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AN EXPLICIT CROOT-LABA-SISASK LEMMA FREE OF PROBABILISTIC LANGUAGE

OLIVIER RAMARÉ

ABSTRACT. We provide an explicit and probabilistic language-free proof of the famous Croot-Laba-Sisask Lemma. In between, we do the same for the Khintchine and Marcinkiewicz-Zygmund inequalities and explicitate the implied constants for the upper bounds.

1. INTRODUCTION

After the fundatory papers of H. Rademacher [10] in 1922 and of A. Khintchine [8] in 1923, the usage of the so-called Rademacher system of functions, described thereafter, has known deep developments in L^p -space theory, then in Banach space theory, harmonic analysis and operator theory, for instance with the introduction of the Rademacher type and cotype. The n -th Rademacher function r_n is simply the function on $[0, 1]$ that takes the value 1 at t when the integer part of $2^n t$ is even and -1 otherwise. They were introduced by Rademacher in [10, Part VI] in an L^2 -setting and by Khintchine in [8, Section 1] in an L^p -setting. It turns out that this alternation of the ± 1 values is deeply connected with sums of Bernoulli variables and this introduces probability theory. We refer the reader to the book [1] by S. Astashkin. As the material stated in such a fashion may be difficult to grasp for a non probabilist, we propose here a fully elementary presentation of some part of it, where elementary means that any generic mathematical background should do.

Our main aim is to provide a proof of (a variant of) [3, Lemma 3.2] by E. Croot, I. Laba & O. Sisask. We state this result in their notation and in particular, when z is a complex number, z° is defined to be $z/|z|$ when $z \neq 0$, and 0 when $z = 0$.

Theorem 1. *Let (X, μ) be a probability space, let $p \geq 2$ and a function f given in the form*

$$f = \sum_{k \leq K} \lambda_k g_k$$

where $(g_k)_k$ is a collection of measurable functions on X of $L^p(\mu)$ -norm at most 1. Let finally $\varepsilon > 0$. There exists an L -tuple $(k_1, \dots, k_L) \in \{1, \dots, K\}^L$ of length $L \leq 20p/\varepsilon^2$ such that

$$\int_X \left| \frac{f(x)}{\|\lambda\|_1} - \frac{1}{L} \sum_{\ell \leq L} \lambda_{k_\ell}^\circ g_{k_\ell}(x) \right|^p d\mu \leq \varepsilon^p,$$

where $\|\lambda\|_1 = \sum_{k \leq K} |\lambda_k|$.

Thus L can be taken uniformly bounded, whatever rate of convergence (with respect to K) of the initial representation of f . This theorem has its origin in the paper [4] by E. Croot & O. Sisask. Since the Croot-Laba-Sisask Lemma has

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important consequences, we thought it was worth presenting an elementary and self-contained proof of it.

We prove the upper Khintchine Inequality in Theorem 2 and the Marcinkiewicz-Zygmund Inequality in Theorem 3. We refer to the paper [9] by L. Pierce for more refined background on the Khintchine and Marcinkiewicz-Zygmund inequalities. Our treatment is far from being comprehensive and we should mention to the readers another important tool in this landscape: the Kahane-Salem-Zygmund Inequality, see for instance [5] by A. Defant & M. Mastyło and [11, Section 4] by A. Raposo, Jr. & D. Serrano-Rodríguez. The followings proofs borrow from several authors.

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2. AN UPPER EXPLICIT KHINTCHINE INEQUALITY

Here is the main result of this section.

Theorem 2. *We have, when $p \geq 1$,*

$$(1/2^N) \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \left| \sum_{n \leq N} c_n \varepsilon_n \right|^p \leq p^{p/2} \left(\sum_{n \leq N} |c_n|^2 \right)^{p/2}.$$

This is only half of the Khintchine Inequality and in a special context, but it is explicit and will be enough for us. We followed [2, Chapter 10, Theorem 1, page 354] by Y.S. Show & H. Teicher. S. Astashkin in [1, Theorem 1.3] gives also a complete proof which is furthermore valid as soon as $p > 0$, up to a modification of the constant. To see the link between both results, let us mention that

$$(1/2^N) \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \left| \sum_{n \leq N} c_n \varepsilon_n \right|^p = \int_0^1 \left| \sum_{n \leq N} c_n r_{n-1}(t) \right|^p dt$$

where the (r_n) are the Rademacher functions defined in the introduction. This equality may be proved by considering the dyadic expansion of $2^N t$, for each $t \in [0, 1]$.

