Optimal supports for rational invariants separating orbits of a C-torus

Martin Jalard

Optimal supports for rational invariants separating orbits of a $\mathbb{C}$-torus

Martin Jalard*
Inria Côte d'Azur, France

February 10, 2025

Abstract

I compute a minimal rational separating set of invariants for a representation of a torus $(\mathbb{G}_\times)^d$. That is, these rational invariants separate as many orbits as the invariant field. In particular, they separate more orbits than the invariant algebra.

To that purpose, I predefine a set of optimal supports for the invariants. The size of these optimal separating supports is bounded by $d+1$. I thus obtain a separating set of cardinality bounded by $\binom{n}{d+1} \sim n^{d+1}$, where n is the dimension of the representation. This is highly competitive when compared to other separating or generating sets, as exhibited in the final explicit examples.

The proofs rely on technical considerations of arithmetic and linear algebra, addressing the intersections of families of affine hyperplanes over $\mathbb{Q}$ that I call foliages.

Keywords— Invariants, Tori, Hyperplanes, Smith normal form, Arithmetic

*Contact: martin.jalard@inria.fr

Contents

1 Introduction

It can be relevant to study algebraic objects up to a group action. Indeed, several properties are invariant under this group action and concern more an orbit than a specific point. Thus arises the need for invariant functions that determine, as much as possible, the orbit of the points of a representation. We can cite the study through invariants of the isotropy classes [PS85] or the piezoelectricity and elasticity of given materials [Azz+22].

For a compact group, invariant polynomials separate all the orbits. This is no longer the case for non compact groups, whose non closed orbits cannot be separated by smooth invariants. This led [Kem07] to generalise the notion of separation in the following way: a set of invariant polynomials is said separating when it separates as many orbits as the invariant algebra. Inspired by this, we formulate the same for rational invariants. We say that a set of invariants is a *rational separating set* if it separates as many orbits as the invariant field.

In this article we compute a rational separating set for representations of the d-dimensional $\mathbb{C}$-torus $(\mathbb{G}_\times)^d$. This set is minimal for the inclusion. As an abelian group, irreducible representations of $(\mathbb{G}_\times)^d$ are of dimension 1. They are determined by a column vector $m \in \mathbb{Z}^d$. We notice that in an irreducible representation, the set of elements mapping a point to another is homeomorphic to a sequence of periodically spaced affine hyperplanes, orthogonal to m. I call these set of hyperplanes a foliage. For a summand of irreducible representations $M = (m_1, ..., m_n)$, we thus address the intersection of the corresponding family of foliages $(\mathcal{F}_1, ..., \mathcal{F}_n)$. Two points are in the same orbit when these foliages globally intersect.

Through a blend of linear algebra and arithmetic, I isolate an optimal set $\gamma(M)$ of supports in $[\![1, n]\!]$ testing the emptiness of that intersection:

$$\bigcap_{s=1}^{n} \mathcal{F}_s \neq \emptyset \Leftrightarrow \forall S \in \gamma(M), \bigcap_{s \in S} \mathcal{F}_s \neq \emptyset.$$

These supports are the minimal $S \subset [\![1, n]\!]$ such that the columns $\{m_s, s \in S\}$ are linearly dependent. One notices that the size of these supports is bounded by $d + 1$. We prove that our set of testing supports is of cardinality $\#\gamma(M) \leq \binom{n}{d+1} \sim n^{d+1}$.

Next, for each $S \in \gamma(M)$, we compute a unique rational monomial P_S separating orbits of support exactly S. That is, testing whether the foliages $\{\mathcal{F}_s,\ s \in S\}$ intersect. This computation is allowed by a classical technique in invariant theory, based on Smith Normal Forms [Bür+21]. In general, to calculate a Smith Normal form of a matrix in $\mathcal{M}_{d,n}(\mathbb{Z})$ has a polynomial cost in n, d. But since we reduced to families $\{m_s,\ s \in S\}$ of corank 1, this complexity is reduced to solve a linear system of size $d+1$. Finally, we get a rational separating set of cardinality $\gamma(M) \sim n^{d+1}$:

$$\mathcal{E}_{sep} = \{P_S, S \in \gamma(M)\}.$$

The most classical approach in invariant theory is to compute a set of invariant polynomials generating the invariant algebra. General algorithms allow to compute them, for which the book [Kem07] is a good entry point. For toric actions, such a set is provided by an algorithm based on Groebner bases [Stu08, p. 1.4.5]. Polynomial separating sets have the advantage of being significantly smaller [Kem07, Section 2.4], whereas they are sufficient for many applications (see for instance [BV23]). This motivated various research for different specific circumstances [DG24; Che+19; Duf09]. Nevertheless, it seems more challenging to find general results to compute them. As far as I know, there is no polynomial separating sets for tori representations in the literature to compare with my rational one.

Because of the inclusion $\mathbb{C}[V]^{\mathrm{G}} \subset \mathbb{C}(\mathcal{V})^{\mathrm{G}}$, rational invariants separate at least as many orbits than the polynomials one. But in fact, they separate far more orbits, including non closed ones. For toric actions, the invariant field separates any point $v \in \mathcal{V}$ such that $\{m_s, s \in \mathrm{supp}\,(v)\}$ is linearly dependent. This is far more accurate than the invariant algebra as illustrated in Remark 3.9. To summarize, one could dress the following empirical comparison, confirmed by explicit computations on random cases exhibited in the final examples 5.4 and 5.5:

$$\underbrace{\#\,(\ \textit{rational separating set}\ )}_{\textit{separates far more orbits}} << \underbrace{\#\,(\ \textit{polynomial generating set}\ )}_{\textit{separates closed orbits}}$$

Another approach is to compute a set generating the invariant field $\mathbb{C}(\mathcal{V})^{\left(\mathbb{G}_\times\right)^d}$, as [HL13] does through Hermite Normal Forms. Its cardinality, lower than n, is highly competitive. Although, from a separation point of view, these are less accurate. Rosenlicht Theorem ensures only that they separate orbits almost everywhere. The set in [HL13] only separates orbits of full support in $[\![1, n]\!]$.

[Bür+21] develops a more complete separation algorithm based on this first step. After testing the equality of the supports of two given points, they apply the process to the representation restricted to that support. This is highly competitive in the sense that it separates these two specific points with at most n monomials. But the huge number (2^n) of different supports in $[\![1, n]\!]$ discourages to announce a total separating set. Combining the separating sets regarding all the supports, the final separating set would have a cardinality in $n \times 2^n$. It highlights the point of this paper: to select optimal and small supports. Bigger supports indeed induce too many invariants.

2 Intersection of foliages

Let us start by defining foliages and studying their intersections, this part being (apparently) an independent problem. A foliage is a subset of $\mathbb{C}^d$ composed of regularly spaced, parallel hyperplanes. This section is entirely devoted to establishing a minimal criterion ensuring that a family of foliages globally intersects (Theorem 2.8). This problem carries the larger part of the difficulties of the article, the other sections resulting naturally from this point.

Definition 2.1. *Let $\mathcal{V}$ be a $\mathbb{C}$-affine space of finite dimension n. Choose an initial point $f \in \mathcal{V}$ and a non-zero direction $m \in \mathbb{Q}^n$. The associated foliage is the set*

$$\mathcal{F}\,(f, m) = f + m^{\perp} + m\mathbb{Z}$$

where $m^{\perp}$ is the hyperplane orthogonal to m, and $m\mathbb{Z}$ is the one dimensional $\mathbb{Z}$-module generated by m.

