\infty L^2} \lesssim \|\partial_d^2 z\|_{X_{\text{tan}}^{2n-2}}^{1/2} \|z\|_{X_{\text{tan}}^{2n}}^{1/2} \leq F(\|z\|_{X_{\text{tan}}^{2n}}) / \varepsilon^2,$$

we conclude

$$\frac{1}{\varepsilon} \|u_d \partial_d^{2k-1} \partial^\alpha z\|_{L^2(\mathbb{R}_+^d)} \leq \frac{F(\|z\|_{X_{\text{tan}}^{2n}})}{\varepsilon^{4k}} + \frac{\|\partial_d^{2k} z\|_{X_{\text{tan}}^{2n-2k}}}{C}.$$

Choosing C large enough, we can absorb $\frac{\|\partial_d^{2k} z\|_{X_{\text{tan}}^{2n-2k}}}{C}$ in the left hand side and complete the induction. \square

A rough estimate on the time of existence The bounds from lemma 4.1 require a L^∞ bound on ρ to be "self closing", it is easily obtained (on very short time scale) as follows : apply the method of characteristics to the equation of mass conservation : for any $\alpha > 0$, $|\inf_{\mathbb{R}_+^d}(\rho(t)) - \inf \rho_0| + |\sup_{\mathbb{R}_+^d}(\rho(t)) - \sup \rho_0| \leq \alpha$ on a time interval $[0, T]$ such that $\int_0^t \|\text{div} u\|_\infty ds \leq \ln(1 + \alpha/2)$.

$$\forall j > d/2 - 1, \|f\|_{L^\infty(\mathbb{R}_+^d)} \lesssim (\|f\|_{X_{\text{tan}}^j} \|\partial_d f\|_{X_{\text{tan}}^j})^{1/2}.$$

Denoting $E_{2n, \text{tan}} = \sum_{2\alpha_0 + \alpha_1 + \dots + \alpha_{d-1}} E_\alpha$, we use lemma 4.1 and Sobolev's embedding

$$\begin{aligned} \frac{d}{dt} E_{2n, \text{tan}}(t) &\lesssim C(\rho\|_\infty + \|\rho^{-1}\|_\infty) (\|(\nabla z, \nabla \rho)\|_{X_{\text{tan}}^{2n-2}} \|(\nabla \partial_d z, \nabla \partial_d \rho)\|_{X_{\text{tan}}^{2n-2}})^{1/2} \|z\|_{X_{\text{tan}}^{2n}}^2, \\ \|\partial_d^{2j} z\|_{X_{\text{tan}}^{2n-j}} &\lesssim \frac{1}{\varepsilon^{4j}} F(\|z\|_{X_{\text{tan}}^{2n}}), \quad 1 \leq j \leq n. \end{aligned}$$

hence there exists a continuous function F_1 such that

$$\begin{aligned} \frac{d}{dt} E_{2n, \text{tan}}(t) &\lesssim C(\rho\|_\infty + \|\rho^{-1}\|_\infty) \frac{F_1(\|z\|_{X_{\text{tan}}^{2n}})}{\varepsilon^{8n+3}}, \\ \|\partial_d^{2j} z\|_{X_{\text{tan}}^{2n-j}} &\lesssim \frac{1}{\varepsilon^{4j}} F(\|z\|_{X_{\text{tan}}^{2n}}), \quad 1 \leq j \leq n. \end{aligned}$$

Of course, $\|z(t)\|_{X_{\text{tan}}^{2n}}^2 \sim E_{2n, \text{tan}}(t)$ with constants depending on $\|\rho\|_\infty, \|1/\rho\|_\infty$. It is now clear that on a timescale $O(\varepsilon^{8n+3})$, the bounds are self-closing.

This is not relevant in the limit $\varepsilon \rightarrow 0$, nevertheless for $\varepsilon = O(1)$ we recover an estimate similar to the \mathbb{R}^d case.

