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CONVOLUTION OF PERIODIC MULTIPLICATIVE FUNCTIONS AND THE DIVISOR PROBLEM

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Abstract. We study a certain class of arithmetic functions that appeared in Klurman's classification of ± 1 multiplicative functions with bounded partial sums, c.f., Comp. Math. 153 (8), 2017, pp. 1622-1657. These functions are periodic and 1-pretentious. We prove that if f_1 and f_2 belong to this class, then $\sum_{n\leq x}(f_1 * f_2)(n) = \Omega(x^{1/4})$. This confirms a conjecture by the first author made in Bull. Braz. Math. Soc. 2022, 53 (4), 1317-1329, 2022. As a byproduct of our proof, we studied the correlation

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_0^X \Delta(x)\Delta(\theta x)dx,$$

where $\Delta(x)$ is the error term in the classical Dirichlet divisor problem. We prove that this limit is positive and non-trivial when θ is rational, and 0 when θ is irrational. Moreover, if θ has a finite degree of irrationality, then we can make it quantitative the limit above in terms of this degree.

1. Introduction

1.1. **Main result and background.** A question posed by Erdős [6], known as the Erdős discrepancy problem, states that whether for all arithmetic functions $f : \mathbb{N} \to \{-1, 1\}$ we have that the discrepancy

$$\sup_{x,d} \left| \sum_{n\leq x} f(nd) \right| = \infty. \tag{1}$$

When in addition f is assumed to be completely multiplicative, then this reduces to whether f has unbounded partial sums.

Date: May 8, 2023.

In 2015, Tao [12] proved that (1) holds for all $f : \mathbb{N} \to \{-1, 1\}$, and a key point of its proof is that it is sufficient to establish (1) only in the class of completely multiplicative functions f taking values in the unit (complex) circle.

When $f : \mathbb{N} \to \{-1, 1\}$ is assumed to be only multiplicative, then not necessarily f has unbounded partial sums. For example, $f(n) = (-1)^{n+1}$ is multiplicative and clearly has bounded partial sums. In this case, $f(2^k) = -1$ for all positive integers k. It was observed by Coons [5] that this rigidity on powers of 2 is actually necessary under suitable conditions on the values that f takes at the remaining primes. Later, in the same paper [12], Tao gave a partial classification of multiplicative functions taking values ± 1 with bounded partial sums: They must satisfy the previous rigidity condition on powers of 2 and they must be 1-pretentious (for more on pretentious Number Theory we refer reader to [7]), that is,

$$\sum_p \frac{1 - f(p)}{p} < \infty.$$

Later, Klurman [9] proved that the only multiplicative functions f taking ± 1 values and with bounded partial sums are the periodic multiplicative functions with sum 0 inside each period, and thus, closing this problem for ± 1 multiplicative functions.

Building upon the referred work of Klurman, the first author proved [1] that if we allow values outside the unit disk, a M-periodic multiplicative function f with bounded partial sums such that $f(M) \neq 0$ satisfies

i. For some prime $q|M$, $\sum_{k=0}^{\infty} \frac{f(q^k)}{q^k} = 0$.
ii. For each $p^a \| M$, $f(p^k) = f(p^a)$ for all $k \geq a$.
iii. For each $\gcd(p, M) = 1$, $f(p^k) = 1$, for all $k \geq 1$.

Conversely, if $f : \mathbb{N} \to \mathbb{C}$ is multiplicative and the three conditions above are satisfied, then f has period M and has bounded partial sums. Therefore, these three conditions above give examples of multiplicative functions with values outside the unit disk with bounded partial sums, despite of the fact that $f(M)$ is zero or not.

Remark 1.1. It is interesting to observe that when it is assumed that $|f| \leq 1$, the only way to achieve condition i. is with $q = 2$ and $f(2^k) = -1$ for all $k \geq 1$.

Remark 1.2. What makes the difference between a multiplicative function f satisfying i-ii-iii from a non-principal Dirichlet character χ is that χ neither satisfies i. nor iii.

Here we are interested in the convolution $f_1 * f_2$ for f_1 and f_2 satisfying i-ii-iii above. It was proved [1] that

$$\sum_{n\leq x}(f_1 * f_2)(n) \ll x^{\alpha+\epsilon},$$

where α is the infimum over the exponents $a > 0$ such that $\Delta(x) \ll x^a$, where $\Delta(x)$ is the classical error term in the Dirichlet divisor problem:

$$\sum_{n\leq x}\tau(n) = x\log x + (2\gamma - 1)x + \Delta(x).$$

It was conjectured in [1] that the partial sums of $f_1 * f_2$ obey the same Ω bound for $\Delta(x)$, that is, $\sum_{n\leq x}(f_1 * f_2)(n) = \Omega(x^{1/4})$. Here we establish this conjecture.

Theorem 1.1. *Let f_1 and f_2 be periodic multiplicative functions satisfying i-ii-iii above, Then $\sum_{n\leq x}(f_1 * f_2)(n) = \Omega(x^{1/4})$.*

Example 1.1. The results from [1] give that for each prime q there exists a unique q-periodic multiplicative function f with bounded partial sums and such that $f(q) \neq 0$. In the case $q = 2$, the corresponding function is $f(n) = (-1)^{n+1}$. Therefore, in this particular case we have that $\sum_{n\leq x}(f*f)(n) = \Omega(x^{1/4})$. In particular, this establishes the conjecture in an uncovered case by Proposition 3.1 of [1].

1.2. **Proof idea.** To proof Theorem 1.1, our starting point is the following formula from [1]:

$$\sum_{n\leq x}(f_1 * f_2)(n) = \sum_{n|M_1M_2}(f_1 * f_2 * \mu * \mu)(n)\Delta(x/n), \tag{2}$$

where μ is the Möbius function. Therefore, the partial sums of $f_1 * f_2$ can be written as a finite linear combination of the quantities $(\Delta(x/n))_n$. Apart from the fact that

$\Delta(x) = \Omega(x^{1/4})$, we cannot, at least by a direct argument, prevent a conspiracy among the large values of $(\Delta(x/n))_n$ in such a way that always has a cancellation among a linear combination of them.