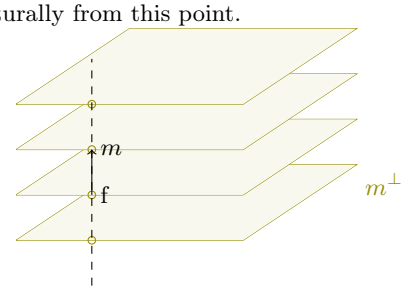


Figure 1: A foliage for $d = 3$

The intersection of a family of foliages $\{\mathcal{F}_s = \mathcal{F}\,(f_s, m_s)\,,\ 1 \leq s \leq n\}$ can vanish from the moment that $\bigcap_{s=1}^{n} m_s^{\perp}$ has dimension greater than $d - n$, that is the family of directions $\{m_s,\ 1 \leq s \leq n\}$ is linearly dependent. Choosing m over $\mathbb{Q}$ forces the eventual intersection to respect a periodicity property in any direction, and thus to be not included in any hyperplane:

Lemma 2.2. *Let $\{\mathcal{F}_s,\ 1 \leq s \leq n\}$ be a family of foliages in $\mathbb{C}^d$ which globally intersects: $\bigcap_{s=1}^{n} \mathcal{F}_s \neq \emptyset$. Then, $\bigcap_{s=1}^{n} \mathcal{F}_s$ is not contained in an affine hyperplane.*

Proof. Take $f \in \bigcap_{s=1}^{n} \mathcal{F}_s$ and $\mathcal{H}$ an affine hyperplane containing f. Choose $v \in \mathbb{Q}^n$ a direction out of $\mathcal{H}$. For any $1 \leq s \leq n$, there exists $r_s \in \mathbb{Q}$ such that $f + v\,(r_s\mathbb{Z}) \subset \mathcal{F}_s$. Since for any $1 \leq s \leq n$, $r_s \in \mathbb{Q}$, we know that $\bigcap_{s=1}^{n} r_s\mathbb{Z} = r\mathbb{Z}$ for some $r \in \mathbb{Q}$. That is $\bigcap_{s=1}^{n} \mathcal{F}_s$ contains infinitely many points in the line $f + \mathbb{Q}v$, and $\bigcap_{s=1}^{n} \mathcal{F}_s \not\subset \mathcal{H}$. □

In that case, $\bigcap_{s=1}^{n} \mathcal{F}_s = f + \mathcal{L} + \bigcap_{s=1}^{n} m_s^{\perp}$, where $\mathcal{L}$ is a module satisfing $\dim(\mathcal{L}) + \dim\left(\bigcap_{s=1}^{n} m_s^{\perp}\right) = d$. There are immediate counter examples when the directions m_s are not rational, like the following one in dimension 1:

$$\mathcal{F}_1 = \mathbb{Z}, \;\; \mathcal{F}_2 = \mathbb{Z}\pi \;\; \Rightarrow \;\; \mathcal{F}_1 \cap \mathcal{F}_2 = \{0\}.$$

2.1 Intersection of foliages for $d = 1$

In this section, we recall a result in dimension $d = 1$ already provided by [Jal24]. Up to a similarity, we can embed a foliage $\mathcal{F} \subset \mathbb{C}$ into $\mathbb{Z}$. This enables us to apply arithmetical techniques and get a criterion for the intersection of several foliages in Lemma 2.4. Let us note $\wedge$ and $\vee$ the classical gcd and lcm over $\mathbb{Z}$. We start by the intersection of two foliages:

Lemma 2.3. *Take two foliages $\mathcal{F} = f + m\mathbb{Z}$ and $\mathcal{F}' = f' + m'\mathbb{Z}$ included in $\mathbb{Z}$. There is the equivalence:*

$$\mathcal{F} \cap \mathcal{F}' \neq \emptyset \Leftrightarrow (m \wedge m') \text{ divides } (f - f')$$

Proof. One has $\mathcal{F} \cap \mathcal{F}' \neq \emptyset \Leftrightarrow \exists \begin{pmatrix} k \\ k' \end{pmatrix} \in \mathbb{Z}^2$, $\begin{pmatrix} m & m' \end{pmatrix} \begin{pmatrix} k \\ k' \end{pmatrix} = f - f'$. However, one computes the Smith Normal Form

$$\begin{pmatrix} m & m' \end{pmatrix} = \underbrace{(1)}_{U} \begin{pmatrix} m \wedge m' & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{m}{m\wedge m'} & \frac{m'}{m\wedge m'} \\ b & b' \end{pmatrix}}_{W}.$$

where b, b' are the Bezout integers such that $bm + bm' = m \wedge m'$. Since W is unimodular, the multiplication map $W : \mathbb{Z}^2 \to \mathbb{Z}^2$ is bijective. Hence the system

$$\begin{pmatrix} m \wedge m' & 0 \end{pmatrix} W \begin{pmatrix} k \\ k' \end{pmatrix} = f - f'$$

has solutions iff $m \wedge m'$ divides $f - f'$. □

[Jal24] uses this lemma to get a criterion of a larger family of foliages for $d = 1$, checking only pairwise intersections:

Lemma 2.4. *[Jal24, Lemma3.4] Let $\mathcal{F}_s = f_s + m_s\mathbb{Z}$, $1 \leq s \leq n$ a family of foliages in an affine line $\mathcal{D}$. Suppose that for any $1 \leq s < j \leq n$, $\mathcal{F}_s \cap \mathcal{F}_j \neq \emptyset$. Then, $\bigcap_{s=1}^{n} \mathcal{F}_s \neq \emptyset$.*

Proof. Up to a similarity $\mathcal{D} \to \mathbb{C}$, we can suppose that for all $1 \leq s \leq n$, $\mathcal{F}_s$ is contained in $\mathbb{Z}$. We then proceed by induction on the number n. The lemma holds for $n \leq 2$. Take $n > 2$ such that the lemma holds for $n - 1$. Then $\bigcap_{s=1}^{n-1} \mathcal{F}_s \neq \emptyset$, and up to translation, we can suppose that $f_s = 0$ for $s \leq n - 1$. Then, $\bigcap_{s=1}^{n-1} \mathcal{F}_s = \left\{ k \left(\bigvee_{s=1}^{n-1} m_s \right), k \in \mathbb{Z} \right\}$ where $\left(\bigvee_{s=1}^{n-1} m_s \right) = \mathrm{lcm}(m_1, ..., m_{n-1})$.

