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Compositional Taylor expansion in cartesian differential categories

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Abstract—This paper provides a compositional approach to Taylor expansion, in the setting of cartesian differential categories. Taylor expansion is captured here by a functor that generalizes the tangent bundle functor to higher order derivatives. The fundamental properties of Taylor expansion then boils down to naturality equations that turns this functor into a monad. This monad provides a categorical approach to higher order dual numbers and the jet bundle construction used in automated differentiation.

The combination of the theory of the differential calculus with the theory of programming languages has seen a tremendous growth in the last decades, most notably in the fields of automated differentiation (AD) [1] and of the differential λ -calculus [2]. Both AD and the differential λ -calculus aim at computing the derivative of a program in a compositional way, this compositionality is crucial to scale those methods to complex assemblies of programs. Because of this interplay between derivatives and composition, category theory provides a strong mathematical basis for both of those fields. Categorical semantics provides critical proofs methods of correctness of the AD algorithm [3], [4], and the differential lambda calculus is deeply related to the categorical semantics of Linear Logic (LL) from its very inception [2], [5]. Among those categorical approaches, cartesian differential categories [6] provide a direct axiomatization of derivatives in any cartesian category. As such, they serve as a framework to understand the differential calculus through the lenses of compositionality.

The compositionality of the derivative is expressed by the *chain rule*: $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$. The issue of the chain rule is that it is not entirely compositional, because the derivative $(g \circ f)'$ also depends on f , and not only on g' and f' . For this reason, one often consider the *tangent bundle* operator T that intuitively maps $f : X \rightarrow Y$ to the function $Tf : (x, u) \mapsto (f(x), f'(x) \cdot u)$. This operator exists in any cartesian differential category, and the chain rule boils down to a functoriality equation on T : $T(g \circ f) = Tg \circ Tf$. Furthermore, the other axioms of the differential calculus (such as the linearity of the derivative or the symmetry of the higher order derivatives) turn out to be equivalent to naturality equations [7], [8] that turn T into a *monad* [8] whose algebraic structure is similar to that of dual numbers widely used in AD. This suggests that differentiation is an effect in the sense of Moggi [9], further cementing the use of category theory as a strong mathematical foundation for the differential calculus.

Contribution of the paper: This article provides a similar

compositional approach to Taylor expansion, in any cartesian differential category. The computation of the Taylor expansion of a composition $g \circ f$ is not straightforward because it requires to compute the higher order derivatives $(g \circ f)^{(n)}$. These derivatives are given by the Faà di Bruno formula [10] (4) which is notoriously combinatorial. We solve this issue by defining an operator T_n that maps $f : X \rightarrow Y$ to $T_n f : X \times X^n \rightarrow Y \times Y^n$ such that the i -th component of $T_n f(x, u_1, \dots, u_n)$ provides the term of order ε^i in the Taylor expansion of $f(x + \varepsilon u_1 + \dots + \varepsilon^n u_n)$. The order n Taylor expansion of f can be recovered from $T_n f$ simply by computing $T_n f(x, u, 0, \dots, 0)$, but generalizing this to any vector (x, u_1, \dots, u_n) is the key to turn this into a compositional operation such that $T_n(g \circ f) = T_n g \circ T_n f$. We prove that T_n is a functor, and satisfies naturality equations that are clear analogues of the naturality equations of the tangent bundle T . These equations turn T_n into a monad, whose algebraic structure is similar to that of higher order dual numbers recently used in AD [11]. We explore some consequences of these observations, such as expressing the partial Taylor expansion with regard to one variable in terms of monad strengths, or by combining multiple Taylor expansions using distributive laws [12].

Proofs methods: The combinatorics of Taylor expansion usually involves quite a lot of heavy lifting. A key contribution of this article is to hide all this combinatorics behind categorical equations. In particular, we will heavily rely on functoriality and naturality equations on the iterated tangent bundle functor $T^n f : X^{2^n} \rightarrow Y^{2^n}$ rather than direct style equations involving the higher order derivatives $f^{(n)} : X \times X^n \rightarrow Y$. More precisely, we prove that T^n has a canonical monad structure that arises from a *distributive law* [12] of the monad T on itself. Then, we express $T_n f : X \times X^n \rightarrow Y \times Y^n$ using $T^n f$ instead of $f^{(n)}$. To this end, we generalize to any cartesian differential category the notion of *higher order directional derivative* of [13], that we tweak with coefficients in order to erase redundancy. Then, it is quite straightforward to check that the functoriality and naturality equations on T^n induce similar equations on T_n . This process also yields a family of morphisms $(L_n)_X : T_n X \rightarrow T^n X$ that turns out to be a monad morphism [14], or monad transformer [15], from T_n to T^n . This gives an insight on the link between the higher order derivatives as studied in [16], [17] and Taylor expansion.

Links with the differential λ -calculus: The operator T_n is

directly inspired from the *coherent Taylor expansion* of [18]. In this article, Ehrhard and Walch provide a new categorical semantics of the differential λ -calculus [2] and of Taylor expansion [19] that accounts for the deterministic nature of computation, as opposed to the previously known models of [5]. This determinism is modeled by a notion of partial sums that prevents the summation of incompatible information (such as summing `true` and `false` in the type of `boolean`). Taylor expansion is axiomatized in their work as a bimonad, whose monad structure is an infinitary counterpart of our monad T_n . There are however two main differences with our work. First, they deal with infinite partial sums which are inherently positives in the sense that $x + y = 0$ implies $x = y = 0$. This clashes with the cartesian differential categories arising from traditional analysis. More importantly, Ehrhard and Walch do not provide any explicit description of their Taylor expansion in terms of derivatives outside of examples. This sharply contrasts with our work in which we explicitly define Taylor expansion from the derivative, in any cartesian differential category. We explore in [sec. IV-A](#) and [VII](#) the relation between their Taylor expansion and ours. In particular, we use our construction T_n to prove that any model of the differential lambda calculus and of Taylor expansion [5] is an instance of coherent Taylor expansion [18]. This solves an open problem of [18], and ensures that coherent Taylor expansion is a generalization of the previous axiomatization.

Related work: The idea behind T_n is very similar to higher order dual numbers [11] and jet bundles that have found recent applications in AD for computing higher order derivatives [20], [21]. It is quite likely that a generalization of the content of this article to tangent categories [7] would provide a categorical abstraction of jet bundles.

Outline of the paper: [Sec. I](#) reintroduces necessary material on cartesian differential categories and higher order derivatives. [Sec. II](#) introduces the tangent bundle T , and reviews how the axioms of cartesian differential categories boil down to functoriality and naturality equations on T . [Sec. III](#) exhibits the canonical monad structure of T^n . [Sec. IV](#) defines T_n , explains why it corresponds to a Taylor expansion, and proves its functoriality as a consequence of the functoriality of T^n . [Sec. V](#) proves that T_n is a monad, by using the monad structure of T^n . [Sec. VI](#) proves that any functor satisfying the same naturality equations as T_n is the Taylor expansion associated to a derivative, meaning that the categorical equations of T_n exactly capture Taylor expansions. Finally, [sec. VII](#) considers infinitary Taylor expansions in cartesian differential categories featuring arbitrary countable sums. We prove that in that setting, the derivative induces a coherent Taylor expansion as defined in [18]. Thus, every model of [5] is a model of [18].

I. CARTESIAN DIFFERENTIAL CATEGORIES

We recall necessary material on cartesian differential categories, with the specificity that the homset are not only commutative monoids but k -semimodules over a commutative semiring k . We mainly recall material from [17], with the difference that we use the word *additive* instead of linear in

order to stick to the conventions of [6], [18]. A commutative semiring k is a set equipped with two commutative monoid structures $(k, 0, +)$ and $(k, 1, \cdot)$ such that $a \cdot 0 = 0$ and $a \cdot (b + c) = a \cdot b + a \cdot c$. For example, the set of natural numbers \mathbb{N} is a commutative semiring, with the usual addition and multiplication of natural numbers. For any semiring k , there is a canonical semiring morphism $\phi : \mathbb{N} \rightarrow k$ given by $\phi(n) = 1_k + \dots + 1_k$.

Definition 1. A semiring k has multiplicative inverses for integers if for all $n \in \mathbb{N}^*$, there exists an element $\frac{1}{n} \in k$ such that $\frac{1}{n} \cdot \phi(n) = 1_k$.

A k -semimodule consists of the data of a commutative monoid M and a multiplicative action $_ \cdot _ : k \times M \rightarrow M$ that respects addition in each variable, and such that for all $a, b \in k$ and $x \in M$, $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ (this is why the multiplicative action has the same name as the multiplication in k). A map $f : M \rightarrow L$ between two k -semimodules is k -additive if $f(x + y) = f(x) + f(y)$ and $f(a \cdot x) = a \cdot f(x)$ (the case $a = 0$ ensures that $f(0) = 0$).

Notations 1. For any cartesian category \mathcal{C} , we write the categorical product $X_0 \times X_1$, the pairing of $f \in \mathcal{C}(X, Y_0)$ with $g \in \mathcal{C}(X, Y_1)$ as $\langle f, g \rangle \in \mathcal{C}(X, Y_0 \times Y_1)$, and the projections as $p_0 \in \mathcal{C}(X_0 \times X_1, X_0)$ and $p_1 \in \mathcal{C}(X_0 \times X_1, X_1)$. We write $X^{\times n}$ and $f^{\times n}$ the cartesian product of X and f with itself taken n -times.

Definition 2. A left k -additive category is a category \mathcal{C} such that each hom-set $\mathcal{C}(X, Y)$ has a k -semimodule structure and such that for all object Z of \mathcal{C} and $f \in \mathcal{C}(X, Y)$, the map $_ \circ f : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$ is k -additive.

A morphism $h \in \mathcal{C}(Y, Z)$ is k -additive if for all object X , the map $h \circ _ : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ is k -additive. A k -additive category is a left k -additive category in which all the morphisms are k -additive.

A cartesian left k -additive category is a left k -additive category that is cartesian and such that the projections of the cartesian product are k -additive. This last condition means that the k -module structure is compatible with the tupling, in the sense that $\langle f_0, g_0 \rangle + \langle f_1, g_1 \rangle = \langle f_0 + f_1, g_0 + g_1 \rangle$ and for all $r \in k$, $r \cdot \langle f, g \rangle = \langle r \cdot f, r \cdot g \rangle$.

Given any cartesian left k -additive category, it is straightforward to check that there exists a k -additive category \mathcal{C}^{add} whose objects are the objects of \mathcal{C} , and whose morphisms are the k -additive morphisms of \mathcal{C} . The composition of $f \in \mathcal{C}^{\text{add}}(X, Y)$ with $g \in \mathcal{C}^{\text{add}}(Y, Z)$ is $g \circ f$, but we write this composition gf to stress the "linear" nature of the morphisms. By k -additivity of the projections, \mathcal{C}^{add} is also cartesian.

Definition 3. A map $f \in \mathcal{C}(X_0 \times \dots \times X_n, Y)$ is k -additive in an argument i if the map $g \mapsto f \circ (\prod_{l=0}^{i-1} \text{id}_{X_l} \times g \times \prod_{l=i+1}^n \text{id}_{X_l})$ is k -additive.

Let S_n be a functor on \mathcal{C}^{add} defined as $S_n X = X \times X^{\times n}$ and $S_n f = f \times f^{\times n}$. In particular, let $S = S_1$, so that $SX = X \times X$ and $Sf = f \times f$. For all $i \in [0, n]$, we write $\pi_i = p_i :$

$S_n \Rightarrow \text{Id}$ (π_i and p_i are the same, but we use different names to stress their respective role). Intuitively, an element of $S_n X$ is a vector (x, u_1, \dots, u_n) , where x is the base point, and u_i is an order i variation around x . The following definition corresponds to Def 2.4 of [17], with a slight variation on **D-lin** and **D-sym** that are equivalent to items (vi) and (vii) of [17] thanks to Prop. 4.2 of [7].

Definition 4. A derivative in a cartesian left k -additive category \mathcal{C} is a family of operators $D : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(S_n X, Y)$ for all objects X, Y such that

- D-proj** for any $p_i \in \mathcal{C}(X_0 \times \dots \times X_n, X_i)$, $Dp_i = p_i \pi_1$;
- D-sum** D is k -additive;
- D-chain** $\text{Did} = \pi_1$ and $D(g \circ f) = Dg \circ \langle f \circ \pi_0, Df \rangle$;
- D-add** Df is k -additive in its second argument ;
- D-lin** $DDf \circ \langle x, 0, 0, u \rangle = Df \circ \langle x, u \rangle$;
- D-sym** $DDf \circ \langle x, u, v, w \rangle = DDf \circ \langle x, v, u, w \rangle$.

A cartesian k -differential category is a cartesian left k -additive category equipped with a derivative.

We refer to sec. 2.2 of [17] for a more involved description of those axioms. As explained in [17], any monoid is a \mathbb{N} -semimodule, so the cartesian differential categories of [6] based on left additive categories where the hom-sets are only assumed to be monoids are instances of this definition with $k = \mathbb{N}$. The most fundamental example of cartesian differential category is the category **Smooth** whose objects are the Euclidean vector spaces \mathbb{R}^n and whose maps are the smooth functions. The derivative D in **Smooth** is the usual derivative of functions, $Df(x, u) = f'(x) \cdot u$.

Proposition 1 (Lemma 2.6 of [16]). *For any operator D that satisfies **D-proj** and **D-chain**, $D\langle f, g \rangle = \langle Df, Dg \rangle$.*

Definition 5. A map $f \in \mathcal{C}(X, Y)$ is D -linear if $Df = f \circ \pi_1$.

In **Smooth**, this definition means that $f'(x) \cdot u = f(u)$ so the D -linear maps are the linear maps between Euclidean spaces. There exists a category $\mathcal{C}^{\text{Dlin}}$ with the same objects as \mathcal{C} , whose morphisms are the D -linear morphisms of \mathcal{C} , and whose composition coincides with the composition in \mathcal{C} . It follows from **prop. 1** that $\mathcal{C}^{\text{Dlin}}$ is cartesian, with the same projections and tupling as in \mathcal{C} . By **D-add**, any D -linear morphism is also k -additive, but the converse is not true in general, see [6]. Thus, $\mathcal{C}^{\text{Dlin}}$ is a subcategory of \mathcal{C}^{add} . We write the composition of f with g as $g \circ f$, as in \mathcal{C}^{add} . **Smooth** is a particular case in which $\mathbf{Smooth}^{\text{Dlin}} = \mathbf{Smooth}^{\text{add}}$.

Definition 6. Given $f \in \mathcal{C}(X_0 \times \dots \times X_n, Y)$ and $i \in \llbracket 0, n \rrbracket$, the i -th partial derivative of f is the morphism $D_i f \in \mathcal{C}(X_0 \times \dots \times X_n \times X_i, Y)$ defined as $D_i f = Df \circ \langle p_0, \dots, p_n, 0^{\times i}, p_{n+1}, 0^{\times n-i} \rangle$.