Proof. Let us start with $p = 2k \geq 2$, so that we may open the inner sum and get

$$\begin{aligned} 2^N S(2k) &= \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \left| \sum_{n \leq N} c_n \varepsilon_n \right|^p \\ &= \sum_{\substack{s_1 + s_2 + \dots + s_N = 2k, \\ s_n \geq 0}} \binom{2k}{s_1, s_2, \dots, s_N} \prod_{1 \leq n \leq N} c_n^{s_n} \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \prod_{n \leq N} \varepsilon_n^{s_n} \end{aligned}$$

by the multinomial theorem. The inner summand vanishes as soon as some s_n is odd, whence, by letting $2t_n = s_n$, we get

$$\begin{aligned} 2^N S(2k) &= \sum_{\substack{t_1 + t_2 + \dots + t_N = k, \\ t_n \geq 0}} \binom{2k}{2t_1, 2t_2, \dots, 2t_N} \prod_{n \leq N} (c_n^2)^{t_n} \\ &\leq C \sum_{\substack{t_1 + t_2 + \dots + t_N = k, \\ t_n \geq 0}} \binom{k}{t_1, t_2, \dots, t_N} \prod_{n \leq N} (c_n^2)^{t_n} = C \left(\sum_{n \leq N} c_n^2 \right)^k \end{aligned}$$

where

$$\begin{aligned} C &= \max \binom{2k}{2t_1, 2t_2, \dots, 2t_N} \binom{k}{t_1, t_2, \dots, t_N}^{-1} \\ &\leq \max \frac{2k(2k-1) \cdots (k+1)}{\prod_j 2t_j(2t_j-1) \cdots (t_j+1)} \leq \max \frac{k^k}{2^{t_1+\dots+t_k}} = (k/2)^k. \end{aligned}$$

As $S(p)$ is increasing, we simply choose $k = \lceil p/2 \rceil$ (the upper integer part of $p/2$). This gives us $2k \geq p+2$ and thus

$$(k/2)^k \leq (p+2)^{1+p/2} \leq (30p)^{p/2}.$$

This concludes the main part of the proof, except for the constant 30. We will not continue the proof but simply refer to the paper [7] by U. Haagerup who shows that best constant is (be careful: the abstract of this paper misses a closing parenthesis for the value of B_p , but the value of B_p displayed in the middle of page 232 misses a squareroot-sign around the π , as an inspection of the proof at the end the paper rapidly reveals)

$$\begin{cases} 1 & \text{when } 0 < p \leq 2, \\ \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p} & \text{when } 2 < p. \end{cases}$$

We readily check that this implies that the constant 1 rather than 30 is admissible. \square

3. AN UPPER EXPLICIT MARCINKIEWICZ-ZYGMUND INEQUALITY

Here is the main result of this section.

Theorem 3. *Let (X, μ) be a probability space. When $p \geq 1$, let $(f_n)_{n \leq N}$ be a system of functions such that $\int_X f_n(x) d\mu = 0$. We have*

$$\begin{aligned} \int_{(x_n) \in X^N} \left| \sum_{1 \leq n \leq N} f_n(x_n) \right|^p d(x_n) \\ \leq (4p)^{p/2} \int_{(x_n) \in X^N} \left(\sum_{1 \leq n \leq N} |f_n(x_n)|^2 \right)^{p/2} d(x_n). \end{aligned}$$

The power of this inequality is that the implied constant does not depend on N , the effect of some orthogonality. Again, this is only half of the Marcinkiewicz-Zygmund Inequality and in a special context, but the constants are explicit. This will be enough for us. We followed [2, Chapter 10, Theorem 2, page 356] by Y.S. Show & H. Teicher. The relevant constant is the subject of [12] by Y.-F. Ren & H.-Y. Liang (their value is slightly worse than ours) and [6] by D. Ferger, where the best constant is determined provided the f_n 's are "symmetric".

Proof. We first notice that, since $\int_0^1 f_n(x) dx = 0$, we may introduce a symmetrization through

$$\sum_{n \leq N} f_n(x_{2n-1}) = - \int_{(x_{2n}) \in X^N} \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) d(x_{2n}).$$

Jensen's inequality gives us that

$$\begin{aligned} \int_{(x_{2n-1}) \in X^N} \left| \int_{(x_{2n}) \in X^N} \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) d(x_{2n}) \right|^p d(x_{2n-1}) \\ \leq \int_{(x_{2n-1}) \in X^N} \int_{(x_{2n}) \in X^N} \left| \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_{2n}) d(x_{2n-1}) \end{aligned}$$

from which we deduce that the L^p -norm of the symmetrization controls the one of the initial sum:

$$\int_{(x_{2n-1}) \in X^N} \left| \sum_{n \leq N} f_n(x_{2n-1}) \right|^p d(x_{2n-1}) \leq \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n).$$

We next notice that, for any $(\varepsilon_n) \in \{\pm 1\}^N$, we have

$$\begin{aligned} \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} \varepsilon_{\lceil n/2 \rceil} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n) \\ = \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n). \end{aligned}$$

Indeed, consider the indices $n \in \{2k-1, 2k\}$. When $\varepsilon_k = 1$, we do not do anything while, when $\varepsilon_k = -1$, we exchange $n = 2k-1$ and $n' = 2k$. This enables us to introduce the Rademacher system:

$$\begin{aligned} (1/2^N) \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} \varepsilon_{\lceil n/2 \rceil} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n) \\ = \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n). \end{aligned}$$