By Lemma 2.3 we have $\mathcal{F}_s \cap \mathcal{F}_n \neq \emptyset \Leftrightarrow (m_s \wedge m_n)$ divides $(f_n - f_s) = f_n$. For all prime number p and integer f we note $\nu_p(f) = \max\{k \in \mathbb{N} \,|\, p^k \text{ divides } f\}$ the p-adic valuation of f. We have

$$\nu_p(f_n) \quad \geq \quad \max_{s \leq n-1} [\nu_p(m_s \wedge m_n)] \quad = \quad \max_{s \leq n-1} [\min(\nu_p(m_s), \nu_p(m_n)]$$

If there is some $1 \leq s \leq n - 1$ such that $\nu_p(m_s) > \nu_p(m_n)$, then

$$\max_{s \leq n-1} [\min(\nu_p(m_s), \nu_p(m_n)] = \nu_p(m_n) = \min[\max_{s \leq n-1} (\nu_p(m_s)), \nu_p(m_n)].$$

Otherwise $\nu_p(m_n) \geq \nu_p(m_s)$ for any $1 \leq s \leq n-1$ and

$$\max_{s\leq n-1}[\min(\nu_p(m_s),\nu_p(m_n)] = \max_{s\leq n-1}[\nu_p(m_s)] \geq \min[\max_{s\leq n-1}(\nu_p(m_s)),\nu_p(m_n)].$$

It provides

$$\nu_p(f_n) \geq \min[\max_{s\leq n-1}(\nu_p(m_s)),\nu_p(m_n)] = \nu_p\left(\left(\bigvee_{s=1}^{n-1} m_s\right)\wedge m_n\right)$$

This is, $\left(\bigvee_{s=1}^{n-1} m_s\right)\wedge m_n$ divides f_n. Hence, apply Lemma 2.3 again to have $\mathcal{F}_n \cap \bigcap_{s=1}^{n-1}\mathcal{F}_s \neq \emptyset$. □

Lemma 2.4 does not hold anymore when the directions m_s are not rational. For instance, we can complete the counter example evoked in Section 2:

Example 2.5. *Let* $\begin{cases}\mathcal{F}_1 = \mathbb{Z}\\ \mathcal{F}_2 = \pi\mathbb{Z}\\ \mathcal{F}_3 = 1+(\pi-1)\mathbb{Z}\end{cases}$ *be three foliages in* $\mathbb{C}$ *with non necessarily rational directions.*
Then, $\begin{cases}\mathcal{F}_1\cap\mathcal{F}_2 = \{0\}\neq\emptyset\\ \mathcal{F}_1\cap\mathcal{F}_3 = \{1\}\neq\emptyset\\ \mathcal{F}_2\cap\mathcal{F}_3 = \{\pi\}\neq\emptyset\end{cases}$ *while* $\mathcal{F}_1\cap\mathcal{F}_2\cap\mathcal{F}_3 = \emptyset$.

2.2 A generalisation for $d \geq 1$

In this section, we give a first generalisation of Lemma 2.4 in larger dimension $d \geq 1$. It work by induction, using Lemma 2.4 as a key to increase the dimension one by one.

Theorem 2.6. *Let* $\mathcal{V}$ *be a affine space of dimension* d *endowed with a family* $\{\mathcal{F}_s,\ 1\leq s\leq n\}$ *of foliages. Suppose that for any support* $S\subset[\![1,n]\!]$ *of cardinality* $d+1$, $\bigcap_{s\in S}\mathcal{F}_s\neq\emptyset$. *Then,* $\bigcap_{s=1}^{n}\mathcal{F}_s\neq\emptyset$.

Proof. Lemma 2.4 ensures that Theorem 2.6 holds for $(1,n)$, $n\in\mathbb{N}$. For any $d\in\mathbb{N}$ and $1\leq n\leq d+1$, it also holds for (d,n). We thus proceed by induction, taking some (d,n) such that Theorem 2.6 holds for $(d,n-1)$ and $(d-1,n-1)$. Consider $\{\mathcal{F}_s = f_s +_s +m_s\mathbb{Z},\ 1\leq s\leq n\}$ a family of foliages in $\mathbb{C}^d$ satisfying the theorem hypothesis::

$$\forall S\subset[\![1,n]\!] \text{ of cardinality } d+1,\ \bigcap_{s\in S}\mathcal{F}_s\neq\emptyset.$$

We shall isolate $\mathcal{F}_n$ and consider the intersections of the $n-1$ first foliages. We define $\pi:\mathcal{V}\to\mathcal{D}$ the projection along $m_n^\perp$ onto the affine line $\mathcal{D} = m_n\mathbb{C}+f_n$. We take a support $S\subset[\![1,n-1]\!]$ of cardinality d. There is the two following possibilities:

1. The projection $\pi\left(\bigcap_{s\in S}\mathcal{F}_s\right)$ is equal to the whole $\mathcal{D}$.
2. The inclusion $\pi\left(\bigcap_{s\in S}\mathcal{F}_s\right)\not\subseteq\mathcal{D}$ is strict. Lemma 2.2 ensures that $\bigcap_{s\in S}\mathcal{F}_s$ is not contained in a single hyperplane. Hence $\pi\left(\bigcap_{s\in S}\mathcal{F}_s\right)$ is not reduced to a point. It is a foliage in $\mathcal{D}$.

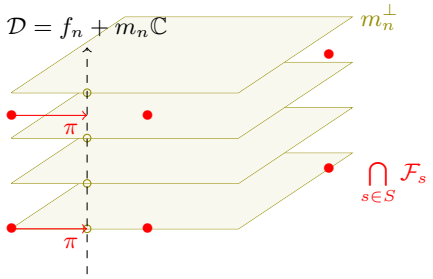


Figure 2: The projection on $\mathcal{D}$

Whereas, $\pi(\mathcal{F}_n) = f_n + m_n\mathbb{Z}$ is also a foliage in $\mathcal{D}$. We thus obtain a family

$$\left\{\pi\left(\bigcap_{s\in S}\mathcal{F}_s\right),\ S\subset[\![1,n-1]\!] \text{ of cardinality } d\right\}\cup\{\pi(\mathcal{F}_n)\}$$

of foliages in the line $\mathcal{D}$ which intersect pairwise:

- Take some $S \subset [\![1, n-1]\!]$ of cardinality d and $\pi(\mathcal{F}_n)$. Then $S \cup \{n\} \subset [\![1, n]\!]$ has cardinality $d+1$ and by assumption, $\mathcal{F}_n \cap \bigcap\limits_{s\in S} \mathcal{F}_s \neq \emptyset$. Thus
$$\pi\left(\bigcap_{s\in S} \mathcal{F}_s\right) \cap \pi(\mathcal{F}_n) \neq \emptyset.$$
- Take S, T supports in $[\![1, n-1]\!]$ of cardinality d. By assumption, Theorem 2.6 holds for $(d, n-1)$. It gives that $\bigcap\limits_{s=1}^{n-1} \mathcal{F}_s \neq \emptyset$. So does $\left(\bigcap\limits_{s\in S} \mathcal{F}_s \cap \bigcap\limits_{t\in T} \mathcal{F}_t\right) \supset \bigcap\limits_{s=1}^{n-1} \mathcal{F}_s \neq \emptyset$. We deduce that
$$\pi\left(\bigcap_{s\in S} \mathcal{F}_s\right) \cap \pi\left(\bigcap_{t\in T} \mathcal{F}_t\right) \neq \emptyset.$$

We apply Lemma 2.4 to this family of one dimensional foliages. It ensures the non-emptiness of the intersection
$$\pi(\mathcal{F}_n) \cap \bigcap_{\substack{S\subset[\![1,n-1]\!]\\ \#S=d}} \pi\left(\bigcap_{s\in S} \mathcal{F}_s\right) \neq \emptyset.$$

Select a point f in this intersection, and the affine hyperplane $\mathcal{V}' = f + m_n^{\perp}$. For any $1 \leq s \leq n-1$, the intersection $\mathcal{F}'_s = \mathcal{F}_s \cap \mathcal{V}'$ is non empty. It is either

- The whole $\mathcal{V}'$,
- or a foliage in $\mathcal{V}'$.