A morphism $f \in \mathcal{C}(X_0 \times \dots \times X_n, Y)$ is D -linear in its i -th argument if $D_i f = f \circ \langle p_0, \dots, p_{i-1}, p_{i+1}, p_{i+1}, \dots, p_n \rangle$.

In **Smooth**, this partial derivative D_i coincides with the usual partial derivative ∂_i , this is an immediate consequence of the fact that for all smooth map $f : X_0 \times \dots \times X_n \rightarrow Y$, $f'(x) \cdot (u_0, \dots, u_n) = \sum_{i=0}^n \partial_i f(x) \cdot u_i$.

Definition 7. The n -th derivative of $f \in \mathcal{C}(X, Y)$ is the map $f^{(n)} = (D_1)^n f \in \mathcal{C}(S_n X, Y)$.

Let us detail the relationship between the n -th derivative $f^{(n)} \in \mathcal{C}(S_n X, Y)$ and the n -th total derivative $D^n f \in \mathcal{C}(S^n X, Y)$. For any set I , let B_I be the set of partitions of I , and B_I^o the set of ordered partitions of I . More precisely, an element of B_I is a set $\{I_1, \dots, I_k\}$ and an element of B_I^o is a vector (I_1, \dots, I_n) such that I_1, \dots, I_n is a partition of I . This notation is based on (unordered) Bell numbers, that give the numbers of (unordered) partitions of a set.

Notations 2. Let $[n] = \{1, \dots, n\}$. There is a bijection between words $w \in \{0, 1\}^*$ of length $l(w) = n$ and subsets of $I_w \subseteq [n]$, given by $I_w = \{i \in [n] \mid w_i = 1\}$. The weight $|w|$ is the number of 1 in w (and not its length as it is often standard), so that the weight $|w|$ matches the size $|I_w|$ of its corresponding set. We will often consider sets and words up to this bijection. For example, B_w is a set whose elements are the sets $\{w^1, \dots, w^k\}$ of words of weight at least one and such that for all $i \in [l(w)]$, $\sum_{j=1}^k w_i^j = w_i$. For any word $w = w_1 \dots w_n$ of length n , we write $\pi_w = \pi_{w_1} \dots \pi_{w_n}$. Let 0^n and 1^n be the words $0 \dots 0$ and $1 \dots 1$ of length n (the context in which this notation is used will make clear that this corresponds to a word).

Intuitively, $S^n X$ can be seen as the set of polynomials of degree 1 over X , with n formal indeterminates $\varepsilon_1, \dots, \varepsilon_n$ such that for all i , $\varepsilon_i^2 = 0$. The projection $\pi_{w_1 \dots w_n}(a)$ provides the coefficient in a of the monomial $\varepsilon_1^{w_1} \dots \varepsilon_n^{w_n}$.

Proposition 2 (Lemma 3.2 of [17]). *For all $f \in \mathcal{C}(X, Y)$:*

- (1) $f^{(n)}$ is symmetric and D -linear in its last n variables;
- (2) $D^n f = \sum_{\{w^1, \dots, w^k\} \in B_{[n]}^o} f^{(k)} \circ \langle \pi_{0^n}, \pi_{w^1}, \dots, \pi_{w^k} \rangle$, this is well-defined by symmetry of the $f^{(k)}$;
- (3) if k has multiplicative inverses for integers, then $D^n f = \sum_{(w^1, \dots, w^k) \in B_{[n]}^o} \frac{1}{k!} f^{(k)} \circ \langle \pi_{0^n}, \pi_{w^1}, \dots, \pi_{w^k} \rangle$.

Proof: Proofs for **items (1) and (2)** are in [17]. **Item (3)** is a direct consequence of **item (2)**. \square

For example, **items (2) and (3)** for $n = 2$ implies that

$$DDf \circ \langle x, u, v, w \rangle = f^{(2)} \circ \langle x, u, v \rangle + Df \circ \langle x, w \rangle \quad (1)$$

which can be understood in **Smooth** by the fact that the derivative of the map $g : (x, u) \mapsto f'(x) \cdot u$ is equal to

$$\begin{aligned} g'(x, u) \cdot (v, w) &= \partial_0 g(x, u) \cdot v + \partial_1 g(x, u) \cdot w \\ &= f^{(2)}(x) \cdot (u, v) + f'(x) \cdot w \end{aligned}$$

by definition of $f^{(2)}(x)$ and linearity of $u \mapsto f'(x) \cdot u$.

Example 1. Cartesian differential categories notably include the models of the differential λ -calculus [22] and its syntactical Taylor expansion [5]. One example of such model is the weighted relational model $\mathbf{Rel}_!(k)$ [23]. Let k be a complete semiring (a semiring with arbitrary countable sums), for example $k = \overline{\mathbb{R}_{\geq 0}} = \mathbb{R}_{\geq 0} \cup \infty$. A finite multiset over A is a function $m : A \rightarrow \mathbb{N}$ such that $\{a \in A \mid m(a) \neq 0\}$ is finite. The objects of the category $\mathbf{Rel}_!(k)$ (it arises as the coKleisli

category of the ! comonad of a model of Linear Logic) are the (countable) sets, and the morphism from A to B are the matrices $k^{\mathcal{M}_{\text{fin}}(A) \times B}$, where $\mathcal{M}_{\text{fin}}(A)$ is the set of all finite multisets over A .

The category $\mathbf{Rel}(k)$ is cartesian, the product of A and B is the disjoint union: $A \uplus B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$. In particular, we have $\mathcal{M}_{\text{fin}}(A \uplus B) = \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B)$. The category $\mathbf{Rel}(k)$ is a cartesian differential category. For all $f \in \mathbf{Rel}_1(k)(A, B)$, the higher order derivative $f^{(n)} \in \mathbf{Rel}_1(k)(\uplus_{i=0}^n A, B)$ can be described by a matrix $f^{(n)} \in k^{\mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(A)^n \times B}$ such that

$$f_{m, [a_1], \dots, [a_n], b} = \frac{(m + [a_1, \dots, a_n])!}{m!} f_{m + [a_1, \dots, a_n], b}$$

and all other coefficients are set to 0. In the above, $[a]$ is the multiset with one element, and $m! = \prod_{a \in A} m(a)!$. A morphism $s \in \mathbf{Rel}_1(k)(\uplus_{i=1}^n A_i, B)$ is D-linear in a coordinate i if and only if $s_{p_1, \dots, p_n} \neq 0 \Rightarrow p_i = [a_i]$ for some $a_i \in A_i$, so D-linearity coincides with the linearity of linear logic.

II. THE TANGENT BUNDLE CONSTRUCTION

We now introduce the tangent bundle functor T and prove that the equations of derivatives are equivalent to naturality equations that turns T into a monad. The functorialization of differentiation can be found in two (orthogonal) generalizations of cartesian differential categories: tangent categories [7], [24] and cartesian coherent differential categories [8] that differ by their action on objects. Both of those theories end up identical when restricted to cartesian differential categories, with action on object $TX = X \times X$.

There is a bijection between operators $D : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(SX, Y)$ (not assumed to be derivatives) and operators $T : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(SX, SY)$ such that $\pi_0 \circ Tf = f \circ \pi_0$.

$$Tf = \langle f \circ \pi_0, Df \rangle \quad Df = \pi_1 \circ Tf \quad (2)$$

Proposition 3. *The following assertions are equivalent.*

- (1) D satisfies **D-chain**: $\text{Did} = \pi_1$ and $D(g \circ f) = Dg \circ Tf$.
- (2) T is a functor with action on objects $TX = SX$.

Proof: Observe that $T(g \circ f) = \langle g \circ f \circ \pi_0, D(g \circ f) \rangle$ and $Tg \circ Tf = \langle g \circ \pi_0, Dg \rangle \circ \langle f \circ \pi_0, Df \rangle = \langle g \circ f \circ \pi_0, Dg \circ Tf \rangle$ so $T(g \circ f) = Tg \circ Tf$ if and only if $D(g \circ f) = Dg \circ Tf$. Similarly, $\text{Tid} = \text{id}$ if and only if $\text{Did} = \pi_1$. \square

This functorial point of view on the derivative is used in [16] to provide an alternative to the heavy combinatorics of the Faà di Bruno formula. Indeed, by definition $D^n f = \pi_1^n \circ T^n f$, so by functoriality of T^n it immediately follows that

$$D^n(g \circ f) = D^n g \circ T^n f. \quad (3)$$

The simplicity of this equation sharply contrasts with the combinatorial Faà di Bruno formula (cor. 3.2.3 of [25])

$$(g \circ f)^{(n)} = \sum_{\{I_1, \dots, I_k\} \in B[n]} g^{(k)} \circ \langle f^{(I_1)}, \dots, f^{(I_k)} \rangle \quad (4)$$

where for any set $I = \{i_1, \dots, i_m\} \subseteq [n]$, $f^{(I)} = f^{(m)} \circ \langle \pi_0, \pi_{i_1}, \dots, \pi_{i_m} \rangle$ (by symmetry of n -th derivative, this value does not depend on the choice of ordering of I).

Proposition 4. *A morphism $f \in \mathcal{C}(X, Y)$ is D-linear iff $Tf = Sf$. Thus, T coincides with S when restricted to $\mathcal{C}^{\text{Dlin}}$.*

Proof: If f is D linear then $Tf = \langle f \circ \pi_0, Df \rangle = \langle f \circ \pi_0, f \circ \pi_1 \rangle = Sf$. Conversely, if $Tf = Sf$ then $Df = \pi_1 \circ Tf = \pi_1 \circ Sf = f \circ \pi_1$. \square

We now define natural transformations in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$.

$$\sigma = \pi_0 + \pi_1 : S \Rightarrow \text{Id} \quad (5)$$

$$\text{for all } r \in k, h(r) = r \cdot \text{id} : \text{Id} \Rightarrow \text{Id}$$

that consists of a sum operation and the homotheties. These are natural transformations by k -additivity of the morphisms in \mathcal{C}^{add} . We also define the following natural transformations

$$z = \langle \text{id}, 0 \rangle : \text{Id} \Rightarrow S$$

$$\theta = \langle \pi_0 \pi_0, \pi_0 \pi_1 + \pi_1 \pi_0 \rangle : S^2 \Rightarrow S$$

$$c = \langle \langle \pi_0 \pi_0, \pi_0 \pi_1 \rangle, \langle \pi_1 \pi_0, \pi_1 \pi_1 \rangle \rangle : S^2 \Rightarrow S^2 \quad (6)$$

$$l = \langle \langle \pi_0, 0 \rangle, \langle 0, \pi_1 \rangle \rangle : S \Rightarrow S^2$$

$$\text{for all } r \in k, t(r) = \langle \pi_0, r \cdot \pi_1 \rangle : S \Rightarrow S$$

The naturality follows from the fact that the pairing $\langle \alpha, \beta \rangle$ of two natural transformations $\alpha, \beta : F \Rightarrow G$ (where F, G are functors whose codomain is \mathcal{C}^{add} or $\mathcal{C}^{\text{Dlin}}$) is a natural transformation $F \Rightarrow SG$, and that their sum is a natural transformation $F \Rightarrow G$. Observe that those natural transformations are natural with regard to S , and not T . In fact, the naturality with regard to T of the families of (6) corresponds *exactly* to the axioms of cartesian differential categories, as discovered in [8]. The forward implication of this observation was first discovered in sec. 2.4 of [7] in the proof that every cartesian differential category is a tangent category.

Theorem 1. *Let $D : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(SX, Y)$ be an operator that satisfies **D-proj** and **D-chain**. Then:*

- (1) D satisfies **D-sum** iff $T\sigma = S\sigma$ and $T(h(r)) = S(h(r))$ for all $r \in k$ (that is, $h(r)$ and σ are D-linear);
- (2) D satisfies **D-add** iff $z : \text{Id} \Rightarrow T$, $\theta : T^2 \Rightarrow T$ and $t(r) : T \Rightarrow T$ (for all $r \in k$) are natural in \mathcal{C} ;
- (3) Assuming that $z : \text{id} \Rightarrow T$ is natural in \mathcal{C} , D satisfies **D-lin** iff $l : T \Rightarrow T^2$ is natural in \mathcal{C} ;
- (4) D satisfies **D-sym** iff $c : T^2 \Rightarrow T^2$ is natural in \mathcal{C} .

Thus, there is a bijection between derivative operators D and functors T such that $TX = SX$, $Tp_i = Sp_i$, $T\sigma = S\sigma$, $Th(r) = Sh(r)$ and such that $\pi_0, z, \theta, t(r)$ (for all $r \in k$), l and c are natural in \mathcal{C} with regard to T .

This theorem and its proof, see **sec. A**, are a slight adaptation of thm. 7 of [8]. The difference is that [8] deals with partial sums but does not feature a k -module structure. The proof idea is similar to that of **prop. 3**: the equations of **def. 4** correspond to the rightmost projections of the equations on T , while the other projections always hold by definition of T .

Proposition 5. *(S, z, θ) is a monad on \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$, meaning that the following diagram commute.*

$$\begin{array}{ccc} S & \xrightarrow{zS} S^2 & \xleftarrow{Sz} S \\ & \searrow \theta & \swarrow \theta \\ & S & \end{array} \quad \begin{array}{ccc} S^3 & \xrightarrow{S\theta} S^2 & \\ \theta S \downarrow & & \downarrow \theta \\ S^2 & \xrightarrow{\theta} S & \end{array}$$

(T, z, θ) is a monad on \mathcal{C} (same diagrams, replacing S by T).

Proof: Let us prove the rightmost diagram.

$$\begin{aligned}\pi_i \theta_X \theta_{SX} &= \sum_{i_1+i_2=i} \pi_{i_1} \pi_{i_2} \theta_{SX} = \sum_{j_1+j_2+j_3=i} \pi_{j_1} \pi_{j_2} \pi_{j_3} \\ \pi_i \theta_X S\theta_X &= \sum_{i_1+i_2=i} \pi_{i_1} \theta_X \pi_{i_2} = \sum_{j_1+j_2+j_3=i} \pi_{j_1} \pi_{j_2} \pi_{j_3}\end{aligned}$$

so $\theta\theta S = \theta S\theta$. The proof of the left diagram is very similar, and we conclude that (S, z, θ) is a monad. We also conclude that (T, z, θ) is a monad on \mathcal{C} by naturality of $z : \text{Id} \Rightarrow T$ and $\theta : T^2 \Rightarrow T$ (see [thm. 1](#)) and because T coincides with S on $\mathcal{C}^{\text{Dlin}}$ by [prop. 4](#). \square

The *Kleisli category* \mathcal{C}_T of the monad T is the category whose objects are the objects of \mathcal{C} , and such that $\mathcal{C}_T(X, Y) = \mathcal{C}(X, TY)$. The composition of $f \in \mathcal{C}_T(X, Y)$ with $g \in \mathcal{C}_T(Y, Z)$ is given by $\theta \circ Tg \circ f$. A morphism $f \in \mathcal{C}_T(X, Y)$ corresponds to a polynomial $f_0 + f_1\varepsilon$ with coefficients in $\mathcal{C}(X, Y)$ and a formal indeterminate ε such that $\varepsilon^2 = 0$. The composition of f with $g = g_0 + g_1\varepsilon$ is equal to $g_0 \circ f_0 + (g_1 \circ f_0 + Dg_0 \circ \langle f_0, f_1 \rangle)\varepsilon$. This monad captures the concept of dual numbers widely used in automated differentiation [1].