To circumvent this, our approach is inspired by an elegant result of Tong [13]:

$$\int_1^X \Delta(x)^2 dx = (1 + o(1)) \left(\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{3/2}} \right) X^{3/2}.$$

By (2), the limit

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \left| \sum_{n\leq x} (f_1 * f_2)(n) \right|^2 dx$$

can be expressed as a quadratic form with matrix $(c_{a,b})_{a,b|M_1M_2}$ where $c_{a,b}$ is the correlation

$$c_{a,b} := \lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x/a)\Delta(x/b)dx.$$

These correlations does not vanish. We generalized Tong's result and proved that, if $\lambda = \gcd(a, b)$, $c = a/\lambda$ and $d = b/\lambda$, then

$$c_{a,b} = \frac{1}{6\pi^2\sqrt{\lambda}cd} \sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{n^{3/2}}.$$

The divisor sum above is a multiplicative function in the variable cd. With that on hand, the matrix correlation-term $c_{a,b}$ can be expressed as

$$\frac{1}{\sqrt{\gcd(a, b)}} \varphi \left(\frac{\operatorname{lcm}(a, b)}{\gcd(a, b)} \right), \tag{3}$$

for some multiplicative function φ.

This matrix entanglement is hard to analyze directly. In section 4 we explore sufficient conditions for a matrix of the form (3) to be positive definite. When this happens, this ensures the referred Ω-bound. Thanks to the Selberg diagonalization procedure, we showed that when φ is completely multiplicative and satisfies other conditions, then this matrix is positive definite. Then, the proof of the main result consists in to conjugate our original matrix and reaching to another related to a completely multiplicative function. With standard linear algebra of Hermitian matrices we conclude that our matrix $(c_{a,b})_{a,b|M_1M_2}$ is positive definite.

1.3. **Byproduct study.** Motivated by Nyman's reformulation of the Riemann hypothesis [11], in recent papers [2, 3, 4] it has been study the correlation

$$A(\theta) := \int_0^\infty \{x\}\{\theta x\}\frac{dx}{x^2},$$

where $\theta > 0$ is a real number and $\{x\}$ stands for the fractional part of x. Several analytic properties for the function $A(\theta)$ have been show.

In the course of the proof of Theorem 1.1 we study the "*divisor*" analogue

$$I(\theta) = \lim_{X\to\infty} \frac{1}{X^{3/2}} \int_0^X \Delta(x)\Delta(\theta x)dx.$$

As mentioned before, when $\theta = p/q$ is a rational number, the limit above is described by a positive multiplicative function depending on p and q. Somewhat surprisingly, when θ is irrational we have the following result.

Proposition 1.1. *Let $\theta > 0$ be an irrational number with degree of irrationality η, that is, for each $\epsilon > 0$ there is a constant $C > 0$ such that the inequality*

$$|n - m\theta| \geq \frac{C}{m^{\eta+\epsilon}}$$

is violated only for a finite number of positive integers n and m. Then

$$\int_0^X \Delta(x)\Delta(\theta x)dx = O(X^{3/2-1/(18\eta)+\epsilon}).$$

In the other cases of irrationals θ, the integral above is $o(X^{3/2})$.

This shows that we have decorrelation among the values $\Delta(x)$ and $\Delta(\theta x)$ when θ is irrational, and moreover, this gives that the function $I(\theta)$ is discontinuous everywhere.

It is important to mention that a similar decorrelation also has been obtained by Ivić and Zhai in [8]. In this paper they show decorrelation between $\Delta(x)$ and $\Delta_k(x)$, where $\Delta_k(x)$ is the error term related to the k-fold divisor function, and $k = 3$ or 4.

2. Notation

2.1. **Asymptotic notation.** We employ both Vinogradov's notation $f \ll g$ or $f = O(g)$ whenever there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$, for all x in a set of parameters. When not specified, this set of parameters is $x \in (a, \infty)$ for sufficiently large $a > 0$. We employ $f = o(g)$ when $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. In this case a can be a complex number or $\pm\infty$. Finally, $f = \Omega(g)$ when $\limsup_{x\to a} \frac{|f(x)|}{g(x)} > 0$, where a is as in the previous notation.

2.2. **Number-theoretic notation.** Here p stands for a generic prime number. We sometimes denote the least common multiple between a, b as $\mathrm{lcm}(a, b)$. The greatest common divisor is denoted by $\gcd(a, b)$. $*$ stands for Dirichlet convolution between two arithmetic functions: $(f * g)(n) = sum_{d|n} f(d)g(n/d)$.

3. Multiplicative auxiliaries

We begin the proof with the following Lemma.

Lemma 3.1. *Let a, b be positive integers, $\lambda = \gcd(a, b)$, $c = a/\lambda$ and $d = b/\lambda$. Then*

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x/a)\Delta(x/b)dx = \frac{1}{6\pi^2\sqrt{\lambda}cd} \sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{n^{3/2}}.$$

Proof. Let $N > 0$ and $\epsilon > 0$ be a small number that may change from line after line. We proceed with Voronoï's formula for $\Delta(x)$ in the following form (see [10])

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\leq N} \frac{\tau(n)}{n^{3/4}} \cos(\sqrt{nx} - \pi/4) + R_N(x)$$

where, for every positive ϵ, we have

$$R_N(x) \ll x^{\epsilon} + \frac{x^{1/2+\epsilon}}{N^{1/2}}.$$

We select N at the end. With this formula we have that in the range $1 \leq x \leq X$,

$$\Delta(x/a) = \frac{(x/a)^{1/4}}{\pi\sqrt{2}} \sum_{n\leq N} \frac{\tau(n)}{n^{3/4}} \cos(\sqrt{nx/a} - \pi/4) + R_N(x/a) = U_N(x/a) + R_N(x/a)$$

say.