4.2 BKW expansion : notations

The estimates from the previous section are only obtained on a very short time interval, with a rapid growth of the norm of derivatives in the normal direction. A common explanation is that in the limit $\varepsilon \rightarrow 0$, the boundary conditions of the formal limit problem are not compatible with the one of the original one. Here the limit problem is the Euler equation with non penetration boundary condition :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (x, t) \in \mathbb{R}_+^d \times \mathbb{R}_t^+, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = 0, & (x, t) \in \mathbb{R}_+^d \times \mathbb{R}_t^+ \\ u \cdot e_d = 0, & (x, t) \in (\partial \mathbb{R}_+^d) \times \mathbb{R}_t^+. \end{cases}$$

The solutions of this problem do not satisfy the boundary condition $\rho_{\partial \mathbb{R}_+^d \times \mathbb{R}_t^+} = 1$, even if the initial data do, hence the presence of boundary layers is expected, leading to the growth of the Sobolev norms of the solution.

It is therefore natural to consider of a formal expansion in ε similarly to the full space case, but with the addition of correctors rapidly varying in x_d . As is common, we search an approximate solution (ρ_a, u_a) , with $u_a = \nabla \phi_a$ irrotational, in the form of a two scale expansion

$$\begin{cases} \rho_a(x, t) = 1 + \sum_0^N \varepsilon^n r^n(x, t) + \varepsilon^n R^n(x', x_d/\varepsilon, t), \\ \phi_a(x, t) = \sum_0^N \varepsilon^n (\phi^n(x, t) + \Phi^n(x', x_d/\varepsilon, t)). \end{cases}$$

We shall denote $\rho^0 = 1 + r^0$.

The terms (R^n, Φ^n) are the so-called boundary layer terms, they will belong to the set \mathcal{E}_T , we recall its definition:

$$\mathcal{E}_T = \{F \in H^\infty(\mathbb{R}_+^d \times [0, T]), \exists \gamma > 0 : \forall j \in \mathbb{N}, e^{\gamma x_d} \partial_d^j F \in L^\infty(\mathbb{R}_{x_d}^+, H^\infty(\mathbb{R}^{d-1} \times [0, T]))\}.$$

The terms (r^n, ϕ^n) are the interior terms. Since we work with the potential ϕ^n , it is convenient to introduce the integrated version of (1.1)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla \phi) = 0, & (x, t) \in \mathbb{R}_+^d \times \mathbb{R}_t^+, \\ \partial_t \phi + |\nabla \phi|^2/2 + g(\rho) = \varepsilon^2 \left(\frac{\Delta \rho}{\rho} - \frac{1}{2\rho^2} |\nabla \rho|^2 \right), & (x, t) \in \mathbb{R}_+^d \times \mathbb{R}_t^+. \end{cases} \quad (4.8)$$

In the following, we denote $\underline{f} = f|_{x_d=0}$.

4.3 The cascade of equations for the BKW expansion

The usual way to obtain equations for (R^n, Φ^n) , (r^n, ϕ^n) is to split the analysis between x_d large with respect to ε , where the boundary layer terms are neglected and we have to solve evolutionary equations on (r^n, ϕ^n) , and conversely for x_d small we obtain ordinary differential equations on the correctors (R^n, Φ^n) . An important tool is the following observation (see Grenier-Guès [15]) : for φ smooth, $(a, B) \in H^\infty \times \mathcal{E}$

$$f(a(x) + B(x', x_d/\varepsilon)) = f(a(x)) + f(a(x'), 0) + B(x', x_d/\varepsilon) + \varepsilon R, \quad R \in \mathcal{E}_T. \quad (4.9)$$

Inserting the ansatz in (1.1), and sorting by powers in ε , it is readily seen that $\Phi^0 = 0$, indeed the power ε^{-2} in the (integrated) momentum equation gives

$$(\partial_d \Phi^0)^2 = 0 \Rightarrow \Phi^0 = 0.$$

The first equations for the inner domain are

$$\begin{cases} \partial_t r^0 + \operatorname{div}(r^0 \nabla \phi^0) = 0, \\ \partial_t \phi^0 + \frac{1}{2} |\nabla \phi^0|^2 + g(r^0) = 0, \\ \partial_d \phi^0|_{x_d=0} = 0. \end{cases} \quad (4.10)$$