Notations 3. Define $\delta_i^j = 1$ if $i = j$, and $\delta_i^j = 0$ otherwise.

Proposition 6. The natural transformation c is a distributive law [12] of the monad S over itself. This means that the following diagrams commute in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$.

$$\begin{array}{ccc} S & \xrightarrow{Sz} & S^2S \\ zS \downarrow & \searrow & \downarrow \theta S \\ SS & \xrightarrow{c} & SS \end{array} \quad \begin{array}{ccc} S^2S & \xrightarrow{Sc} & SSS \\ \theta S \downarrow & \searrow & \downarrow \theta \\ SS & \xrightarrow{c} & SS \end{array}$$

$$\begin{array}{ccc} S & \xrightarrow{Sz} & S^2S \\ zS \downarrow & \searrow & \downarrow \theta S \\ SS & \xleftarrow{c} & SS \end{array} \quad \begin{array}{ccc} S^2S & \xleftarrow{Sc} & SSS \\ \theta S \downarrow & \searrow & \downarrow \theta \\ SS & \xleftarrow{c} & SS \end{array}$$

Similarly, c is a distributive law of T over itself in \mathcal{C} .

Proof: We prove that $c_X(zS) = Sz$ as follows.

$$\begin{aligned}\pi_i \pi_j c_X zS &= \pi_j \pi_i z = \delta_0^i \pi_j \\ \pi_i \pi_j Sz &= \pi_i z \pi_j = \delta_0^j \pi_i\end{aligned}$$

We prove that $c(\theta S) = (S\theta)(cS)Sc$ as follows.

$$\pi_i \pi_j c_X \theta_{SX} = \pi_j \pi_i \theta_{SX} = \pi_j \sum_{i_1+i_2=i} \pi_{i_1} \pi_{i_2} = \sum_{i_1+i_2=i} \pi_j \pi_{i_1} \pi_{i_2}$$

$$\begin{aligned}\pi_i \pi_j S\theta_X c_{SX} S c_X &= \pi_i \theta_X \pi_j c_{SX} S c_X \\ &= \sum_{i_1+i_2=i} \pi_{i_1} \pi_{i_2} \pi_j c_{SX} S c_X = \sum_{i_1+i_2=i} \pi_j \pi_{i_1} \pi_{i_2}\end{aligned}$$

The two other diagrams hold upon observing that c is an involution: $cc = \text{id}$. We conclude that c is also a distributive law of T over itself, by naturality of $c : TT \Rightarrow TT$ ([thm. 1](#)) and because T coincides with S on $\mathcal{C}^{\text{Dlin}}$. \square

III. THE ITERATED TANGENT BUNDLE MONAD

This section presents the following new result: T^n has a canonical monad structure that arises from the distributive law c of T over itself. The naturality equations of this structure provide a higher order variant of the axioms of cartesian differential categories, in the same way that the functoriality of T^n provides a higher order variant of the chain rule (3). Let us provide some intuition on $T^n f$. An element $a \in S^n X$ intuitively corresponds to polynomials over X in n formal indeterminates $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i^2 = 0$,

$$a(\varepsilon_1, \dots, \varepsilon_n) = \sum_{w_1, \dots, w_n \in \{0,1\}} a_{w_1, \dots, w_n} \varepsilon_1^{w_1} \dots \varepsilon_n^{w_n}$$

where $a_{w_1, \dots, w_n} = \pi_{w_1, \dots, w_n}(a)$. Then, $T^n f \in \mathcal{C}(S^n A, S^n B)$ can be seen as a map between polynomials defined as follows. Let $b = a - a_0^n$, so that by Taylor expansion

$$f(a) \simeq f(a_0^n) + \sum_{k=1}^n \frac{1}{k!} f^{(k)}(a_0^n)(b, \dots, b) + O(b^{n+1}).$$

But then, $b^{n+1} = 0$ because $\varepsilon_i^2 = 0$ for all i . So all the higher order term of the Taylor expansion vanish. This expression can then be developed by linearity and multilinearity of the derivatives, and simplified thanks to the equalities $\varepsilon_i^2 = 0$. This yields a polynomial over $\varepsilon_1, \dots, \varepsilon_n$. This polynomial precisely corresponds to the polynomial $T^n f(a)$.

For any functors $F, F' : \mathcal{C} \rightarrow \mathcal{D}$ and $G, G' : \mathcal{D} \rightarrow \mathcal{E}$ and natural transformations $\alpha : F \Rightarrow F'$, $\beta : G \Rightarrow G'$, we write the horizontal composition $\alpha * \beta : GF \Rightarrow G'F'$.

$$(\alpha * \beta)_X = \beta_{F'X} \circ G\alpha_X = F'\alpha_X \circ \beta_{FX}$$

As it is standard [12], the existence of the distributive law c of S over itself induces a monad structure on S^2 given by

$$\begin{aligned}z^{[2]} &= \text{Id} \xrightarrow{z * z} SS \\ \theta^{[2]} &= SSS \xrightarrow{ScS} SSS \xrightarrow{\theta * \theta} SS.\end{aligned}$$

We describe a similar structure for any $n \in \mathbb{N}$. The natural transformation c satisfies the *Yang-Baxter equation*.

$$\begin{array}{ccccc} SSS & \xrightarrow{cS} & SSS & \xrightarrow{Sc} & SSS \\ & \searrow & & & \searrow \\ & SSS & \xrightarrow{cS} & SSS & \xrightarrow{Sc} & SSS \end{array}$$

By [thm 2.1](#) of [26], this induces a canonical monad structure on S^n for any $n \in \mathbb{N}$, and a canonical distributive law of S^n over S^m . Let us provide an explicit description of this structure. Define natural transformations $z^{[n]} : \text{Id} \Rightarrow S^n$ and $\theta^{[n]} : S^{2n} \Rightarrow S^n$, $c^{[n,m]} : S^n S^m \Rightarrow S^m S^n$ and $t^{[n]}(r) : S^n \Rightarrow S^n$ in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$ by

$$\begin{aligned}\pi_w z^{[n]} &= \delta_w^{0^n} \text{id} & \pi_w \theta^{[n]} &= \sum_{(w^1, w^2) \in B_w^n} \pi_{w^1} \pi_{w^2} \\ \pi_w t^{[n]}(r) &= r^{|w|} \pi_w & \pi_u \pi_v c^{[n,m]} &= \pi_v \pi_u\end{aligned} \quad (7)$$

Proposition 7. For all $n \in \mathbb{N}$, S^n is a monad with unit $z^{[n]}$ and multiplication $\theta^{[n]} : S^n S^n \Rightarrow S^n$. Furthermore, $c^{[n,m]}$ is a distributive law of S^n over S^m .

Proof: The proofs of the monad diagrams is the same as the proof of [prop. 5](#), replacing the integers in the indices by words. The proof that $c^{[n,m]}$ is a distributive law is the same proof as [prop. 6](#), upon observing that $c^{[m,n]}c^{[n,m]} = \text{id}$. \square

Theorem 2. *For all $n \in \mathbb{N}$, $z^{[n]} : \text{Id} \Rightarrow T^n$, $\theta^{[n]} : T^n T^n \Rightarrow T$ and $c^{[n,m]} : T^n T^m \Rightarrow T^m T^n$ are natural in \mathcal{C} . Thus, T^n is a monad and $c^{[n,m]}$ is a distributive law of T^n over T^m .*

Proof: First, we prove that $z^{[n]}$ is a natural transformations $\text{Id} \Rightarrow T^n$. We can prove by a straightforward induction that $z^{[n]}$ is the horizontal composition of $z : \text{Id} \Rightarrow S$ in $\mathcal{C}^{\text{Dlin}}$ with itself, taken n times

$$z^{[n]} = \text{Id} \cdots \text{Id} \xrightarrow{z \ast \cdots \ast z} S \cdots S$$

But z is linear, so $Tz = Sz$. Thus, z is also the horizontal composition of $z : \text{Id} \Rightarrow T$ in \mathcal{C} with itself, taken n times. Thus, it is a natural transformation $\text{Id} \Rightarrow T^n$ in \mathcal{C} . The proof of naturality of $t^{[n]}(r)$ is the same. Then, we prove that $c^{[n,m]}$ is a natural transformation $T^n T^m \Rightarrow T^n T^m$ for all n, m . We can prove by a straightforward computation that

$$\begin{aligned} c^{[1,m+1]} &= SSS^m \xrightarrow{cS^m} SSS^m \xrightarrow{Sc^{[1,m]}} SS^m S \\ c^{[n+1,m]} &= SS^n S^m \xrightarrow{Sc^{[n,m]}} SS^m S^n \xrightarrow{c^{[1,m]}S^n} S^m SS^n, \end{aligned}$$

this corresponds to a decomposition of a cyclic permutation into a product of transpositions. But we know that $c^{[n,m]}$ is D-linear for all $n, m \in \mathbb{N}$, so we get that

$$c^{[1,n+1]} = Tc^{[1,m]} \circ (cT^m) \quad c^{[n+1,m]} = (c^{[1,m]}T^n) \circ Tc^{[n,m]}.$$

We conclude that $c^{[n,m]} : T^n T^m \Rightarrow T^m T^n$ is a natural transformation by a straightforward induction, using the naturality of $c : TT \Rightarrow TT$. Finally, we prove that $\theta^{[n]}$ is a natural transformation $T^n T^n \Rightarrow T^n$. Observe that

$$\theta^{[n+1]} = SS^n SS^n \xrightarrow{Sc^{[n,1]}S^n} SSS^n S^n \xrightarrow{\theta S^n} SS^n S^n \xrightarrow{S\theta^{[n]}} SS^n$$

By D-linearity of $c^{[n,1]}$ and $\theta^{[n]}$, we get that $\theta^{[n+1]} = T\theta^{[n]} \circ (\theta T^n) \circ (Tc^{[n,1]}T^n)$. We conclude that $\theta^{[n]}$ is a natural transformation $T^n T^n \Rightarrow T$ by a straightforward induction, using the naturality of θ and $c^{[1,n]}$. \square

IV. TAYLOR EXPANSION AS A FUNCTOR

In this section, we address the central question of the paper. We define an operator T_n that performs an order n Taylor expansion, and we prove its compositionality. Remember that the Taylor expansion of a smooth function is given by

$$f(x + \varepsilon u) = \sum_{k=0}^n \frac{\varepsilon^k}{k!} f^{(k)}(x) \cdot (u, \dots, u) + o(\varepsilon^n) \quad (8)$$

Intuitively, we would like to define T_n such that the i -th component of $T_n f(x, u_1, \dots, u_n)$ provides the term of order ε^i in the Taylor expansion of $f(x + \varepsilon u_1 + \dots + \varepsilon^n u_n)$. For example, $T_2 f(x, u_1, u_2)$ should be equal to

$$\left(f(x), f'(x) \cdot u_1, \frac{1}{2} f^{(2)}(x) \cdot (u_1, u_1) + f'(x) \cdot u_2 \right). \quad (9)$$

Formally, let us assume that \mathcal{C} is a cartesian differential category, and that its associated semiring k has multiplicative inverses for integers (see [def. 1](#)). For any $f \in \mathcal{C}(X, Y)$, define $T_n f \in \mathcal{C}(S_n X, S_n Y)$ by the following equation:

$$\pi_j \circ T_n f = \sum_{k=1}^j \sum_{\substack{i_1, \dots, i_k > 0 \\ i_1 + \dots + i_k = j}} \frac{1}{k!} f^{(k)} \circ \langle \pi_0, \pi_{i_1}, \dots, \pi_{i_k} \rangle. \quad (10)$$

This functor corresponds to the push forward of higher order dual numbers [\[11\]](#). Intuitively, an element $e \in S_n X$ is a degree n polynomial over a formal variable ε such that $\varepsilon^{n+1} \neq 0$. Then, $T_n f$ maps this polynomial to the best polynomial approximation of $f(e_0 + (e - e_0))$ given by Taylor expansion.

Example 2. We can check in the relational model (see [example 1](#)) that the functor T_n maps $f \in \mathbf{Rel}_!(k)(A, B)$ to a matrix $T_n f \in \mathbf{Rel}_!(k)(\bigoplus_{i=0}^n A, \bigoplus_{i=0}^n B) \simeq k^{\mathcal{M}_{\text{fin}}(A)^{n+1} \times \bigoplus_{i=0}^n B}$ whose coefficients of degree i are given by

$$(T_n f)_{(m_0, \dots, m_n), (i, b)} = \frac{(m_0 + \dots + m_n)!}{m_0! \cdots m_n!} f_{m_0 + \dots + m_n, b}$$

if $1|m_1| + \dots + n|m_n| = i$, and are equal to 0 otherwise (here, $|m| = \sum_{a \in A} m(a)$ is the size of the multiset). As explained in [Sec. 9.4.2](#) of [\[18\]](#), those coefficients correspond to a Taylor expansion on formal power series.

A. Alternative definition of Taylor expansion

The issue with [\(10\)](#) is that it is hard to manipulate. The computation of $T_n(g \circ f)$ would intensively require the Faà di Bruno formula, and the computation of $T_n^2 f$ seems nightmarish (we want to show that T_n is a monad, this implies intensive uses of $T_n^2 f$). We solve this issue by relating T_n with the higher order total derivative $D^n f$ rather than the higher order derivative $f^{(n)}$. For example, [\(1\)](#) and [\(9\)](#) imply that

$$\pi_2 \circ T_2 f \circ \langle x, u, v \rangle = DDf \circ \langle x, u, \frac{1}{2}u, v \rangle. \quad (11)$$

We generalize this alternative definition to all n and prove in [thm. 3](#) that this coincides with [\(10\)](#). This provides simpler equations on T_n , that we use to prove that T_n is a functor using the functoriality of T^n . We use a similar proof technique in [sec. V](#) in order to prove that T_n is a monad, using the monad structure on T^n of [sec. III](#). For any $n \in \mathbb{N}^*$, define two natural transformations $k_n, k'_n : S_n \Rightarrow SS_{n-1}$ in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$ by

$$\begin{aligned} k'_{n,X} &= \langle \langle \pi_0, \dots, \pi_{n-1} \rangle, \langle \pi_1, \dots, \pi_n \rangle \rangle \\ k_{n,X} &= \left\langle \langle \pi_0, \dots, \pi_{n-1} \rangle, \left\langle \frac{1}{n} \pi_1, \dots, \frac{n}{n} \pi_n \right\rangle \right\rangle. \end{aligned} \quad (12)$$

The subscript X is omitted when writing $k_{n,X}$ and $k'_{n,X}$. For any $n \in \mathbb{N}$ and $f \in \mathcal{C}(X, Y)$, define $\Delta_n f \in \mathcal{C}(S_n X, Y)$ and $\Delta'_n f \in \mathcal{C}(S_n X, Y)$ by induction on n : $\Delta_0 f = \Delta'_0 f = f$ and

$$\begin{aligned} \Delta_{n+1} f &= S_{n+1} X \xrightarrow{k_{n+1}} SS_n X \xrightarrow{D\Delta_n f} Y \\ \Delta'_{n+1} f &= S_{n+1} X \xrightarrow{k'_{n+1}} SS_n X \xrightarrow{D\Delta'_n f} Y. \end{aligned} \quad (13)$$

Observe that $\Delta_1 f = \Delta'_1 f = Df$. The operator Δ'_n is a generalization to differential categories of the *higher order*

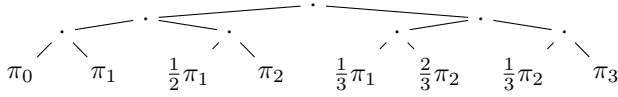


Figure 1: Graphical representation of L_3

directional derivative of def. 1 of [13]. On the other hand, **thm. 3** proves that Δ_n is equal to the sum in (10). For example, $\Delta_2 f \circ \langle x, u, v \rangle = \text{DD}f \circ \langle x, u, \frac{1}{2}u, v \rangle$ coincides with (11).