Now,

$$\int_1^X \Delta(x/a)\Delta(x/b)dx = \int_1^X U_N(x/a)U_N(x/b)dx + \int_1^X U_N(x/a)R_N(x/b)dx + \int_1^X U_N(x/b)R_N(x/a)dx + \int_1^X R_N(x/a)R_N(x/b)dx = \int_1^X U_N(x/a)U_N(x/b)dx + O\left(X^{1+1/4+\epsilon} + \frac{X^{1+3/4+\epsilon}}{\sqrt{N}}\right),$$

where we used the Cauchy-Schwarz inequality in the last equality. Let now $\lambda = \gcd(a,b)$, $c = a/\lambda$ and $d = b/\lambda$. By making the change of variable $u = x/\lambda$, we reach

$$\int_1^X U_N(x/a)U_N(x/b)dx = \lambda \int_1^{X/\lambda} U_N(x/c)U_N(x/d)dx$$
$$= \frac{\lambda}{2\pi^2(cd)^{1/4}} \sum_{n,m\leq N} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \int_1^{X/\lambda} x^{1/2}\cos(\sqrt{nx/c} - \pi/4)\cos(\sqrt{mx/d} - \pi/4)$$
$$= \frac{\lambda}{\pi^2(cd)^{1/4}} \sum_{n,m\leq N} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \int_1^{(X/\lambda)^{1/2}} u^2\cos(u\sqrt{n/c} - \pi/4)\cos(u\sqrt{m/d} - \pi/4)du,$$

where in the last equality above we made a change of variable $u = \sqrt{x}$. We claim now that the main contribution comes when $n/c = m/d$. Since c and d are coprime, this implies that $m = dk$ and $n = ck$. Therefore the sum over these n and m can be written as

$$\frac{\lambda}{\pi^2 cd}\sum_{k=1}^{\infty} \frac{\tau(ck)\tau(dk)}{k^{3/2}} \int_1^{(X/\lambda)^{1/2}} u^2\cos^2(u\sqrt{k} - \pi/4)^2du + O\left(\frac{X^{3/2+\epsilon}}{\sqrt{N}}\right). \tag{4}$$

We recall now that $\cos^2(v) = \frac{1+\cos(2v)}{2}$, and hence the integral above is

$$\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos^2(\sqrt{n}x - \pi/4)dx = \frac{X^{3/2}}{6\lambda^{3/2}} + O(X), \tag{5}$$

where the big-oh term is uniform in n. Now we will show that the sum over those n and m such that $n/c \neq m/d$ will be $o(X^{3/2})$. With this the proof will be complete by combining (4) and (5).

We recall the identity $2\cos(u)\cos(v) = \cos(u-v)+\cos(u+v)$. Thus, for $\sqrt{n/c} \neq \sqrt{m/d}$, we find that

$$\begin{aligned}
&\int_1^{X^{1/2}/\lambda^{1/2}} x^2 \cos(\sqrt{n/c}x - \pi/4)\cos(\sqrt{m/d}x - \pi/4)dx \\
&= \int_1^{X^{1/2}/\lambda^{1/2}} x^2 \cos((\sqrt{n/c} - \sqrt{m/d})x)dx + \int_1^{X^{1/2}/\lambda^{1/2}} x^2 \sin((\sqrt{n/c} + \sqrt{m/d})x)dx \\
&\ll \frac{X}{\sqrt{n/c} - \sqrt{m/d}} \\
&\ll \frac{\sqrt{n/c} + \sqrt{m/d}}{nd - mc} X.
\end{aligned}$$

Let $\mathbb{1}_P(n)$ be the indicator that n has property P. We find that

$$\begin{aligned}
&\sum_{\substack{n,m\leq N \\ nd-mc\neq 0}} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \int_1^{X/\lambda} x^{1/2}\cos(\sqrt{nx/c} - \pi/4)\cos(\sqrt{mx/d} - \pi/4)dx \\
&\ll XN^{\epsilon} \sum_{\substack{n,m\leq N \\ nd-mc\neq 0}} \frac{\sqrt{n/c} + \sqrt{m/d}}{(nm)^{3/4}|nd-mc|} \\
&= XN^{\epsilon} \sum_{\substack{n,m\leq N \\ nd-mc\neq 0}} \frac{\sqrt{n/c} + \sqrt{m/d}}{(nm)^{3/4}|nd-mc|} \sum_{\substack{k=-N\max(c,d) \\ k\neq 0}}^{N\max(c,d)} \mathbb{1}_{nd-mc=k}.
\end{aligned}$$

On calling this sum S, we readily continue with

$$\begin{aligned}
S &\ll XN^{\epsilon} \sum_{k=1}^{N\max(c,d)} \frac{1}{k} \sum_{m\leq N} \frac{\sqrt{m}+\sqrt{k}}{((k+mc)m)^{3/4}} \\
&\ll XN^{\epsilon} \left(O(\log N)^2 + \sum_{k\leq N} \frac{1}{\sqrt{k}} \sum_{m\leq N} \frac{1}{(m^2+mk)^{3/4}} \right) \\
&\ll XN^{\epsilon} \left(O(\log N)^2 + \sum_{k\leq N} \frac{1}{\sqrt{k}} \left(\sum_{k\leq m\leq N} \frac{1}{m^{3/2}} + \frac{1}{k^{3/4}} \sum_{m\leq k} \frac{1}{m^{3/4}} \right) \right) \\
&\ll XN^{\epsilon}(\log N)^2.
\end{aligned}$$

Finally, by selecting $N = X^2$, we arrive at

$$\int_1^X \Delta(x/a)\Delta(x/b)dx = \frac{1}{6\pi^2\sqrt{\lambda}cd}\left(\sum_{n=1}^{\infty}\frac{\tau(cn)\tau(dn)}{n^{3/2}}\right)X^{3/2} + O(X^{3/2-1/4+\epsilon}),$$

where the main contribution in the O-term above comes from the usage of Cauchy-Schwarz in the beggining of the proof.

The proof is complete. □

Now we deviate from the main line and prove Proposition 1.1.