Those in the second category form a family of foliages satisfying
$$\forall S[\![1, n-1]\!] \text{ of cardinality } d, \bigcap_{s\in S} \mathcal{F}'_s = \mathcal{V}' \cap \bigcap_{s\in S} \mathcal{F}_s \neq \emptyset.$$

Whereas, we assumed that Theorem 2.6 is true in dimension $d-1 = \dim(\mathcal{V}')$. It gives $\bigcap\limits_{s=1}^{n-1} \mathcal{F}'_s \neq \emptyset$. However $\bigcap\limits_{s=1}^{n-1} \mathcal{F}'_s = \underbrace{\mathcal{V}'}_{\subset \mathcal{F}_n} \cap \bigcap\limits_{s=1}^{n-1} \mathcal{F}_s$. It proves that $\bigcap\limits_{s=1}^{n} \mathcal{F}_s \neq \emptyset$. □

This result is my favourite one, because of its elegant formulation. It can be shared around a table, as an amusing enigma. Its hypotheses are nevertheless not optimal, and must be refined to be successfully applied to the separation problem in Section 3.3.

2.3 The refined criterion

In this section we refine Theorem 2.6. We obtain optimal supports to check the non emptyness of the intersection of foliages (Theorem 2.8). This refinement makes the criterion applicable to our separation purpose.

Definition 2.7. *Let M be a matrix in $\mathcal{M}_{d,n}(\mathbb{Z})$. We note $\gamma(M)$ the set of minimal supports $S \subset [\![1, n]\!]$ such that the family of columns $\{m_s,\ s \in S\}$ is linearly dependent.*

The maximal cardinality of supports in $\gamma(M)$ is $d+1$. Furthermore, each support in $[\![1, n]\!]$ of cardinality $d+1$ contains at least a support in $\gamma(M)$.

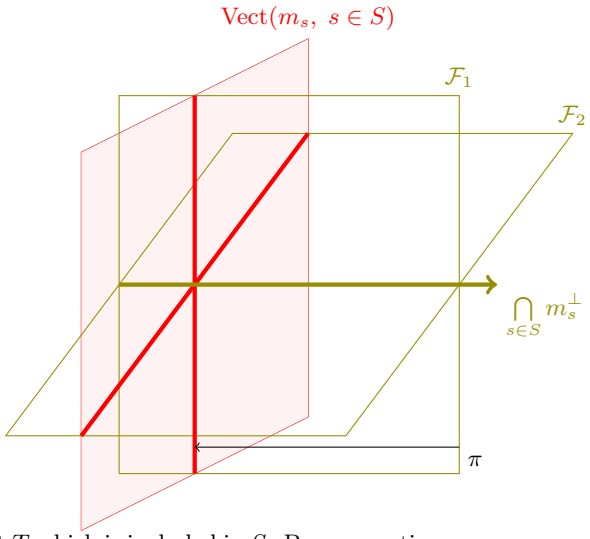
Theorem 2.8. *Let (d, n) be positive integers. Let $\mathcal{V}$ be an affine space of dimension d endowed with a family $\{\mathcal{F}(f_s, m_s),\ 1 \leq s \leq n\}$ of foliages. Consider the matrix M whose columns are the directions $\{m_1, ..., m_n\}$. Suppose that for any $S \in \gamma(M)$, $\bigcap\limits_{s\in S} \mathcal{F}_s \neq \emptyset$. Then, $\bigcap\limits_{s=1}^{n} \mathcal{F}_s \neq \emptyset$.*

Proof. For $d = 1$, $\gamma(M)$ is exactly the set of support of cardinality 2. Thus, Lemma 2.4 provides the theorem for $(1, n)$, $n \in \mathbb{N}$. We proceed again by induction, choosing d such that the theorem is true for (d', n) for any lower

dimension $d' < d$ and $n \in \mathbb{N}$. Consider $\mathcal{F} = \{\mathcal{F}_s, 1 \leq s \leq n\}$ a family of foliages in an affine space $\mathcal{V}$ of dimension d satisfying

$$\forall S \in \gamma(M), \bigcap_{s\in S} \mathcal{F}_s \neq \emptyset.$$

Let $S \subset [\![1, n]\!]$ be a support of cardinality $d+1$. There is two possibilities:



• The family $\{m_s, s \in S\}$ is of rank $r \leq d-1$. Up to a change of basis we note

$$M = \left(\begin{array}{c|c} \dfrac{M_S}{0} & M' \end{array} \right)$$

where $M_S \in \mathcal{M}_{r,d+1}(\mathbb{Z})$. We introduce π_S the pojection onto Vect$(m_s,\ s \in S)$ along $\bigcap_{s\in S} m_s^\perp$. The images $\{\pi(\mathcal{F}_s),\ s \in S\}$ are foliages in Vect$(m_s,\ s \in S) \cong \mathbb{C}^r$ whose directions are the columns of M_S. Note that $\gamma(M_S) \subset \gamma(M)$. Then, by assumption, for all $T \in \gamma(M_S)$, $\bigcap_{t\in T} \pi_S(\mathcal{F}_t) \neq \emptyset$. Since we assumed the theorem true in dimension $r \leq d-1$, it gives $\bigcap_{s\in S} \pi(\mathcal{F}_s) \neq \emptyset$. Since $\bigcap_{s\in S} \pi(\mathcal{F}_s) = \text{Vect}(m_s,\ s \in S) \cap \bigcap_{s\in S} \mathcal{F}_s$, we proved that $\bigcap_{s\in S} \mathcal{F}_s \neq \emptyset$.

• Otherwise, the family $\{m_s,\ s \in S\}$ is of rank d. $\gamma(M)$ admits a support T which is included in S. By assumption then, $\bigcap_{t\in T} \mathcal{F}_t \neq \emptyset$. This intersection is of the form

$$\bigcap_{t\in T} \mathcal{F}_t = f_T + \bigcap_{s\in T} m_s^\perp + \mathcal{L}_T,$$

where $\bigcap_{s\in T} m_s^\perp$ is of dimension $d - \#T + 1$ and $\mathcal{L}_T$ a $\mathbb{Z}$-module of dimension $\#T - 1$. Whereas, the complementary support $K = S \setminus T$ brings a linearly independent family of directions $\{m_k,\ k \in K\}$ of cardinality $d + 1 - \#T$. Then, $\bigcap_{k\in K} \mathcal{F}_k \neq \emptyset$ is of the form

$$\bigcap_{k\in K} \mathcal{F}_k = f_K + \bigcap_{s\in K} m_s^\perp + \mathcal{L}_K,$$

where $\bigcap_{s\in K} m_s^\perp$ is of dimension $d - \#K = \#T - 1$. Since $\{m_s,\ s \in S\}$ is of total rank $d = \dim\left(\bigcap_{t\in T} \mathcal{F}_t\right) + \dim\left(\bigcap_{k\in K} \mathcal{F}_k\right) = \dim(\mathcal{V})$, we have $\left(\bigcap_{t\in T} \mathcal{F}_t\right) \cap \left(\bigcap_{k\in K} \mathcal{F}_k\right) = \bigcap_{s\in S} \mathcal{F}_s \neq \emptyset$.