Remark 1. Both k_n and k'_n duplicate values. As such, they do not exist in the setting of [8], [18] in which sums are partial.

There is an alternative definition of Δ_n and Δ'_n that refers directly to the iterated derivative D^n . Define for any $n \in \mathbb{N}$ two natural transformations $L_n, L'_n : S_n \Rightarrow S^n$ in $\mathcal{C}^{\text{Dlin}}$ by induction on n : $L_{0,X} = L'_{0,X} = \text{id}_X \in \mathcal{C}(X, X)$ and

$$\begin{aligned} L_{n+1,X} &= S_{n+1}X \xrightarrow{k_{n+1,X}} SS_nX \xrightarrow{SL_{n,X}} S^{n+1}X \\ L'_{n+1,X} &= S_{n+1}X \xrightarrow{k'_{n+1,X}} SS_nX \xrightarrow{SL'_{n,X}} S^{n+1}X. \end{aligned} \quad (14)$$

Again, the subscript X is omitted when writing L_n and L'_n . **Fig. 1** provides a graphical description of L_3 . The following result is the counterpart of Theorem 1 of [13].

Proposition 8. $\Delta_n f = D^n f \circ L_n$ and $\Delta'_n f = D^n f \circ L'_n$

Proof: By induction on n . The base case $n = 0$ is trivial, and the inductive case consists of the following computation.

$$\begin{aligned} D^{n+1}f \circ L_{n+1} &= \text{DD}^n f \circ \text{TL}_n \circ k_{n+1} \quad \text{by linearity of } L_n \\ &= D(D^n f \circ L_n) \circ k_{n+1} \quad \text{by D-chain} \\ &= D\Delta_n f \circ k_{n+1} = \Delta_{n+1}f \quad \text{I.H.} \end{aligned}$$

The proof for the equation on $\Delta'_n f$ is the same. \square

We now give an explicit computation of $\Delta_n f$ that involves the higher order derivatives $f^{(k)}$ in order to ensure that $\Delta_n f$ corresponds to (10). First, let us describe explicitly what are the components of L_n . For any word $w \in \{0, 1\}^*$, let $w! = \prod_{i \in w} i$ (recall that w can be seen as the set $\{i \mid w_i = 1\}$).

Proposition 9. For any word w of length n ,

$$\pi_w L_n = \frac{|w|!}{w!} \pi_{|w|} \quad \pi_w L'_n = \pi_{|w|}.$$

Proof: We do the proof for L_n , by induction on n . The base case $n = 0$ is trivial (remember that $S_0 = \text{Id}$). Let $w = v \cdot 0$, where v is a word of length n .

$$\begin{aligned} \pi_w L_{n+1} &= \pi_v \pi_0 L_{n+1} \quad \text{by definition of } \pi_w \\ &= \pi_v L_n \langle \pi_0, \dots, \pi_n \rangle \quad \text{by definition of } L_n \\ &= \frac{|w|!}{w!} \pi_{|w|} \quad \text{by inductive hypothesis.} \end{aligned}$$

Now, let $w = v \cdot 1$, where v is a word of length n .

$$\begin{aligned} \pi_w L_{n+1} &= \pi_v \pi_1 L_{n+1} = \pi_v L_n \pi_1 k_n \quad \text{by (14)} \\ &= \frac{|v|!}{v!} \pi_{|v|} \pi_1 k_n \quad \text{by inductive hypothesis} \\ &= \frac{|v|! (|v| + 1)}{v! (n + 1)} \pi_{|v|+1} = \frac{|w|!}{w!} \pi_{|w|} \end{aligned}$$

This concludes the proof. The proof for L'_n is a simpler version of this proof, not involving coefficients. \square

Theorem 3. For all $f \in \mathcal{C}(X, Y)$,

$$\Delta_n f = \sum_{k=1}^n \sum_{\substack{i_1, \dots, i_k > 0 \text{ s.t.} \\ i_1 + \dots + i_k = n}} \frac{1}{k!} f^{(k)} \circ \langle \pi_0, \pi_{i_1}, \dots, \pi_{i_k} \rangle \quad (15)$$

Proof: We start the proof with the following computation.

$$\begin{aligned} \Delta_n f &= D^n f \circ L_n \quad \text{prop. 8} \\ &= \sum_{(w^1, \dots, w^k) \in B_{[n]}^o} \frac{1}{k!} f^{(k)} \circ \langle \pi_{0^n}, \pi_{w^1}, \dots, \pi_{w^k} \rangle \circ L_n \quad \text{prop. 2} \\ &= \sum_{(w^1, \dots, w^k) \in B_{[n]}^o} \frac{1}{k!} f^{(k)} \circ \langle \pi_0, \frac{|w^1|!}{w^1!} \pi_{|w^1|}, \dots, \frac{|w^k|!}{w^k!} \pi_{|w^k|} \rangle \\ &= \sum_{(w^1, \dots, w^k) \in B_{[n]}^o} \frac{|w^1|! \dots |w^k|!}{n! k!} f^{(k)} \circ \langle \pi_0, \pi_{|w^1|}, \dots, \pi_{|w^k|} \rangle \end{aligned}$$

We now use a well known result in combinatorics.

Fact 1. For all $i_1, \dots, i_k > 0$ such that $i_1 + \dots + i_k = n$, the number of partitions w^1, \dots, w^k of $[n]$ such that for all j , $|w^j| = i_j$ is equal to the multinomial coefficient $\frac{n!}{i_1! \dots i_k!}$.

Thus, we can regroup terms of the sum based on the number of words k and on the sizes $|w^1|, \dots, |w^k|$ to conclude that

$$\Delta_n f = \sum_{k=1}^n \sum_{\substack{i_1, \dots, i_k > 0 \text{ s.t.} \\ i_1 + \dots + i_k = n}} \frac{1}{k!} f^{(k)} \circ \langle \pi_0, \pi_{i_1}, \dots, \pi_{i_k} \rangle.$$

This concludes the proof. \square

Remark 2. Observe that by similar arguments invoking **props. 2, 8** and **9**,

$$\Delta'_n f = \sum_{\{w^1, \dots, w^k\} \in B_{[n]}} f^{(k)} \circ \langle \pi_0, \pi_{|w^1|}, \dots, \pi_{|w^k|} \rangle$$

This equation is a clear counterpart of Thm 7.7 of [27] shown in the setting of the abelian functor calculus.

For any $n, m \in \mathbb{N}$ such that $m \leq n$, we define a natural transformation $q_{n,m} : S_n \Rightarrow S_m$ in $\mathcal{C}^{\text{Dlin}}$ and \mathcal{C}^{add} by $q_{n,m} = \langle p_0, \dots, p_m \rangle$. For any $f \in \mathcal{C}(X, Y)$ and any $n \in \mathbb{N}$, define two morphisms $T_n f$ and $T'_n f$ in $\mathcal{C}(S_n X, S_n Y)$ by

$$\pi_i \circ T_n f = \Delta_i f \circ q_{n,i} \quad \pi_i \circ T'_n f = \Delta'_i f \circ q_{n,i} \quad (16)$$

By **Thm. 3**, this definition coincides with the definition in (10).

B. Functoriality of the Taylor expansion

The goal of this section is to prove the following result.

Theorem 4. For any $n \in \mathbb{N}$, T_n, T'_n are functors, defining the action on objects as $T_n X = T'_n X = S_n X$.

The proof consists in two steps. First, we show that k_n, k'_n, L_n, L'_n and $q_{n,m}$ satisfy equations that will eventually turn them (once we have shown the functoriality of T_n and T'_n) into natural transformations in \mathcal{C} :

- (1) $q_{n,m} : T_n \Rightarrow T_m$ and $q_{n,m} : T'_n \Rightarrow T'_m$.
- (2) $k_n : T_n \Rightarrow TT_{n-1}$ and $k'_n : T'_n \Rightarrow TT'_{n-1}$;
- (3) $L_n : T_n \Rightarrow T^n$ and $L'_n : T'_n \Rightarrow T^n$;

Then, we use those commutations and the functoriality of T^n to prove the functoriality of T_n and T'_n .

Proposition 10. *The following diagrams commute.*

$$\begin{array}{ccc} S_n X & \xrightarrow{q_{n,m}} & S_m X \\ \tau_n f \downarrow & & \downarrow \tau_m f \\ S_n Y & \xrightarrow{q_{n,m}} & S_m Y \end{array} \quad \begin{array}{ccc} S_n X & \xrightarrow{q_{n,m}} & S_m X \\ \tau'_n f \downarrow & & \downarrow \tau'_m f \\ S_n Y & \xrightarrow{q_{n,m}} & S_m Y \end{array}$$

Proof: First, observe that for any $k \leq m \leq n$, $q_{m,k} \circ q_{n,m} = q_{n,k}$. Thus, for any $k \in \llbracket 0, m \rrbracket$,

$$\begin{aligned} \pi_k \circ T_m f \circ q_{n,m} &= \Delta_k f \circ q_{m,k} \circ q_{n,m} = \Delta_k f \circ q_{n,k} \\ &= \pi_k \circ T_n f = \pi_k \circ q_{n,m} \circ T_n f \end{aligned}$$

so we conclude that $T_m f \circ q_{n,m} = q_{n,m} \circ T_n f$. The proof that $T'_m f \circ q_{n,m} = q_{n,m} \circ T'_n f$ is the same. \square

Proposition 11. *The following diagrams commute.*

$$\begin{array}{ccc} S_n X & \xrightarrow{k_n} & S_{n-1} X \\ \tau_n f \downarrow & & \downarrow \tau_{n-1} f \\ S_n Y & \xrightarrow{k_n} & S_{n-1} Y \end{array} \quad \begin{array}{ccc} S_n X & \xrightarrow{k'_n} & S_{n-1} X \\ \tau'_n f \downarrow & & \downarrow \tau'_{n-1} f \\ S_n Y & \xrightarrow{k'_n} & S_{n-1} Y \end{array}$$

Proof: We prove the commutation on k_n . Observe that

$$\begin{aligned} \pi_0 \circ TT_{n-1} f \circ k_n &= T_{n-1} f \circ \pi_0 \circ k_n = T_{n-1} f \circ q_{n,n-1} \\ \pi_0 \circ k_n \circ T_n f &= q_{n,n-1} \circ T_n f \end{aligned}$$

so $\pi_0 \circ TT_{n-1} f \circ k_n = \pi_0 \circ k_n \circ T_n f$ by **prop. 10**. Then,

$$\begin{aligned} \pi_1 \circ TT_{n-1} f \circ k_n &= D \langle \Delta_i f \circ q_{n-1,i} \rangle_{i=0}^{n-1} \circ k_n \\ &= \langle D(\Delta_i f \circ q_{n-1,i}) \rangle_{i=0}^{n-1} \circ k_n \quad \text{prop. 1} \\ &= \langle D\Delta_i f \circ Tq_{n-1,i} \rangle_{i=0}^{n-1} \quad \text{D-chain} \\ &= \langle D\Delta_i f \circ Sq_{n-1,i} \rangle_{i=0}^{n-1} \end{aligned}$$

by D-linearity of $q_{n-1,i}$. But, for any $i \neq 0$,

$$\begin{aligned} Sq_{n-1,i} k_n &= \left\langle q_{n-1,i} q_{n,n-1}, q_{n-1,i} \left\langle \frac{1}{n} \pi_1, \dots, \frac{n}{n} \pi_n \right\rangle \right\rangle \\ &= \left\langle q_{n,i}, \left\langle \frac{1}{n} \pi_1, \dots, \frac{i+1}{n} \pi_{i+1} \right\rangle \right\rangle \\ &= \left\langle q_{n,i}, \frac{i+1}{n} \left\langle \frac{1}{i+1} \pi_1, \dots, \frac{i+1}{i+1} \pi_{i+1} \right\rangle \right\rangle \end{aligned}$$

Thus, $Sq_{n-1,i} k_n = t(\frac{i+1}{n}) k_{i+1} q_{n,i+1}$. But then,

$$\begin{aligned} D\Delta_i f \circ Sq_{n-1,i} \circ k_n &= D\Delta_i f \circ t(\frac{i+1}{n}) \circ k_{i+1} \circ q_{n,i+1} \\ &= \frac{i+1}{n} D\Delta_i f \circ k_{i+1} \circ q_{n,i+1} \quad \text{D-add} \\ &= \frac{i+1}{n} \Delta_{i+1} f \circ q_{n,i+1} \end{aligned}$$

and it follows that $\pi_1 \circ TT_{n-1} f \circ k_n = \pi_1 \circ k_n \circ T_n f$. This concludes the proof for k_n . The proof for k'_n is similar. \square

Proposition 12. *The following diagrams commute.*

$$\begin{array}{ccc} S_n X & \xrightarrow{L_n} & S^n X \\ \tau_n f \downarrow & & \downarrow \tau^n f \\ S_n Y & \xrightarrow{L_n} & S^n Y \end{array} \quad \begin{array}{ccc} S_n X & \xrightarrow{L'_n} & S^n X \\ \tau'_n f \downarrow & & \downarrow \tau'^n f \\ S_n Y & \xrightarrow{L'_n} & S^n Y \end{array}$$

Proof: By induction. The equation holds trivially for $L_0 = L'_0 = \text{id}$. Now, observe that $L_{n+1} = SL_n k_{n+1} = TL_n \circ k_{n+1}$ by D-linearity of L_n . We conclude that the commutation on L_{n+1} holds by a straightforward diagram chase.

$$\begin{array}{ccc} S_{n+1} X & \xrightarrow{k_{n+1}} & SS_n X \xrightarrow{TL_n} SS^n X \\ \tau_{n+1} f \downarrow & \text{prop. 11} & \downarrow \tau^n f \text{ I.H.} \quad \downarrow \tau'^n f \\ S_{n+1} Y & \xrightarrow{k_{n+1}} & SS_n Y \xrightarrow{TL_n} SS^n Y \end{array}$$

The proof for L'_n is the same. \square

Proposition 13. *L_n and L'_n are monic: if $f, g \in \mathcal{C}(X, S_n Y)$ satisfy $L_n \circ g = L_n \circ f$ or $L'_n \circ g = L'_n \circ f$ then $g = f$.*

Proof: Assume that $L_n g = L_n f$. Let $i \in \llbracket 0, n \rrbracket$, and let w be any word such that $|w| = i$. Then $\pi_w \circ L_n \circ g = \pi_w \circ L_n \circ f$. That is, $\frac{|w|!}{w!} \pi_i \circ g = \frac{|w|!}{w!} \pi_i \circ f$ by **prop. 9**. Thus, $\pi_i \circ f = \pi_i \circ g$ for all i (k has multiplicative inverses for integers), so $f = g$. The proof that L'_n is monic is similar. \square

Proof of thm. 4: Let $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$.

$$\begin{aligned} L_n \circ T_n(g \circ f) &= T^n(g \circ f) \circ L_n && \text{by prop. 12} \\ &= T^n g \circ T^n f \circ L_n && \text{by functoriality of } T^n \\ &= T^n g \circ L_n \circ T_n f && \text{by prop. 12} \\ &= L_n \circ T_n g \circ T_n f && \text{by prop. 12} \end{aligned}$$

so $T_n(g \circ f) = T_n g \circ T_n f$ by monicity of L_n . We prove that $T_n \text{id}_X = \text{id}_{T_n X}$ in a similar way, using that $T^n \text{id}_X = \text{id}_{T^n X}$. The proof that T'_n is a functor is similar. \square

Remark 3. It immediately follows from **thm. 4** that for all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$,

$$\Delta_n(g \circ f) = \Delta_n g \circ T_n f, \quad \Delta'_n(g \circ f) = \Delta'_n g \circ T'_n f \quad (17)$$

This equation is a clear counterpart of Theorem 3 of [13].