Proof of Proposition 1.1. By the proof of Lemma 3.1 we have that

$$\begin{aligned} I_\theta(X) :=& \int_0^X \Delta(x)\Delta(\theta x)dx \\ =& \frac{1}{\pi^2}\sum_{n,m\leq N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_0^{X^{1/2}} x^2\cos(x\sqrt{n}-\pi/4)\cos(x\sqrt{m\theta}-\pi/4)dx \\ &+ O\left(X^{1+1/4+\epsilon} + \frac{X^{1+3/4+\epsilon}}{\sqrt{N}}\right). \end{aligned}$$

Now, by appealing to the identity $2\cos(u)\cos(v) = \cos(u-v)+\cos(u+v)$, we reach at

$$I_\theta(X) = \frac{1}{2\pi^2}\sum_{n,m\leq N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_0^{X^{1/2}} x^2\cos(x(\sqrt{n}-\sqrt{m\theta}))dx + O\left(X^{1+1/4} + \frac{X^{1+3/4+\epsilon}}{\sqrt{N}}\right).$$

On calling the sum above $S_\theta(X)$, $a_{n,m} := \sqrt{n}-\sqrt{m\theta}$, we obtain that

$$S_\theta(X) = X^{3/2}\sum_{n,m\leq N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}),$$

where $\Lambda(0) := 1/3$ and for $u \neq 0$

$$\Lambda(u) := \frac{\sin(u)}{u} + 2\frac{\cos(u)}{u^2} - 2\frac{\sin(u)}{u^3}.$$

A careful inspection shows that Λ is continuous and for large $|u|$, $\Lambda(u) \ll |u|^{-1}$.

Now, for a large parameter T to be chosen later, we split

$$S_\theta(X) = X^{3/2}\sum_{\substack{n,m\leq N\\ |a_{n,m}\sqrt{X}|\leq T}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}) + X^{3/2}\sum_{\substack{n,m\leq N\\ |a_{n,m}\sqrt{X}|> T}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}).$$

We call the first sum in the right hand side above by *diagonal* contribution and the second sum by *non-diagonal* contribution. We select $T = X^{1/2-\delta}$ and $N = X^{1/2+\delta}$, for some small $\delta > 0$.

The diagonal contribution. We have that

$$D(X) = X^{3/2} \sum_{\substack{n,m \leq N \\ |a_{n,m}\sqrt{X}| \leq T}} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \Lambda(a_{n,m}\sqrt{X}) \tag{6}$$

$$\ll X^{3/2} N^{\epsilon} \sum_{m \leq N} \frac{1}{m^{3/4}} \sum_{n=m\theta - \frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}}}^{m\theta + \frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}}} \frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}}. \tag{7}$$

The inner sum above we split accordingly $\frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}}$ is below and above 1. In the case that this quantity is great or equal to 1, we have that $m \geq ((2\theta)^{-1} + o(1))X^{2\delta}$, and hence

$$\begin{aligned} & X^{3/2} N^{\epsilon} \sum_{((2\theta)^{-1}+o(1))X^{2\delta} \leq m \leq N} \frac{1}{m^{3/4}} \sum_{n=m\theta - \frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}}}^{m\theta + \frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}}} \frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} \\ & \ll X^{3/2} N^{\epsilon} \sum_{((2\theta)^{-1}+o(1))X^{2\delta} \leq m \leq N} \frac{1}{m^{3/4}} \cdot \frac{1}{m^{3/4}} \frac{\sqrt{m}}{X^{\delta}} \\ & \ll X^{3/2-\delta} N^{\epsilon}. \end{aligned}$$

In the case that $\frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}} \leq 1$, we have that $m \leq ((2\theta)^{-1} + o(1))X^{2\delta}$, and now the Diophantine properties of θ come in to play. If the degree of irrationality of θ is η, we have that for each ϵ there is a constant $C > 0$ such that the inequality

$$|n - m\theta| \geq \frac{C}{m^{\eta+\epsilon}}$$

is violated only for a finite number of positive integers n and m. In our case, this allows us to lower bound $|a_{n,m}\sqrt{X}|$ for all but a finite number of n and m such that

$1 \le m \ll X^{2\delta}$ and $\sqrt{n} \approx \sqrt{m\theta}$:

$$\begin{aligned} |a_{n,m}\sqrt{X}| \cdot \frac{\sqrt{n}+\sqrt{m\theta}}{\sqrt{n}+\sqrt{m\theta}} &= \sqrt{X}\frac{|n-m\theta|}{\sqrt{n}+\sqrt{m\theta}} \\ &\ge \frac{\sqrt{X}}{m^{\eta+\epsilon}(\sqrt{n}+\sqrt{m\theta})} \\ &\gg X^{1/2-(2\eta+1)\delta-\epsilon}. \end{aligned}$$

Observe that the diagonal contribution from those exceptional n and m will be at most $O(X)$. With these estimates on hand and recalling that $\Lambda(u) \ll |u|^{-1}$, we obtain

$$\begin{aligned} & X^{3/2}N^{\epsilon} \sum_{m \le ((2\theta)^{-1}+o(1))X^{2\delta}} \frac{1}{m^{3/4}} \sum_{n=m\theta-\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}}^{m\theta+\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}} \frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} \\ & \ll X^{3/2}N^{\epsilon} \sum_{m \le ((2\theta)^{-1}+o(1))X^{2\delta}} \frac{1}{m^{3/2}} \cdot \frac{1}{X^{1/2-(2\eta+1)\delta-\epsilon}} + O(X) \\ & \ll X^{1+(2\eta+1)\delta+\epsilon}. \end{aligned}$$

Therefore, the diagonal contribution is at most

$$D(X) \ll X^{1+(2\eta+1)\delta+\epsilon} + X^{3/2-\delta+\epsilon}.$$

The non-diagonal contribution. Now, we reach

$$\begin{aligned} X^{3/2} \sum_{\substack{n,m \le N \\ |a_{n,m}\sqrt{X}|>T}} \frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}) &\ll \frac{X^{3/2}N^{1/2+\epsilon}}{T} \\ &= X^{3/2+1/4+(\delta+\epsilon)/2+\epsilon\delta-1/2+\delta} \\ &= X^{1+1/4+3\delta/2+\epsilon/2+\epsilon\delta}. \end{aligned}$$

We choose $\delta = \frac{1}{3(2\eta+1)}$ and obtain

$$I_{\theta}(X) = X^{3/2-1/(18\eta)}.$$

The proof of the first part of Proposition 1.1 is complete.

