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Non Null-Controllability Properties of the Grushin-Like Heat Equation on 2D-Manifolds

Roman Vanlaere *

February, 2025

Abstract

We study the internal non null-controllability properties of the heat equation on 2-dimensional almost-Riemannian manifolds with an interior singularity, and under the assumption that the closure of the control zone does not contain the whole singularity. We show that if locally, around the singularity, the sub-Riemannian metric can be written in a Grushin form, or equivalently the sub-Laplacian writes as a generalized Grushin operator, then, achieving null-controllability requires at least a minimal amount of time. As locally the manifold looks like a rectangular domain, we consequently focus ourselves on the non null-controllability properties of the generalized Grushin-like heat equation on various Euclidean domains.

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1 Introduction

Over the past decade, there has been significant attention devoted to exploring the controllability properties of evolution equations associated with degenerate elliptic operators. These are parabolic operators whose symbol can vanish. For the heat equation, this interest has been spurred by positive answers to the question of controllability of the heat equation associated to the Laplace-Beltrami operator on Riemannian manifolds ([FI96; FR71; LR95; Mil05]). A pioneer work for the sub-elliptic heat equation is very recent ([BCG14]), and shows that in this case, at least a minimum amount of time is required. As for the sub-elliptic wave equation on compact manifold, it has been very recently shown in [LS23] that it is never null-controllable. However, this question for the sub-elliptic heat in a general context still lacks of answers.

1.1 The Sub-Riemannian Setting

We set here the geometric settings of the present paper. For details on sub-Riemannian geometry, see for instance [ABB19; Jea14].

Let \mathbb{M} be an orientable two-dimensional manifold, with or without boundary, and let $\mathcal{Z} \subset \mathbb{M}$ be a connected embedded one-dimensional submanifold. We will assume throughout this work that \mathbb{M} and \mathcal{Z} satisfy the following.

H0 Let $\gamma \in \mathbb{N}^*$. \mathbb{M} is a complete almost-Riemannian manifold, with a degree of non-holonomy equal to 1 on $\mathbb{M} \setminus \mathcal{Z}$, and $\gamma+1$ on \mathcal{Z} . Moreover, \mathcal{Z} is non-characteristic (in the sense of [FPR20, Definition 2.3]), the injectivity radius from \mathcal{Z} , that we denote by $\text{inj}(\mathcal{Z})$, is bounded below by some $C > 0$, and $\partial\mathcal{Z} \subset \partial\mathbb{M}$.

Observe that due to the assumption on the injectivity radius, any neighborhood \mathcal{O} of \mathcal{Z} must satisfy $\mathcal{O} \setminus \mathcal{Z} = \mathcal{O}^- \sqcup \mathcal{O}^+$. We make the following complementary assumption to **H0**. Denote the distance to \mathcal{Z} by δ .

H0' For any $0 < R < \text{inj}(\mathcal{Z})$, letting $\mathcal{O} = \{p \in \mathbb{M}, \delta(p) < R\}$, δ is smooth in $\mathcal{O}^\pm \cup \mathcal{Z}$.

Remark 1.1. *The assumption on the injectivity radius is automatically verified whenever \mathcal{Z} is compact, which is always the case when \mathbb{M} is compact, since \mathcal{Z} is a closed embedded smooth submanifold of \mathbb{M} . The smoothness of δ is automatically verified if $\partial\mathbb{M} = \emptyset$ (note that in this case, $\partial\mathcal{Z} = \emptyset$). In the case with boundary, it is sufficient to ensure our arguments to hold under **H1-glob**, but can be relaxed under **H1-loc** (see the assumptions below and Lemma 5.3). Moreover, the assumption $\partial\mathcal{Z} \subset \partial\mathbb{M}$ is to ensure the right geometrical setting under which we can work in the case $\gamma > 1$.*

The condition on the degree of nonholonomy implies the Hörmander condition, also known as bracket-generating condition. Namely, the sub-Riemannian structure on \mathbb{M} is locally generated

by a family of smooth vector fields such that, at each point, their iterated Lie brackets span the tangent space to \mathbb{M} . In our case, we need γ iterations of Lie brackets on \mathcal{Z} , while none are needed on $\mathbb{M} \setminus \mathcal{Z}$. Given such a family \mathcal{D} of vector fields, we can consider a metric G which is defined to make \mathcal{D} an orthonormal frame on $\mathbb{M} \setminus \mathcal{Z}$. The metric G is Riemannian almost everywhere (except on \mathcal{Z}). Thanks to hypothesis **H0**, Chow-Raschevskii theorem applies, and the notion of (sub-Riemannian) distance associated to the metric G is well-defined on \mathbb{M} . We denote it by d_{sR} , and (M, d_{sR}) is a metric space. Hence, we write, for $p \in \mathbb{M}$,

$$\delta(p) = \inf\{d_{sR}(p, q), q \in \mathcal{Z}\}.$$

The assumption **H0** ensures the existence of a double-sided tubular neighborhood around \mathcal{Z} , that we denote by $\mathcal{Z}_L \subset \{p \in \mathbb{M}, \delta(p) < L\}$, for some $0 < L < \text{inj}(\mathcal{Z})$, and $\mathcal{Z}_L \simeq (-L, L) \times \mathcal{Z}$. Thanks to the smoothness assumption on δ , we actually have that $\mathcal{Z}_L = \{p \in \mathbb{M}, \delta(p) < L\}$ (see Lemma 5.3). Whenever \mathcal{Z} is compact, such a neighborhood always exists under the only assumption that \mathcal{Z} is non-characteristic (see e.g. [FPR20, Proposition 3.1]). When \mathcal{Z} is non-compact, its existence is ensured by the assumption on the injectivity radius (see e.g. [Ros22, Theorem 3.7] combined with [FPR20, Proposition 3.1]).

We now introduce ω to be an open set of \mathbb{M} that satisfies one of the following.

H1-loc $\mathbb{M} \setminus \bar{\omega}$ contains a point $p \in \mathcal{Z}$.

H1-glob $d_{sR}(\omega, \mathcal{Z}) > 0$.

We observe that there exists an open set $\mathcal{U} \subset \mathbb{M} \setminus \bar{\omega}$, that is either

- (i) a neighborhood of the point p appearing in **H1-loc**,
- (ii) a neighborhood of \mathcal{Z} if we are under **H1-glob**,

such that \mathcal{U} is diffeomorphic to $(-L, L) \times \Omega_y$, where $\Omega_y = \mathbb{S}^1, \mathbb{R}$, or a bounded interval (see Figure 1). Indeed, in the setting of **H1-glob**, it is sufficient to choose $\mathcal{U} = \mathcal{Z}_L$, with $0 < L < \min(d_{sR}(\omega, \mathcal{Z}), \text{inj}(\mathcal{Z}))$. In the setting of **H1-loc**, we can choose \mathcal{U} to be a neighborhood of such a point p , such that it is diffeomorphic to $(-L, L) \times \mathcal{W}$, for some $L > 0$, with $\mathcal{W} \subset \mathcal{Z} \setminus \omega$ a relatively compact neighborhood of p in \mathcal{Z} . In either case, since \mathcal{Z} is one-dimensional and connected, \mathcal{U} is always diffeomorphic to a set of the form $(-L, L) \times \Omega_y$. From now on, we fix \mathcal{U} to be the open set prescribed above.

In such a tubular neighborhood, the sub-Riemannian structure is always generated by smooth vector fields of the form $X = \partial_x$ and $Y = \tilde{q}(x, y)\partial_y$ (see [ABS08, Lemma 1, Theorem 1]).

We endow \mathbb{M} with a smooth non-singular measure μ , and we assume that in \mathcal{U} , the vector fields X, Y , and the measure μ , write as

H2 $X = \partial_x$ and $Y = q(x)r(y)\partial_y$, with $r > 0$ and identically one whenever $\Omega_y = \mathbb{R}$, and q satisfying

$$\partial_x^k q(0) = 0 \text{ for all } k \in \{0, \dots, \gamma - 1\}, \partial_x^\gamma q(0) > 0, \text{ and } q(x) \neq 0 \text{ for every } x \neq 0,$$

H3 $\mu = h(x)dxdy$, with $h > 0$.

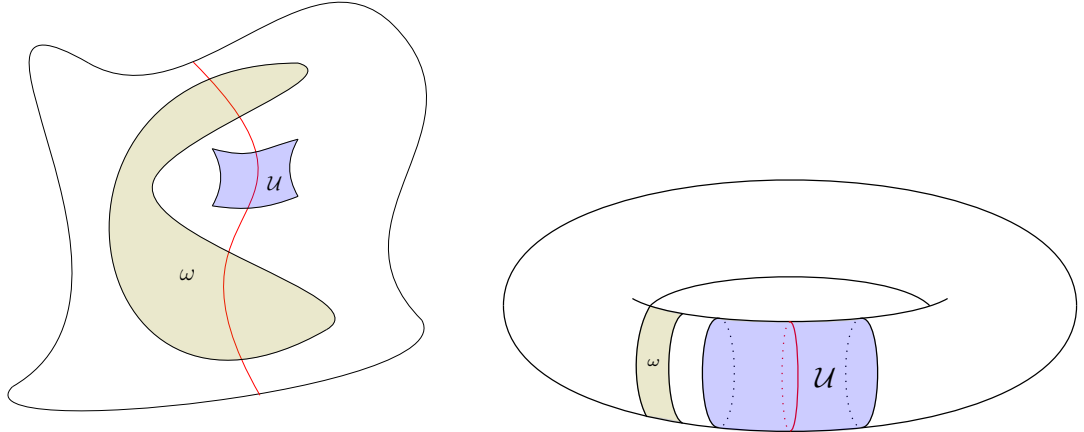
Observe that given the form of Y in **H2**, the assumptions on q and the fact that we need γ iteration of Lie brackets only at the singularity $\mathcal{Z} \simeq \{0\} \times \Omega_y$ in **H0** are equivalent.

We also emphasize that in particular, **H2** says that the sub-Riemannian structure on \mathbb{M} is generated, at least locally, by the smooth vector fields $\{\partial_x, q(x)r(y)\partial_y\}$. Moreover, it implies the assumption on the injectivity radius from $\mathcal{U} \cap \mathcal{Z}$ given in **H0**.

Remark 1.2. Under **H1-glob**, our results below will hold true if we write $\mu = h_1(x)h_2(y)dxdy$. In this case, the exact same computations as for $h_2(\cdot) = 1$ lead to the same result. For the simplicity of the presentation, we make the choice of taking μ as prescribed by **H3**.

We denote by Δ the sub-Laplacian with respect to μ . That is, for every sufficiently regular function f , in \mathcal{U} we have

$$\Delta f = \operatorname{div}_\mu(\nabla f) = \frac{1}{h(x)}\partial_x(h(x)\partial_x f) + q(x)^2\partial_y(r(y)^2\partial_y f). \quad (1)$$



(a) An abstract manifold under **H1-loc**.

(b) The Torus under **H1-glob**.

Figure 1: The singularity is in red, the control zone in green, and the tubular neighborhood in blue.

We emphasize that the choice of a single connected singularity is for simplicity of the presentation. In the case of multiple singularities, or \mathcal{Z} has multiple connected components, each satisfying **H0** to **H3**, it is sufficient to focus independently around each component.

1.2 The Control Problem and Main Result

The goal of this paper is to study the null-controllability properties of the following system. Let $T > 0$,

$$\begin{cases} \partial_t f - \Delta f &= u(t,p)\mathbf{1}_\omega(p), & (t,p) \in (0,T) \times \mathbb{M}, \\ f(t,p) &= 0, & (t,p) \in (0,T) \times \partial\mathbb{M}, \text{ if } \partial\mathbb{M} \neq \emptyset, \\ f(0,p) &= f_0(p), & p \in \mathbb{M}, \end{cases} \quad (2)$$

where $f_0 \in L^2(\mathbb{M})$, $f \in L^2((0, T) \times \mathbb{M})$ is the state, and $u \in L^2((0, T) \times \mathbb{M})$ is the control supported in ω .

Remark 1.3. From *H0*, the operator Δ with domain $D(\Delta) = \{f \in L^2(\mathbb{M}), \Delta f \in L^2(\mathbb{M})\}$ is densely defined, self-adjoint on $L^2(\mathbb{M})$, and generates a smooth semigroup of contractions $(e^{t\Delta})_{t \geq 0}$ on $L^2(\mathbb{M})$ ([Str86, p. 261]), where $L^2(\mathbb{M}) := L^2(\mathbb{M}, \mu)$ is the set of square integrable, with respect to μ , real-valued functions on \mathbb{M} . Moreover it is hypoelliptic ([Hör67, Theorem 1.1]). From [Paz12, Sec. 4.2]), system (2) is well posed in the sense that for all $f_0 \in L^2(\mathbb{M})$, there exists a unique solution $f \in C^0([0, T], L^2(\mathbb{M})) \cap L^2((0, T), D(\Delta))$ given by the Duhamel formula.

Definition 1.4 (Null-Controllability). We say that system (2) is null-controllable in time T from ω if, for every $f_0 \in L^2(\mathbb{M})$, there exists $u \in L^2((0, T) \times \mathbb{M})$ such that the associated solution f of system (2) satisfies $f(T, \cdot, \cdot) = 0$ on \mathbb{M} .

Let us introduce respectively the minimal time for null-controllability and the Agmon distance as

$$T(\omega) := \inf\{T > 0, \text{ such that the system is null-controllable in time } T \text{ from } \omega\}, \quad (3)$$

$$d_{\text{agm}} : x \in (-L, L) \mapsto \int_0^x q(s) ds. \quad (4)$$

Observe that $T(\omega) = \infty$ if and only if the system of interest is not null-controllable independently of the final time $T > 0$.

Theorem 1.5. Assume *H0*, and that *H2-H3* holds in each of the settings of *H1*. Depending on the value of $\gamma \geq 1$, we have the following statements for system (2).