Proposition 14. *A morphism $f \in \mathcal{C}(X, Y)$ is D-linear if and only if $T_n f = S_n f$.*

Proof: If $T_n f = S_n f$ then by (16), $\Delta_1 f \circ q_{n,1} = f \circ \pi_1$. It immediately follows that $Df = \Delta_1 f = f \circ \pi_1$, so f is D-linear. Conversely, if f is D-linear, then $D^n f = f \circ \pi_{1^n}$ (by a straightforward induction). Thus,

$$\Delta_n f = D^n f \circ L_n = f \circ \pi_{1^n} \circ L_n = f \circ \pi_n$$

and by (16), $\pi_i \circ T_n f = f \circ \pi_i = \pi_i \circ S_n f$ so $T_n f = S_n f$. \square

V. THE MONADIC STRUCTURE OF TAYLOR EXPANSION

We have proved the existence of a functor T_n that performs an order n Taylor expansion in a compositional way. We now prove that T_n has a monad structure similar to the monad structure of T . Define natural transformations $z_n : \text{Id} \Rightarrow S_n$, $\theta_n : S_n^2 \Rightarrow S_n$, $c_{n,m} : S_n S_m \Rightarrow S_m S_n$, $l_n : S_n \Rightarrow S_n^2$ and

$t_n(r) : S_n \Rightarrow S_n$ (for all $r \in k$) in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$ as the tuplings characterized by the following equations:

$$\begin{aligned} \pi_k z_n &= \delta_k^0 \text{id} & \pi_k t_n(r) &= r^k \pi_k & \pi_k \theta_n &= \sum_{i+j=k} \pi_i \pi_j \\ \pi_i \pi_j c_{n,m} &= \pi_j \pi_i & \pi_i \pi_j l &= \delta_i^j \pi_i \end{aligned} \quad (18)$$

These natural transformations are the counterpart of the natural transformations defined in (6).

A. The natural transformations of Taylor expansion

The goal of this section is to prove that $z_n, \theta_n, c_{n,m}, l_n$ and $t_n(r)$ are also natural transformations on T_n . As mentioned in [sec. IV](#), we avoid heavy combinatorics by making use of the canonical monad structure of T^n . This proof is similar to the proof of functoriality. First, we show in [prop. 15](#) that $L_n : T_n \Rightarrow T^n$ satisfies equations that will eventually turn L_n (once we have shown that T_n is a monad) into a monad morphism [14] (also called monad transformer [15]) from T_n to T^n that sends the distributive law $c_{n,m}$ to $c^{[n,m]}$. Then, we use these commutations and the naturality equations on T^n to prove the naturality equations on T_n .

Proposition 15. *The following diagrams commute.*

$$\begin{array}{ccc} \text{Id} & \xlongequal{\quad} & \text{Id} \\ z_n \downarrow & & \downarrow z^{[n]} \\ S_n & \xrightarrow{L_n} & S^n \end{array} \quad \begin{array}{ccc} S_n S_n & \xrightarrow{L_n * L_n} & S^n S^n \\ \theta_n \downarrow & & \downarrow \theta^{[n]} \\ S_n & \xrightarrow{L_n} & S^n \end{array}$$

$$\begin{array}{ccc} S_n & \xrightarrow{L_n} & S^n \\ t_n(r) \downarrow & & \downarrow t^{[n]}(r) \\ S_n & \xrightarrow{L_n} & S^n \end{array} \quad \begin{array}{ccc} S_n S_m & \xrightarrow{L_n * L_m} & S^n S^m \\ c_{n,m} \downarrow & & \downarrow c^{[n,m]} \\ S_m S_n & \xrightarrow{L_m * L_n} & S^m S^n \end{array}$$

The same diagrams commute when replacing S_n by T_n .

Proof: We prove the diagram on $z_n/z^{[n]}$ as follows.

$$\pi_w L_n z_n = \frac{|w|!}{w!} \pi_{|w|} z_n = \delta_{|w|}^0 \text{id} = \delta_w^0 = \pi_w z^{[n]}$$

The proof on $t_n(r)/t^{[n]}(r)$ is similar. We now prove the diagram on $c_{n,m}/c^{[n,m]}$. Observe that for any words u, v , $\pi_u * \pi_v = \pi_v \pi_u$, so the projections of the two path are

$$\begin{aligned} (\pi_u * \pi_v) (L_m * L_n) c_{n,m} &= \frac{|u|! |v|!}{u! v!} (\pi_{|u|} * \pi_{|v|}) c_{n,m} \\ &= \frac{|u|! |v|!}{u! v!} (\pi_{|v|} * \pi_{|u|}) \\ (\pi_u * \pi_v) c^{[n,m]} (L_n * L_m) &= (\pi_v * \pi_u) (L_n * L_m) \\ &= \frac{|u|! |v|!}{u! v!} (\pi_{|v|} * \pi_{|u|}) \end{aligned}$$

Finally, we prove the diagram involving θ_n and $\theta^{[n]}$.

$$\pi_w L_n \theta_n = \frac{|w|!}{w!} \pi_{|w|} \theta_n = \frac{|w|!}{w!} \sum_{i_1+i_2=|w|} \pi_{i_1} \pi_{i_2}$$

$$\begin{aligned} \pi_w \theta^{[n]} (L_n * L_n) &= \sum_{(w^1, w^2) \in B_w^o} (\pi_{w^2} * \pi_{w^1}) (L_n * L_n) \\ &= \sum_{(w^1, w^2) \in B_w^o} \frac{|w^1|! |w^2|!}{w!} \pi_{|w^1|} \pi_{|w^2|} \end{aligned}$$

We conclude that the two operands are equal, upon observing that there are exactly $\frac{|w|!}{i_1! i_2!}$ partitions (w^1, w^2) of w such that $|w^j| = i_j$ by [fact 1](#).

Finally, the same diagrams commute when replacing S_n by T_n , by D-linearity of L_n and [prop. 14](#). \square

Theorem 5. *The following families of morphisms are natural transformations in \mathcal{C} : $z_n : \text{Id} \Rightarrow T_n$, $\theta_n : T_n^2 \Rightarrow T_n$, $c_{n,m} : T_n T_m \Rightarrow T_m T_n$ and $t_n(r) : T_n \Rightarrow T_n$.*

Remark 4. On the other hand, θ_n is not a natural transformation $T'_n T'_n \Rightarrow T'_n$ because of the lack of coefficients that erase redundancy. This can be checked by an explicit computation in [Smooth](#) for $n = 2$.

Proof: We prove the naturality of $t_n(r)$.

$$\begin{aligned} L_n \circ T_n f \circ t_n(r) &= T^n f \circ L_n \circ t_n(r) && \text{naturality of } L_n \\ &= T^n f \circ t^{[n]}(r) \circ L_n && \text{by prop. 15} \\ &= t^{[n]}(r) \circ T^n f \circ L_n && \text{naturality of } t^{[n]}(r) \\ &= t^{[n]}(r) \circ L_n \circ T_n f && \text{naturality of } L_n \\ &= L_n \circ t_n(r) \circ T_n f && \text{by prop. 15} \end{aligned}$$

so by monicity of L_n , $T_n f \circ t_n(r) = t_n(r) \circ T_n f$ and $t_n(r)$ is natural. Similarly,

$$\begin{aligned} (L_m * L_n) \circ T_m T_n f \circ c_{n,m} &= T^m T^n f \circ (L_m * L_n) \circ c_{n,m} && \text{naturality of } L_n \\ &= T^m T^n f \circ c^{[n,m]} \circ (L_n * L_m) && \text{by prop. 15} \\ &= c^{[n,m]} \circ T^n T^m f \circ (L_n * L_m) && \text{naturality of } c^{[n,m]} \\ &= c^{[n,m]} \circ (L_n * L_m) \circ T_n T_m f && \text{naturality of } L_n \\ &= (L_m * L_n) \circ c_{n,m} \circ T_n T_m f && \text{by prop. 15} \end{aligned}$$

so $T_m T_n f \circ c_{n,m} = c_{n,m} \circ T_n T_m f$ by monicity of $L_m * L_n$ (the composition of two monos is also a mono) and $c_{n,m}$ is natural. The proof that z and θ_n are natural follow a very similar pattern. \square

Those equations rewrite as direct equations on Δ_n , in the same way that naturality equations on T rewrite as equation on the derivative D .

Corollary 1. *For all $i \in \llbracket 0, n \rrbracket$ and $j \in \llbracket 0, m \rrbracket$,*

$$\pi_i \pi_j \circ T_m T_n f = \Delta_j \Delta_i f \circ (q_{m,j} * q_{n,i}) \quad (19)$$

Thus, Δ_n satisfies the following equations:

- (1) $\Delta_n f \circ z_n = 0$;
- (2) $\Delta_n f \circ \theta_n = \sum_{i=0}^n \Delta_{n-i} \Delta_i f \circ (q_{n,n-i} * q_{n,i})$;
- (3) $\Delta_n \Delta_m f \circ c_{n,m} = \Delta_m \Delta_n f$;
- (4) $\Delta_n f \circ t_n(r) = r^n \Delta_n f$.

Proof: We prove (19) first,

$$\begin{aligned} \pi_i \pi_j \circ T_m T_n f &= \pi_j \circ T_m \pi_i \circ T_n f && \text{D-linearity of } \pi_i \\ &= \pi_j \circ T_m (\pi_i \circ T_n f) \\ &= \pi_j \circ T_m (\Delta_i f \circ q_{n,i}) \\ &= \pi_j \circ T_m (\Delta_i f) \circ T q_{n,i} \\ &= \Delta_j \Delta_i f \circ q_{m,j} \circ T q_{n,i}. \end{aligned}$$

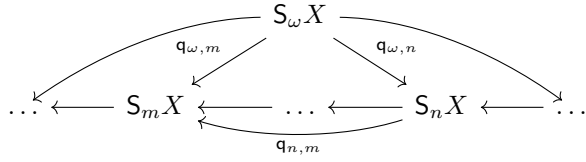


Figure 2: The cone of Taylor expansions

Then, all of the equations above are simply the projections π_n of the naturality equations on T_n . For example, $\pi_i \circ \theta_n \circ T_n f \circ \theta_n = \Delta_i f \circ q_{n,i} \circ \theta_n = \Delta_i f \circ \theta_i \circ (q_{n,i} * q_{n,i})$, and

$$\begin{aligned} \pi_i \circ \theta_n \circ T_n^2 f &= \sum_{k=0}^i \pi_{i-k} \pi_k \circ T_n^2 f \\ &= \sum_{k=0}^i \Delta_{i-k} \Delta_k f \circ (q_{i,i-k} * q_{i,k}) \quad \text{by (19)} \end{aligned}$$

so **item (2)** holds by naturality of θ_n and by straightforward equations on the $q_{n,m}$. The other proofs are similar. \square

We now prove that l_n is also a natural transformation, so that T_n satisfies the same naturality equations as T . Using the same proof technique as above, we would like to define a natural transformation $l^{[n]} : S^n \Rightarrow S^{2n}$ characterized by $\pi_u \pi_v l^{[n]} = \delta_u^v \pi_u$ and to prove that

$$\begin{array}{ccc} S_n & \xrightarrow{L_n} & S^n \\ l_n \downarrow & & \downarrow l^{[n]} \\ S_n S_n & \xrightarrow{L_n * L_n} & S^n S^n \end{array}$$

commutes. However, this diagram does not commute, we can check that $(\pi_u * \pi_v)(L_n * L_n)l_n \neq 0$ if and only if $|u| = |v|$, whereas $(\pi_u * \pi_v)l^{[n]}L_n \neq 0$ if and only if $u = v$. Thus, we need to perform a more explicit (and combinatorial) computation, making use of the relationship between Δ_n and D^n . The proof is done in **sec. B**.

Proposition 16. *For all n, i, j such that $n \geq i, n \geq j$,*

$$\Delta_j \Delta_i f \circ (q_{n,j} * q_{n,i}) \circ l_n = \begin{cases} \Delta_i f \circ q_{n,i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

and l_n is a natural transformation $T_n \Rightarrow T_n^2$.

B. An infinitary Taylor expansion

This section defines a functor T_ω performing Taylor expansion at all order. Assume that for all object X of \mathcal{C} , there exists a countable cartesian product $S_\omega X = \prod_{i=0}^\infty X$. Let $\pi_k : S_\omega \Rightarrow \text{Id}$, as it is now standard. Let $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$. All the natural transformations in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$ defined in (18) generalize to $n, m \in \mathbb{N}_\omega$. Similarly, let $q_{\omega,n} : S_\omega \Rightarrow S_n$ be the natural transformation in $\mathcal{C}^{\text{Dlin}}$ and \mathcal{C}^{add} given by $q_{\omega,n} = \langle \pi_0, \dots, \pi_n \rangle$. Observe that this yields a cone in \mathcal{C} , \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$, see **fig. 2**.