The next equations for the boundary layer are obtained using $\underline{\partial_d \phi^0} = 0$

$$\begin{cases} (1 + \underline{r^0} + R^0) \partial_d^2 \Phi^1 + (\partial_d \Phi^1 + \underline{\partial_d \phi^0}) \partial_d R^0 = 0, \\ \frac{\partial_d^2 R^0}{1 + R^0 + \underline{r^0}} - \frac{1}{2} \frac{(1 + \partial_d R^0)^2}{(R^0 + \underline{r^0})^2} = \partial_d \Phi^1 \underline{\partial_d \phi^0} + \frac{1}{2} (\partial_d \Phi^1)^2 + g(1 + R^0 + \underline{r^0}), \\ R^0|_{x_d=0} + \underline{r^0} = 0. \end{cases} \\ \Leftrightarrow \begin{cases} (1 + \underline{r^0} + R^0) \partial_d^2 \Phi^1 + (\partial_d \Phi^1) \partial_d R^0 = 0, \\ \frac{\partial_d^2 R^0}{1 + R^0 + \underline{r^0}} - \frac{1}{2} \frac{(\partial_d R^0)^2}{(1 + R^0 + \underline{r^0})^2} = \frac{1}{2} (\partial_d \Phi^1)^2 + g(1 + R^0 + \underline{r^0}), \\ R^0|_{x_d=0} + \underline{r^0} = 0, \end{cases} \quad (4.11)$$

Similarly to the full space case, the higher order equations for the interior terms are

$$\forall k \geq 1, \begin{cases} \partial_t r^k + \operatorname{div}(\rho^0 \nabla \phi^k + r^k \nabla \phi^0) = f_1^k, \\ \partial_t \phi^k + \nabla \phi^0 \cdot \nabla \phi^k + g'(\rho^0) r^k = f_2^k, \\ \partial_d \phi^k|_{x_d=0} + \partial_d \Phi^{k+1}|_{x_d=0} = 0. \end{cases} \quad (4.12)$$

where f_1^k, f_2^k only depend on $(\nabla \phi^j, r^j)_{j \leq k-1}$ and their derivatives.

The higher order boundary layer equations are

$$\begin{cases} \partial_d((R^0 + \rho^0) \partial_d \Phi^{k+2}) = F_1^k, \text{ mass, order } \varepsilon^k, \\ \frac{\partial_d^2(R^k)}{\rho^0 + R^0} - \frac{\partial_d R^k \partial_d R^0}{(\rho^0 + R^0)^2} = g'(\rho^0 + R^0) R^k + F_2^k, \text{ momentum, order } \varepsilon^k, \\ R^k|_{x_d=0} = \underline{r^k}. \end{cases} \quad (4.13)$$

where F_1^k , respectively F_2^k , depends on $(\Phi^j)_{j \leq k+1}, (R^j, \underline{r^j}, \underline{\varphi^j})_{j \leq k}$, respectively $(R^j)_{j \leq k-1}, (\Phi^j, \underline{r^j}, \underline{\varphi^j})_{j \leq k}$, and are exponentially fast decaying. We underline here that it is important for solvability that F_1^k does not depend on R^{k+1} , this is due to the fact that $\Phi^0 = \Phi^1 = \underline{\partial_d \phi^0} = 0$.