(i) Assume that $\gamma = 1$, that we are in the setting of *H1-loc*, and r is identically one. Then,

$$T(\omega) \geq \frac{1}{q'(0)} \min\{d_{\text{agm}}(-L), d_{\text{agm}}(L)\}. \quad (5)$$

(ii) Assume that we are in the setting of *H1-glob*. If $\gamma > 1$, then $T(\omega) = \infty$, and if $\gamma = 1$, then

$$T(\omega) \geq \frac{1}{q'(0)} \min\{d_{\text{agm}}(-L), d_{\text{agm}}(L)\}. \quad (6)$$

1.3 State of the Art

Around the seventies, V.V. Grushin, in [Gru70; Gru71], and M.S. Baouendi, in [Bao67], introduced a class of degenerate hypoelliptic operators $-\partial_x^2 - x^{2\gamma}\partial_y^2$, with $\gamma > 0$, now commonly referred to as the classical (Baouendi-)Grushin operators. The first controllability result was given in [BCG14], on the rectangle $(-1, 1) \times (0, \pi)$ with Dirichlet boundary conditions. In this study, the control was supported on a vertical strip in $(0, \pi) \times (0, \pi)$, with its closure not intersecting the singularity $\{0\} \times (0, \pi)$. It is showed that when $\gamma < 1$, the equation is null-controllable in any time $T > 0$, that when $\gamma > 1$, null-controllability never occurs, and that when $\gamma = 1$, a minimal time is required to have null-controllability. This minimal time, for $\gamma = 1$, was latter obtained in [BMM15], with the control supported on two vertical strips, in each side of the singularity, and for which the closure does not intersect the latter. The first result for the classical Grushin equation on $(-1, 1) \times \mathbb{S}^1$, and on the rectangle, with the control supported on the complementary of a horizontal strip, was given in [Koe17]. It is proved that when $\gamma = 1$, null-controllability is never achieved. In this similar setting for the control zone, but this time with the equation posed on \mathbb{R}^2 , the same result was obtained in

[Lis22]. In [DK20], the minimal time and non null-controllability for the classical Grushin equation were thoroughly explored, considering a broader range of control support configurations.

The Grushin operator was then subsequently generalized by replacing x^γ by a sufficiently regular function q behaving like x^γ as $x \rightarrow 0$. Similar results as for the classical operator were obtained. On the rectangle, the problem of boundary null-controllability has been investigated in [BDE20]. For the problem of internal null-controllability, for a wide range of considerations for the control zone, results were obtained in [DKR23]. On the Grushin sphere, endowed with the canonical measure inherited by \mathbb{R}^3 , analogous results were obtained in [Tam22].

Finally, considering the fractional Grushin equation, positive results were obtained very recently on $\mathbb{R}^n \times \mathbb{R}^m$ or $\mathbb{R}^n \times \mathbb{T}^m$ in [JW24] using spectral inequalities, and in [LS23] by means of resolvent estimates.

Although the question of controllability properties of the generalized Grushin operator was mainly investigated in some precise settings, the question of giving a geometric interpretation of these results on manifolds remains, as for a geometric interpretation of the null-controllability results of the sub-elliptic heat equation in general.

Concerning other results of null-controllability of a class of degenerate parabolic equation, we can also cite [BC17] that study the heat equation on the Heisenberg group, or results concerning the Kolmogorov equation [Bea+15; Bea14; Koe20; LM16] among others.

1.4 Structure of the Paper

In Section 2, we outline the proofs of our main result, and those of our complementary results.

In Section 3, we collect some results concerning the spectral analysis of the operators under consideration. In particular, in Section 3.1, we provide the spectral analysis for the perturbed Laplacian $\partial_y(r(y)^2\partial_y)$ on Ω_y . In Section 3.2.1 and 3.2.2, we respectively remind the behaviour of the first eigenvalue, and exponential decay of the associated first eigenfunction, of the Fourier-Grushin operator $-\partial_x^2 + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2}|\xi|^2x^{2\gamma}$, $\xi \in \mathbb{R} \setminus \{0\}$. In Section 3.3, we provide the asymptotic behaviour of a sequence of eigenvalues of the operator $G_V = -\partial_x^2 - q(x)^2\partial_y(r(y)^2\partial_y) + V(x)$. The study of the exponential decays for the associated eigenfunctions are encapsulated within the proofs of the theorems. In Section 3.3.2, we study the behaviour of the first eigenvalue of the operator $-\partial_x^2 - q(x)^2\xi^2 + V(x)$ for $\xi \in \mathbb{C}$.

In Section 4, we prove some non null-controllability results for the generalized Grushin equation posed on Euclidean domains. When the control acts on vertical strips at non-negative distance from the singularity, for $\gamma \geq 1$, we prove a non null-controllability result in Section 4.1. For $\gamma = 1$, and the control acting on the complementary of a rectangle, we provide a negative result in Section 4.2.

Finally, in Section 5, we prove Theorem 1.5.

2 Sketch of Proofs

Let us begin by outlining the proof of Theorem 1.5. We take \mathcal{U} the largest tubular neighborhood prescribed by **H1-loc** or **H1-glob**. We define $\mathcal{U}^{-\varepsilon} := \{p \in \mathcal{U}, d_{sR}(p, \partial\mathcal{U}) > \varepsilon > 0\}$. The idea is to say that, through trace arguments, internal null-controllability in time $T > 0$ yields boundary null-controllability in time T for the system posed in $\mathcal{U}^{-\varepsilon}$. Subsequently, employing cutoff arguments, it in turn implies internal null-controllability for the system posed in \mathcal{U} , with Dirichlet boundary conditions, and the control being supported in $\mathcal{U} \setminus \mathcal{U}^{-\varepsilon}$. We can then address the latter within its coordinates representation, where the restriction of Δ to \mathcal{U} is expressed as (1). After the transformation $f \mapsto \sqrt{h}f$, we are left with addressing the null-controllability properties of the following system on $L^2(\Omega, dx dy)$, where $\Omega = \Omega_x \times \Omega_y$ is given in the sub-Riemannian setting, and the control zone satisfies one of the conditions outlined in **H1**. Let $T > 0$,

$$\begin{cases} \partial_t f - \partial_x^2 f - q(x)^2 \partial_y (r(y)^2 \partial_y f) + V(x)f &= u(t, x, y) \mathbf{1}_{\tilde{\omega}}(x, y), & (t, x, y) \in (0, T) \times \Omega, \\ f(t, x, y) &= 0, & (t, x, y) \in (0, T) \times \partial\Omega, \\ f(0, x, y) &= f_0(x, y), & (x, y) \in \Omega, \end{cases} \quad (7)$$

Consequently, Theorem 1.5(i) and 1.5(ii) follow, respectively, from Theorem 2.1 and Theorem 2.2 below. The *reduction process* is treated in Section 5.1, while Theorem 1.5(i) and 1.5(ii) are both proved in Section 5.2.

Theorem 2.1 (Control in the complementary of a rectangle). *Let $\gamma = 1$. Consider system (7) with $\Omega_x = (-L, L)$, $\Omega_y = (0, \pi)$, and $\tilde{\omega}$ the complementary of a rectangle $[-a, b] \times \bar{I}$, $a, b > 0$, where \bar{I} is a proper closed interval of Ω_y . Assume that q satisfies **H2**, r is identically one on Ω_y , and $V \in L^\infty(\Omega)$. Then,*

$$T(\omega) \geq \frac{1}{q'(0)} \min(d_{agm}(-a), d_{agm}(b)), \quad (8)$$

where $d_{agm}(x)$ is introduced in (4).

Theorem 2.2 (Control on vertical strips). *Consider system (7) with $\Omega_x = (-L, L)$, $\Omega_y = (0, \pi)$, \mathbb{S}^1 or \mathbb{R} . Assume that $\tilde{\omega} = \tilde{\omega}_x \times \Omega_y$, with $\text{dist}(\tilde{\omega}_x, 0) > 0$, q satisfies **H2**, and $V \in L^\infty(\Omega)$.*

(i) *If $\gamma = 1$, then $T(\tilde{\omega}) \geq \frac{1}{q'(0)} d_{agm}(\tilde{\omega}_x, 0)$,*

(ii) *If $\gamma > 1$, then system (7) is never null-controllable.*

Recall that when $\Omega_y = \mathbb{R}$, the function r is assumed to be identically one.

Remark 2.3. *In the case $V = 0$, Theorem 2.1 is already known from [DKR23, Theorem 1.3].*

Remark 2.4. *While Theorems 2.1 and 2.2 assume that Ω_x is symmetric with respect to zero, they remain valid when substituting $(-L, L)$ with any other interval containing 0 in its interior (see Section 6). Same remark goes for Ω_y , that can be replaced by any bounded interval instead of $(0, \pi)$.*

As customary, we will adopt the observability perspective. By linearity of our systems, and using duality arguments, thanks to a lemma due to Douglas, Dolecki and Russell (see [Cor07, Proposition 2.48]), the null-controllability in time $T > 0$ of system (7) is equivalent to the observability of its adjoint system.

Definition 2.5 (Observability). *Let $T > 0$. The adjoint system*

$$\begin{cases} \partial_t f - \partial_x^2 f - q(x)^2 \partial_y (r(y)^2 \partial_y f) + V(x)f & = 0, & (t, x, y) \in (0, T) \times \Omega, \\ f(t, x, y) & = 0, & (t, x, y) \in (0, T) \times \partial\Omega, \\ f(0, x, y) & = f_0(x, y), & (x, y) \in \Omega, \end{cases} \quad (9)$$

is observable from $\tilde{\omega}$ in time T if there exists $C > 0$ such that for every $f_0 \in L^2(\Omega)$, the associated solution satisfies

$$\int_{\Omega} |f(T, x, y)|^2 dx dy \leq C \int_0^T \int_{\tilde{\omega}} |f(t, x, y)|^2 dx dy dt. \quad (10)$$

For Theorem 2.1, we follow [DKR23]. Much of the work is already carried out in [DKR23], so we simply show that their strategy still holds in our case. Let us present the idea when $\Omega_y = \mathbb{S}^1$. By separation of variables, we can choose a sequence of solutions of the form $g_N(t, x, y) = \sum_{n \geq N} a_n v_n(x) e^{iny} e^{-\lambda_n t}$, where the sum is finite. Here, v_n is chosen to lie within the first eigenspace of the operator $-\partial_x^2 + n^2 q(x)^2 + V(x)$, associated to the first eigenvalue λ_n of the latter. Since near $x = 0$ we have that $-\partial_x^2 + n^2 q(x)^2 + V(x) \approx -\partial_x^2 + n^2 q'(0)^2 x^2 + V(x)$, in the limit $n \rightarrow +\infty$ classical arguments from complex perturbation theory will show that $\lambda_n \sim nq'(0)$, the effect of the potential V becoming more and more negligible in front of $n^2 q^2$. Choosing v_n as the spectral projection of $\tilde{v}_n(x) = n^{1/4} e^{-nq'(0)x^2/2}$, we show using an Agmon-type argument that it concentrates near zero as $v_n \approx e^{-nd_{\text{agm}}(x)}$ (which can also be shown by a standard WKB argument). Hence, for N large enough, our solutions behave like $g_N(t, x, y) \approx \sum_{n \geq N} a_n e^{-nd_{\text{agm}}(x)} e^{iny} e^{-nq'(0)t}$, or more precisely

$$g_N(t, x, y) = \sum_{n \geq N} a_n \gamma_{t,x}(n) e^{n(iy - q'(0)t - (1-\varepsilon)d_{\text{agm}}(x))},$$

where the $\gamma_{t,x}(n)$ are error terms.

We observe then that after a change of variables, our solutions resemble complex polynomials of the form $\sum_{n \geq N} a_n z^n$, up to the error terms $\gamma_{t,x}(n)$. The idea is then to show that observability implies an $L^2 - L^\infty$ inequality on these polynomials with a zero of order N at zero, which cannot hold in small times. This implication is carried out up to the possibility of estimating the error term, and is based on rather complicated complex analysis arguments that we shall not exhibit here (see [DK20; DKR23]). We refer the reader to Section 4.2 (and [DK20; DKR23]) for more details. The complex spectral analysis, and the proof of Theorem 2.1, are provided respectively in Section 3.3.2 and 4.2.

For Theorem 2.2 in the case Ω_y bounded, we proceed by separation of variables. We choose a sequence of eigenfunctions $g_n(x, y) = v_n(x) \phi_n(y)$ of our generalized Grushin operator, where ϕ_n is the n -th eigenfunction of $-\partial_y (r(y)^2 \partial_y)$ associated to ν_n , and v_n an eigenfunction of $-\partial_x^2 + \nu_n q(x)^2 + V(x)$ associated to λ_n . We show that the sequence of solutions $g_n(t, x, y) = v_n(x) \phi_n(y) e^{-\lambda_n t}$ disproves the observability in any time $T > 0$.

The idea is to see $-\partial_x^2 + \nu_n q(x)^2 + V(x)$ as a perturbation of the classical Grushin operator $-\partial_x^2 + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} \nu_n x^{2\gamma}$. Since the classical eigenfunctions concentrate near the singularity, where the generalized operator behaves like the classical one, we expect λ_n to behave asymptotically the same way as the sequence of first eigenvalues of the classical operator in Fourier. The proof culminates with the application of an Agmon-type arguments, showing that the eigenfunctions v_n concentrate outside the control zone significantly faster than they decay across the entire domain. This behaviour

occurs due to the degeneracy of q . We show that V exerts minimal influence on the behavior of the eigenfunctions, as with increasing n , the term $\frac{V(x)}{n^2}$ becomes negligible in front of $q(x)^2$. The Fourier decomposition, with the estimates on ν_n , is done in Section 3.1.

The spectral analysis for the classical operator is encapsulated within the analysis of the operator $-\partial_x^2 + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} |\xi|^2 q(x)^2$ in Section 3.2, and for the generalized operator, see Section 3.3. The proof is provided along with the Agmon-type argument in Section 4.1. When $\Omega_y = \mathbb{R}$, we treat the problem in Fourier. We choose as solutions the sequence $g_n(t, x, y) = v(x, \xi) e^{-\lambda(\xi)t} \psi_n$, where $v(\cdot, \xi)$ is an eigenfunction of $-\partial_x^2 + |\xi|^2 q(x)^2 + V(x)$, associated to $\lambda(\xi)$, and ψ_n is a sequence of cutoffs that localize in high frequencies. The strategy is then the same as for the case Ω_y bounded.