Let $T_\omega X = S_\omega X$, and for all $f \in \mathcal{C}(X, Y)$, let $T_\omega f \in \mathcal{C}(T_\omega X, T_\omega Y)$ be the unique morphism characterized by the equation $\pi_i \circ T_\omega f = \Delta_n f \circ q_{\omega,n}$.

Proposition 17. *T_ω is a functor, $\pi_0 : T_\omega \Rightarrow \text{Id}$, $z_\omega : \text{Id} \Rightarrow T_\omega$, $\theta_\omega : T_\omega^2 \Rightarrow T_\omega$, $l_\omega : T_\omega \Rightarrow T_\omega^2$, and $t_\omega(r) : T_\omega \Rightarrow T_\omega$ are natural transformations, and $c_{n,m} : T_n T_m \Rightarrow T_m T_n$, $q_{\omega,n} : T_\omega \Rightarrow T_n$ are natural transformations for all $n, m \in \mathbb{N}_\omega$. Finally, $f \in \mathcal{C}(X, Y)$ is D-linear if and only if $T_\omega f = S_\omega f$.*

Proof: The functoriality and the naturality equations on T_ω are equivalent, by projections, to equations on $\Delta_n f$ for all $n \in \mathbb{N}$. These equations are in turn equivalent to the functoriality and the naturality equations on T_n for all $n \in \mathbb{N}$. For example, the functoriality of T_ω is equivalent to the equations of (17), for all $n \in \mathbb{N}$. These equations were proved as a consequence of the functoriality of T_n . Similarly, the naturality of z_ω , θ_ω , $t_\omega(r)$ and $c_{n,m}$ is a consequence of **cor. 1**, because the equations on Δ_n given in **cor. 1** are the projections of the naturality equations on T_ω . Similarly, the naturality of $l_\omega : T_\omega \Rightarrow T_\omega^2$ is a consequence of **prop. 16**. The proof that $q_{\omega,n}$ is natural is the same as the proof of **prop. 10**. Finally, the characterization of the D-linear morphisms is the same as the proof of **prop. 14**. \square

This functor T_ω is the counterpart of the coherent Taylor expansion T of [18], more detail in **sec. VII**.

C. The monad structure of Taylor expansion

In this section, we only consider indexes $n, m \in \mathbb{N}_\omega$ such that S_n is defined (so n and m can be equal to ω , provided that the category has countable products).

Proposition 18. *(S_n, z_n, θ_n) is a monad in \mathcal{C}^{add} and $\mathcal{C}^{\text{Dlin}}$, and $c_{n,m}$ is a distributive law of S_n over S_m .*

Proof: Same proof as the proofs of **props. 5** and **6**, using that $c_{n,m} c_{m,n} = \text{id}$. \square

Corollary 2. *(T_n, z_n, θ_n) is a monad in \mathcal{C} and $c_{n,m}$ is a distributive law of T_n over T_m .*

Proof: Direct consequence of **prop. 18** and **prop. 14**. \square

The Kleisli category \mathcal{C}_{T_n} of the monad T_n formalizes a notion of higher order dual numbers [11]. For all $n \in \mathbb{N}$, a morphism $f \in \mathcal{C}_{T_n}(X, Y)$ corresponds to a polynomial $\sum_{k=0}^n f_k \varepsilon^k$ of degree n over a formal indeterminate ε such that $\varepsilon^{n+1} = 0$. When $n = \omega$, a morphism is a power series over a formal indeterminate ε that satisfies no equation. The composition of f with $g = \sum_{k=0}^n g_k \varepsilon^k \in \mathcal{C}_{T_n}(Y, Z)$ consists of performing a Taylor expansion of each g_k around f_0 on the variation $f - f_0$, and returning the induced polynomial (or formal power series, when $n = \omega$) over ε , taking into account that $\varepsilon^{n+1} = 0$ if $n \in \mathbb{N}$.

As in **sec. III**, the distributive laws $c_{n,m}$ satisfy the Yang Baxter equations, so by thm 2.1 of [26], there exists a canonical monad structure on $T_{i_1} \cdots T_{i_n}$ for any $i_1, \dots, i_n \in \mathbb{N}_\omega$ as well as canonical distributive laws between those monads. The associated natural transformations are generalizations of the natural transformations given in (7), with projections taken on words on integers instead of words on $\{0, 1\}$. The intuition behind these monads is very similar to the intuition behind T^n given in **sec. III**. Intuitively, an element $e \in T_{i_1} \cdots T_{i_n} X$ is a

formal power series over n formal variables $\varepsilon_1, \dots, \varepsilon_n$ such that for all j , $\varepsilon_j^{i_j+1} = 0$ (if $i_j = \omega$, then this is simply a formal variable, with no equation on it). The projection $\pi_{w_1 \dots w_n}(e)$ gives the coefficient in e of the monomial $\varepsilon_1^{w_1} \dots \varepsilon_n^{w_n}$. Then, $T_{i_1} \dots T_{i_n} f(e)$ pushes forward this power series using the power series given by the Taylor expansion of f , taking into account that the monomials eventually vanish at high order, since $\varepsilon_j^{i_j} = 0$ for all $i_j \neq \omega$.

By \mathbb{D} -linearity of the projections p_i , the natural iso

$$c_\times = \langle S_n p_1, S_n p_2 \rangle \in \mathcal{C}^{\text{add}}(S_n(X_1 \times X_2), S_n X_1 \times S_n X_2)$$

is also a natural isomorphism $T_n(_ \times _) \Rightarrow T_n(_) \times T_n(_)$. Then, both S_n and T_n equipped with c_\times are *strong monoidal monads*, which turns them into *commutative monads* [28], [29]. Concretely, it means that there exist two natural transformations ψ^1 and ψ^2 called *strengths* defined as

$$\begin{aligned} \psi^1 &= c_\times^{-1} \langle \text{id}, z_n \rangle \in \mathcal{C}(T_n X_0 \times X_1, T_n(X_0 \times X_1)) \\ \psi^2 &= c_\times^{-1} \langle z_n, \text{id} \rangle \in \mathcal{C}(X_0 \times T_n X_1, T_n(X_0 \times X_1)). \end{aligned} \quad (21)$$

These strengths induce two operators $T_n^{\partial_1}$ and $T_n^{\partial_2}$ that map $f \in \mathcal{C}(X_0 \times X_1, Y)$ to $T_n^{\partial_0} f = T_n f \circ \psi^1 \in \mathcal{C}(T_n X_0 \times X_1, T_n Y)$ and $T_n^{\partial_1} f = T_n f \circ \psi^2 \in \mathcal{C}(X_0 \times T_n X_1, T_n Y)$. These operators perform a Taylor expansion of f with regard to only one of its argument. The commutativity of the monad means that

$$\theta_n \circ T_n^{\partial_0} T_n^{\partial_1} f = \theta_n \circ T_n^{\partial_1} T_n^{\partial_0} f = T_n f \circ c_\times.$$

It corresponds to the fact that the Taylor expansion of $f \in \mathcal{C}(X_0 \times X_1, Y)$ can either be computed directly, or by expanding f with regard to its two arguments separately (in any order) by using the partial derivatives. Note that those fundamentals categorical equations capture a complex underlying combinatorics.

VI. A DIRECT AXIOMATIZATION OF TAYLOR EXPANSION

In this section, \mathcal{C} is a cartesian left k -additive category where k has multiplicative inverses for integers. Let $n \in \mathbb{N}_\omega$. We may assume $n = \omega$ or not, depending on the existence of $S_\omega X$ in \mathcal{C} . The work of the previous section suggest a direct categorical axiomatization of Taylor expansion. This axiomatization is very similar to def. 116 of [18], except that their work considers a setting with infinite partial sums.

Definition 8. An order n Taylor expansion in \mathcal{C} is a functor U such that $UX = S_n X$ and

- (1) $U p_i = S_n p_i$, $U \sigma = S_n \sigma$ and $U h(r) = S_n h(r)$;
- (2) the families $\pi_0 : U \Rightarrow \text{Id}$, $z_n : \text{Id} \Rightarrow U$, $\theta_n : U^2 \Rightarrow U$, $t_n(r) : U \Rightarrow U$, $c_{n,n} : UU \Rightarrow UU$ and $l_n : U \Rightarrow U^2$ are natural transformations in \mathcal{C} .

Proposition 19. Any cartesian k -differential category such that k has multiplicative inverses for integers has an order n Taylor expansion, given by T_n .

Proof: T_n satisfies the equations of item (1) by prop. 14. T_n satisfies the naturality equations, by thm. 5 and prop. 16 when $n \in \mathbb{N}$, and by prop. 17 when $n = \omega$. \square

We now show that def. 8 only describes functors that are indeed Taylor expansions. More precisely, we prove that any order n Taylor expansion U induces a differential \mathbb{D} and that $U = T_n$ where T_n is the Taylor expansion induced by \mathbb{D} . First, we show that the term of degree i in Uf only depends on the first i coordinates. For all $m \leq n$, let $r_{m,n} : S_m \Rightarrow S_n$ and $s_{n,m} : S_n \Rightarrow S_m$ be natural transformations defined by

$$\pi_k r_{m,n} = \begin{cases} \pi_k & \text{if } k \leq m \\ 0 & \text{otherwise,} \end{cases} \quad \pi_k s_{n,m} = \begin{cases} \pi_k & \text{if } k \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $q_{n,m}$ and $r_{m,n}$ are sections and retractions since $q_{n,m} r_{m,n} = \text{id}_{S_m X}$, and that $s_{n,m} = r_{m,n} q_{n,m}$.

Lemma 1. For all Taylor expansion U , $\pi_i \circ Uf = \pi_i \circ Uf \circ s_{n,i}$.

The proof of this statement is in sec. C. Let $\Delta_i^U f = \pi_i \circ Uf \circ r_{i,n} \in \mathcal{C}(S_i X, Y)$. By lemma 1, $\pi_i \circ Uf = \Delta_i^U f \circ q_{n,i}$, so $\Delta_i^U f$ is to U what $\Delta_i f$ is to T_n . As such, the naturality equations on U imply equations on Δ_n^U that are very similar to the equations of cor. 1.

Proposition 20. For all Taylor expansion U ,

$$\pi_i \pi_j \circ UUf = \Delta_j^U \Delta_i^U f \circ (q_{n,j} * q_{n,i}) \quad (22)$$

Thus, the naturality equations on U are equivalent to the following equations:

- (1) $\Delta_0^U f = f$ and $\Delta_i^U f \circ z_i = 0$ for $i \geq 1$;
- (2) $\Delta_i^U f \circ \theta_i = \sum_{k=0}^i \Delta_{i-k}^U \Delta_k^U f \circ (q_{i,i-k} * q_{i,k})$;
- (3) $\Delta_i^U \Delta_j^U f \circ c_{i,j} = \Delta_j^U \Delta_i^U f$;
- (4) $\Delta_i^U f \circ t_i(r) = r^i \Delta_i^U f$;
- (5) $\Delta_j^U \Delta_i^U f \circ (q_{n,j} * q_{n,i}) \circ l_n = \begin{cases} \Delta_i^U f \circ q_{n,i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$.

and Δ_1^U is a derivative, in the sense of def. 4.

Proof: The proof of (22) is the same as the proof of (19) in cor. 1. The other equations are simply the projections of the naturality equations on U , we also refer to the proof of cor. 1. We conclude that Δ_1^U is a derivative, taking those equations on $i = j = 1$, and by projections of the other equations of def. 8 to recover **D-sum**, **D-proj** and **D-chain**. \square

Thus, there are two maps $\phi : U \mapsto \Delta_1^U$ and $\psi : \mathbb{D} \mapsto T_n$ between differentials and Taylor expansions, and by definition $\phi \circ \psi = \text{id}$. We want to prove that $\psi \circ \phi = \text{id}$. First, we prove that this is the case when restricting ϕ on Taylor expansions that are "functorializations" of an actual Taylor expansion.

Proposition 21. The maps ϕ and ψ induce a bijection between differentials and order n Taylor expansions U such that

$$U \circ \langle x, u, 0, \dots \rangle = \left\langle \frac{1}{k!} f^{(k)} \circ \langle x, u, \dots, u \rangle \right\rangle_{k=0}^n \quad (23)$$

where $f^{(k)}$ is the higher order derivative defined from the derivative $\Delta_1^U = \phi(U)$.

The proof is in sec. C. We now prove that (23) always hold when the additive structure of \mathcal{C} is *cancellative*, in the sense that $f + g = f + h$ implies $g = h$. We do not have a proof in the general case.

Remark 5. Unfortunately, categories in which the additive monoid is not cancellative include the categories with total sums that we will see in [sec. VII](#). Those include the weighted relational model of [example 1](#). Indeed, the completeness of a semiring k induces the existence of an infinite coefficient $\infty = \sum_{x \in k} x$, and for all element x $\infty + x = \infty$. Still, [prop. 21](#) is already a satisfactory result, because it shows that the "functorialization" of a Taylor expansion is unique.;

Theorem 6. *Assume that the additive structure of \mathcal{C} is cancellative. For all $n \in \mathbb{N}_\omega$ such that $n \geq 1$, there is a bijection between differentials D and order n Taylor expansions U , given by the maps $\phi : U \mapsto \Delta_1^U$ and $\psi : D \mapsto T_n$.*

The proof is in [sec. C](#).

VII. TAYLOR EXPANSION AND COUNTABLE SUMS

We now assume that \mathcal{C} admits countable sums. Formally, this means that k is a complete semiring, that $\mathcal{C}(X, Y)$ is a complete monoid, and that the multiplicative action $_ \cdot _ : k \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Y)$ respects countable sums in each variable. We call such semimodules *complete semimodules*. A map between complete k -semimodules is k -additive if it preserves all countable sums, and the multiplicative action. The notions of cartesian left k -additive categories and cartesian differential categories directly carry to this new setting. We refer the reader to [Sec. 6](#) of [\[30\]](#) for a more detailed definition.

Definition 9 (Prop. 6.3 of [\[30\]](#)). A cartesian differential category with countable sums is *Taylor* if k has multiplicative inverses for integers and if for all $f \in \mathcal{C}(X, Y)$ and $x, u \in \mathcal{C}(Z, X)$,

$$f \circ (x + u) = f(x) + \sum_{k=1}^{\infty} \frac{1}{k!} f^{(k)} \circ \langle x, u, \dots, u \rangle \quad (24)$$

One main examples of such Taylor categories is the weighted relational model of [example 1](#) over semirings with multiplicative inverses for integers. Typically, those categories are closed, and the differential is assumed to be compatible with the closed structure, see [def. 4.4](#) of [\[22\]](#). As observed in [\[30\]](#), a cartesian differential category with countable sums which is closed is Taylor if and only if it *models Taylor expansion*, in the sense of [def. 5.19](#) of [\[5\]](#). As such, those categories serve as models of the differential λ -calculus and its associated Taylor expansion [\[5\]](#), [\[19\]](#).