4.4 Solvability of the BKW expansion

The order in which we solve the equations is as follows

$$\Phi^{k+1} \rightarrow (\varphi^k, r^k) \rightarrow R^k \rightarrow \Phi^{k+2} \dots$$

The existence of the expansion will be a consequence of the following three lemmas :

Lemma 4.2 (Inner expansion). *For smooth initial data $(\rho_0^0, \varphi_0^0) \in (1 + H^\infty) \times H^\infty$ that satisfy the compatibility conditions, there exists a time $T > 0$ and a unique smooth solution such that $(\rho^0 - 1, \nabla \varphi^0) \in C([0, T], H^\infty)$ to (4.10).*

For such (r^0, φ^0) , T , and (r_0, φ_0) that satisfy the compatibility conditions, the linear problem

$$\forall k \geq 1, \begin{cases} \partial_t r + \operatorname{div}(\rho^0 \nabla \phi + r \nabla \phi^0) = f_1 \in H^\infty([0, T] \times \mathbb{R}_+^d), \\ \partial_t \phi + \nabla \phi^0 \cdot \nabla \phi + g'(\rho^0) r = f_2 \in H^\infty([0, T] \times \mathbb{R}_+^d), \\ \partial_d \phi|_{x_d=0} = b(x', t) \in H^\infty([0, T] \times \mathbb{R}^{d-1}), \\ (\varphi, r)|_{t=0} = (\varphi_0, r_0) \in H^\infty(\mathbb{R}_+^d), \end{cases} \quad (4.14)$$

has a unique solution with $(\nabla \phi, r) \in H^\infty([0, T] \times \mathbb{R}_+^d)$.

Proof. Define $u^0 = \nabla \varphi^0$, and take the gradient of the second equation. The new system is the Euler equations with non penetration boundary condition. The existence of a smooth solution is due to Schochet [21]. Then we obtain φ^0 simply with the formula

$$\varphi^0 = \varphi^0(t=0) + \int_0^t |u^0|^2/2 + g(\rho) ds.$$

The system (4.14) is a hyperbolic maximal dissipative problem, the general solvability can be found in [20], as for the smoothness of solution the method of proof of Schochet³ works also in this case. \square

Lemma 4.3 (Boundary layer, first order). *There exists $T > 0$ such that the system (4.11) has a unique solution*

$$\Phi^1 = 0, \quad R^0 \in \mathcal{E}_T.$$

Proof. By integration of the first equation and decay at infinity, $(R^0 + \underline{r}^0) \partial_d \Phi^1 = 0$, hence $\Phi^1 = 0$ provided $R^0 + \underline{r}^0 \neq 0$.

The second equation rewrites

$$\frac{1}{\sqrt{\underline{\rho}^0 + R^0}} \partial_d^2 \sqrt{\underline{\rho}^0 + R^0} = g(\underline{r}^0 + R^0) + \partial_t \varphi^0 + |\nabla \varphi^0|^2/2 = g(\underline{\rho}^0 + R^0) - g(\underline{\rho}^0).$$

Setting $A^0 = \sqrt{\underline{\rho}^0 + R^0}$, this rewrites

$$\partial_d^2 A^0 = A^0 (g((A^0)^2) - g(\underline{\rho}^0)) := f(A^0), \quad \text{with } A^0|_{x_d=0} = \underline{\rho}^0 + R^0|_{x_d=0} = 1.$$

Note that $f'(\sqrt{\underline{\rho}^0}) = 2\underline{\rho}^0 g'(\underline{\rho}^0) > 0$ if $\underline{\rho}^0(x', t) = 1 + \underline{r}^0$ is close enough to 1, so standard ODE arguments ensure for any x', t the existence of A^0 converging exponentially fast to 0 with $(A^0)^2(x', 0, t) = 1$. By continuity of $\underline{\rho}^0$ and the compatibility condition $\underline{\rho}^0(x', 0, 0) = 1$, this is true on some time interval $[0, T]$, T small enough. \square

³The problem is characteristic, but near the boundary one can trade tangential regularity to estimate $\partial_d u_d, \partial_d r$, then the regularity of $(u_j)_{1 \leq j < d-1}$ is obtained by considering $\operatorname{curl}(u)$, which also satisfies a dissipative hyperbolic system.