3 Spectral Analysis

We want to extract a sequence of eigenvalues of the operator $G_V = -\partial_x^2 - q(x)^2 \partial_y(r(y)^2 \partial_y) + V(x)$ on $L^2(\Omega)$, with $\Omega_y = (0, \pi)$ or \mathbb{S}^1 , for which we can get sufficiently precise estimates, and for which we can estimate the decay of some associated eigenfunctions in the control zone. We are therefore interested by the eigenvalue problem

$$\begin{cases} -\partial_x^2 f - q(x)^2 \partial_y(r(y)^2 \partial_y f) + V(x)f & = \lambda f, & (x, y) \in \Omega, \\ f(x, y) & = 0, & (x, y) \in \partial\Omega. \end{cases} \quad (11)$$

3.1 Fourier Decomposition

We consider the Fourier decomposition of $f \in L^2(\Omega)$, with respect to $y \in \Omega_y$. For almost every $x \in \Omega_x$, we can write f as

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) \phi_n(y), \quad (12)$$

$$f_n(x) := \int_{\Omega_y} f(x, y) \phi_n(y) dy, \text{ for every } n \in \mathbb{N}^*, \quad (13)$$

where ϕ_n is the normalized eigenfunction of $-\partial_y(r(y)^2 \partial_y)$ in $L^2(\Omega_y)$, with Dirichlet boundary conditions or periodicity conditions depending on the form of Ω_y , associated to the eigenvalue ν_n . It is well-known that the set $\{\phi_n, n \in \mathbb{N}^*\}$ forms an orthonormal basis of $L^2(\Omega_y)$.

We need to compute the eigenvalues of $-\partial_y(r(y)^2 \partial_y)$ for the realization of the latter in $L^2(\Omega_y)$. That is, we are looking at the eigenvalue problem

$$\begin{cases} -\partial_y(r(y)^2 \partial_y \phi) & = \nu \phi, & y \in \Omega_y, \\ \phi(y) & = 0, & y \in \partial\Omega_y, \text{ if } \partial\Omega_y \neq \emptyset. \end{cases} \quad (14)$$

Using the well-known Courant-Hilbert transformation (see for instance [CH89, section V.3.3]), we set

$$\phi = \frac{\Phi}{\sqrt{r}}. \quad (15)$$

Hence, (14) becomes

$$\begin{cases} -r^2 \partial_y^2 \Phi - r \partial_y r \partial_y \Phi + \left[\frac{r \partial_y^2 r}{2} + \frac{(\partial_y r)^2}{4} \right] \Phi & = \nu \Phi, & y \in \Omega_y, \\ \Phi(y) & = 0, & y \in \partial\Omega_y, \text{ if } \partial\Omega_y \neq \emptyset. \end{cases} \quad (16)$$

Now, setting

$$z = \int_0^y \frac{1}{r(s)} ds, \quad \int_{\Omega_y} \frac{1}{r(s)} ds = \pi,$$

with the normalization done without loss of generality, and for simplicity of the presentation, (16) becomes

$$\begin{cases} -\partial_z^2 \tilde{\Phi} + \rho(z) \tilde{\Phi} & = \nu \tilde{\Phi}, & z \in \Omega_y, \\ \tilde{\Phi}(z) & = 0, & z \in \partial\Omega_y, \text{ if } \partial\Omega_y \neq \emptyset, \end{cases} \quad (17)$$

with $\tilde{\Phi}(z) = \Phi(y)$, and $\rho(z) = \frac{\partial_{zz} \tilde{r}}{\tilde{r}}(z)$ with $\tilde{r}(z) = r(y)$.

Remark 3.1. We choose to normalize the value $L(\mathcal{Z}) := \int_{\Omega_y} \frac{1}{r(s)} ds$ for simplicity of the presentation, avoiding constants in the computations. This choice has no impact on the result of Theorem 2.2 (see Remark 4.2).

The eigenvalue problem (17) is a well-known problem in the literature. It is discussed for example in [Pos87] in the case $(0, 1)$ with Dirichlet boundary conditions, and in [Eas73] when we are on any interval of the form $[0, a]$, with periodic boundary conditions. Set $C(\rho) := \int_{\Omega_y} \rho(z) dz$. In the first case, since ρ is square integrable, [Pos87] ensures that the eigenvalues ν_n form a sequence of reals of the form

$$\nu_n = n^2 + \frac{C(\rho)}{\pi^2} + l^2(n), \quad (18)$$

with $l^2(n)$ a square-summable sequence of reals. In the second case, [Eas73, Theorem 4.2.3] ensures that asymptotically, ν_{2n+1} and ν_{2n+2} both satisfy, setting $a = 2\pi$,

$$\sqrt{\nu} = n + 1 + \frac{C(\rho)}{4\pi(n+1)} + o(1/n). \quad (19)$$

It follows that as $n \rightarrow +\infty$, we have

$$\nu_n = n^2 + o(n^2). \quad (20)$$

Since the asymptotic behaviour in both case is the same, we shall keep the same notation ν_n .

3.2 Preliminaries on the Classical Grushin Operator

We introduce, for every $\xi \in \mathbb{R} \setminus \{0\}$, and for every $\gamma \geq 1$, the operator $G_{0,\xi}$ on $L^2(\Omega_x)$, defined by

$$\begin{aligned} D(G_{0,\xi}) &:= H^2 \cap H_0^1(\Omega_x), \\ G_{0,\xi} u &:= -u'' + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} |\xi|^2 |x|^{2\gamma} u. \end{aligned} \quad (21)$$

$G_{0,\xi}$ has compact resolvent, is self-adjoint, positive definite and has discrete spectrum [BS91]. Hence, its first eigenvalue is given by the Rayleigh formula

$$\mu_\xi = \min\{\mathcal{Q}_{\xi,L}(u, u); u \in D(G_{0,\xi}), \|u\| = 1\}, \quad (22)$$

with

$$\mathcal{Q}_{\xi,L}(u, v) = \int_{-L}^L u'(x)v'(x) + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} |\xi|^2 |x|^{2\gamma} u(x)v(x) dx. \quad (23)$$

3.2.1 Dissipation Speed

The following Proposition is already known [BCG14, Proposition 4], but we propose a slightly different proof with a more precise upper bound. This gain of precision will not be of use for our proofs, but we provide it for its own interest. First, we need the following Lemma.

Lemma 3.2. [PRS18, Lemma 3.5.] *Let χ be a real-valued Lipschitz function with compact support in \mathbb{R} . Let $u \in H^1(\mathbb{R})$. Then, we have*

$$\mathcal{Q}_{1,\infty}(\chi u, \chi u) = \mathcal{Q}_{1,\infty}(u, \chi^2 u) + \langle u, |\nabla \chi|^2 u \rangle. \quad (24)$$

We can now estimate μ_ξ .

Proposition 3.3. *There exists $C > 0$ such that*

$$\mu_\xi \geq C |\xi|^{\frac{2}{1+\gamma}}, \text{ for all } |\xi| > 0. \quad (25)$$

Moreover, for every $\varepsilon > 0$ sufficiently small, there exists $R > 0$, such that for all $|\xi| \geq R$, we have

$$\mu_\xi \leq (1 + \varepsilon)(C + 1) |\xi|^{\frac{2}{1+\gamma}}, \quad (26)$$

where the constant C coincides with the one appearing in the lower bound.

Proof. We start with the lower bound. Let $\tau_\xi := |\xi|^{1/(1+\gamma)}$. Making the change of variable $s = \tau_\xi x$, and setting $v(s) = u(\tau_\xi^{-1}y)\tau_\xi^{-1/2}$, (22) becomes

$$\mu_\xi = \tau_\xi^2 \min\{\mathcal{Q}_{1,\tau_\xi L}(v, v), v \in H^2 \cap H_0^1(-\tau_\xi L, \tau_\xi L), \|v\| = 1\}. \quad (27)$$

Thus, we have

$$\mu_\xi \geq \tau_\xi^2 \cdot \bar{\mu}, \quad (28)$$

with $\bar{\mu} := \min\{\mathcal{Q}_{1,\infty}(v, v), v \in H^2(\mathbb{R}), |x|^\gamma v \in L^2(\mathbb{R}), \|v\| = 1\} > 0$.

Now we treat the upper bound. Let $0 < \delta < L$. Set $\chi_{\xi,1}, \chi_{\xi,2}$ to be two smooth functions such that

- $0 \leq \chi_{\xi,i} \leq 1$, for $i = 1, 2$,
- $\chi_{\xi,1} = 1$ on $[-\delta\tau_\xi, \delta\tau_\xi]$ and $\chi_{\xi,1} = 0$ on $\mathbb{R} \setminus [-\tau_\xi, \tau_\xi]$,
- $\chi_{\xi,2} = 1$ on $\mathbb{R} \setminus [-\tau_\xi, \tau_\xi]$ and $\chi_{\xi,2} = 0$ on $[-\delta\tau_\xi, \delta\tau_\xi]$,
- $\chi_{\xi,1}^2 + \chi_{\xi,2}^2 = 1$, for all $|\xi| > 0$,

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- $\sup_{\mathbb{R}} |\chi'_{\xi,i}| \leq \frac{1+\delta}{(1-\delta)\tau_{\xi}}$, for $i = 1, 2$.

Let $v \in H^1(\mathbb{R}) \cap L^2(|s|^{2\gamma} ds)$, $\|v\|_{L^2(\mathbb{R})} = 1$, be the minimizer of $\mathcal{Q}_{1,\infty}(v, v)$. From Lemma 3.2, we derive the following

$$\mathcal{Q}_{1,\infty}(v, v) = \sum_{i=1}^2 \mathcal{Q}_{1,\infty}(\chi_{\xi,i}v, \chi_{\xi,i}v) - \sum_{i=1}^2 \int_{\mathbb{R}} |\nabla \chi_{\xi,i}|^2 |v|^2 \geq \mathcal{Q}_{1,\infty}(\chi_{\xi,1}v, \chi_{\xi,1}v) - c(\xi)\|v\|_{L^2(\mathbb{R})}, \quad (29)$$

where $c(\xi) := \left(\frac{1+\delta}{(1-\delta)\tau_{\xi}}\right)^2$.

Henceforth, by definition of μ_{ξ} , with the minimum taken on all non-zero elements, we have that

$$\begin{aligned} \mu_{\xi} &\leq \tau_{\xi}^2 \frac{\mathcal{Q}_{1,\tau_{\xi}L}(\chi_{\xi,1}v, \chi_{\xi,1}v)}{\|\chi_{\xi,1}v\|_{L^2(-\tau_{\xi}L, \tau_{\xi}L)}} \\ &= \tau_{\xi}^2 \frac{\mathcal{Q}_{1,\infty}(\chi_{\xi,1}v, \chi_{\xi,1}v)}{\|\chi_{\xi,1}v\|_{L^2(\mathbb{R})}} \\ &\leq \frac{\tau_{\xi}^2}{\|\chi_{\xi,1}v\|_{L^2(\mathbb{R})}} [\mathcal{Q}_{1,\infty}(v, v) + c(\xi)\|v\|_{L^2(\mathbb{R})}] \\ &= \frac{\tau_{\xi}^2}{\|\chi_{\xi,1}v\|_{L^2(\mathbb{R})}} \left(\bar{\mu} + \frac{(1+\delta)^2}{(1-\delta)^2} \right), \end{aligned}$$

which proves the upper bound since $\|\chi_{\xi,1}v\|_{L^2(\mathbb{R})} \rightarrow 1$ as $|\xi|$ tends to infinity, and δ can be chosen arbitrary small. \square

In the case $\gamma = 1$, we can actually get very much more precise estimates as we know exactly the first eigenfunction of the harmonic oscillator on \mathbb{R} . This is the idea of [BCG14, Lemma 4].

Proposition 3.4. *Let $\gamma = 1$. Then,*

$$\mu_{\xi} \sim q'(0)|\xi|, \text{ as } \xi \rightarrow +\infty. \quad (30)$$

3.2.2 L^2 and Pointwise Exponential Decay for the Classical Grushin Operator

In this section, we recall the exponential decay for the first eigenfunctions of $G_{0,\xi}$, which follows from the above estimates on the first eigenvalues. The propositions in this section are already proved in [BCG14] for $n \in \mathbb{N}^*$, $\Omega_x = (-1, 1)$, and the operator $-\partial_x^2 + n^2\pi^2x^{2\gamma}$, but holds in our case. So we state them, but omit their proofs. We first have the following Lemma, which is proved as [BCG14, Lemma 2].

Lemma 3.5. *There exists a unique non-negative function $v_{\xi} \in L^2(\Omega_x)$ such that $\|v_{\xi}\| = 1$ and solving the problem*

$$\begin{cases} -\partial_x^2 v_{\xi} + \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} |\xi|^2 |x|^{2\gamma} v_{\xi} &= \mu_{\xi} v_{\xi}, & \text{if } x \in \Omega_x, \\ v_{\xi}(\pm L) &= 0, & \text{if } \Omega_x = (-L, L), \\ \lim_{|x| \rightarrow +\infty} v_{\xi}(x) &= 0, & \text{if } \Omega_x = \mathbb{R}, \end{cases} \quad (31)$$

Moreover, v_{ξ} is even.

We now have the following asymptotic $L^2(\Omega_x)$ exponential decay of the v_ξ 's, as $|\xi| \rightarrow +\infty$, outside any neighborhood of 0 in $(-L, L)$. It is proved in [BCG14, Lemma 3].