We proved that for any $S \subset [\![1, n]\!]$ of cardinality $d+1$, $\bigcap_{s\in S} \mathcal{F}_s \neq \emptyset$. Theorem 2.6 gives then $\bigcap_{s=1}^{n} \mathcal{F}_s \neq \emptyset$. □

3 Separation of representations of tori

In this section, we apply the results regarding foliages intersections to invariant theory. We first describe representations of tori and the classical way to compute their rational invariants. Next, we define clearly what we mean by *rational separating set*. We finally get the announced result of the paper: a minimal rational separating set of invariants for a representation of a torus (Theorem 3.13).

3.1 $\mathbb{K}$-tori, their representations, their rational invariants

In this section, we recall the link between representations of $\mathbb{C}$ tori and matrices of integers. We quickly derive the existing methods based on linear algebra, including Smith Normal forms, to compute rational invariants.

Definition 3.1. *A d-dimensional $\mathbb{C}$-torus is a group of the form $\mathrm{G} \cong (\mathbb{G}_\times)^d$ where $\mathbb{G}_\times = \mathbb{C}^*$ is the multiplicative group in $\mathbb{C}$.*

Since tori are abelian and $\mathbb{C}$ is algebraically closed, the linear representations of $(\mathbb{G}_\times)^d$ are of dimension one. They are all of the form $(\rho_m, \mathbb{C})$ determined by a column vector m in $\mathbb{Z}^d$:

$$\forall g = (g_1, ..., g_d) \in (\mathbb{G}_\times)^d \,,\ \rho_m(g) = \prod_{i=1}^{d} g_i^{m_i}.$$

Thus, a representation $(\rho_M, \mathcal{V})$ of dimension n is determined by a matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$:

$$\forall v = (v_1, ..., v_n) \in \mathcal{V},\ g = (g_1, ..., g_d) \in (\mathbb{G}_\times)^d \,,\ \rho(g)(v) = \left(v_s \prod_{i=1}^{d} g_i^{M_{is}} \right)_{s=1..n}.$$

We note $\{m_1, ..., m_n\}$ the columns of M, corresponding to each irreducible representations. The matricial view upon representations is helpful computing invariant monomials. There is a natural correspondence between monomials on $\mathcal{V}$ and vectors in Z^n: to a vector $e \in \mathbb{Z}^n$ is associated the monomial $\prod_{i=1}^{n} v_i^{e_i} \in \mathbb{C}(\mathcal{V})$.

Lemma 3.2. *[Stu08] Let $(\rho_M, \mathcal{V})$ be the representation of $(\mathbb{G}_\times)^d$ associated to the matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. Invariant rational monomials correspond to the vectors of the module*

$$\mathcal{M} = \{e \in \mathbb{Z}^n,\ Me = 0\}$$

These monomials generate the invariant field $\mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}$. Since $\mathcal{M}$ is of dimension $\dim(\mathcal{M}) = n - \mathrm{rank}(M)$, the invariant field is generated by a basis of $n - \mathrm{rank}(M)$ monomials. This basis can be computed through Smith Normal Form:

Proposition 3.3. *[Smi61] Let $M \in \mathcal{M}_{d,n}(\mathbb{Z})$ be a matrix of rank r. Then, there exists a sequence $\alpha_1, ..\alpha_r \in \mathbb{Z}$ with α_i dividing α_{i+1}, $U \in \mathrm{GL}_d(\mathbb{Z})$ and $W \in \mathrm{GL}_n(\mathbb{Z})$ such that*

$$UMW = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \alpha_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{d,n}(\mathbb{Z})$$

This form is computed in polynomial time up to d and n. It is classically used to get a basis of rational invariants [Bür+21]:

Theorem 3.4. *[Bür+21] Let ρ_M be the representation of $(\mathbb{G}_\times)^d$ associated to a matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. Compute UMW the Smith normal form of M. Then, the $n - \mathrm{rank}(M)$ last columns of W generate the module $\mathcal{M}$. The corresponding monomials generate the invariant field $\mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}$.*

3.2 Orbit separation

Let us define precisely what we mean by a *rational separating set*. For a compact group, all the orbits are closed, and the invariant algebra separates all of them. This is not the case for general algebraic groups. A generalisation of the definition of separation is thus proposed in [Kem07]:

Definition 3.5. *[Kem07, Definition 2.4.1] Let G be an algebraic group and $\mathcal{V}$ a G-variety. We say that $\mathcal{E}_{sep} \subset \mathbb{C}[\mathcal{V}]^{\mathrm{G}}$ is a polynomial separating set iff for any $v, v' \in \mathcal{V}$ admitting some $P \in \mathbb{C}[\mathcal{V}]^{\mathrm{G}}$, $P(v) \neq P(v')$, there exists some $S \in \mathcal{E}_{sep}$ with $S(v) \neq S(v')$.*

Inspired by this definition, we imagine the same for rational functions:

Definition 3.6. *Let G be an algebraic group and $\mathcal{V}$ a G-variety. We say that $\mathcal{E}_{sep} \subset \mathbb{C}(\mathcal{V})^{\mathrm{G}}$ is a* rational *separating set iff for any $v, v' \in \mathcal{V}$ admitting some $P \in \mathbb{C}(\mathcal{V})^{\mathrm{G}}$ separating v and v', there exists some $S \in \mathcal{E}_{sep}$ separating v and v'.*

Because of the inclusion $\mathbb{C}[V]^{\mathrm{G}} \subset \mathbb{C}(\mathcal{V})^{\mathrm{G}}$, rational invariants separate at least as many orbits than polynomial invariants. But it can occur that rational invariants separate far more orbits. Since they are not continuous, they might separate non closed orbits, even when their closures intersect. This happens broadly in our tori actions. We shall describe where orbits of a torus are separated by rational invariants. We define the support of a point $v \in \mathcal{V}$ as the set of non vanishing indices $\mathrm{supp}\,(v) = \{1 \leq s \leq n,\ v_s \neq 0\}$. For a support $S \subset [\![1, n]\!]$, we note $\mathcal{V}_S$ the set of points $v \in \mathcal{V}$ such that $\mathrm{supp}\,(v) = S$.

Definition 3.7. *Let M be a matrix in $\mathcal{M}_{d,n}(\mathbb{Z})$. We note $\Gamma(M)$ the set of supports $S \subset [\![1, n]\!]$ such that the family of columns $\{m_s,\ s \in S\}$ is linearly dependent.*

One could also see $\Gamma(M)$ as the set above $\gamma\,(M)$, that is the set of supports containing an element in $\gamma\,(M)$.

Lemma 3.8. *Let $(\rho_M, \mathcal{V})$ be the representation of $(\mathbb{G}_\times)^d$ associated to the matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. Then, the invariant field separates any orbits in $\bigsqcup_{S \in \Gamma(M)} \mathcal{V}_S$.*

Proof. Take v and v' in $\bigsqcup_{S \in \Gamma(M)} \mathcal{V}_S$.

• Suppose that $\mathrm{supp}\,(v) \neq \mathrm{supp}\,(v')$. Then, up to permuting v and v', there exists some $S \in \Gamma(M)$ such that $\begin{cases} S \subset \mathrm{supp}\,(v)\,. \\ S \not\subset \mathrm{supp}\,(v')\,. \end{cases}$ In that case, Smith Normal forms provides monomials $P \in \mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}$ of support in S with a pole or a root at v'. However, these monomials are defined at v and satisfy $P(v) \neq 0$. They separate v and v'.