Now we assume that θ is a Liouville number, *i.e.*, θ doesn't have finite degree of irrationality. We see that the non-diagonal argument does not depend on the Diophantine properties of θ. Let $\eta > 0$ be a large fixed number, $t > 0$ a small number that will tend to 0. For $D(X)$ as in (6), by repeating verbatim the estimates above we have that

$$D(X) \ll X^{3/2} \sum_{m \leq ((2\theta)^{-1}+o(1))X^{2\delta}} \frac{\tau(m)}{m^{3/4}} \sum_{n=m\theta-\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}}^{m\theta+\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}} \frac{\tau(n)|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} + O(X^{3/2-\delta}N^{\epsilon}).$$

Let $\|x\|$ be the distance from x to the nearest integer. We split the sum over m above into two sums: One over those m such that $\|m\theta\| > tm^{-\eta}$ and the other over m such that $\|m\theta\| \leq tm^{-\eta}$.

Repeating the argument above for non-Liouville numbers, we have that the contribution over those m such that $\|m\theta\| > tm^{-\eta}$ is $O(t^{-1}X^{1+\delta(2\eta+1)})$. Therefore

$$D(X) \ll X^{3/2} \sum_{\substack{m=1 \\ \|m\theta\| \leq tm^{-\eta}}}^{\infty} \frac{1}{m^{3/2-\epsilon}} + O(t^{-1}X^{1+\delta(2\eta+1)} + X^{3/2-\delta+\epsilon}).$$

Combining all these estimates, we see that

$$\limsup_{X\to\infty} \frac{1}{X^{3/2}} \left| \int_0^X \Delta(x)\Delta(\theta x)dx \right| \ll \sum_{\substack{m=1 \\ \|m\theta\| \leq tm^{-\eta}}}^{\infty} \frac{1}{m^{3/2-\epsilon}}.$$

Since the upper bound above holds for all $t > 0$, we have that as $t \to 0^+$, the sum above converges to 0 and thus implying that the lim sup is 0. The proof is complete. □

Our next task is to evaluate $\sum_{n\geq 1} \tau(cn)\tau(dn)/n^{3/2}$ for coprime positive integers c and d.

Lemma 3.2. *Let c be fixed positive number and $f(n)$ be a multiplicative function with $f(c) \neq 0$. Then $n \mapsto \frac{f(cn)}{f(c)}$ is multiplicative.*

Proof. For positive integers u, v, we have

$$f(u)f(v) = f(\gcd(u,v))f(\mathrm{lcm}(u,v)).$$

Let $u = cn$, $v = cm$ with $\gcd(n, m) = 1$. Then $f(cn)f(cm) = f(c)f(cnm)$.

Therefore, we obtained

$$\frac{f(cm)}{f(c)}\frac{f(cn)}{f(c)} = \frac{f(cnm)}{f(c)}.$$

□

Lemma 3.3. *Let c, d be two fixed positive integers with $\gcd(c, d) = 1$. Then*

$$\sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{n^s} = \tau(cd)\frac{\zeta(s)^4}{\zeta(2s)} \prod_{p^k \| cd} \left(1 + p^{-s}\right)^{-1} \left(1 - \tfrac{(k-1)}{(k+1)}p^{-s}\right).$$

Proof. Note that $\frac{\tau(cn)}{\tau(c)}$ is a multiplicative function in the variable n, and so is $\frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)}$. Therefore, for $\Re(s) > 1$ we have the following Euler factorization

$$\sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)n^s} = \prod_{p \nmid cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tau(p^\ell)^2}{p^{\ell s}}\right) \prod_{p \mid cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tau(cp^\ell)\tau(dp^\ell)}{\tau(c)\tau(d)p^{\ell s}}\right).$$

For $|x| < 1$, we know that

$$\sum_{\ell=0}^{\infty} (\ell+1)x^\ell = \frac{1}{(1-x)^2}, \qquad \sum_{\ell=0}^{\infty} (\ell+1)^2 x^\ell = \frac{(1+x)}{(1-x)^3},$$

from which we also derive that

$$\sum_{\ell=0}^{\infty} \ell(\ell+1)x^\ell = \frac{2x}{(1-x)^3}.$$

Now,

$$\begin{aligned}
\prod_{p \nmid cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tau(p^\ell)^2}{p^{\ell s}}\right) &= \prod_{p} \left(1 + \sum_{\ell=1}^{\infty} \frac{(\ell+1)^2}{p^{\ell s}}\right) \prod_{p \mid cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{(\ell+1)^2}{p^{\ell s}}\right)^{-1} \\
&= \prod_{p} \frac{(1+p^{-s})}{(1-p^{-s})^3} \prod_{p \mid cd} \frac{(1-p^{-s})^3}{(1+p^{-s})} \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_{p \mid cd} \frac{(1-p^{-s})^3}{(1+p^{-s})}.
\end{aligned}$$

If $\gcd(c,d)=1$

$$\prod_{p|cd}\left(1+\sum_{\ell=1}^{\infty}\frac{\tau(cp^\ell)^2\tau(dp^\ell)^2}{\tau(c)\tau(d)p^{\ell s}}\right)=\prod_{p^k\|cd}\left(1+\sum_{\ell=1}^{\infty}\frac{(k+1+\ell)(\ell+1)}{(k+1)p^{\ell s}}\right)$$
$$=\prod_{p^k\|cd}\left(1+\sum_{\ell=1}^{\infty}\frac{(\ell+1)}{p^{\ell s}}+\frac{1}{k+1}\sum_{\ell=1}^{\infty}\frac{\ell(\ell+1)}{p^{\ell s}}\right)$$
$$=\prod_{p^k\|cd}\left(1-p^{-s}\right)^{-3}\left(1-\tfrac{(k-1)}{(k+1)}p^{-s}\right).$$

□

4. Quadratic forms auxiliaries

The main proof will lead to considering the quadratic form attached to a matrix of the form

$$M_{S,\varphi}=\left(\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)\right)_{a,b\in S} \tag{8}$$

where S is some finite set of integers while φ is a *non-negative multiplicative function such that* $\varphi(p^k)\leq 1$. So we stray somewhat from the main line and investigate this situation. Our initial aim is to find conditions under which the associated quadratic form is positive definite, but we shall finally restrict our scope. GCD-matrices have received quite some attention, but it seems the matrices occuring in (8) have not been explored. We obtain results in two specific contexts.

Completely multiplicative case. Here is our first result.