Proposition 3.6. *For every $|\xi| > 0$ large enough, set*

$$x_\xi = \left(\frac{(\gamma!)^2 \mu_\xi}{q^{(\gamma)}(0)^2 |\xi|^2} \right)^{1/2\gamma}. \quad (32)$$

Then, there exists $R > 0$, such that for every $|\xi| > R$, there exists a function W_ξ of the form

$$W_\xi(x) = A_\xi e^{-B_\xi x^{\gamma+1}}, \quad (33)$$

with, for some $C > 0$,

$$A_\xi = 2 \frac{\sqrt{x_\xi} \mu_\xi}{(\gamma+1) B_\xi x_\xi^\gamma} e^{B_\xi x_\xi^{\gamma+1}} \text{ and } B_\xi \sim C |\xi|, \text{ as } |\xi| \rightarrow +\infty, \quad (34)$$

such that for every $x \geq x_\xi$,

$$v_\xi(x) \leq W_\xi(x). \quad (35)$$

Moreover, $W_\xi \in C^2(\{x \geq x_\xi\}, \mathbb{R})$ is solution of

$$\begin{cases} -\partial_x^2 W_\xi + \left(\frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} |\xi|^2 |x|^{2\gamma} - \mu_\xi \right) W_\xi & \geq 0, & \text{if } x > x_\xi, \\ W_\xi(L) & \geq 0, \\ \partial_x W_\xi(x_\xi) & < -\sqrt{x_\xi} \mu_\xi. \end{cases} \quad (36)$$

Remark 3.7. *Since v_ξ is even, we can extend W_ξ on $\Omega_x \setminus (-x_\xi, x_\xi)$ by $W_\xi(-x) = W_\xi(x)$, and for every $x \in \Omega_x \setminus (-x_\xi, x_\xi)$, the proposition still holds.*

Remark 3.8. *Due to the upper bound (26) on μ_ξ , we have that $x_\xi \rightarrow 0$. Moreover, due to the behaviour of μ_ξ , and B_ξ , easy computations show that in the limit at infinity for $|\xi|$, we have for some $C_1, C_2, C_3 > 0$,*

$$\begin{aligned} x_\xi &\sim C_1 |\xi|^{-1/(\gamma+1)} \\ A_\xi &\sim C_2 |\xi|^{\frac{1}{2(\gamma+1)}} \\ B_\xi x_\xi^{\gamma+1} &\sim C_3, \text{ since } B_\xi \sim C_3 |\xi|. \end{aligned} \quad (37)$$

3.3 Dissipation Speed for the Operator G_V in Fourier

We split this section into two parts. Its first part consists in the real spectral analysis of $G_{V,\xi}$, introduced below, which will be of use when the control zone stays at non-negative distance from the singularity. The second part consists in the complex spectral analysis, which will be of use for Theorem 2.1.

We introduce, for every $\xi > 0$, and for every $\gamma \geq 1$, the operator $G_{V,\xi}$ on $L^2(\Omega_x)$, defined by

$$\begin{aligned} D(G_{V,\xi}) &:= H^2 \cap H_0^1(\Omega_x), \\ G_{V,\xi} u &:= -u'' + |\xi|^2 q(x)^2 u + V(x)u, \end{aligned} \quad (38)$$

with q and V satisfying the appropriate assumptions of Theorem 2.1 and 2.2.

We stress that $G_{V,\xi}$ has compact resolvent, and admits an increasing sequence of eigenvalues $\lambda_{k,\xi} \xrightarrow[k \rightarrow +\infty]{} +\infty$ (see [BS91]).

3.3.1 Real Spectral Analysis for $\gamma \geq 1$

Let us recall the following well-known Lemma for self-adjoint operators.

Lemma 3.9. *Let G be a self-adjoint operator with domain $D(G)$, and $\lambda \in \mathbb{R}$. Then*

$$\text{dist}(\lambda, \sigma(G)) \leq \frac{\|(G - \lambda)u\|}{\|u\|}, \quad (39)$$

for every $u \in D(G)$.

We can now investigate the behaviour of some eigenvalues λ_ξ of the operator of interest, that will be shown to behave asymptotically like the μ_ξ 's introduced in Section 3.2.

Proposition 3.10. *For every $|\xi| > 0$ large enough, there exists a constant $C > 0$ and an eigenvalue λ_ξ of $G_{V,\xi}$ that satisfies*

$$|\lambda_\xi - \mu_\xi| \leq C|\xi|^{\frac{1}{\gamma+1}}. \quad (40)$$

Namely, for $|\xi|$ large enough, there exists $C_2 > C_1 > 0$ such that

$$C_1|\xi|^{2/(\gamma+1)} \leq \lambda_\xi \leq C_2|\xi|^{2/(\gamma+1)}. \quad (41)$$

Proof. In the present proof, we shall denote by C any non-negative constant that does not depend on ξ . We denote by v_ξ the first eigenfunction, normalized in L^2 -norm, of $G_{0,\xi}$. We have, applying $G_{V,\xi}$ to v_ξ ,

$$G_{V,\xi}v_\xi = \mu_\xi v_\xi + Qv_\xi + Vv_\xi,$$

with

$$Qv_\xi = \left(q(x)^2|\xi|^2 - \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2}|\xi|^2 x^{2\gamma} \right) v_\xi.$$

We want to estimate $(G_{V,\xi} - \mu_\xi)v_\xi$ in L^2 -norm, and the Proposition will follow by Lemma 3.9. Let $\varepsilon > 0$ sufficiently small. First, we split the integral as

$$\begin{aligned} \|Qv_\xi\|^2 &= \int_{\Omega_x} |Qv_\xi|^2 dx \\ &= \int_{\Omega_x \setminus (-\varepsilon, \varepsilon)} |Qv_\xi|^2 dx + \int_{(-\varepsilon, \varepsilon)} |Qv_\xi|^2 dx. \end{aligned}$$

We remark that by assumption H2 on q near the singularity, all the derivatives of q^2 up to $2\gamma - 1$ are zero at $x = 0$. Hence, near zero,

$$q(x)^2 = \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} x^{2\gamma} + O(|x|^{2\gamma+1}).$$

Therefore, by normalization and parity of the v_ξ 's, using the supersolutions of Proposition 3.6, and since $x_\xi \rightarrow 0$, for $|\xi|$ large enough we have

$$\begin{aligned} \int_{(-\varepsilon, \varepsilon)} |Qv_\xi|^2 dx &\leq C|\xi|^4 \int_{(-\varepsilon, \varepsilon)} |x|^{4\gamma+2} |v_\xi|^2 dx \\ &= C|\xi|^4 \left(\int_0^{x_\xi} |x|^{4\gamma+2} |v_\xi|^2 dx + \int_{x_\xi}^\varepsilon |x|^{4\gamma+2} |v_\xi|^2 dx \right) \\ &\leq C|\xi|^4 \left(x_\xi^{4\gamma+2} + \int_{x_\xi}^\varepsilon |x|^{4\gamma+2} W_\xi(x)^2 dx \right). \end{aligned}$$

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Using the definition of W_ξ , and the change of variable $z = x/x_\xi$, we get

$$\begin{aligned} \int_{x_\xi}^\varepsilon |x|^{4\gamma+2} W_\xi(x)^2 dx &= \int_{x_\xi}^\varepsilon |x|^{4\gamma+2} A_\xi^2 e^{-2B_\xi x^{\gamma+1}} dx \\ &= \int_1^{\varepsilon/x_\xi} x_\xi^{4\gamma+2} |z|^{4\gamma+2} x_\xi A_\xi^2 e^{-2B_\xi x_\xi^{\gamma+1} z^{\gamma+1}} dz \\ &\leq x_\xi^{4\gamma+2} \int_1^{+\infty} |z|^{4\gamma+2} x_\xi A_\xi^2 e^{-2B_\xi x_\xi^{\gamma+1} z^{\gamma+1}} dz. \end{aligned}$$

Now, thanks to Remark 3.8, for $|\xi|$ large enough,

$$C \leq B_\xi x_\xi^{\gamma+1} \text{ and } x_\xi A_\xi^2 \leq C.$$

Hence,

$$\int_{x_\xi}^\varepsilon |x|^{4\gamma+2} A_\xi^2 e^{-2B_\xi x^{\gamma+1}} dx \leq C x_\xi^{4\gamma+2} \int_1^{+\infty} |z|^{4\gamma+2} e^{-Cz^{\gamma+1}} dz.$$

with the integral on the right-hand side being finite. We therefore have

$$\int_{(-\varepsilon, \varepsilon)} |Qv_\xi|^2 dx \leq C |\xi|^4 x_\xi^{4\gamma+2}. \quad (42)$$

On the other hand,

$$\begin{aligned} \int_{\Omega_x \setminus (-\varepsilon, \varepsilon)} |Qv_\xi|^2 dx &\leq C |\xi|^4 \sup_{\Omega_x} \left| q(x)^2 - \frac{q^{(\gamma)}(0)^2}{(\gamma!)^2} x^{2\gamma} \right|^2 \int_{\Omega_x \setminus (-\varepsilon, \varepsilon)} W_\xi(x)^2 dx \\ &= C |\xi|^4 \int_\varepsilon^1 A_\xi^2 e^{-2B_\xi x^{\gamma+1}} dx \\ &\leq C |\xi|^4 A_\xi^2 e^{-2B_\xi \varepsilon^{\gamma+1}}. \end{aligned}$$

Obviously, by Remark 3.8, the right-hand term decays exponentially fast as $|\xi| \rightarrow +\infty$. Therefore, there exists a constant $C > 0$ such that

$$\|Qv_\xi\| \leq C \xi^2 x_\xi^{2\gamma+1}. \quad (43)$$

Using once again Remark 3.8, we have that

$$\xi^2 x_\xi^{2\gamma+1} \leq C \xi^{2-\frac{2\gamma+1}{\gamma+1}} = C \xi^{\frac{1}{\gamma+1}}.$$

Finally, we obviously have

$$\|Vv_\xi\| \leq \sup_{\Omega_x} |V(x)| < \infty,$$

and (40) follows. Coupling (40) with Proposition 3.3, we directly obtain (41). \square

3.3.2 Complex Spectral Analysis for $\gamma = 1$

In this subsection, we show that in the case $\gamma = 1$, we can obtain more precise estimates on λ_ξ . The results here directly follow from the spectral analysis provided in [DKR23], in the case $V = 0$. As a matter of fact, we show that the spectral analysis from [DKR23] still holds for our operator.

We first need to reintroduce some objects from [DKR23], for which we keep as much as possible the same notations. Set $I := (-L, L)$. Denote by $\mathbb{1}_I$ the operator that maps $v \in L^2(I)$ to its extension by zero on \mathbb{R} , and by $\mathbb{1}_I^*$ its adjoint, which maps $v \in L^2(\mathbb{R})$ to its restriction on I . Set

$$\Sigma_{\theta_0} := \{\nu \in \mathbb{C}, |\nu| \geq 1, \arg(\nu) \leq \theta_0\}, \text{ for some } \theta_0 \in [0, \pi/2). \quad (44)$$

Let $R > 0$ and $\varepsilon > 0$. For $\nu \in \Sigma_{\theta_0}$, set

$$\tilde{\mathcal{Z}}_\nu = \mathcal{Z}_{q'(0)\nu, R, \varepsilon} := \{z \in \mathbb{C}, |z| \leq Rq'(0)\nu, \text{dist}(z, \sigma(H_{q'(0)\nu})) \geq \varepsilon q'(0)|\nu|\}. \quad (45)$$

We shall write $G_{V, \nu}$, $\nu \in \mathbb{C}$, to be $G_{V, \xi}$ with ξ complex, to keep up with the notations of [DKR23], and G_ν to be $G_{V, \nu}$ with $V = 0$, not to be mistaken with the classical Grushin operator. Also, we denote, for $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$, the non-self-adjoint harmonic oscillator $H_\beta = -\partial_x^2 + \beta^2 x^2$ on \mathbb{R} .

Proposition 3.11. *There exists $\nu_0 \geq 1$ such that for every $\nu \in \Sigma_{\theta_0}$, $|\nu| \geq \nu_0$ and $z \in \tilde{\mathcal{Z}}_\nu$, we have $z \in \rho(G_{V, \nu})$, and*

$$\sup_{z \in \tilde{\mathcal{Z}}} \left\| (G_{V, \nu} - z)^{-1} - \mathbb{1}_I^*(H_{q'(0)\nu} - z)^{-1} \mathbb{1}_I \right\| = \underset{\substack{|\nu| \rightarrow +\infty \\ \nu \in \Sigma_{\theta_0}}}{o} \left(\frac{1}{|\nu|} \right). \quad (46)$$

Proof. From [DKR23, Proposition 4.1], we know that such a z introduced in the proposition is in fact in $\rho(G_\nu)$. For such a z , we write

$$G_{V, \nu} - z = G_\nu + V - z = [1 + V(G_\nu - z)^{-1}](G_\nu - z),$$

where the potential V can be understood as a bounded linear operator on $L^2(I)$. Since we know from [DKR23, Proposition 4.1] that

$$\|(G_\nu - z)^{-1}\| = \underset{\substack{|\nu| \rightarrow +\infty \\ \nu \in \Sigma_{\theta_0}}}{o} \left(\frac{1}{|\nu|} \right), \quad (47)$$

it follows that $1 + V(G_\nu - z)^{-1}$ is an invertible linear operator on $L^2(I)$ for ν large enough, and so is $G_{V, \nu} - z$. Moreover,

$$\|[1 + V(G_\nu - z)^{-1}]^{-1}\| = \frac{1}{1 - \|V(G_\nu - z)^{-1}\|} \rightarrow 1,$$

as $|\nu| \rightarrow +\infty$ by (47). Hence, since

$$(G_{V, \nu} - z)^{-1} = (G_\nu - z)^{-1}[1 + V(G_\nu - z)^{-1}]^{-1},$$

it follows that we also have

$$\|(G_{V, \nu} - z)^{-1}\| = \underset{\substack{|\nu| \rightarrow +\infty \\ \nu \in \Sigma_{\theta_0}}}{o} \left(\frac{1}{|\nu|} \right). \quad (48)$$

By triangular inequality, since we already know from [DKR23, Proposition 4.1] that

$$\sup_{z \in \tilde{\mathcal{Z}}} \left| |(G_\nu - z)^{-1} - \mathbb{1}_I^*(H_{q'(0)\nu} - z)^{-1} \mathbb{1}_I| \right| = \underset{\substack{|\nu| \rightarrow +\infty \\ \nu \in \Sigma_{\theta_0}}}{o} \left(\frac{1}{|\nu|} \right),$$

it is sufficient to check that

$$\left| |(G_{V,\nu} - z)^{-1} - (G_\nu - z)^{-1} \mathbb{1}_I| \right| = \underset{\substack{|\nu| \rightarrow +\infty \\ \nu \in \Sigma_{\theta_0}}}{o} \left(\frac{1}{|\nu|} \right),$$

but this is a direct consequence of (47) and (48), and the proposition follows. \square

Now, using basic tools from perturbation theory (see for instance [Kat95]), or following the exact same proofs of [DKR23, Proposition 3.6, Proposition 4.2], we get the following corollary.

Corollary 3.12. *In the limit $|\xi| \rightarrow +\infty$, $\lambda_\xi = |\xi|q'(0) + o(|\xi|)$.*

4 The Grushin Equation on $\Omega_x \times \Omega_y$

In this section, we prove the non null-controllability of our generalized Grushin operators for various considerations of the control zone in some Euclidean domains.