• Assume that $\mathrm{supp}\,(v) = \mathrm{supp}\,(v') = S$. By localisations, we can realise $\mathcal{V}_S$ as an algebraic $(\mathbb{G}_\times)^d$-variety whose orbits are all closed [Bür+21]. Since $(\mathbb{G}_\times)^d$ is reductive, the invariant algebra $\mathbb{R}[\mathcal{V}_S]^{(\mathbb{G}_\times)^d}$ separates closed orbits. The inclusion $\mathbb{C}[\mathcal{V}_S]^{(\mathbb{G}_\times)^d} \subset \mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}$ ensures that $\mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}$ separates v and v'. □

Remark 3.9. *The greater accuracy of rational invariants appears already in dimension $d = 1$. Consider the torus $\mathbb{C}^*$ and the representation $(\rho_M, \mathcal{V})$ associated to a matrix $n \times 1$ matrix $M = (m_1, .., m_n)$. Since the torus is reductive, the invariant algebra separates the closed orbits. That is, orbits whose support is not included in $\{1 \leq s \leq n,\ m_s < 0\}$ or $\{1 \leq s \leq n,\ m_s > 0\}$ [Jal24, Lemma 4.3]. Whereas, Lemma 3.8 claims that rational invariants separate any orbits of supports of cardinality at least two.*

3.3 The separating set

In this section we exhibit the link between foliages and representations of tori. It results in a natural criterion for two points to belong to the same orbit. We next use Smith Normal Form to build a set of rational invariants checking that criterion. It provides a minimal rational separating set in Theorem 3.13, main result of the paper.

Lemma 3.10. *Let $(\rho_m, \mathbb{C})$ be the irreducible representation of $(\mathbb{G}_\times)^d$ associated to a vector $m \in \mathbb{Z}^d$. Take v and v' be two points in $\mathbb{C}^*$. Then, the set of elements $\{g \in (\mathbb{G}_\times)^d\,,\ \rho_m(v) = v'\}$ mapping v to v' is homeomorphic to the foliage $\mathcal{F} \in \mathbb{C}^d$ of initial point $f = \frac{v'm}{2\mathrm{i}\pi v||m||_2^2}$ and direction $\frac{m}{||m||_2^2}$.*

Proof. We consider the complex logarithm $\log : \mathbb{C}^* \to \frac{\mathbb{C}}{2\mathrm{i}\pi}$ defined as the inverse of $\exp : \frac{C}{2\mathrm{i}\pi} \to \mathbb{C}^*$. We compute that

$$\begin{aligned} \rho_m(v) = v' \quad &\Leftrightarrow \quad \prod_{i=1}^{d} g_i^{m_i} = \tfrac{v'}{v} \\ &\Leftrightarrow \quad \sum_{i=1}^{d} m_i \log(g_i) = \tfrac{v'}{v}[2\mathrm{i}\pi] \\ &\Leftrightarrow \quad (\log(g_i))_i \in \tfrac{v'm}{v||m||_2^2} + m^\perp + 2\mathrm{i}\pi \tfrac{m}{||m||_2^2}\mathbb{Z} \subset \mathbb{C}^d \end{aligned}$$

Thus, the homeomorphism $\phi = \frac{\exp}{2\mathrm{i}\pi} : \mathbb{C}^d \to (\mathbb{G}_\times)^d$ maps $\mathcal{F}\left(f, \frac{m}{||m||_2^2}\right)$ to $\{g \in (\mathbb{G}_\times)^d\,,\ \rho_m(v) = v'\}$. □

Let us now consider the representation $(\rho_M, \mathcal{V})$ of $(\mathbb{G}_\times)^d$ associated to a matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. For two points $v, v' \in \mathcal{V}$ and $s \in \mathrm{supp}\,(v) \cap \mathrm{supp}\,(v')$, we note $\mathcal{F}_s$ the foliage associated to the elements of the torus mapping v_s to v'_s. As in Definition 2.7, we note $\gamma\,(M)$ the set of minimal supports S such that the columns $\{m_s, s \in S\}$ are linearly dependent. The work about intersection of foliages achieved in Section 2 provides the following criterion for orbit equivalence:

Proposition 3.11. *Let v and v' be two points in $\mathcal{V}$.*

$$\exists g \in (\mathbb{G}_\times)^d,\ \rho_M(g)(v) = v' \Leftrightarrow \left\{ \begin{array}{l} \mathrm{supp}\,(v) = \mathrm{supp}\,(v') \\ \forall S \in \gamma(M)\ \ with\ S \subset \mathrm{supp}\,(v),\ \bigcap\limits_{s\in S} \mathcal{F}_s \neq \emptyset \end{array} \right.$$

Proof. Firstly, the existence of such a $g \in (\mathbb{G}_\times)^d$ implies that $\mathrm{supp}\,(v) = \mathrm{supp}\,(v')$. Suppose furthermore that for any support $S \in \gamma(M)$ such that $S \subset \mathrm{supp}\,(v)$, $\bigcap\limits_{s\in S} \mathcal{F}_s \neq \emptyset$. Since the columns $\{m_s,\ s \in S\}$ wear the directions of the foliages $\{\mathcal{F}_s,\ s \in S\}$, Theorem 2.8 ensures that $\bigcap\limits_{s\in \mathrm{supp}(v)} \mathcal{F}_s \neq \emptyset$. Take then some f in that intersection, and the element $g = \phi^{-1}(f) \in (\mathbb{G}_\times)^d$. Lemma 3.10 claims that $\forall s \in \mathrm{supp}\,(v)$, $\rho_{m_s}(g)(v_s) = v'_s$. We proved that $\rho_M(g)(v) = v'$. □

The thing is taht the intersection with respect to the optimal supports in $\gamma(M)$ are easily checked by rational monomials. These monomials are provided by Smith Normal Forms:

Definition 3.12. *Let S be a support in $\gamma(M)$. Note M_S the matrix consisting of columns m_s, $s \in S$, and U_S, W_S giving the Smith Normal Form (Theorem 3.3). Take w_S the last column of W_S. We note P_S the rational invariant monomial corresponding to w_S.*

Since $\mathrm{rank}\,(M_S) = \#S - 1$, w_S is uniquely determined up to ± 1, and so is P_S up to the inverse. We guess also that its support $\mathrm{supp}\,(P_S)$ is exactly S. Indeed, for any $T \not\subseteq S$, the family $\{m_t,\ t \in T\}$ is linearly independent. By Lemma 3.2, the invariant field $\mathbb{C}(\mathcal{V}_T)^{(\mathbb{G}_\times)^d}$ is reduced to constant functions. We get finally the main theorem:

Theorem 3.13. *Let $(\rho_M, \mathcal{V})$ the representation of $(\mathbb{G}_\times)^d$ associated to the matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. The following set of invariant monomials is a rational separating set:*

$$\mathcal{E}_{sep} = \{P_S,\ S \in \gamma(M)\}.$$

Proof. Take $v, v' \in \mathcal{V}$ two points in the area separated by the invariant field. That is, the families $\{M_s,\ s \in \mathrm{supp}\,(v)\}$ and $\{M_s,\ s \in \mathrm{supp}\,(v')\}$ are linearly dependent (Lemma 3.8).

• Suppose that $\mathrm{supp}\,(v) \neq \mathrm{supp}\,(v')$. Up by permuting v and v', one chooses an indice $s \in \mathrm{supp}\,(v) \setminus \mathrm{supp}\,(v')$. There exists a support $S \in \gamma(M)$ containing s and included in $\mathrm{supp}\,(v)$. Consider then the monomial $P_S \in \mathcal{E}_M$ of supports S. It is defined at v and satisfies $P_S(v) \neq 0$. Whereas, it admits a pole or a singularity at v'.