Lemma 4.1. *When φ is completely multiplicative, the matrix $M_{S,\varphi}$ is non-negative. When $p^{1/4}\varphi(p)\in(0,1)$ and S is divisor closed, this matrix is positive definite. The determinant in that case is given by the formula*

$$\det\left(\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)\right)_{a,b\in S}=\prod_{d\in S}\varphi(d)^2(\mu*\psi)(d),$$

where ψ is the completely multiplicative function given by $\psi(p)=1/(\sqrt{p}\varphi(p)^2)$.

By *divisor closed*, we mean that every divisor of an element of S also belongs to S.

Proof. We write

$$\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) = \frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) = \varphi(a)\varphi(b)\psi(\gcd(a,b))$$

where $\psi(n) = 1/(\varphi(n)^2\sqrt{n})$ is another non-negative multiplicative function. We introduce the auxiliary function $h = \mu * \psi$. Notice that this function is multiplicative and non-negative, as $\psi(p) \geq 1$. We use Selberg's diagonalization process to write

$$\begin{aligned}\sum_{a,b\in S}\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)x_a x_b &= \sum_{a,b\in S}\psi(\gcd(a,b))\varphi(a)x_a\varphi(b)x_b \\ &= \sum_{a,b\in S}\sum_{d|(a,b)}h(d)\varphi(a)x_a\varphi(b)x_b \\ &= \sum_{d}h(d)\left(\sum_{\substack{a\in S\\ d|a}}\varphi(a)x_a\right)^2\end{aligned}$$

from which the non-negativity follows readily. When φ verifies the more stringent condition that $p^{1/4}\varphi(p) \in (0,1)$, we know that both φ and h are strictly positive. Let us define $y_d = \sum_{\substack{a\in S\\ d|a}}\varphi(a)x_a$. The variable d varies in the set D of divisors of S. We assume that S is divisor closed, so that $D = S$. We can readily invert the triangular system giving the y_d's as functions of the x_a's into

$$\varphi(a)x_a = \sum_{a|b}\mu(b/a)y_b$$

Indeed, the fact that the mentioned system is triangular ensures that a solution y is unique if it exists. We next verify that the proposed expression is indeed a solution by:

$$\sum_{\substack{a\in S\\ d|a}}\varphi(a)x_a = \sum_{\substack{a\in S\\ d|a}}\sum_{a|b}\mu(b/a)y_b = \sum_{\substack{b\in S\\ d|b}}y_b\sum_{d|a|b}\mu(b/a) = y_d$$

as the last inner sum vanishes when $d \neq b$. We thus have a writing as a linear combination of squares of independant linear forms. In a more pedestrian manner, if our quadratic form vanishes, then all y_d's do vanish, hence so do the x_a's. □

Here is a corollary.

Lemma 4.2. *When the set S contains solely squarefree integers, the matrix $M_{S,\varphi}$ is non-negative.*

Proof. Simply apply Lemma 4.1 to completely multiplicative function φ' that have the same values on primes as φ. □

An additive-like situation. Let us restrict our attention to the case

$$S = \{1, p, p^2, \cdots, p^K\}.$$

In that case, the matrix has the form

$$\mathcal{M}_{\varphi,K} = \left(\frac{1}{p^{\min(i,j)/2}} \varphi(p^{\max(i,j)-\min(i,j)}) \right)_{i,j \leq K}.$$

We have not been able to get general results like Lemma 4.2 in that case. We may however work out some criterium that is simple to verify in our case. We first recall the following theorem of Frobenius.

Lemma 4.3. *A hermitian complex valued matrix $M = (m_{i,j})_{i,j \leq K}$ defines a positive definite form if and only if all its principal minors $\det(m_{i,j})_{i,j \leq k}$ for $k \leq K$ are positive.*

So in our case, here is the list of conditions to verify, where we have set $q = 1/\sqrt{p}$:

- $q - \varphi(p)^2 > 0$.
- $(q - \varphi(p^2))(\varphi(p^2) - 2\varphi(p)^2 + q) > 0$.
- $q^2\varphi(p)^4 - 2q\varphi(p^3)\varphi(p)^3 + (4q^2\varphi(p^2) - 2q\varphi(p^2)^2 + \varphi(p^3)^2 - 3q^3)\varphi(p)^2 + 2(2q - \varphi(p^2))\varphi(p^3)\varphi(p^2)\varphi(p) + \varphi(p^2)^4 - 2q^2\varphi(p^2)^2 - q\varphi(p^3)^2 + q^4 > 0$.

The first condition is equivalent to the condition $p^{1/4}\varphi(p) < 1$ that we have already met in Lemma 4.1. We conclude to the next lemma.

Lemma 4.4. *Recall that $\varphi(1) = 1$, that $\varphi(p) \in (0, p^{-1/4})$. We have*

(1) The matrix $\mathcal{M}_{\varphi,1}$ is positive definite.
(2) The matrix $\mathcal{M}_{\varphi,2}$ is positive definite if and only if $\varphi(p^2) - 2\varphi(p)^2 + 1/\sqrt{p} > 0$.

(3) *The matrix* $\mathcal{M}_{\varphi,3}$ *is positive definite if and only if* $\mathcal{M}_{\varphi,2}$ *is positive definite and if*

$$q^2\varphi(p)^4 - 2q\varphi(p^3)\varphi(p)^3 + (4q^2\varphi(p^2) - 2q\varphi(p^2)^2 + \varphi(p^3)^2 - 3q^3)\varphi(p)^2 + 2(2q - \varphi(p^2))\varphi(p^3)\varphi(p^2)\varphi(p) + \varphi(p^2)^4 - 2q^2\varphi(p^2)^2 - q\varphi(p^3)^2 + q^4 > 0$$

where $q = 1/\sqrt{p}$.

(4) *The matrix* $\mathcal{M}_{\varphi,K}$ *is positive definite if and only if* $\mathcal{M}_{\varphi,K-1}$ *is positive definite and if* $\det \mathcal{M}_{\varphi,K} > 0$.

Here is another situation where we are able to conclude.