4.1 Proof of Theorem 2.2: Control on Vertical Strips, $\Omega_x = (-L, L)$, $\Omega_y = \mathbb{S}^1, (0, \pi)$ or \mathbb{R}

Assume that $\Omega_y = \mathbb{S}^1$ or $(0, \pi)$. In this case, we consider a sequence of solutions of the form

$$g_n(t, x, y) = e^{-\lambda_n t} v_n(x) \phi_n(y),$$

for n large, where ϕ_n is an eigenfunction of $\partial_y(r(y)^2 \partial_y)$ on Ω_y , associated to ν_n , introduced in Section 3.1, and v_n a normalized eigenfunction of $G_n := G_{V, \sqrt{\nu_n}}$, associated to λ_n that satisfies the estimate of Proposition 3.10 when $\gamma \geq 1$, and Corollary 3.12 when $\gamma = 1$. We abuse the notation here by denoting λ_n and ν_n instead of λ_{ν_n} and ν_{ν_n} . We write the control zone as $\omega := \omega_x \times \Omega_y$, with $\overline{\omega_x} \cap \{x = 0\} = \emptyset$.

The proof of Theorem 2.2 follows directly from the following proposition for which we follow the ideas of Agmon estimates (see e.g. [Agm82; Hel88]).

Proposition 4.1. *For n large enough, there exists $C_1, C_2 > 0$ such that*

$$\int_{\omega_x} v_n(x)^2 dx \leq C_1 e^{-n C_2}. \quad (49)$$

Proof. We drop every notation concerning the variable x to simplify the reading. Let $v_n = \theta_n e^{-n\psi}$ for some sufficiently regular function ψ . Then, we have

$$\begin{aligned} v_n' &= \theta_n' e^{-n\psi} - n\psi' \theta_n e^{-n\psi}, \\ v_n'' &= \theta_n'' e^{-n\psi} - 2n\theta_n' \psi' e^{-n\psi} - n\psi'' \theta_n e^{-n\psi} + n^2 \psi'^2 \theta_n e^{-n\psi}. \end{aligned}$$

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Since $(G_n - \lambda_n)v_n = 0$, we get that θ_n must satisfy

$$-\theta_n'' + 2n\theta_n'\psi' + (n\psi'' - n^2\psi'^2 + \nu_n q^2 + V - \lambda_n)\theta_n = 0. \quad (50)$$

Integrating by parts, we first remark that

$$\int_{\Omega_x} \theta_n' \psi' \theta_n = - \int_{\Omega_x} \theta_n (\psi'' \theta_n + \psi' \theta_n'),$$

which implies that

$$\int_{\Omega_x} \theta_n' \psi' \theta_n = -\frac{1}{2} \int_{\Omega_x} \psi'' \theta_n^2.$$

Therefore, multiplying (50) by θ_n and integrating by parts we get

$$\int_{\Omega_x} \theta_n'^2 + (\nu_n q^2 + V - n^2\psi'^2 - \lambda_n)\theta_n^2 = 0,$$

which implies, dividing by n^2 ,

$$\frac{1}{n^2} \int_{\Omega_x} \theta_n'^2 + \int_{\Omega_x} \left(\frac{\nu_n}{n^2} q^2 - \psi'^2 + \frac{V}{n^2} - \frac{\lambda_n}{n^2} \right) \theta_n^2 = 0,$$

and

$$\int_{\Omega_x} \left(\frac{\nu_n}{n^2} q^2 - \psi'^2 + \frac{V}{n^2} - \frac{\lambda_n}{n^2} \right) \theta_n^2 \leq 0.$$

From the estimates on ν_n from Section 3.1, we have in the limit $n \rightarrow +\infty$,

$$\int_{\Omega_x} \left(q^2 + \frac{o(n^2)}{n^2} q^2 - \psi'^2 + \frac{V}{n^2} - \frac{\lambda_n}{n^2} \right) \theta_n^2 \leq 0. \quad (51)$$

Since the only zero for the function q is at $x = 0$ by assumption, there exists $\delta > 0$ sufficiently small such that the set $F_\delta := \{x \in \Omega_x, q(x)^2 \leq \delta\}$ is a neighborhood of zero, which is decreasing for the inclusion as $\delta \rightarrow 0^+$. Moreover, as we know that $\frac{\lambda_n}{n^2}$ tends to zero as n tends to infinity thanks to Proposition 3.10 coupled to the estimate on ν_n given in Section 3.1, and the fact that $V \in L^\infty(\Omega)$, for n large enough we have $\{x \in \Omega_x, q(x)^2 \leq \frac{o(n^2) + \lambda_n - V}{n^2}\} \subset F_\delta$. Finally, since $\overline{\omega_x} \cap \{0\} = \emptyset$, we can choose δ sufficiently small so that we additionally have $F_\delta \cap \overline{\omega_x} = \emptyset$.

Choose such a $\delta > 0$. Thus, for n large enough,

$$\int_{\Omega_x} (q^2 - \psi'^2 - \delta)\theta_n^2 \leq 0.$$

Choose $\psi(x) := (1 - \delta)d_{agm}(x, F_\delta)$, where d_{agm} is the degenerated distance induced by the degenerated Agmon metric $(q^2 - \delta)_+ dx^2$, where dx^2 is the standard Euclidean metric on Ω_x . We note that we have

$$|\psi'(x)|^2 \leq (1 - \delta)^2 (q^2 - \delta)_+. \quad (52)$$

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Therefore, on the set F_δ we have that $q^2 - \psi'^2 - \delta = q^2 - \delta \leq 0$, and on the complementary of F_δ we have

$$\begin{aligned} q^2 - \psi'^2 - \delta &\geq (1 - (1 - \delta)^2)(q^2 - \delta)_+ \\ &= (2\delta - \delta^2)(q^2 - \delta)_+ \\ &\geq 0. \end{aligned}$$

It follows that

$$(2\delta - \delta^2) \int_{\Omega_x \setminus F_\delta} (q^2 - \delta) \theta_n^2 \leq - \int_{F_\delta} (q^2 - \delta) \theta_n^2.$$

Replacing θ_n by its expression $v_n(x)e^{n(1-\delta)d_{\text{agm}}(x, F_\delta)}$, using the fact that $\overline{\omega_x} \cap F_\delta = \emptyset$, and that $d_{\text{agm}}(x, F_\delta) = 0$ on F_δ , we get that on one hand

$$\begin{aligned} - \int_{F_\delta} (q^2 - \delta) \theta_n^2 &= - \int_{F_\delta} (q^2 - \delta) v_n^2 \\ &\leq \sup_{F_\delta} |q^2 - \delta|, \end{aligned}$$

since v_n is normalized in L^2 -norm. On the other hand,

$$\begin{aligned} (2\delta - \delta^2) \int_{\Omega_x \setminus F_\delta} (q^2 - \delta) \theta_n^2 &= (2\delta - \delta^2) \int_{\Omega_x \setminus F_\delta} (q^2 - \delta) v_n^2 e^{2n(1-\delta)d_{\text{agm}}(x, F_\delta)} \\ &\geq (2\delta - \delta^2) \int_{\omega_x} (q^2 - \delta) v_n^2 e^{2n(1-\delta)d_{\text{agm}}(x, F_\delta)} \\ &\geq (2\delta - \delta^2) \min_{\omega_x} (q^2 - \delta) e^{2n(1-\delta)d_{\text{agm}}(\omega_x, F_\delta)} \int_{\omega_x} v_n^2, \end{aligned}$$

where we stress that $\min_{\omega_x} (q^2 - \delta) > 0$. Therefore, we finally get

$$\int_{\omega_x} v_n^2 \leq \frac{\sup_{F_\delta} |q^2 - \delta|}{(2\delta - \delta^2) \min_{\omega_x} (q^2 - \delta)} e^{-2n(1-\delta)d_{\text{agm}}(\omega_x, F_\delta)},$$

which proves the proposition. □

We can now proceed to the proof of Theorem 2.2.

Proof of theorem 2.2 in the case $\Omega_y = \mathbb{S}^1$ or $(0, \pi)$. Let us split the proof in two parts.

Proof in the case $\gamma > 1$

Choose the sequence of solutions g_n introduced at the beginning of the section. Assume that system (7) is observable from ω in time $T > 0$. Then, there exists $C > 0$ such that for any $n \in \mathbb{N}^*$,

$$\begin{aligned} e^{-2\lambda_n T} \int_{\Omega_x} v_n(x)^2 dx &\leq C \int_0^T \int_{\omega_x} e^{-2\lambda_n t} v_n(x)^2 dx \\ &\leq CT \int_{\omega_x} v_n(x)^2 dx. \end{aligned}$$

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As n tends to infinity, applying Proposition 4.1, this implies, since the integral in the left-hand side values one,

$$1 \leq C_1 C T e^{2\lambda_n T} e^{-nC_2}.$$

Using now the fact that $\lambda_n = \mathcal{O}(n^{\frac{2}{\gamma+1}})$ by Proposition 3.10, we have that for some $C_3 > 0$

$$1 \leq C_1 C T e^{2C_3 n^{\frac{2}{\gamma+1}} T} e^{-nC_2}.$$

We therefore simply have to show that for any time $T > 0$, the right hand side tends to zero as n tends to infinity. This is straightforward since

$$2C_3 n^{\frac{2}{\gamma+1}} T - nC_2 = n(n^{\frac{1-\gamma}{\gamma+1}} 2C_3 T - C_2),$$

becomes negative for any time $T > 0$ as n tends to infinity since $\gamma > 1$. This disproves the observability inequality for any time $T > 0$.

Proof in the case $\gamma = 1$.

In this case, the above strategy still works to disprove the observability in small time, and give analogous results as the ones known in [BCG14; DKR23]. We use that $\lambda_n = q'(0)\sqrt{\nu_n} + o(\sqrt{\nu_n})$ for large values of n from Corollary 3.12, with $\nu_n = n^2 + o(n^2)$. Therefore, in the proof of Proposition 4.1, the inequality

$$\int_{\omega_x} v_n^2 \leq \frac{\sup_{F_\delta} |q^2 - \delta|}{(2\delta - \delta^2) \min_{\omega_x} (q^2 - \delta)} e^{-2n(1-\delta)d_{agm}(\omega_x, F_\delta)},$$

still holds true, for any $\delta > 0$ sufficiently small. Thus, for any $T > 0$, observability cannot hold if

$$e^{2\lambda_n T - 2n(1-\delta)d_{agm}(\omega_x, F_\delta)} \rightarrow 0, \text{ as } n \text{ tends to infinity.}$$

Letting $\varepsilon > 0$, for n large enough, from the estimates on λ_n , ν_n , and the above estimate, the observability inequality cannot hold if, for n large,

$$2(1 + \varepsilon)q'(0)nT - 2n(1 - \delta)d_{agm}(\omega_x, F_\delta) < 0.$$

That is, we cannot have

$$T < \frac{(1 - \delta)d_{agm}(\omega_x, F_\delta)}{(1 + \varepsilon)q'(0)}.$$

Since this is true for any $\varepsilon > 0$, $\delta > 0$ sufficiently small, we get in the case $\gamma = 1$ that

$$T(\omega) \geq \frac{d_{agm}(\omega_x, 0)}{q'(0)}. \tag{53}$$

□

Remark 4.2. Assume that Ω_y is any other interval, with Dirichlet or periodic boundary conditions, and we do not normalize by π the value $L(\mathcal{Z}) := \int_{\Omega_y} \frac{1}{r(s)} ds$ in Section 3.1. Then, from [Eas73;

[Pos87], in the limit $n \rightarrow +\infty$, $\nu_n = Cn^2 + o(n^2)$, with $C = \frac{\pi^2}{L(\bar{z})^2}$ or $\frac{4\pi^2}{L(\bar{z})^2}$ depending on the boundary condition. In this case, the proof of Proposition 4.1 changes slightly. Indeed, for the proof to remain true, one has to choose for the Agmon metric $\sigma := (Cq^2 - \delta)_+ dx^2$. Hence, the exponential decay remains true modulo the multiplicative constant C in the exponential. This constant then also appears in the left-hand side of the observability since $\lambda_n = C(1 + \varepsilon)q'(0)n$ in the limit $n \rightarrow +\infty$, and C ends up getting simplified. Thus, the result is left unchanged.

Let us now proceed to the case of $\Omega_y = \mathbb{R}$ and show that it is analogous to the previous case.

Proof of theorem 2.2 in the case $\Omega_y = \mathbb{R}$. Recall that we set $r(y) = 1$ on \mathbb{R} . Due to the absence of restriction in the y -direction for some x , the (equivalent) problems of null-controllability and observability are equivalent to their formulation in Fourier components. Namely, (10) is equivalent to

$$\int_{\mathbb{R}^2} |\hat{g}(T, x, \xi)|^2 dx d\xi \leq C \int_0^T \int_{\mathbb{R}} \int_{\tilde{\omega}_x} |\hat{g}(t, x, \xi)|^2 dx d\xi dt, \quad (54)$$

with \hat{g} the solution of

$$\begin{cases} \partial_t \hat{g} - \partial_x^2 \hat{g} + |\xi|^2 q(x)^2 \hat{g} + V(x) \hat{g} &= 0, & (t, x, \xi) \in (0, T) \times (-L, L) \times \mathbb{R}, \\ \hat{g}(t, \pm L, \xi) &= 0, & (t, \xi) \in (0, T) \times \mathbb{R}, \\ \hat{g}(0, x, \xi) &= \hat{g}_0(x, \xi), & (x, \xi) \in (-L, L) \times \mathbb{R}, \end{cases} \quad (55)$$

and where \hat{g} is the partial Fourier transform of g with respect to y .