• Otherwise, $\mathrm{supp}\,(v) = \mathrm{supp}\,(v')$. Suppose then that for all $P \in \mathcal{E}_{sep}$ defined at v, $P(v) = P(v')$. Consider a support $S \in \gamma(M)$ included in $\mathrm{supp}\,(v)$. Orbits of support S are closed in $\mathcal{V}_S$ and are separated by the invariant algebra $\mathbb{C}[\mathcal{V}_S]^{(\mathbb{G}_\times)^d}$. The latter, of dimension 1, is generated by P_S. That is, there exists $g \in (\mathbb{G}_\times)^d$ such that for all $s \in S$, $\rho_M(g)(v_s) = v'_s$. We have $\phi^{-1}(g) \in \bigcap\limits_{s\in S} \mathcal{F}_s$. We obtain that for any $S \in \gamma(M)$ included in $\mathrm{supp}\,(v)$, $\bigcap\limits_{s\in S} \mathcal{F}_s \neq \emptyset$. Proposition 3.11 claims then that v and v' belong to the same orbit. □

Proposition 3.14. *The separating set $\mathcal{E}_{sep}$ defined in Theorem 3.13 is minimal for the inclusion.*

Proof. Suppose that we remove a monomial P_S from $\mathcal{E}_{sep}$ for some $S \in \gamma(M)$. Take then v and v' in distinct orbits of $\mathcal{V}_S$ (the variety of points of support exactly S). Then for any monomial $P \in \mathcal{E}_{sep}$, $\mathrm{supp}\,(P) \not\subset S$. Thus P admits a pole at v and v', or a root at v and v'. The points v and v' are not separated. □

Remark 3.15. *In general, the computation of the Smith Normal Form has a polynomial cost in d, n [KB79]. But another advantage of our method is that this difficulty is largely avoided. Indeed, we managed to reduce to matrices of corank 1. In these conditions, the computation of the Smith Normal Form is the computation of a one dimensional null space, that is solving a linear system of size $\#S \leq d + 1$.*

4 The cardinality of our separating set

In this section, we discuss the cardinality of $\gamma(M)$, which is the cardinality of the minimal rational separating set given by Theorem 3.13. In particular, we give a polynomial upper bound when the matrix M is of full rank in Theorem 4.1.

Suppose that for any supports S such that $\#S \leq d$, the family of columns $\{m_s,\ s \in S\}$ is linearly independent. Then, $\gamma(M)$ is exactly the set of supports $S \subset [\![1,n]\!]$ of cardinality $d+1$, and $\#(\mathcal{E}_{sep}) = \binom{n}{d+1} \sim n^{d+1}$. From a computational point of view, this case is pleasant, the set $\gamma(M)$ being easily visualisable and computable. But concerning the cardinality, it is in fact proven to be the worst, as soon as M is of full rank:

Theorem 4.1. *Suppose that $M \in \mathcal{M}_{d,n}(\mathbb{Z})$ is of full rank d. Then, $\#(\mathcal{E}_{sep}) \leq \binom{n}{d+1} \sim_{n\to\infty} n^{d+1}$.*

Proof. For $d=1$, $\#\gamma(M) = \binom{n}{2}$ and Theorem 4.1 holds. Furthermore, for any $d \in \mathbb{N}$ and $M \in \mathcal{M}_{d,d}(\mathbb{Z})$ of full rank d, $\#\gamma(M) = 0$ and Theorem 4.1 holds. We then proceed by induction. Consider some integers d and $n \geq d+1$ such that the theorem holds at $(d, n-1)$ and $(d-1, n-1)$.

Consider a matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$ of full rank d. Up to a permutation, we assume that the $n-1$ last columns are of full rank d. A basis change allows then to write

$$M = \left(\begin{array}{c|c} m_1 & L \\ \hline 0 & \\ \vdots & M' \\ 0 & \end{array}\right)$$

where M' is a matrix in $\mathcal{M}_{d-1,n-1}(\mathbb{Z})$ of full rank $d-1$ and $\left(\frac{L}{M'}\right)$ is a matrix in $\mathcal{M}_{d,n-1}(\mathbb{Z})$ of full rank d. Take S a support in $\gamma(M)$. There is the following two possibilities:

- $1 \in S$. Then, the family of columns $\{M'_s,\ s \in S \setminus \{1\}\}$ is linearly dependent, and minimal for that property. That is, $S \setminus \{1\} \in \gamma(M')$.
- Otherwise, $1 \notin S$. Then, $S \in \gamma\left(\left(\frac{L}{M'}\right)\right)$.

We thus have $\gamma(M) \subset \gamma(M') \cup \gamma\left(\left(\frac{L}{M'}\right)\right)$. We apply Theorem 4.1 (assumed true for $(d-1, n-1)$ and $(d, n-1)$) to M' and $\left(\frac{L}{M'}\right)$. It gives

$$\#\gamma(M) \leq \#\gamma(M') + \#\gamma\left(\left(\frac{L}{M'}\right)\right) \leq \binom{n-1}{d} + \binom{n-1}{d+1}.$$

Pascal's rule completes in $\gamma(M) \leq \binom{n}{d+1}$. □

When the matrix M is not of full rank, the situation is less clear. For instance, if one considers the representation associated to the matrix

$$\mathcal{M}_{d,n}(\mathbb{Z}) = \begin{pmatrix} 1 & \dots & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{d,d+1}(\mathbb{Z}),$$

one has $\binom{d+1}{d+1} = 1$ whereas $\#(\gamma(M)) = \binom{d+1}{2}$. In fact, we could give the following formulation:

Theorem 4.2. *Suppose that $M \in \mathcal{M}_{d,n}(\mathbb{Z})$ is of rank $k \leq d$. Then, $\#(\gamma(M)) = \#(\mathcal{E}_{sep}) \leq \binom{n}{k+1} \sim n^{k+1}$.*

But a basis change reduces in fact these representations to representations of the torus of dimension k. We thus prefer to assume that any representations has full rank.

5 Comparison to generating sets

In this section we compare the obtained separating sets with sets generating the invariant algebra or the invariant field. Beyond the considerations regarding the cardinality, we discuss the accuracy of their separation and the complexity of their computation.

Definition 5.1. *Let* G *be a complex reductive algebraic group and* $\mathcal{V}$ *a* G*-variety.*

- *A subset* $\mathcal{E}_{gen} \subset \mathbb{C}(\mathcal{V})^{\mathrm{G}}$ *is a* rational *generating set if it generates the invariant* field*:* $\mathbb{C}(\mathcal{E}_{gen}) = \mathbb{C}(\mathcal{V})^{\mathrm{G}}$.
- *A subset* $\mathcal{E}_{gen} \subset \mathbb{C}[\mathcal{V}]^{\mathrm{G}}$ *is a* polynomial *generating set if it generates the invariant algebra:* $\mathbb{C}[\mathcal{E}_{gen}] = \mathbb{C}[\mathcal{V}]^{\mathrm{G}}$.