Lemma 4.5. *If for every* $i \leq K$, *we have*

$$\sum_{1\leq \ell \leq i-1} \varphi(p^\ell) p^{\ell/2} + \sum_{1\leq \ell \leq K-i} \varphi(p^\ell) < 1,$$

then the matrix $\mathcal{M}_{\varphi,K}$ *is definite positive.*

Proof. By Gershgorin Disks' Theorem (see the book [14] by Varga), we know that each eigenvalue of $\mathcal{M}_{\varphi,K}$ lies inside one of the Gershgorin's disks. As these eigenvalues are real numbers, the disk reduce to segments. They are the K intervals of center $1/p^{i/2}$ and radius $\sum_{j\neq i} \frac{1}{p^{\min(i,j)/2}} \varphi(p^{\max(i,j)-\min(i,j)})$. When this radius is strictly less than the center, we are sure that each eigenvalue is positive. We massage a bit this condition to get the one stated in the lemma, hence completing the proof. □

A tensor product-like situation. Lemma 4.2 is enough to solve our main problem when M_1 and M_2 are coprime squarefree integers. We need to go somewhat further. Let S be a divisor closed set. We consider the quadratic form

$$\sum_{a,b\in S} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) x_a x_b \tag{9}$$

where the variables x_a's are also multiplicatively split, i.e.

$$x_a = \prod_{p^k \| a} x_{p^k}. \tag{10}$$

Let $S(p)$ the subset of S made only of 1 and of prime powers. We extend S so that it contains every products of integers from any collection of distinct $S(p)^*$. We then find that

(11)
$$\sum_{a,b\in S}\frac{1}{\sqrt{\gcd(a,b)}}\varphi\Big(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\Big)x_ax_b=\prod_{p\in S}\Big(\sum_{p^k,p^\ell\in S(p)}\frac{\varphi\big(p^{\max(k,\ell)-\min(k,\ell)}\big)}{p^{\min(k,\ell)/2}}x_{p^k}x_{p^\ell}\Big).$$

We check this identity simply by opening the right-hand side and seeing that every summand from the left-hand side appears one and only one time. Then Lemma 4.4 and 4.5 apply.

5. Proof of the main result

Proof. By [1, Theorem 1.4], we have

$$S(x)=\sum_{n\le x}(f_1*f_2)(n)=\sum_{a|M_1M_2}g(a)\Delta(x/a)$$

where $g=f_1*f_2*\mu*\mu$. We infer from this formula that

$$\int_1^X|S(x)|^2dx=\sum_{a,b|M_1M_2}g(a)g(b)\int_1^X\Delta(x/a)\Delta(x/b)dx$$
$$=\frac{(1+o(1))}{6\pi^2}X^{3/2}\sum_{a,b|M_1M_2}g(a)g(b)\frac{\gcd(a,b)^{3/2}}{ab}\sum_{n=1}^{\infty}\frac{\tau(an/\gcd(a,b))\tau(bn/\gcd(a,b))}{n^{3/2}}$$

by Lemma 3.1. We next use Lemma 3.3 to infer that

$$\lim_{X\to\infty}\frac{1}{X^{3/2}}\int_1^X|S(x)|^2dx=\frac{\zeta(3/4)^4}{6\pi^2\zeta(3/2)}\sum_{a,b|M_1M_2}g(a)g(b)\frac{1}{\sqrt{\gcd(a,b)}}\varphi\Big(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\Big)$$

*This is not automatically the case, as the example $S=\{1,2,3,5,6,10\}$ shows, since 30 does not belong to S

where φ is multiplicative and at prime powers:

$$\begin{aligned}\varphi(p^k) &= \frac{(k+1)}{p^k}\frac{1}{1+p^{-3/2}}\left(1-\frac{(k-1)}{(k+1)p^{3/2}}\right)\\ &= \frac{1}{p^k(1+p^{-3/2})}\left((k+1)-(k-1)p^{-3/2}\right)\\ &= \frac{1}{p^k(1+p^{-3/2})}\left(k(1-p^{-3/2})+(1+p^{-3/2})\right)\\ &= \frac{k\beta(p)+1}{p^k},\end{aligned} \tag{12}$$

where

$$\beta = \beta(p) = \frac{1-p^{-3/2}}{1+p^{-3/2}}.$$

As we are dealing with a given prime p, we call it β.

Now, we can write

$$\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) = \frac{1}{(ab)^{1/4}}\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^{1/4}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right).$$

Since the terms $a^{-1/4}$ and $b^{-1/4}$ can be absorbed into the variables $g(a)$ and $g(b)$ of the quadratic form, it is enough to consider the quantity

$$\varphi^*\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right), \quad \text{where } \varphi^*(n) = n^{1/4}\varphi(n).$$

We note that

$$\varphi^*(p^k) = p^{k/4}\varphi(p^k) = \frac{k\beta+1}{p^{3k/4}}. \tag{13}$$

Due to (11) and the discussion before it, we now restrict to the prime power case, that is, we look to matrices of the form

$$\mathcal{M}_K = \left(\varphi^*(p^{|i-j|})\right)_{i,j\le K}.$$

Since φ^* is not completely multiplicative, it is not clear how to handle the matrix $\mathcal{M}_K$ directly. So, our aim will be to transform this into another matrix which, in some way associates with a completely multiplicative function. So, let us consider

$$\mathcal{A}_K = \mathcal{U}_K^\top \mathcal{M}_K \mathcal{U}_K,$$

where,

$$\mathcal{U}_K(i,j) = \begin{cases} \dfrac{\mu(p^{|i-j|})}{p^{3(|i-j|)/4}}, & \text{when } i \geq j \text{ or } (i,j) = (K-1,K), \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Simply speaking, $\mathcal{U}_K$ is 1 on the diagonal and $-p^{-3/4}$ on all $(i+1,i)$ as well as $(K-1,K)$. Also

$$\det(\mathcal{U}_K) = 1 - p^{-3/2}.$$

We now calculate the entries of the matrix $\mathcal{A}_K$. We have the following:

Proposition 5.1. *The matrix $\mathcal{A}_K$ above is given by:*

$$\mathcal{A}_K(i,j) = \beta(1-p^{-3/2}) \cdot \begin{cases} p^{-3|i-j|/4}, & \textit{when } 1 \leq i,j \leq K-1 \textit{ or } i=j=K, \\ 0, & \textit{otherwise.} \end{cases}$$

We begin with the following lemma:

Lemma 5.1. *We have*

$$\varphi^*(p^m) - p^{-3/4}\varphi^*(p^{|m-1|}) = p^{-3m/4}\beta, \quad \textit{for all } m \geq 0.$$

Proof. First, assume $m \geq 1$. We have

$$\varphi^*(p^m) - p^{-3/4}\varphi^*(p^{m-1}) = \frac{m\beta+1}{p^{3m/4}} - p^{-3/4}\frac{(m-1)\beta+1}{p^{3(m-1)/4}} = p^{-3m/4}\beta.$$

When $m = 0$, we have

$$1 - p^{-3/4}\varphi^*(p) = 1 - p^{-3/2}(\beta+1) = 1 - \frac{2p^{-3/2}}{1+p^{-3/2}} = \beta.$$

□

Now, we shall proceed with the proof of the Proposition 5.1.