As we do not have to come back to the functions in the y variable, we can drop the hat notation. We also write the dependence in ξ as a variable and not as a subscript. Consider the first eigenfunction $g(\cdot, \xi)$ of

$$-\partial_x^2 + q(x)^2 |\xi|^2 + V(x) \quad (56)$$

with eigenvalue $\lambda(\xi) \simeq |\xi|^{2/\gamma+1}$. Due to the form of (55), we can localize this eigenfunctions around high frequencies by multiplying them by a smooth compactly supported function. Let $\delta > 0$, and define, for $n > 0$, $\psi_n \in C_c^\infty(\mathbb{R})$ verifying

$$\text{supp}(\psi_n) = [n - \delta, n + \delta], \quad 0 \leq \psi_n \leq 1, \quad \psi_n(x) = 1, \quad \forall x \in [n - \delta/2, n + \delta/2]. \quad (57)$$

Hence, the sequence of functions

$$g_n(t, x, \xi) = e^{-\lambda(\xi)t} g(x, \xi) \psi_n(\xi) \quad (58)$$

are solutions of (55). In the observability inequality (54), we get

$$\int_{\mathbb{R}^2} e^{-2\lambda(\xi)T} g(x, \xi)^2 \psi_n(\xi)^2 dx d\xi \leq C \int_0^T \int_{\mathbb{R}} \int_{\tilde{\omega}_x} e^{-2\lambda(\xi)t} g(x, \xi)^2 \psi_n(\xi)^2 dx d\xi dt.$$

By normalization of $g(\xi, \cdot)$, for n large enough, the left-hand term is bounded below by

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-2\lambda(\xi)T} g(x, \xi)^2 \psi_n(\xi)^2 dx d\xi &\geq \int_{n-\delta/2}^{n+\delta/2} \int_{\mathbb{R}} e^{-2\lambda(\xi)T} g(x, \xi)^2 dx d\xi \\ &= \int_{n-\delta/2}^{n+\delta/2} e^{-2\lambda(\xi)T} d\xi. \end{aligned}$$

For the right-hand term, for n large enough, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \int_{\tilde{\omega}_x} e^{-2\lambda(\xi)t} g(x, \xi)^2 \psi_n(\xi)^2 dx d\xi dt &\leq \int_0^T \int_{n-\delta}^{n+\delta} \int_{\tilde{\omega}_x} e^{-2\lambda(\xi)t} g(x, \xi)^2 dx d\xi \\ &\leq T \int_{n-\delta}^{n+\delta} \int_{\tilde{\omega}_x} g(x, \xi)^2 dx d\xi. \end{aligned}$$

Notice that Proposition 4.1 holds replacing n with $|\xi|$, for $|\xi|$ large enough. Hence, we obtain the same results as in the case Ω_y bounded, since the sequence of cutoffs can be chosen arbitrarily concentrated around the frequencies $\xi_n = n$. \square

4.2 Proof of Theorem 2.1: Control in the Complementary of a Rectangle, $\Omega_x = (-L, L)$, $\Omega_y = \mathbb{S}^1$ or a Bounded Interval

The proof of this theorem is the same as [DKR23, Theorem 1.3] in the case $V = 0$, up to some remarks. We therefore mainly only state the modifications to take into account. We set ourselves on $L^2(\Omega, \mathbb{C})$, for which non null-controllability implies the same on $L^2(\Omega, \mathbb{R})$. Indeed, the Grushin operator G_V having real-valued coefficients, it acts on complex-valued functions as $\hat{G}_V(f + ig) = G_V f + iG_V g$. It inherits all the properties of G_V , has domain $D(\hat{G}_V) = D(G_V) + iD(G_V)$, and generates a continuous semigroup $(T(t))_t$, which satisfies, denoting $(S(t))_t$ the one of G_V , $T(t)(f + ig) = S(t)f + iS(t)g$. Hence, it is not hard to see that observability in time $T > 0$ for G_V implies the same for \hat{G}_V . In particular, given a sequence of solutions refuting the observability for \hat{G}_V , the real-valued functions given by either the real part or imaginary part will disprove the observability on $L^2(\Omega, \mathbb{R})$ for G_V . We shall drop the hat notation for the complexification of G_V as there shall not be any confusion.

Before proceeding to the proofs, for the sake of the reader not to be lost in the references and technical results therein, we summarize the results and their applications:

[Koe17, Proposition 16] is a technical result on holomorphic extensions of entire series given a sequence of complex numbers \implies [Koe17, Theorem 18] which is also [DKR23, Theorem 4.7] is an abstract result on the L^∞ norm of perturbed entire series \implies [DKR23, Lemma 3.5] which applies the previous Theorem to a particular family of sequence of complex numbers of interest \implies [DKR23, Lemma 3.7] which links the previous Lemma to the observability inequality.

Proof of Theorem 2.1. We prove the theorem in the case $\Omega_y = \mathbb{S}^1$. The theorem on $(0, \pi)$ then follows from [DKR23, Appendix A]. Given a sequence of complex numbers $(a_n)_{n>0}$, a solution of system (7) is given by

$$g(t, x, y) = \sum_{n>0} a_n g(t, x, n) e^{iny}, \quad (59)$$

where the sum is taken finite, and $g(t, x, n)$ is the solution of system

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 q(x)^2 + V(x)) g &= 0, & (t, x) \in (0, T) \times (-L, L), \\ g(t, x, n) &= 0, & (t, x) \in (0, T) \times \partial(-L, L), \\ g(0, x, n) &= g_0(x, n), & x \in (-L, L). \end{cases} \quad (60)$$

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Denote by v_n an eigenfunction associated to the first eigenvalue λ_n of the operator $G_{n,V} = -\partial_x^2 + n^2q(x)^2 + V(x)$. Then $g(t, x, n) = e^{-\lambda_n t} v_n(x)$. We follow from now on the proof of [DKR23, Theorem 1.3].

Recall that we denote, for $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) > 0$, the non-self-adjoint harmonic oscillator $H_\beta = -\partial_x^2 + \beta^2 x^2$ on \mathbb{R} . Then $G_{n,V}$ is a perturbation of $H_{q'(0)n}$ restricted to $(-L, L)$.

Set $\tilde{v}_n(x) = n^{1/4} e^{-nq'(0)x^2/2}$, and set

$$v_n := \Pi_n \tilde{v}_n, \quad (61)$$

where Π_n is the spectral projector onto the first eigenspace associated to λ_n of $G_{n,V}$. Recall that the Agmon distance is defined by (4)

$$d_{\text{agm}} : x \in (-L, L) \mapsto \int_0^x q(s) ds. \quad (62)$$

Let $\varepsilon \in (0, \pi)$, and define $\gamma_{t,x}(n)$ by

$$\gamma_{t,x}(n-1) := e^{-t(\lambda_n - q'(0)n)} v_n(x) e^{n d_{\text{agm}}(x)(1-\varepsilon)}. \quad (63)$$

Then, a sequence of solutions can be written as

$$g(t, x, y) = \sum_{n>N} a_{n-1} \gamma_{t,x}(n-1) e^{n(iy - q'(0)t - (1-\varepsilon)d_{\text{agm}}(x))} \quad (64)$$

The change of variable $(x, z) = (x, e^{iy - q'(0)t - (1-\varepsilon)\frac{\pi}{L(\bar{z})} d_{\text{agm}}(x)})$ is a diffeomorphism from $(0, T) \times \Omega$ onto its image (see [DKR23, Fig. 3.4]). Letting $p \in \mathbb{C}[X]$, where X is a compact subset of \mathbb{C} , to be well-chosen later, with a zero of order N at zero, it writes as $p(z) = \sum_{n \geq N} a_n z^n$. Hence, writing $g_{\text{pol}}(z) = zp(z) = \sum_{n > N} a_{n-1} z^n$, we see that

$$g_N(t, x, y) \approx g_{\text{pol}}(e^{iy - q'(0)t - (1-\varepsilon)d_{\text{agm}}(x)}), \quad (65)$$

and we must control the error term $\gamma_{t,x}(n-1)$. The change of variable in the observability changes the measure as $dx dy dt = \frac{1}{q'(0)|z|^2} dx dm(z)$, where $dm(z)$ is the complex Lebesgue measure. The choice of writing $zp(z)$ instead of only $p(z)$ is henceforth technical and due to the change of variable.

Assume that observability holds. Thanks to Proposition 3.11, [DKR23, Proposition 4.2, Proposition 3.6] holds true, and $\|v_n\|$ is bounded below uniformly in n . Then, thanks to Corollary 3.12 and the uniform lower bound on $\|v_n\|$, for some $\varepsilon > 0$ we have

$$\begin{aligned} \int_{\Omega} |g(T, x, y)|^2 dx dy &= \int_{\Omega} \left| \sum_{n>N} a_{n-1} v_n(x) \phi_n(y) e^{-\lambda_n T} \right|^2 dx dy \\ &= C \sum_{n>N} |a_{n-1}|^2 \|v_n\|_{L^2(\Omega_x)}^2 e^{-2\lambda_n T} \\ &\geq C e^{-2C_\varepsilon T} \sum_{n>N} |a_{n-1}|^2 e^{-2\frac{n\pi}{L(\bar{z})} q'(0)(1+\varepsilon)T} \\ &\geq C \|p\|_{L^2\left(D\left(0, e^{-\frac{q'(0)\pi}{L(\bar{z})}(1+\varepsilon)T}\right)\right)}^2. \end{aligned}$$

On the other hand, we have

$$\int_0^T \int_{\omega} |g(t, x, y)|^2 dx dy dt = \frac{1}{q'(0)} \int_U \left| \sum_{n>N} a_{n-1} \gamma_{t,x}(n-1) z^n \right|^2 \frac{1}{|z|^2} dx dm(z), \quad (66)$$

where (see [DKR23, Fig. 3.4])

$$U = D(0, e^{-(1-\varepsilon)d_{\text{agm}}(-a)}) \cup D(0, e^{-(1-\varepsilon)d_{\text{agm}}(b)}) \cup \{z \in \mathbb{C}, |z| < 1, \arg(z) \notin I\}. \quad (67)$$

The error term is to be understood as a (pseudo-differential) operator $\gamma_{t,x}$ in the sense that it acts on complex polynomials as

$$\gamma_{t,x}(z\partial_z) \left(\sum_n a_n z^n \right) = \sum_n a_n \gamma_{t,x}(n) z^n. \quad (68)$$

It is introduced in [DKR23, eq. (3.19)] in the case $V = 0$, and its properties are given by [DKR23, Lemma 3.5]. Hence, assuming that observability holds, it implies that there exists a constant $C > 0$ such that

$$\begin{aligned} \|p\|_{L^2(D(0, e^{-q'(0)(1+\varepsilon)T})} &\leq \frac{C}{q'(0)} \int_U |\gamma_{t,x}(z\partial_z)(p)(z)|^2 dx dm(z) \\ &\leq C \sup_{(t,x) \in (0,T) \times \Omega_x} \|\gamma_{t,x}(z\partial_z)(p)\|_{L^\infty(U)}. \end{aligned}$$

All the above computations are carried out in [DKR23, Lemma 3.7]. The key point is then to show that the right-hand side can be controlled by the L^∞ -norm of p in a neighborhood of U . This is the core of [DKR23, Lemma 3.5] which follows from [DKR23, Theorem 4.7], proved in [Koe17, Theorem 18]. This is a problem of holomorphic interpolation. The following arguments treat this problem.

From Section 3.3.2, for every $\theta \in [0, \frac{\pi}{2})$ there exists $r_\theta \geq 1$ such that Proposition 3.11 holds for any $\nu \in \Sigma_{\theta_0} \setminus D(0, r_{\theta_0})$. Following [DKR23, Definition 4.6], we henceforth define

$$D := \cup_{\theta \in [0, \pi/2)} \Sigma_\theta \setminus D(0, r_\theta),$$

and $\mathcal{S}(D)$ to be the set of holomorphic function f on D such that for every $\theta \in [0, \pi/2), \delta > 0$ we have

$$\sup_{D \cap \Sigma_\theta} |f(z) e^{-\delta|z|}| < +\infty.$$

We must show that the sequence $\gamma_{t,x}(n)$ can be interpolated by an element of $\mathcal{S}(D)$. That is, show that [DKR23, Lemma 3.5] still holds in our case, and then the arguments of [DKR23] hold true. Observe that $\mathcal{S}(D)$ is stable under multiplication. Hence, it is sufficient to show separately that the terms $e^{-t(\lambda_n - q'(0)n)}$, and $v_n(x) e^{nd_{\text{agm}}(x)(1-\varepsilon)}$ can be interpolated by an element of $\mathcal{S}(D)$.

Let us treat the first term $e^{-t(\lambda_n - q'(0)n)}$. Write $\lambda(z)$ to be the first eigenvalue of $-\partial_x^2 + z^2 q(x)^2 + v(x)$ obtained in Section 3.3.2. With this notation, we have $\lambda_n = \lambda(n)$. Hence, the first term is interpolated by the function $z \mapsto e^{-t(\lambda(z) - q'(0)z)}$. Similarly, denoting $v_z = \Pi_z \left(z^{1/4} e^{-zq'(0)x^2/2} \right)$, the second term is interpolated by $z \mapsto v_z(x) e^{zd_{\text{agm}}(x)(1-\varepsilon)}$. Thanks to Proposition 3.11, and Proposition 4.3 below, [DKR23, Lemma 3.5 and Lemma 3.7] hold verbatim, that is the two functions above are indeed elements of $\mathcal{S}(D)$. The spectral projection Π_z , and $\lambda(z)$ are both, by standard perturbation theory arguments, holomorphic functions of z .