Lemma 4.4. For F_1, F_2 in \mathcal{E}_T , the problems

$$\partial_d((R^0 + \underline{\rho}^0)\partial_d\Phi) = F_1,$$

and

$$\begin{cases} \frac{\partial_d^2(R)}{\underline{\rho}^0 + R^0} - \frac{\partial_d R \partial_d R^0}{(\underline{\rho}^0 + R^0)^2} = g'(\underline{\rho}^0 + R^0)R + F_2, \\ R|_{x_d=0} + \underline{r} = 0. \end{cases}$$

have unique smooth solutions R, Φ in \mathcal{E}_T .

Proof. From the first equation we get

$$\partial_d\Phi^k(x', x_d, t) = (R^0 + \underline{\rho}^0)^{-1} \int_{\infty}^{x_d} F_1(x', r, t) dr.$$

Since the right hand side belongs to \mathcal{E}_T , another integration gives the unique solution in \mathcal{E}_T . Note that the decay at infinity of Φ does not allow to prescribe its value at $x_d = 0$.

Now R satisfies :

$$\begin{cases} \partial_d \left(\frac{\partial_d R}{\underline{\rho}^0 + R^0} \right) = g'(\underline{\rho}^0 + R^0)R + F_2, \\ R|_{x_d=0} + \underline{r} = 1. \end{cases}$$

This is a boundary value problem of the form $(aX')'(s) = bX(s) + F(s)$, with $a, b > 0$, and F exponentially decaying. The existence of an exponentially decaying solution is a direct consequence of the change of unknown $\tilde{X} = e^{\alpha s} X$ -with some α small enough- and an application of Lax-Milgram theorem. \square

To summarize, from $(\Phi^j)_{j \leq k}$, $(R^j, \varphi^j, r^j)_{j \leq k-1}$, we obtain Φ^{k+1} by solving the first ODE in (4.13) (lemma 4.4). Given Φ^{k+1} the boundary condition in (4.12) is well-defined so we get (φ^k, r^k) with lemma 4.2, then from r^{k+1} we get the boundary condition to compute R^{k+1} in (4.13), using again lemma 4.4.

4.5 Comparison with other boundary conditions

Previous works adressed (for the nonlinear Schrödinger equation) the case of Dirichlet boundary conditions $\varphi|_{x_d=0} = 0, \rho|_{x_d=0} = 1$ (Gui-Zhang [16]), and the case of Neumann boundary conditions $\varphi|_{x_d=0} = 0, \rho|_{x_d=0} = 1$ (Chiron-Rousset [10]). Following this terminology, we label the boundary conditions considered here as “mixed Dirichlet-Neumann” The hierarchy of corrector terms is as follows :

1. Dirichlet : $R^0 \neq 0, \Phi^1 \neq 0$. Existence of the BKW expansion at any order for small smooth data.
2. Neumann : $R^0 = \Phi^1 = \Phi^2 = 0, R^1 \neq 0, \Phi^3 \neq 0$. Existence of the BKW expansion at any order for smooth data.
3. Mixed boundary conditions : $\Phi^1 = 0, R^0 \neq 0, \Phi^2 \neq 0$. Existence of the BKW expansion at any order for smooth data.

The justification that $\Phi^2 \neq 0$ in our case is merely a computation : the equation satisfied by Φ^2 is

$$\partial_d ((\underline{\rho}^0 + R^0(x_d))\partial_d \Phi^2(x_d)) = -\partial_d (R^0(x_d)x_d \underline{\partial_d^2} \varphi) = F_1^2,$$

and F_1^2 is not zero.

Hence the first boundary layer term for the velocity is small for the mixed boundary conditions, but not as small as in the Neumann case. While the mixed boundary conditions seem to lie in between Dirichlet and Neumann in term of the strength of the boundary layers, it has similar difficulty to Dirichlet since it contains R^0 as a $O(1)$ boundary layer term. More importantly, the boundary conditions have no simple expression in the Schrödinger formulation. The use of the Schrödinger formulation is a key point for the convergence analysis in both [16] and [10], and this is what prevents us so far from proving the convergence of the BKW expansion to the exact solution.

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