Proof of Proposition 5.1. Let us first assume that $1 \le i, j \le K-1$. We have

$$\begin{aligned}\mathcal{A}_K(i,j) &= \sum_{k_1,k_2} \mathcal{U}_K^\top(i,k_1)\mathcal{M}_K(k_1,k_2)\mathcal{U}_K(k_2,j)\\ &= \sum_{\substack{k_1-i\in\{0,1\}\\ k_2-j\in\{0,1\}}} \frac{\mu(p^{k_1-i})}{p^{3(k_1-i)/4}}\frac{\mu(p^{k_2-j})}{p^{3(k_2-j)/4}}\varphi^*(p^{|k_1-k_2|}) \qquad (15)\\ &= \left(\varphi^*(p^{|i-j|})\big(1+p^{-3/2}\big) - \frac{\varphi^*(p^{|i-j+1|})+\varphi^*(p^{|i-j-1|})}{p^{3/4}}\right).\end{aligned}$$

Here, we do not have the contribution coming from $\mathcal{U}_K(K-1,K)$ or $\mathcal{U}_K^\top(K,K-1)$ as we have assumed $i,j \le K-1$. This assumption is necessary because we are considering the values $k_1 = i+1$ and $k_2 = j+1$ (both of which should remain $\le K$).

First, let us consider the case $i \ge j$. Letting $i-j=m\ge 0$, (15) becomes

$$\begin{aligned}\mathcal{A}_K(i+m,i) &= \varphi^*(p^m) - p^{-3/4}\varphi^*(p^{|m-1|}) - p^{-3/4}\big(\varphi^*(p^{m+1}) - p^{-3/4}\varphi^*(p^m)\big)\\ &= p^{-3m/4}\beta - p^{-3/4}p^{-3(m+1)/4}\beta\\ &= \beta(1-p^{-3/2})p^{-3m/4}.\end{aligned}$$

Similarly, for $j \ge i$, we will obtain the same expression in terms of $m = j-i$. This proves Proposition 5.1 for $1 \le i, j \le K-1$.

Next, we consider the case when one of i or j equals K.

Claim: $\mathcal{A}_K(i,K) = \mathcal{A}_K(K,j) = 0$, for all $1 \le i, j \le K-1$.

We revert to the first line of the expression (15). Letting $m = K-i \ge 1$, we obtain

$$\begin{aligned}\mathcal{A}_K(i,K) &= \sum_{\substack{k_1\in\{i,i+1\}\\ k_2\in\{K-1,K\}}} \frac{\mu(p^{k_1-i})}{p^{3(k_1-i)/4}}\frac{\mu(p^{K-k_2})}{p^{3(K-k_2)/4}}\varphi^*(p^{|k_1-k_2|})\\ &= -p^{-3/4}\varphi^*(p^{m-1}) + p^{-3/2}\varphi^*(p^{|m-2|}) + \varphi^*(p^m) - p^{-3/4}g(p^{m-1})\\ &= -p^{-3/4}\big(\varphi^*(p^{m-1}) - p^{-3/4}\varphi^*(p^{|m-2|})\big) + \varphi^*(p^m) - p^{-3/4}\varphi^*(p^{m-1})\\ &= -p^{-3/4}p^{-3(m-1)/4}\beta + p^{-3m/4}\beta = 0.\end{aligned}$$

It similarly follows that $\mathcal{A}_K(K,j) = 0$ for $1 \le j \le K-1$, proving the claim.

Next, we see that

$$\begin{aligned}\mathcal{A}_K(K,K) &= \sum_{k_1,k_2\in\{K-1,K\}} \frac{\mu(p^{K-k_1})}{p^{3(K-k_1)/4}}\frac{\mu(p^{K-k_2})}{p^{3(K-k_2)/4}}\,\varphi^*(p^{|k_1-k_2|})\\ &= 1-p^{-3/4}\varphi^*(p)-p^{-3/4}\big(\varphi^*(p)-p^{-3/4}\big)\\ &= \beta - p^{-3/4}\big(p^{-3/4}\beta\big) = \beta(1-p^{-3/2}).\end{aligned}$$

This completes the proof of Proposition 5.1. □

Now, since $n\mapsto n^{-3/4}$ is completely multiplicative, by repeating almost as verbatim the the proof of Lemma 4.1, we obtain that for some $c>0$, the matrix

$$\mathcal{A}_K = c\left(\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^{-3/4}\right)_{a,b\in\{1,\dots,p^K\}}$$

is positive definite for all K. Moreover, since $\mathcal{A}_K=\mathcal{U}_K^\top\mathcal{M}_K\mathcal{U}_K$, we have

$$\det(\mathcal{A}_K)=\det(\mathcal{U}_K)^2\,\det(\mathcal{M}_K)=(1-p^{-3/2})^2\,\det(\mathcal{M}_K).$$

This proves that $\det(\mathcal{M}_K)>0$ and by induction over K in Lemma 4.4, $\mathcal{M}_K$ is positive definite for all K.

The factorization (11) completes the proof of Theorem 1.1. □

Acknowledgements. This project started while MA was a visiting Professor at Aix-Marseille Université, and he is thankful for their kind hospitality. MA is funded by CNPq grant PDE no. 400010/2022-4 (200121/2022-7) and by CNPq grant Universal no. 403037/2021-2.

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