Proposition 4.3. *Let $\theta_0 \in [0, \pi/2)$, and $\varepsilon \in (0, \pi)$. There exists $C > 0$ such that for $\nu \in \Sigma_{\theta_0}$, $|\nu|$ large enough, we have*

$$\int_I \left| v_\nu(x) e^{\nu(1-\varepsilon)d_{agm}(x)} \right|^2 dx \leq C|\nu|, \quad (69)$$

$$\|v_\nu(x) e^{\nu(1-\varepsilon)d_{agm}(x)}\|_{L^\infty(I)} \leq C|\nu|. \quad (70)$$

Proof. We have the following Agmon equality from the exact same computations as in [DKR23, Proposition 4.3],

$$\|v'_\nu\|_{L^2(I)}^2 + \int_I (|\nu|^2 q(x)(1 - (1 - \varepsilon)^2) + V - \mu_\nu) |v_\nu(x)|^2 dx = 0,$$

with

$$\mu_\nu := \frac{\operatorname{Re}(e^{-i \arg(\nu)} \lambda_\nu)}{\cos(\arg(\nu))}.$$

For $|\nu|$ large enough, we have for some $c > 0$

$$|\nu|^2 q(x)(1 - (1 - \varepsilon)^2) + V - \mu_\nu \geq c|\nu|,$$

since in the limit, the potential, whether positive or negative, becomes negligible in front of the other terms. The proposition follows now from the exact same argument as in [DKR23, Corollary 4.5]. \square

Now we can conclude following the exact same line as for the proof of [DKR23, Theorem 1.3, Theorem 3.1]. Assume that observability holds in time T . This implies that there exists $C > 0$ such that

$$\|p\|_{L^2(D(0, e^{-q'(0)(1+\varepsilon)T})}^2 \leq C \|p\|_{L^\infty(\mathcal{V})}^2, \quad (71)$$

where \mathcal{V} a neighborhood of \bar{U} . The proof then concludes with an application of Runge Theorem. For every $\varepsilon > 0$, for every $T > 0$ such that

$$T \leq \frac{1 + \varepsilon}{q'(0)(1 - \varepsilon)} \min(d_{agm}(-a), d_{agm}(b)), \quad (72)$$

there exists $z_0 \in D(0, e^{-q'(0)(1+\varepsilon)T}) \setminus \bar{U}$. Hence, there exists a sequence of polynomials $(\tilde{p}_k)_k$ that converges uniformly on every compact subset of $\mathbb{C} \setminus z_0[1, +\infty)$ to $1/(z - z^0)$. We then choose $p_k = z^{N+1} \tilde{p}_k$, which disproves (71). Namely, the sequence of complex numbers in our solutions that disprove the observability are entirely determined by the sequence of polynomials $(\tilde{p}_k)_k$. Now, since ε can be chosen arbitrary small, the result follows. \square

Remark 4.4. *It would be interesting to obtain this result when r is not identically one, which seems to be a complicated task. Indeed, the holomorphic interpolation argument becomes a challenging issue. In this case, the first eigenvalue λ_n becomes $\lambda_{\sqrt{\nu_n}}$, where we recall that ν_n is the n -th eigenvalue of $\partial_y(r(y)^2 \partial_y)$, and the associated eigenfunctions are now perturbations of sinusoidal, at least for large values of n . Ignoring the eigenfunctions issue as we believe it is not the most important one, for almost every (t, x) we want to find an holomorphic function $F \in \mathcal{S}(D)$ such that $F(n) = \gamma_{t,x}(n)$. This first amounts to interpolate for example the term $e^{-t(\lambda_{\sqrt{\nu_n}} - q'(0)n)}$ at each n . We saw that when $r = 1$,*

the interpolation is directly given by the same formula replacing n by z , as the holomorphy is ensured by the perturbation theory arguments. Now, we must find G , holomorphic in the right half-plane (see [Koe17]), such that $G(n) = \sqrt{\nu_n}$. Indeed, in this case, the composite function $z \mapsto G(z) \mapsto \lambda(G(z))$ interpolates correctly $\lambda_{\sqrt{\nu_n}}$. The existence of an interpolation is not a complicated problem, but one must keep in mind that we have heavy restrictions on the choice of the interpolation since the interpolation of the error term must in the end belong to $\mathcal{S}(D)$.

5 Reduction to a Local Problem and Proof of Theorem 1.5

In this section, we prove our main result. First we prove how the observability of system (2) implies observability of the same system but posed in a tubular neighborhood near the singularity, and then we conclude the proof.

5.1 Reduction process

Proposition 5.1. *Assume that system (2) is null-controllable in time $T > 0$. Let \mathcal{O} be an open subset of \mathbb{M} with Lipschitz and piecewise smooth boundary, such that $\mathcal{O} \subset \mathbb{M} \setminus \omega$. Then, system*

$$\begin{cases} \partial_t f - \Delta f &= 0, & (t, x, y) \in (0, T) \times \mathcal{O}, \\ f(t, p) &= 0, & (t, p) \in (0, T) \times \partial\mathcal{O} \cap \partial\mathbb{M}, \\ f(t, p) &= v(t, p), & (t, p) \in (0, T) \times \partial\mathcal{O} \setminus \partial\mathbb{M}, \\ f(0, p) &= f_0(p), & p \in \mathcal{O}, \end{cases} \quad (73)$$

where $f_0 \in L^2(\mathcal{O})$, f is the state, $v \in L^2((0, T) \times \partial\mathcal{O})$ is the control, is null-controllable in time T .

Proof. Let $f_0 \in L^2(\mathcal{O})$. Extend f_0 by 0 outside of \mathcal{O} , and denote it by \tilde{f}_0 . By assumption of null-controllability of system (2) in time $T > 0$, there exists $u \in L^2((0, T) \times \mathbb{M})$, supported in ω , such that the solution \tilde{f} of (2), with initial state \tilde{f}_0 , satisfies $\tilde{f}(T, x, y) = 0$ on \mathbb{M} . Since we have that $\tilde{f} \in L^2((0, T), H_0^1(\mathbb{M}))$, we can define the L^2 -trace of \tilde{f} on $(0, T) \times \partial\mathcal{O}$.

Define by $(\mathcal{B}_i)_i$ a countable family of smooth open subsets of $\partial\mathcal{O}$ such that $\partial\mathcal{O} = \overline{\bigcup_i \mathcal{B}_i}$.

Denote by $\mathbf{1}_{\mathcal{O}}^* \tilde{f}$ the restriction of \tilde{f} to \mathcal{O} . Denote by Γ_i , the trace operator

$$\Gamma_i : H^1((0, T) \times (\mathbb{M} \setminus \omega)) \longrightarrow L^2((0, T) \times \mathcal{B}_i).$$

Hence, $\mathbf{1}_{\mathcal{O}}^* \tilde{f}$ is a solution of (73), with initial state $\mathbf{1}_{\mathcal{O}}^* \tilde{f}(0, x, y) = f_0(x, y)$, with $v(t, x, y)$ defined piecewise, almost everywhere, on $\partial\mathcal{O} \setminus \partial\mathbb{M}$ by

$$v(t, p) = \sum_i \mathbf{1}_{\mathcal{O}_i \setminus \mathbb{M}} \Gamma_i(\mathbf{1}_{\mathcal{O}}^* \tilde{f})(t, x, y),$$

satisfies $\mathbf{1}_{\mathcal{O}}^* \tilde{f}(t, p) = 0$ on $\partial\mathcal{O} \cap \partial\mathbb{M}$ and $\mathbf{1}_{\mathcal{O}}^* \tilde{f}(T) = 0$ on \mathcal{O} . By uniqueness of the solution, the solution f of (73) with initial state f_0 , and control v defined as above, satisfies $f(T) = 0$. This proves the proposition. \square

Now that we have reduced the problem of internal controllability on \mathbb{M} to a problem of boundary controllability on an open subsets with boundary in the complementary of the control zone, we can show that boundary null-controllability implies internal null-controllability through cutoff

arguments. We focus on the case of rectangles in \mathbb{R}^2 as it is sufficient in our case. The following Proposition is elementary, and can be found in [DK20, Appendix A], up to minor adjustments in the choice of the cutoffs. We therefore state them, but omit the proof.

Proposition 5.2. *Let $T > 0$. Assume that (73) is null-controllable in time T , with $\mathcal{O} = (L_x^-, L_x^+) \times (L_y^-, L_y^+)$, $L_x^+ > 0 > L_x^-$, $L_y^-, L_y^+ \geq 0$. Let $\varepsilon > 0$ sufficiently small, and set $\mathcal{O}^\varepsilon = \{p \in \mathbb{R}^2, d(p, \mathcal{O}) < \varepsilon\}$, and $\omega^\varepsilon = \{p \in \mathcal{O}^\varepsilon, d(p, \partial\mathcal{O}^\varepsilon) < \varepsilon\}$. Then,*

$$\begin{cases} \partial_t g - \partial_x^2 g - q(x)^2 \partial_y (r(y)^2 \partial_y g) &= \mathbf{1}_{\omega^\varepsilon}(x, y) u(t, x, y), & (t, x, y) \in (0, T) \times \mathcal{O}^\varepsilon, \\ g(t, x, y) &= 0, & (t, x, y) \in (0, T) \times \partial\mathcal{O}^\varepsilon, \\ g(0, x, y) &= g_0(x, y), & (x, y) \in \mathcal{O}^\varepsilon, \end{cases} \quad (74)$$

is null-controllable in time T .

Sketch of proof of Proposition 5.2. Let $T > 0$. Let $g_0 \in L^2((0, T) \times \mathcal{O}^\varepsilon)$, and let $g_{0|\mathcal{O}}$ be the initial state in (73). Then, by assumption, there exists a control $v \in L^2((0, T) \times \partial\mathcal{O})$ such that the associated solution g_{bound} satisfies $g_{\text{bound}}(T) = 0$. Denote now by g_{adj} the solution of system (74) with $u = 0$. We set

$$g_{\text{int}} := \eta \theta g_{\text{adj}} + (1 - \theta) g_{\text{bound}},$$

where $\eta \in C^\infty([0, T])$ and $\theta := \theta_x \theta_y \in C^\infty(\mathcal{O})$ are defined by, for $s = x$ or y ,

$$\begin{cases} \theta_s(z) = 1, & z \in [L_s^- - \varepsilon, L_s^- + \frac{\varepsilon}{4}] \cup [L_s^+ - \frac{\varepsilon}{4}, L_s^+ + \varepsilon], \\ \theta_s(z) = 0, & z \in [L_s^- + \frac{\varepsilon}{3}, L_s^+ - \frac{\varepsilon}{3}], \\ \theta_s(z) \in [0, 1], & z \in [L_s^- - \varepsilon, L_s^+ + \varepsilon], \end{cases} \quad \begin{cases} \eta(t) = 1, & t \in [0, \frac{T}{3}], \\ \eta(t) = 0, & t \in [\frac{2T}{3}, T], \\ \eta(t) \in [0, 1], & t \in [0, T]. \end{cases}$$

Applying [DK20, Appendix A], Proposition 5.2 holds. \square

5.2 Proof of Theorem 1.5

We can now proceed to prove Theorem 1.5. Both considerations in the Theorem are proved the same way.

Proof of Theorem 1.5. We illustrate the proof in Figure 2. Assume that system (2) is null-controllable in time $T > 0$ from ω . By assumption H1, we can find in $\mathbb{M} \setminus \omega$ an open set \mathcal{U} which is either a tubular neighborhood of the whole singularity \mathcal{Z} , or of a point of the latter. Under both considerations, for some $L > 0$, \mathcal{U} is diffeomorphic to $\Omega = (-L, L) \times \Omega_y$, with, without loss of generality,

$$\Omega_y = \begin{cases} (0, \pi) & \text{under H1-loc,} \\ (0, \pi), \mathbb{S}^1, \text{ or } \mathbb{R} & \text{under H1-glob.} \end{cases}$$

We emphasize that such a tubular neighborhood has a sufficiently regular boundary to apply Proposition 5.1. Yet, before pursuing the proof, we need to characterize how the boundary of \mathcal{U} intersects the boundary of \mathbb{M} under the assumption H1-glob.

Lemma 5.3. *Let \mathcal{U} be the tubular neighborhood constructed under H1-glob. Then,*

- (i) if $\mathcal{Z} \simeq \mathbb{S}^1$ or \mathbb{R} , for L small enough we have $\partial\mathcal{U} \cap \partial\mathbb{M} = \emptyset$,
- (ii) if $\mathcal{Z} \simeq (0, \pi)$, we have $\partial\mathcal{U} \cap \partial\mathbb{M} \simeq ([-L, L] \times \{0\}) \cup ([-L, L] \times \{\pi\})$.

Proof. We need to show that $\mathcal{U} = \{p \in \mathbb{M}, \delta(p) < L\}$, where we recall that δ denotes the sub-Riemannian distance to \mathcal{Z} . Indeed, assume that this holds. If $\mathcal{Z} \simeq \mathbb{S}^1$, then $\partial\mathcal{U} = \{p \in \mathbb{M}, \delta(p) = L\} \simeq (\{-L\} \times \mathbb{S}^1) \cup (\{L\} \times \mathbb{S}^1)$. In this case, since $\mathcal{Z} \subset \text{Int}(\mathbb{M})$, the lemma directly follows as L can be chosen small enough so that $\mathcal{U} \subset \text{Int}(\mathbb{M})$. The same idea goes for $\mathcal{Z} \simeq \mathbb{R}$. In the second case, let $p = (0, 0)$ and consider, in coordinates, the curve $c : t \in [0, L] \mapsto c(t) = (0, t)$, which belongs to $\partial\mathcal{U}$. Assume that c has exited $\partial\mathbb{M}$ at some time $t_0 \in (0, L)$. Then, there exists an open ball around $c(t_0)$ of radius $\varepsilon > 0$ sufficiently small such that $B(c(t_0), \varepsilon) \subset \text{Int}(\mathbb{M})$, and a point $p' \in B(c(t_0), \varepsilon)$ that does not belong to \mathcal{U} . By continuity of δ , we have $\delta(p') \leq t_0 + \varepsilon < L$. Hence $p' \in \{p \in \mathbb{M}, \delta(p) < L\}$ which is a contradiction. The other parts of $\partial\mathcal{U}$ can be treated the same way.

The set equality is proved in the case $\mathcal{Z} \simeq \mathbb{S}^1$ in [FPR20, Proposition 3.1], [Ros22, Theorem 3.7]. When $\mathcal{Z} \simeq \mathbb{R}$, it still holds thanks to the injectivity radius assumption in H0 (see [FPR20, footnote p.98]).