Definition 10. Let $\sigma : S_\omega \Rightarrow \text{Id}$ be the natural transformation in \mathcal{C}^{add} defined as $\sigma = \sum_{k=0}^{\infty} \pi_k$. An order ω Taylor expansion U_ω is analytic if σ is a natural transformation $U_\omega \Rightarrow \text{Id}$.

The analytic Taylor expansions of [def. 117](#) of [\[18\]](#) coincide, in cartesian differential categories that admit countable sums, with the analytic Taylor expansion of [def. 10](#), hence the following result.

Theorem 7. *A cartesian differential category is Taylor if and only if T_ω is an analytic Taylor expansion. Thus, any model of [\[5\]](#) is a model of [\[18\]](#).*

Proof: Assume that T_ω is analytic. Then, $\sigma \circ T_\omega \circ \langle x, u, 0, \dots \rangle = f \circ \sigma \circ \langle x, u, 0, \dots \rangle = f \circ (x + u)$, this is exactly [\(24\)](#) upon observing that $T_\omega f \circ \langle x, u, 0, \dots \rangle = \langle \frac{1}{n!} f^{(n)} \circ \langle x, u, \dots, u \rangle \rangle_{n=0}^\infty$. Conversely, assume that \mathcal{C} is Taylor. Then,

$$\begin{aligned} f \circ \sigma &= f \circ (\pi_0 + \pi_1) \circ \left\langle \pi_0, \sum_{k=1}^{\infty} \pi_k \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)} \circ \left\langle \pi_0, \sum_{k=1}^{\infty} \pi_k, \dots, \sum_{k=1}^{\infty} \pi_k \right\rangle \quad \text{by (24)} \end{aligned}$$

By multilinearity of the derivative, we get that

$$f \circ \sigma = \sum_{k=0}^{\infty} \sum_{a_1, \dots, a_k \in \mathbb{N}^*} \frac{1}{k!} f^{(k)} \circ \langle \pi_0, \pi_{a_1}, \dots, \pi_{a_k} \rangle.$$

By a reordering of the sum and [thm. 3](#), this sum is equal to $\sum_{n=0}^{\infty} \Delta_n f \circ q_{\omega, n} = \sigma \circ T_n f$. So σ is natural and T_ω is analytic. \square

VIII. CONCLUSION

This paper defined a functor T_n that performs an order n Taylor expansion in a compositional way, in any cartesian differential category (with multiplicative inverse for integers). The fundamental properties of Taylor expansion then boil down to naturality equations that turns T_n into a monad. This monad provides a categorical approach to higher order dual numbers [\[11\]](#) and the jet bundle construction [\[20\]](#), [\[21\]](#) recently developed in automated differentiation. Of particular interest is the canonical monad $T_{i_1} \dots T_{i_n}$, that freely combines various Taylor expansion over different dual numbers, thanks to the theory of distributive laws.

A syntactical approach to this monad should be developed, with applications both in AD and in the differential lambda calculus. In particular, the coherent Taylor expansion of [\[18\]](#) suggest that the coherent differential PCF of [\[31\]](#) based on the tangent bundle T can be generalized to the infinitary Taylor expansion T_ω . This calculus would provide an alternative to the Taylor expansion of programs that accounts for the determinism of computation, and [thm. 7](#) ensures that this calculus would be a generalization of the previous one.

It should also be possible to unify this article with [\[8\]](#), [\[18\]](#) in a single framework, by providing a more general theory of Taylor expansion with partial sums that encapsulate both the finite total sums of this article, and the partial *positive* sums of [\[18\]](#). A good candidate for the sums would be the notion of *partial commutative monoids* [\[32\]](#).

This article also opens up a lot of perspective on the study of the categorical properties of Taylor expansion. A generalization of the Taylor expansion functor to tangent categories [\[7\]](#) should provide a categorical axiomatization of the jet construction. A generalization to reverse derivative [\[33\]](#) and reverse tangent categories [\[34\]](#) would allow a computation of higher order derivatives and Taylor expansion in a reverse mode, which is very well suited for AD because of its efficiency. This should be similar to the notion of covelocity (the dual of jet bundles) mentionned in [\[20\]](#).

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APPENDIX A PROOFS OF SEC. II

The goal of this appendix is to prove **thm. 5**, that we recall below. This proof was left as an appendix, because it mostly consists of an adaptation of the proof of Thm. 7 of [8].

Theorem. *Let $D : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(SX, Y)$ be an operator that satisfies **D-proj** and **D-chain**. Then:*

- (1) *D satisfies **D-sum** iff $T\sigma = S\sigma$ and $T(h(r)) = S(h(r))$ for all $r \in k$ (that is, $h(r)$ and σ are **D-linear**);*
- (2) *D satisfies **D-add** iff $z : \text{Id} \Rightarrow T$, $\theta : T^2 \Rightarrow T$ and $t(r) : T \Rightarrow T$ (for all $r \in k$) are natural in \mathcal{C} ;*
- (3) *Assuming that $z : \text{id} \Rightarrow T$ is natural in \mathcal{C} , D satisfies **D-lin** iff $l : T \Rightarrow T^2$ is natural in \mathcal{C} ;*
- (4) *D satisfies **D-sym** iff $c : T^2 \Rightarrow T^2$ is natural in \mathcal{C} .*

Thus, there is a bijection between derivative operators D and functors T such that $TX = SX$, $Tp_i = Sp_i$, $T\sigma = S\sigma$, $Th(r) = Sh(r)$ and such that $\pi_0, z, \theta, t(r)$ (for all $r \in k$), l and c are natural in \mathcal{C} with regard to T .

Let $D : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, SY)$ be an operator for all X, Y that satisfies **D-proj** and **D-chain**.

Proposition 22. *D satisfies **D-sum** iff $T\sigma = S\sigma$ and $T(h(r)) = S(h(r))$ for all $r \in k$ (that is, $h(r)$ and σ are **D-linear**).*

Proof: Assume that D satisfies **D-sum**. By **prop. 4** it suffices to prove that $D\sigma = \sigma \circ \pi_1$ and $Dh(r) = h(r) \circ \pi_1$

to conclude that $T\sigma = S\sigma$ and $T(h(r)) = S(h(r))$. By **D-lin**, $Dp_i = p_i \circ \pi_1$. Thus, by **D-sum**,

$$D\sigma = D\pi_0 + D\pi_1 = \pi_0 \circ \pi_1 + \pi_1 \circ \pi_1 = (\pi_0 + \pi_1) \circ \pi_1 = \sigma \circ \pi_1.$$

$$Dh(r) = D(r \cdot \text{id}) = r \cdot D\text{id} = r \cdot \pi_1 = h(r) \circ \pi_1$$

and it concludes the proof of the forward implication. Conversely, assume that σ and $h(r)$ are linear. Then,

$$\begin{aligned} D(f + g) &= D(\sigma \circ \langle f, g \rangle) = D\sigma \circ T\langle f, g \rangle \quad \text{by **D-chain**} \\ &= \sigma \circ \pi_1 \circ T\langle f, g \rangle = \sigma \circ D\langle f, g \rangle \quad \text{by assumption} \\ &= \sigma \circ \langle Df, Dg \rangle = Df + Dg \quad \text{by **prop. 1**} \\ D(r \cdot f) &= D(h(r) \circ f) = Dh(r) \circ Tf \quad \text{by **D-chain**} \\ &= h(r) \circ \pi_1 \circ Tf = h(r) \circ Df \quad \text{by assumption} \\ &= r \cdot Df \end{aligned}$$

so D is k -additive. \square

Before moving on to the other results, we describe a bit more precisely the components of T^2f .

Lemma 2. For all $f \in \mathcal{C}(X, Y)$,

$$T^2f = \langle \langle f \circ \pi_0 \pi_0, Df \circ \pi_0 \rangle, \langle Df \circ T\pi_0, DDf \rangle \rangle$$

Proof: Two of the coordinates are given by a straightforward computation, by definition of T .

$$\pi_0 \pi_0 \circ T^2f = \pi_0 \circ Tf \circ \pi_0 = f \circ \pi_0 \pi_0$$

$$\pi_1 \pi_0 \circ T^2f = \pi_1 \circ Tf \circ \pi_0 = Df \circ \pi_0$$

The other two relies on **D-chain** and **D-proj**.

$$\begin{aligned} \pi_1 \pi_1 \circ T^2f &= \pi_1 S\pi_1 \circ T^2f \quad \text{naturality of } \pi_1 \text{ in } \mathcal{C}^{\text{Dlin}} \\ &= \pi_1 \circ T\pi_1 \circ T^2f \quad \text{by **D-proj** and **prop. 4**} \\ &= \pi_1 \circ T(\pi_1 \circ Tf) \quad \text{by **D-chain**} \\ &= \pi_1 \circ T(Df) = DDf \quad \text{by definition of } T \\ \pi_0 \pi_1 \circ T^2f &= \pi_1 S\pi_0 \circ T^2f \quad \text{naturality of } \pi_1 \text{ in } \mathcal{C}^{\text{add}} \\ &= \pi_1 \circ T\pi_0 \circ T^2f \quad \text{by **D-proj** and **prop. 4**} \\ &= \pi_1 \circ T(\pi_0 \circ Tf) \quad \text{by **D-chain**} \\ &= \pi_1 \circ T(f \circ \pi_0) \quad \text{by definition of } T \\ &= Df \circ T\pi_0 \quad \text{by **D-chain** and definition of } T \end{aligned}$$

\square

We now decompose the proof of **item (2)** into 3 different subgoals.

Lemma 3. Let $r \in k$. The following assertions are equivalent:

- (1) $Df \circ \langle x, r \cdot u \rangle = r \cdot Df \circ \langle x, u \rangle$ for all $f \in \mathcal{C}(X, Y)$ and $x, u \in \mathcal{C}(Z, X)$;
- (2) $Df \circ t(r) = r \cdot Df$;
- (3) $t(r) : T \Rightarrow T$ is a natural transformation in \mathcal{C} .

Proof: (1) \Leftrightarrow (2) Observe that (2) is a special instance of (1) in which $x = \pi_0$ and $u = \pi_1$. Conversely, the composition of the equation of (2) by $\langle x, u \rangle$ on the right yields the equation of (1).

(2) \Leftrightarrow (3) Observe that

$$Tf \circ t(r) = \langle f \circ \pi_0, Df \circ t(r) \rangle$$

$$t(r) \circ Tf = \langle f \circ \pi_0, r \cdot Df \rangle$$

so $t(r) : T \Rightarrow T$ is natural if and only if $Df \circ t(r) = r \cdot Df$. \square

Lemma 4. The following assertions are equivalent:

- (1) $Df \circ \langle x, 0 \rangle = 0$ for all $f \in \mathcal{C}(X, Y), x \in \mathcal{C}(Z, X)$;
- (2) $Df \circ z = 0$;
- (3) z is a natural transformation $\text{id} \Rightarrow T$.

Proof: (1) \Leftrightarrow (2) Observe that (2) is a special instance of (1) in which $x = \pi_0$. Conversely, the composition of the equation of (2) by $\langle x, 0 \rangle$ on the right yields the equation of (1).

(2) \Leftrightarrow (3) Observe that

$$Tf \circ z = \langle f, Df \circ z \rangle$$

$$z \circ f = \langle f, 0 \rangle$$

so $z : \text{id} \Rightarrow T$ is natural if and only if $Df \circ z = 0$. \square

Lemma 5. The following assertions are equivalent:

- (1) $Df \circ \langle x, u + v \rangle = Df \circ \langle x, u \rangle + Df \circ \langle x, v \rangle$ for all $f \in \mathcal{C}(X, Y)$ and $x, u, v \in \mathcal{C}(Z, X)$;
- (2) $Df \circ \theta = Df \circ T\pi_0 + Df \circ \pi_0$ for all $f \in \mathcal{C}(X, Y)$;
- (3) θ is a natural transformation $T^2 \Rightarrow T$.

Proof: (1) \Leftrightarrow (2) Observe that (2) is an instance of (1), taking $x = \pi_0 \pi_0$, $u = \pi_0 \pi_1$ and $v = \pi_1 \pi_0$. Conversely, the composition of the equation of (2) by $\langle \langle x, u \rangle, \langle v, 0 \rangle \rangle$ on the right yields the equation of (1).

(2) \Leftrightarrow (3). Observe that

$$\pi_0 \circ \theta \circ T^2f = \pi_0 \pi_0 \circ T^2f = f \circ \pi_0 \pi_0 = f \circ \pi_0 \circ \theta = \pi_0 \circ Tf \circ \theta$$

by **lemma 2**. Thus, $\theta : T^2 \Rightarrow T$ is natural if and only if

$$\pi_1 \circ \theta \circ T^2f = \pi_1 \circ Tf \circ \theta$$

and we can check that this is equivalent to the equation of (2). Indeed,

$$\begin{aligned} \pi_1 \circ \theta \circ T^2f &= \pi_1 \pi_0 \circ T^2f + \pi_0 \pi_1 \circ T^2f \\ &= Df \circ \pi_0 + Df \circ T\pi_0 \quad \text{by **lemma 2**} \\ \pi_1 \circ Tf \circ \theta &= Df \circ \theta \quad \text{by assumption} \end{aligned}$$

so (2) holds if and only if $\theta : T^2 \Rightarrow T$ is natural. \square

Proposition 23. D satisfies **D-add** iff $z : \text{id} \Rightarrow T$, $\theta : T^2 \Rightarrow T$ and $t(r) : T \Rightarrow T$ (for all $r \in k$) are natural in \mathcal{C}

Proof: Direct consequence of **lemmas 3** to **5**. \square

Proposition 24. Assuming that $z : \text{id} \Rightarrow T$ is natural in \mathcal{C} , the following assertions are equivalent:

- (1) D satisfies **D-lin**, that is, $DDf \circ \langle x, 0, 0, u \rangle = Df \circ \langle x, u \rangle$ for all $f \in \mathcal{C}(X, Y)$ and $x, u \in \mathcal{C}(Z, X)$;
- (2) $DDf \circ \text{id} = Df$;
- (3) $\text{id} : T \Rightarrow T^2$ is a natural transformation in \mathcal{C} .

Proof: (1) \Leftrightarrow (2) Observe that (2) is a special instance of (1) in which $x = \pi_0$ and $u = \pi_1$. Conversely, the composition of the equation of (2) by $\langle x, u \rangle$ on the right yields the equation of (1).

(2) \Leftrightarrow (3) By [lemma 2](#)

$$\pi_1 \pi_1 \circ T^2 f \circ l = DDf \circ l \quad \pi_1 \pi_1 \circ l \circ Tf = \pi_1 Tf = Df$$

so the equation of (2) consists of the rightmost projection of the naturality equation on l . Thus, it suffices to show that

$$\pi_i \pi_j \circ T^2 f \circ l = \pi_i \pi_j \circ l \circ Tf$$

for all $(i, j) \in \{0, 1\}^2 \setminus \{(1, 1)\}$ to conclude the equivalence. We make use of [lemma 2](#).