We may now remove the symmetrization since:

$$\begin{aligned} \int_{(x_n) \in X^{2N}} \left| \sum_{n \leq 2N} \varepsilon_{\lceil n/2 \rceil} (-1)^n f_{\lceil n/2 \rceil}(x_n) \right|^p d(x_n) \\ \leq \int_{(x_n) \in X^{2N}} 2^{p-1} \left(\left| \sum_{n \leq N} \varepsilon_n f_n(x_{2n}) \right|^p + \left| \sum_{n \leq N} \varepsilon_n f_n(x_{2n-1}) \right|^p \right) d(x_n) \\ \leq 2^p \int_{(x_{2n}) \in X^N} \left| \sum_{n \leq N} \varepsilon_n f_n(x_{2n}) \right|^p d(x_{2n}). \end{aligned}$$

The Khintchine Inequality from Theorem 2 finally gives us that

$$(1/2^N) \sum_{(\varepsilon_n) \in \{\pm 1\}^N} \left| \sum_{n \leq N} \varepsilon_n f_n(x_{2n}) \right|^p \leq p^{p/2} \left(\sum_{n \leq N} |f_n(x_{2n})|^2 \right)^{p/2}.$$

The proof is then complete. Concerning the constant 4 in the Theorem, the paper [12] by Y.-F. Ren & H.-Y. Liang gives the upper bound $9/2$, which is worse than the above one. \square

4. PROOF OF THEOREM 1

Proof. We define $\Omega = \{1, \dots, K\}$ which we equip with the probability measure defined by $\nu(\{k\}) = |\lambda_k| / \|\lambda\|_1$. It induces a product measure on Ω^L , and, when $\mathbf{u} = (u_h)_{h \leq L} \in \Omega^L$, we shall simply write $d\mathbf{u}$ when integrating with respect to this

measure. Given a positive integer L , we consider the family of functions φ_ℓ , for $\ell \leq L$ given by

$$\begin{aligned} \varphi_\ell : \Omega^L \times X &\rightarrow \mathbb{C} \\ (\mathbf{u}, x) &\mapsto \lambda_{u_\ell}^\circ g_{u_\ell}(x) \end{aligned}$$

so that

$$\int_{\Omega^L} \varphi_\ell(\mathbf{u}, x) d\mathbf{u} = \sum_{k \in \Omega} \frac{|\lambda_k|}{\|\lambda\|_1} \lambda_k^\circ g_k(x) = \frac{f(x)}{\|\lambda\|_1} = f_0(x)$$

say. We aim at showing that $(1/L) \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x)$ closely approximates f_0 for most values of $\mathbf{u} = (u_h)_{h \leq L}$. Selecting one such value gives qualitatively our result. To do so, we write

$$\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx = \frac{1}{L^p} \int_{\Omega^L} \int_X \left| \sum_{\ell \leq L} (\varphi_\ell(\mathbf{u}, x) - f_0(x)) \right|^p d\mathbf{u} dx.$$

We apply the Marcinkiewicz-Zygmung Inequality, i.e. Theorem 3, to this latter expression, getting for fixed \mathbf{u} ,

$$\begin{aligned} \int_X \left| \frac{1}{L} \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p dx &\leq \frac{(4p)^{p/2}}{L^{p/2}} \int_X \left| \frac{1}{L} \sum_{\ell \leq L} |\varphi_\ell(\mathbf{u}, x) - f_0(x)|^2 \right|^{p/2} dx \\ &\leq \frac{(4p)^{p/2}}{L^{p/2}} \int_X \frac{1}{L} \sum_{\ell \leq L} |\varphi_\ell(\mathbf{u}, x) - f_0(x)|^p dx, \end{aligned}$$

the second step having been obtained through the Hölder inequality. We next integrate over \mathbf{u} and notice that $\int_{\Omega^L} |\varphi_\ell(\mathbf{u}, x) - f_0(x)|^p d\mathbf{u}$ is independent of ℓ to infer that

$$\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx \leq \frac{(4p)^{p/2}}{L^{p/2}} \int_X \int_{\Omega^L} |\varphi_1(\mathbf{u}, x) - f_0(x)|^p d\mathbf{u} dx.$$

Concerning the relevant p -norms, we make the following observations:

$$\int_X \int_{\Omega^L} |\varphi_1(\mathbf{u}, x)|^p d\mathbf{u} dx = \int_{\Omega^L} \left(\int_X |\varphi_1(\mathbf{u}, x)|^p dx \right) d\mathbf{u} \leq 1,$$

on the one side while on the other side, by the triangle inequality, we have

$$\|f_0\|_p \leq \sum_{k \leq K} \frac{|\lambda_k|}{\|\lambda\|_1} \|g_k\|_p \leq 1.$$

Therefore

$$\left(\int_{\Omega^L} \int_X \left| \frac{1}{L} \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p d\mathbf{u} dx \right)^{1/p} \leq \frac{(4p)^{1/2}}{L^{1/2}} (1+1) = \sqrt{16p/L}.$$

We deduce from this inequality that the set of \mathbf{u} for which

$$\int_X \left| \frac{1}{L} \sum_{\ell \leq L} \varphi_\ell(\mathbf{u}, x) - f_0(x) \right|^p dx > \varepsilon^p$$

has measure at most $\sqrt{16p/(\varepsilon^2 L)}$ which is strictly less than 1 by our assumption on L . The theorem follows readily. \square

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