Let us focus on the second case. We obviously have the inclusion $\mathcal{U} \subset \{p \in \mathbb{M}, \delta(p) < L\}$. Let $p_0 \in \{q \in \mathbb{M}, \delta(q) < L\} \setminus \mathcal{U}$. Let $q \in \mathcal{Z}$ be the unique point in \mathcal{Z} such that $\delta(p_0) = d_{sR}(p_0, q)$. Then, there exists $c_0 : [0, 1] \rightarrow \mathbb{M}$ a minimizing geodesic such that $c_0(0) = q$ and $c_0(1) = p_0$. Assume that $p_1 \in \mathcal{U}$ is in the same connected component of $\{q \in \mathbb{M}, \delta(q) < L\}$ as p_0 , and such that $\delta(p_1) = d_{sR}(p_0, q) = \delta(p_0)$. We know that p_1 exists since it is either $p_1 = (q, \delta(p_0))$ or $p_1 = (q, -\delta(p_0))$ depending on where p_0 is situated. There exists a minimizing normal geodesic c_1 such that $c_1(0) = q$ and $c_1(1) = p_1$. Observe that such a minimizing geodesic must satisfy $c'_i(t) = \alpha(t)\nabla\delta(c_i(t))$, where $\alpha : [0, 1] \rightarrow \mathbb{R}^+$ is smooth. Since we assumed smoothness of δ in H0', we must have that there exists $\beta > 0$ such that $c'_0(0) = \beta c'_1(0)$. Since c_0 and c_1 are both integral curves of $\nabla\delta$ starting from the same point, we must have that there exists a smooth function $\beta : [0, 1] \rightarrow \mathbb{R}^+$ such that $\beta(0) = \beta$ and $c'_1(t) = \beta(t)c'_0(t)$. That is, c_1 is a reparametrization of c_0 , and hence $c_0(1) = c_1(1)$. Thus, $p_0 = p_1$ and the inclusion follows. \square

Remark 5.4. *What the above proof says, is that assuming smoothness of δ as in H0', we only have a unique way of exiting \mathcal{Z} in each side of it, by means of a minimizing geodesic (minimizing the distance to \mathcal{Z}), which is a normal one, from any point of the singularity. This geodesic must be an integral curve for $\nabla\delta$. If we exit from $q \in \text{Int}(\mathcal{Z})$, this geodesic exists, is unique, and must be normal (see [FPR20, Proposition 2.7] or [BRR24, Proposition 2.2]). However, this is not true in all generalities from $q \in \partial\mathcal{Z}$, as it may be done by means of different normal geodesics, or abnormal ones. However, the existence of a normal geodesic is ensured. Finally, the smoothness assumption from H0' forces all possibilities of leaving \mathcal{Z} from q to be along the same path. That is, even if $q \in \partial\mathcal{Z}$, this geodesic must be unique (modulo reparametrization) and normal. This in turns forces $\mathcal{U} = \{p \in \mathbb{M}, \delta(p) < L\}$.*

Set $\mathcal{U}^{-\varepsilon} \subset \mathcal{U}$ to be diffeomorphic to $\Omega^{-\varepsilon} := \{(x, y) \in \Omega, d((x, y), \partial\Omega) > \varepsilon\}$. Namely, $\Omega^{-\varepsilon} = (-L + \varepsilon, L - \varepsilon) \times \Omega_y^{-\varepsilon}$, with $\Omega_y^{-\varepsilon} = (\varepsilon, \pi - \varepsilon)$, \mathbb{R} , or \mathbb{S}^1 . Therefore, by Proposition 5.1, we have boundary null-controllability in time T for the system

$$\begin{cases} \partial_t f - \Delta f &= 0, & (t, x, y) \in (0, T) \times \mathcal{U}^{-\varepsilon}, \\ f(t, p) &= 0, & (t, p) \in (0, T) \times \partial\mathcal{U}^{-\varepsilon} \cap \partial\mathbb{M}, \\ f(t, p) &= v(t, p), & (t, p) \in (0, T) \times \partial\mathcal{U}^{-\varepsilon}, \\ f(0, p) &= f_0(p), & p \in \mathcal{U}^{-\varepsilon}, \end{cases}, \quad (75)$$

where $f_0 \in L^2(\mathcal{U}^{-\varepsilon})$, f is the state, $v \in L^2((0, T) \times \partial\mathcal{U}^{-\varepsilon})$ is the control.

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The diffeomorphism $\tilde{\phi} : \mathcal{U} \rightarrow \Omega$ induces a unitary transformation

$$\begin{aligned} S : L^2(\mathcal{U}, \mu) &\rightarrow L^2(\Omega, h(x)dxdy) \\ f &\mapsto Tf = f \circ \phi, \end{aligned}$$

and we have the coordinates representation of Δ restricted to \mathcal{U} to be

$$G := S\Delta S^{-1} = \frac{1}{h(x)}\partial_x(h(x)\partial_x f) + q(x)^2\partial_y(r(y)^2 f). \quad (76)$$

We can therefore address the problem of boundary null-controllability in coordinates, with the system

$$\begin{cases} \partial_t f - (\frac{1}{h(x)}\partial_x(h(x)\partial_x) + q(x)^2\partial_y(r(y)^2))f &= 0, & (t, x, y) \in (0, T) \times \Omega^{-\varepsilon}, \\ f(t, p) &= v(t, p), & (t, p) \in (0, T) \times \partial\Omega^{-\varepsilon}, \\ f(0, p) &= f_0(p), & p \in \Omega^{-\varepsilon}, \end{cases} \quad (77)$$

posed in $L^2(\Omega^{-\varepsilon}, h(x)dxdy)$. Thanks to Lemma 5.3, we have that

- (i) under **H1-loc**, the control is supported on the whole boundary,
- (ii) under **H1-glob**, the control is supported on $(\{-L + \varepsilon\} \cup \{L - \varepsilon\}) \times \Omega_y$.

Applying Proposition 5.2, this in turn implies internal null-controllability of system

$$\begin{cases} \partial_t f - (\frac{1}{h(x)}\partial_x(h(x)\partial_x) + q(x)^2\partial_y(r(y)^2))f &= u(t, x, y)\mathbf{1}_\omega(x, y), & (t, x, y) \in (0, T) \times \Omega, \\ f(t, p) &= 0, & (t, p) \in (0, T) \times \partial\Omega, \\ f(0, p) &= f_0(p), & (p) \in \Omega, \end{cases} \quad (78)$$

posed in $L^2(\Omega, h(x)dxdy)$, with the control supported

- (i) on a crown consisting of all point at distance less than ε from the boundary, i.e. complementary of a rectangle intersecting $\{x = 0\}$ in its interior, under **H1-loc**,
- (ii) on two vertical strips of the form $((-L, -L + \varepsilon) \cup (L - \varepsilon, L)) \times \Omega_y$, under **H1-glob**.

The problem of internal null-controllability in Ω is nothing less than the coordinate representation of system (2) restricted to \mathcal{U} , with Dirichlet boundary condition, and the control zone being an open subset of \mathcal{U} .

Consider now the unitary transformation

$$\begin{aligned} T : L^2(h(x)dxdy) &\longrightarrow L^2(dxdy) \\ f &\longmapsto \sqrt{h}f. \end{aligned} \quad (79)$$

Then, setting $G_V := TGT^{-1} = \partial_x^2 f + q(x)^2\partial_y(r(y)^2\partial_y f) + V(x)f$, the observability inequality associated to the problem of internal null-controllability in time $T > 0$ of system (78) is equivalent to the observability inequality (10), associated to the problem of internal null-controllability in time $T > 0$ of system (7), with $V(x) = \frac{\partial_x^2(\sqrt{h})}{\sqrt{h}}$.

Therefore, the non null-controllability results presented in Theorem 1.5 are inherited from both Theorems 2.1 and 2.2.

1. Under **H1-loc**, and $\gamma = 1$, we have for every $\varepsilon > 0$ arbitrary small, that for any time $T < \frac{1}{q'(0)} \min\{d_{agm}(-L + \varepsilon), d_{agm}(L - \varepsilon)\}$, system (7) is not null-controllable by Theorem 2.1. Making ε tend to zero, we get that $T(\omega) \geq \frac{1}{q'(0)} \min\{d_{agm}(-L), d_{agm}(L)\}$.
2. Under **H1-glob**, by Theorem 2.2, when $\gamma > 1$ we obviously never have null-controllability, since system (7) is never null-controllable, and for $\gamma = 1$, we obtain the same lower bound for $T(\omega)$ as above.

□

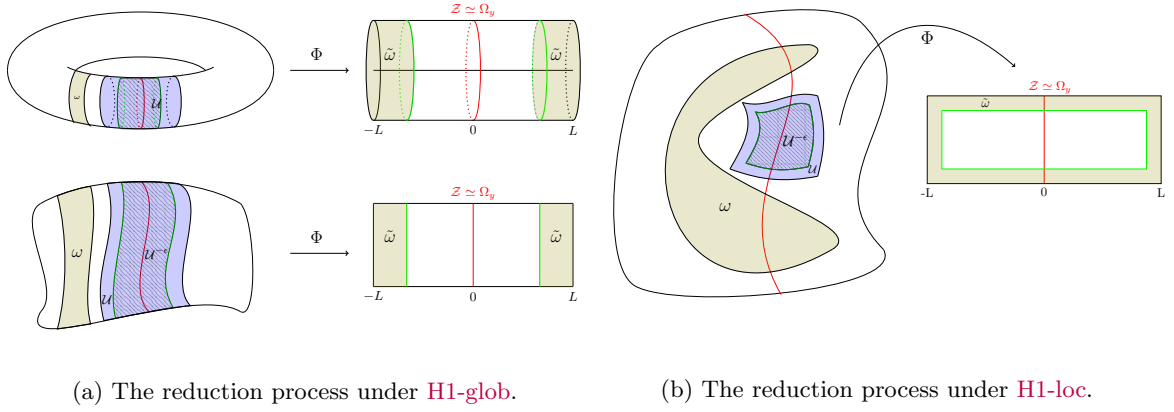


Figure 2: The singularity is in red, the control zone in olive, the tubular neighborhood \mathcal{U} in blue, and the dashed part of \mathcal{U} represents the domain on which we reduce the problem to a boundary control problem, with the boundary control supported on the green paths. The arrows represent the passage into coordinates representation, with in olive the support of the control for the problem of internal controllability.

6 Applications, Comments, and Open Problems

The Grushin sphere. We retrieve with the strategy of Theorem 1.5 the negative result of Tamekue [Tam22]. Namely, consider $\mathbb{M} = \mathbb{S}^2$, endowed with the sub-Riemannian structure generated by the vector fields $X = -z\partial_y + y\partial_z$, $Y = -z\partial_x + x\partial_z$. Observe that they are linearly independent outside $\{z = 0\}$, but $\{X, Y\}$ is bracket-generating since $[X, Y] = -y\partial_x + x\partial_y$. Moreover, **H0** is satisfied. We also endow \mathbb{S}^2 with the restricted Lebesgue measure. Consider system (2) with ω consisting of two symmetric, with respect to $\{z = 0\}$, horizontal crowns as in [Tam22, Fig. 1]. For example $\omega = \{(x, y, z) \in \mathbb{S}^2, |z| > a\}$ for some $a > 0$. Then, reducing the problem in $\mathbb{S}^2 \setminus \omega$, in spherical coordinates, the restriction of the Lebesgue measure writes $\cos(x)dx dy$, and we have that $\Delta = \frac{1}{\cos(x)}\partial_x(\cos(x)\partial_x) + \tan(x)^2\partial_y^2$ on $(-a, a) \times \mathbb{S}^1$. Then, by parity of the function \tan , our result show that $T(\omega) \geq \frac{1}{q'(0)} \int_0^a \tan(s) ds = \ln\left(\frac{1}{\cos(a)}\right)$, which coincides with the negative result in [Tam22]. This strategy would also hold if the two components of the control zone are not at the same nonnegative distance of the singularity.

The Grushin equation on the real line and lack of optimality. Consider the Grushin equation on $\mathbb{M} = \mathbb{R} \times (0, \pi)$, with the Lebesgue measure, controlled on vertical strips. Theorem 1.5

says that we can reduce the analysis of the negative results to the case $(-L, L) \times (0, \pi)$. We can emphasize that when $\lim_{|x| \rightarrow +\infty} |q(x)| = +\infty$, and q satisfies **H2**, the strategy proposed in the case where Ω_x is bounded extends to \mathbb{R} , and also if we perturb the equation by a potential $V \in L_{loc}^\infty(\mathbb{R})$ bounded below. The strategy of Theorem 2.2 extended to \mathbb{R} and Theorem 1.5 both give the same results. Nonetheless, it is important to notice that in both cases we obtain a lower bound for the minimal time of null-controllability, even when the equation is never null-controllable (for example when we do not control in one of the connected component of $\mathbb{M} \setminus \mathcal{Z}$).

The Grushin operator without warped product assumption. Consider $\mathbb{M} = (-1, 1) \times (0, \pi)$ with the Lebesgue measure. Set $X = \partial_x$, $Y = \tilde{q}(x, y)\partial_y$. The sub-Laplacian writes $\Delta = \partial_x^2 + \partial_y(\tilde{q}(x, y)^2\partial_y)$. Although there is no result for the heat equation associated to this operator, Theorem 1.5 says that to obtain a negative result, it is sufficient that \tilde{q} locally writes as $\tilde{q}(x, y) = q(x)r(y)$. Having a result for Δ , even in the particular case of the rectangle, is an open problem. The usual Fourier techniques are not valid in this case.

Non-symmetry of the tubular neighborhood. While Theorems 2.1 and 2.2 assume that Ω_x is symmetric with respect to zero, they remain valid when substituting $(-L, L)$ with any other interval containing 0 in its interior, up to some minor adjustments, in particular for Theorem 2.2. For Theorem 2.1, the result comes from [DKR23] where the symmetry is never assumed. For Theorem 2.2, the argument for exponential decay of the classical eigenfunctions in Proposition 3.6 is no longer valid on the whole interval. That is, by lack of parity, Remark 3.8 does not hold. One can either reapply the proof to negative x , or obtain an exponential decay argument as in Proposition 4.1. In any case, the asymptotic on the eigenvalues still hold, and so does Theorem 2.2. As a consequence, we can choose \mathcal{U} to be diffeomorphic to $(-L_-, L_+) \times \Omega_y$.

Positive results on \mathbb{M} . Although we only focus on negative results, positive results are achievable in our settings when $\gamma = 1$. Indeed, if a positive result is achieved in \mathcal{U} , it is possible to extend it to the whole manifold through cutoffs arguments, as the problem is uniformly parabolic away from the singularity. Nevertheless, we must also require that $\inf_{\mathcal{U}} q'(x) > 0$, which is a stronger assumption than **H2** (see [BDE20]). Proving positive results for the Grushin equation on rectangular domain without the monotony assumption on q is still an open problem. Moreover, we stress that obviously, if $\mathcal{Z} \subset \omega$, system (2) is null-controllable in any time $T > 0$. But a positive result for system (2) in the case $\mathcal{Z} \subset \partial\omega$, and ω belongs to only one connected component of $\mathbb{M} \setminus \mathcal{Z}$, cannot be covered through cutoffs arguments, as the latter will always add a connected component to the control zone.

Complementary of a horizontal strip. The analogue on manifolds of controlling in the complementary of a horizontal strip on a rectangular domain, is to say that there exists a normal geodesic to the singularity that never intersects the control zone. Although we expect to never have null-controllability for any $\gamma \geq 1$, Theorem 1.5 does not cover this case correctly. It only gives a lower bound for $T(\omega)$. It is also important to say that a negative result when $\gamma > 1$ in this setting on the rectangle has never been obtained, and is an open problem. If such a result is obtained, Theorem 1.5(ii) would hold under **H1-loc**.

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