- Case $i = 0, j = 0$: $\pi_0 \circ \pi_0 \circ l \circ Tf = \pi_0 \circ Tf = f \circ \pi_0$ and $\pi_0 \circ \pi_0 \circ T^2 f \circ l = f \circ \pi_0 \circ \pi_0 \circ l = f \circ \pi_0$;
- Case $i = 1, j = 0$: $\pi_0 \circ \pi_1 \circ l \circ Tf = 0 \circ Tf = 0$ and $\pi_1 \circ \pi_0 \circ T^2 f \circ l = Df \circ \pi_0 \circ l = Df \circ z \circ \pi_0 = 0$ by naturality of z ;
- Case $i = 0, j = 1$: $\pi_0 \circ \pi_1 \circ l \circ Tf = 0 \circ Tf = 0$ and $\pi_0 \circ \pi_1 \circ T^2 f \circ l = Df \circ T\pi_0 \circ l = Df \circ z \circ \pi_0 = 0$ by naturality of z .

□

For the last goal, we first prove useful equations on c .

Lemma 6. $\pi_i c = S\pi_i = T\pi_i$ and $T\pi_i c = S\pi_i c = \pi_i$.

Proof: First,

$$\pi_j \pi_i c = \pi_i \pi_j \quad \pi_j S\pi_j = \pi_j \pi_i$$

so $\pi_i c = S\pi_i$. Similarly,

$$\pi_j S\pi_i c = \pi_i \pi_j c = \pi_j \pi_i$$

so $S\pi_i \circ c = \pi_i$. We conclude the proof by [prop. 4](#) and [D-lin](#). □

Proposition 25. *The following assertions are equivalent:*

- (1) D satisfies [D-sym](#), that is, $DDf \circ \langle x, u, v, w \rangle = DDf \circ \langle x, v, u, w \rangle$ for all $f \in \mathcal{C}(X, Y)$ and $x, u, v, w \in \mathcal{C}(Z, X)$;
- (2) $DDf \circ c = DDf$;
- (3) $c : T^2 \Rightarrow T^2$ is a natural transformation.

Proof: (1) \Leftrightarrow (2) Observe that (2) is a special instance of (1) in which $x = \pi_0 \pi_0$, $u = \pi_0 \pi_1$, $v = \pi_1 \pi_0$ and $w = \pi_1 \pi_1$. Conversely, the composition of the equation of (2) by $\langle \langle x, u \rangle, \langle v, w \rangle \rangle$ on the right yields the equation of (1).

(2) \Leftrightarrow (3) By [lemma 2](#),

$$\pi_1 \pi_1 \circ T^2 f \circ c = DDf \circ c \quad \pi_1 \pi_1 \circ c \circ T^2 f = \pi_1 \pi_1 \circ T^2 f = DDf$$

so the equation of (2) consists of the rightmost projection of the naturality equation on c . Thus, it suffices to show that

$$\pi_i \pi_j \circ T^2 f \circ c = \pi_i \pi_j \circ c \circ T^2 f$$

for all $(i, j) \in \{0, 1\}^2 \setminus \{(1, 1)\}$ to conclude the equivalence. We make use of [lemma 2](#) and [lemma 6](#).

- $i = 0, j = 0$: the computation is immediate;

- $i = 1, j = 0$: $\pi_1 \pi_0 \circ c \circ T^2 f = \pi_0 \pi_1 \circ T^2 f = Df \circ T\pi_0$ and $\pi_1 \circ \pi_0 \circ T^2 f \circ c = Df \circ \pi_0 \circ c = Df \circ T\pi_0$;
- $i = 0, j = 1$: $\pi_0 \circ \pi_1 \circ T^2 f \circ c = Df \circ T\pi_0 \circ c = Df \circ \pi_0$ and $\pi_0 \pi_1 \circ c \circ T^2 f = \pi_1 \circ \pi_0 \circ T^2 f = Df \circ \pi_0$.

□

This concludes the proof of [thm. 5](#).

APPENDIX B

PROOF OF [PROP. 16](#)

Lemma 7. *For all $i \in \llbracket 0, n \rrbracket$ and $j \in \llbracket 0, m \rrbracket$, we have $\Delta_j \Delta_i f = D^{i+j} f \circ (L_j * L_i)$.*

Proof: This is a direct consequence of [prop. 8](#) and (3): $\Delta_j \Delta_i f = D^j (D^i f \circ L_i) \circ L_j = D^{i+j} f \circ T^j L_i \circ L_j$. □

Proposition 26. *For all n, i, j such that $n \geq i, n \geq j$,*

$$\Delta_j \Delta_i f \circ (q_{n,j} * q_{n,i}) \circ l_n = \begin{cases} \Delta_i f \circ q_{n,i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

and l_n is a natural transformation $T_n \Rightarrow T_n^2$.

Proof: By [prop. 2](#),

$$D^{i+j} f = \sum_{(u^1 v^1, \dots, u^k v^k) \in B_{[i+j]}^o} \frac{1}{k!} f^{(k)} \circ \langle \pi_{0^{i+j}}, \pi_{v^1} * \pi_{u^1}, \dots, \pi_{v^l} * \pi_{u^l} \rangle$$

where $l(u^l) = i$ and $l(v^l) = j$ for all l . Thus, by [lemma 7](#) and [prop. 9](#),

$$\Delta_j \Delta_i f \circ (q_{m,j} * q_{n,i}) \circ l_n = \sum_{(u^1 v^1, \dots, u^k v^k) \in B_{[i+j]}^o} \frac{\left(\prod_{l=1}^k |u^l|! \right) \left(\prod_{l=1}^k |v^l|! \right)}{i! j! k!} f^{(k)} \circ \langle \pi_0, \pi_{|u^1|} | \pi_{|v^1|}, \dots, \pi_{|u^k|} | \pi_{|v^k|} \rangle \circ l_n$$

By definition of l_n and by additivity of $f^{(k)}$ in its k last variables, the terms of this sum vanish as soon as there is $l \in [k]$ such that $|u^l| \neq |v^l|$. This is always the case when $i \neq j$, because $\sum_{l=1}^k |u^l| = i \neq j = \sum_{l=1}^k |v^l|$. So we get that

$$\Delta_j \Delta_i f \circ (q_{n,j} * q_{n,i}) \circ l_n = 0$$

for $i \neq j$. We now assume that $i = j$. Observe that

$$B_{[i+i]}^o = \{(u^1 v^1, \dots, u^k v^k) \mid (u^1, \dots, u^l) \in B_{[i]}^o \text{ and } (v^1, \dots, v^l) \in B_{[i]}^o\}$$

By [fact 1](#), for any $(u^1, \dots, u^k) \in B_{[i]}^o$, there is $\frac{i!}{|u^1|! \dots |u^k|!}$ elements $(v^1, \dots, v^k) \in B_{[i]}^o$ such that $|v^l| = |u^l|$ for all $l \in [k]$. So we can regroup terms of the sum above to get that $\Delta_i \Delta_i f \circ (q_{m,i} * q_{n,i}) \circ l_n$ is equal to

$$\sum_{(u^1, \dots, u^k) \in B_{[i]}^o} \frac{\prod_{l=1}^k |u^l|!}{k! i!} f^{(k)} \circ \langle \pi_0, \pi_{|u^1|}, \dots, \pi_{|u^k|} \rangle.$$

This quantity is equal to $\Delta_i f$, as shown in the proof of [thm. 3](#). This concludes the proof of (25). Finally, we prove that l_n is natural. Observe that

$$\pi_i \pi_j \circ T_n^2 \circ l_n = \Delta_j \Delta_i f \circ (q_{n,j} * q_{n,i}) \circ l_n.$$

By (25), this is equal to $\delta_i^j \Delta_n f \circ q_{n,i} = \pi_i \pi_j \circ l_n \circ T_n f$, so $T_n^2 f \circ l_n = l_n \circ T_n f$ and l_n is natural. \square

APPENDIX C PROOFS OF SEC. VI

Lemma. For all Taylor expansion U , $\pi_i \circ Uf = \pi_i \circ Uf \circ s_{n,i}$.

Proof: By induction on i , the case $i = 0$ is a direct consequence of the naturality of π_0 . For the inductive hypothesis, assume that $i \geq 1$. Then, let $h_i \in \mathcal{C}(S_n X, S_n X)$ and $g_i \in \mathcal{C}(S_n X, S_n^2 X)$ defined by

$$\pi_k \circ h_i = \begin{cases} \pi_{k+1} & \text{if } k \in \llbracket i, n-1 \rrbracket \\ 0 & \text{otherwise} \end{cases} \quad g_i = \langle s_{n,i}, h_i, 0, \dots \rangle$$

so that $\theta_n \circ g_i = \text{id}$. Then,

$$\pi_i \circ Uf = \pi_i \circ Uf \circ \theta_n \circ g_i = \pi_i \circ \theta_n \circ U^2 f \circ g_i$$

by naturality of θ_n . Thus,

$$\pi_i \circ Uf = \pi_i \pi_0 \circ U^2 f \circ g_i + \sum_{k=1}^i \pi_{i-k} \pi_k \circ U^2 f \circ g_i.$$

Observe now that $\pi_i \pi_0 \circ U^2 f \circ g = \pi_i \circ Uf \circ s_{n,i}$ by naturality of π_0 . Furthermore, for all $k \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} & \pi_{i-k} \pi_k \circ U^2 f \circ g_i \\ &= \pi_k \circ U \pi_{i-k} \circ U^2 f \circ g_i && \text{because } U \pi_{i-k} = S_n \pi_{i-k} \\ &= \pi_k \circ U \pi_{i-k} \circ U^2 f \circ U s_{n,i-k} \circ g_i && \text{inductive hypothesis} \\ &= \pi_k \circ U \pi_{i-k} \circ U^2 f \circ S_n s_{n,i-k} \circ g_i && U s_{n,i-k} = S_n s_{n,i-k} \\ &= \pi_k \circ U \pi_{i-k} \circ U^2 f \circ z_n \circ s_{n,i-k} && s_{n,i-k} \circ h_i = 0 \ (k \geq 1) \\ &= \pi_k \circ z_n \circ \pi_{i-k} \circ Uf \circ s_{n,i-k} && \text{naturality of } z_n \\ &= 0 && k \geq 1 \end{aligned}$$

using the fact that $s_{n,i-k} \circ h_i = 0$, since $k \geq 1$. This conclude the inductive step: $\pi_i \circ Uf = \pi_i \circ Uf \circ s_{n,i}$. \square

Proposition. The maps $\phi : U \mapsto \Delta_1^U$ and $\psi : D \mapsto T_n$ induce a bijection between differentials and order n Taylor expansions U such that

$$U \circ \langle x, u, 0, \dots \rangle = \left\langle \frac{1}{k!} f^{(k)} \circ \langle x, u, \dots, u \rangle \right\rangle_{k=0}^n \quad (26)$$

where $f^{(k)}$ is the higher order derivative defined from the derivative $\Delta_1^U = \phi(U)$.

Proof: We have seen that $\phi \circ \psi = \text{id}$. Let U be an order n Taylor expansion that satisfies (26), we prove that $\psi \circ \phi(U) = U$. Let $D = \Delta_1^U$, and define Δ_n and T_n from D as in sec. IV so that $T_n = \psi(\phi(U))$. We want to prove that $U = T_n$ so it suffices to prove that $\Delta_i^U f = \Delta_i f$ for all $f \in \mathcal{C}(X, Y)$ and $i \leq n$. We proceed by induction on i .

The case $i = 0$ holds by naturality of π_0 , and the case $i = 1$ holds by definition, since $\Delta_1 = D = \Delta_1^U$. Now, assume that $i \geq 2$. Define $h_i \in \mathcal{C}(S_i X, S_i X)$ as $h_i = \langle 0, \pi_2, \dots, \pi_i, 0 \rangle$ so

that $\theta_i \langle s_{i,1}, h_i, 0, \dots, 0 \rangle = \text{id}$. Then, by items (1) and (2) of prop. 20,

$$\begin{aligned} \Delta_i^U f &= \Delta_i^U f \circ \theta_i \circ \langle s_{i,1}, h_i, 0, \dots, 0 \rangle \\ &= \Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle + R_i \end{aligned}$$

with $R_i = \sum_{k=1}^{i-1} \Delta_k^U \Delta_{i-k}^U f \circ (q_{i,k} * q_{i,i-k}) \circ \langle s_{i,1}, h_i, 0, \dots, 0 \rangle$. Similarly, by cor. 1,

$$\Delta_i f = \Delta_i f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle + R'_i$$

with $R'_i = \sum_{k=1}^{i-1} \Delta_k \Delta_{i-k} f \circ (q_{i,k} * q_{i,i-k}) \circ \langle s_{i,1}, h_i, 0, \dots, 0 \rangle$. By induction hypothesis, $R_i = R'_i$. Furthermore, $\Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle = \Delta_i f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle$ by (26). So $\Delta_i^U f = \Delta_i f$, this concludes the inductive case. \square

Theorem. Assume that the additive structure of \mathcal{C} is cancellative. For all $n \in \mathbb{N}_\omega$ such that $n \geq 1$, there is a bijection between differentials D and order n Taylor expansions U , given by the maps $\phi : U \mapsto \Delta_1^U$ and $\psi : D \mapsto T_n$.

Proof: It suffices to prove that all order n Taylor expansion U satisfy (26) in that setting. Let U be an order n Taylor expansion. Then, (26) holds if and only if $\Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots \rangle = \Delta_i f \circ \langle \pi_0, \pi_1, 0, \dots \rangle$ for all $i \geq 1$, where Δ_i is defined from the derivative Δ_1^U as in sec. IV-A. We proceed by induction on i . The case $i = 1$ holds by definition of Δ_1 . By prop. 20,

$$\Delta_i^U f \circ \langle \pi_0, 2\pi_1, 0, \dots \rangle = 2^i \Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots \rangle$$

and, defining $g = \langle \pi_1, 0, \dots \rangle$, we have by item (2)

$$\begin{aligned} & \Delta_i^U f \circ \langle \pi_0, 2\pi_1, 0, \dots \rangle \\ &= \Delta_i^U f \circ \theta_i \circ \langle s_{i,1}, g, 0, \dots \rangle \\ &= 2\Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots \rangle + Q_i \end{aligned}$$

with $Q_i = \sum_{k=1}^{i-1} \Delta_k^U \Delta_{i-k}^U f \circ (q_{i,k} * q_{i,i-k}) \circ \langle s_{i,1}, g, 0, \dots \rangle$. So, if the additive monoid is cancellative, it follows that

$$(2^n - 2)\Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle = Q_i.$$

Similarly, $(2^n - 2)\Delta_i f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle = Q'_i$ with $Q'_i = \sum_{k=1}^{i-1} \Delta_k \Delta_{i-k} f \circ (q_{i,k} * q_{i,i-k}) \circ \langle s_{i,1}, g, 0, \dots \rangle$. By induction hypothesis, $Q_i = Q'_i$ so $\Delta_i^U f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle = \Delta_i f \circ \langle \pi_0, \pi_1, 0, \dots, 0 \rangle$ which concludes the proof. \square