Let $(\rho_M, \mathcal{V})$ be the representation of $(\mathbb{G}_\times)^d$ associated to the matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$. Recall that invariant rational monomial correspond to the $\mathbb{Z}$-module $\mathcal{M} = \{e \in \mathbb{Z}^n,\ Me = 0\}$. Theorem 3.4 provides then a minimal generating set of highly competitive cardinality $n - \mathrm{rank}(M)$. An analogous description is derived for the invariant algebra:

Proposition 5.2. *Monomials in the invariant algebra* $\mathbb{C}[V]^{(\mathbb{G}_\times)^d}$ *correspond to the cone*

$$\mathcal{M}^+ := \{e \in \mathcal{M},\ \forall 1 \leq s \leq n, e_s \geq 0\}.$$

Algebra generators correspond then to the extremal vectors of this cone. [Stu08] provides an algorithm to compute them, based on Groebner bases:

Algorithm 5.3. *[Stu08, p. 1.45]*

Input: A matrix $M \in \mathcal{M}_{d,n}(\mathbb{Z})$.

Output: a set of generators for $\mathbb{C}[\mathcal{V}]^{(\mathbb{G}_\times)^d}$.

- *Compute generators of the kernel* I_M *of the* $\mathbb{C}$*-algebra homomorphism*

$$\left\{ \begin{array}{rcl} \mathbb{C}[v_1, ..., v_n, y_1, ..., y_n] & \to & \mathbb{C}[g_1, ..., g_d, v_1, ..., v_n, y_1, ..., y_n] \\ \forall 1 \leq s \leq n,\ v_s & \mapsto & y_s \prod_{i=1}^{d} g_i^{M_{is}} \\ \forall 1 \leq s \leq n,\ y_s & \mapsto & y_s \end{array} \right.$$

- *Choose the monomial order* $g_1 > ... > g_d > v_1 > ... > v_n > y_1 ... > y_n$.
- *Compute a reduced Groebner Basis* $\mathcal{G}$ *for* I_M *with respect to* $>$.
- *Select the monomials* v^e, $e \in \mathbb{Z}^n$ *such that* $v^e - y^e$ *appears in* $\mathcal{G}$.

The cardinality of such algebra generators is hard to control. It can be reduced to 0, when the representation admits no closed orbits. But in general, this cardinality is quite bigger, most of the 2^n supports in $[\![1, n]\!]$ admitting irreducible monomials. We thus dress the following empirical hierarchy:

$$\underbrace{\#\Big(\ \textit{generating}\ \mathbb{C}(\mathcal{V})^{(\mathbb{G}_\times)^d}\Big)}_{\sim n} \quad << \quad \underbrace{\#(\ \textit{rational separating set}\ )}_{\sim n^{d+1}} \quad << \quad \#\Big(\ \textit{generating}\ \mathbb{C}[\mathcal{V}]^{(\mathbb{G}_\times)^d}\Big)$$

These differences are confirmed by Examples 5.4 and 5.5. I take random integer matrices with $d = 2$ or 3 and $n = 8$. I compute the rational generating set provided by Theorem 3.4, the rational separating set of that paper and the polynomial generating set provided by Algorithm 5.3. All the computations have been implemented in Maple.

Example 5.4. *Consider the representation of* $(\mathbb{G}_\times)^2$ *associated to*

$$M_2 = \begin{pmatrix} 2 & -2 & 5 & -2 & -5 & -3 & 2 & 6 \\ 3 & 5 & 6 & 4 & -4 & 0 & -6 & -3 \end{pmatrix}$$

- *Thanks to Smith Normal Forms, Theorem 3.4 provides a rational generating set of cardinality* 6*:*

$$\mathcal{E}^{(2)}_{gen} = \left\{ \begin{array}{cccccc} \frac{v_2^3 v_3^{16}}{v_1^{37}} & \frac{v_2 v_3^{10} v_4}{v_1^{23}} & \frac{v_2^2 v_3^{13} v_5}{v_1^{28}} & \frac{v_3^3 v_6}{v_1^6} & \frac{v_2^3 v_3^{10} v_7}{v_1^{23}} & \frac{v_2^3 v_3^8 v_8}{v_1^{20}} \end{array} \right\}$$

- *Any* 2×2 *submatrix in* M_2 *is invertible. Theorem 3.13 provides thus a rational separating set consisting of* $\binom{8}{3} = 56$ *monomials:*

$$\mathcal{E}^{(2)}_{sep} = \left\{ \begin{array}{cccccccccc} \frac{v_2^3 v_3^{16}}{v_1^{37}} & \frac{v_1 v_4^8}{v_1^7} & \frac{v_1^{33} v_5^{16}}{v_1^7} & \frac{v_1^{15} v_6^{16}}{v_2^9} & v_1 v_2^9 v_7^8 & \frac{v_2^3 v_8^2}{v_3^3} & \frac{v_3^{14} v_4^3}{v_1^{32}} & \frac{v_3^7 v_5^3}{v_1^{10}} & \frac{v_3^3 v_6}{v_1^6} & \frac{v_1^{14} v_7}{v_3^6} \\ \frac{v_1^{17} v_8}{v_1^8} & \frac{v_1^4 v_5^2}{v_4} & \frac{v_1^{12} v_6^{14}}{v_1^9} & v_1^2 v_4^9 v_7^7 & \frac{v_4^{12} v_8^7}{v_1^9} & \frac{v_6^7}{v_1^{12} v_2^9} & v_1^{38} v_5^{18} v_7^7 & v_1^{39} v_5^{24} v_8^7 & v_1^2 v_6^2 v_7 & v_1^3 v_6^8 v_8^3 \\ \frac{v_3^3}{v_1^5 v_8^7} & \frac{v_3^2 v_4^{37}}{v_2^{32}} & \frac{v_3^{33} v_5^{37}}{v_2^{10}} & \frac{v_3^{15} v_6^{37}}{v_2^{18}} & v_2^{42} v_3^2 v_7^{37} & \frac{v_2^{51} v_8^{37}}{v_3^{24}} & \frac{v_2^{28} v_5^2}{v_4^{33}} & \frac{v_2^{12} v_6^2}{v_4^{15}} & \frac{v_2^2 v_7}{v_4} & \frac{v_4^{12} v_8}{v_2^9} \\ \frac{v_6^{11}}{v_2^4 v_5^5} & \frac{v_2^{38} v_7^{33}}{v_5^2} & v_2^{13} v_5^8 v_8^{11} & \frac{v_2^{18} v_7^{15}}{v_6^2} & v_2^3 v_6^8 v_8^5 & v_2^{15} v_7^{12} v_8 & \frac{v_3^{14} v_5^{16}}{v_4^5} & \frac{v_3^6 v_6^{16}}{v_4^9} & v_3^2 v_4^{21} v_7^{16} & \frac{v_4^{51} v_8^{32}}{v_3^{18}} \\ \frac{v_6^5}{v_3^6 v_5^9} & v_3^{19} v_5^{21} v_7^5 & v_3^{39} v_5^{51} v_8^{10} & v_3^3 v_6^7 v_7^3 & v_3^3 v_6^{17} v_8^6 & \frac{v_8^{14}}{v_3^{10} v_7^{17}} & \frac{v_6^7}{v_4^3 v_5^3} & \frac{v_4^{19} v_7^{14}}{v_5^2} & v_4^{39} v_5^{18} v_8^{28} & \frac{v_4^9 v_6^6}{v_4^2 v_6^2} \\ v_4^3 v_6^6 v_8^4 & v_4^{15} v_7^9 v_8^2 & \frac{v_6^{19} v_7^6}{v_5^9} & \frac{v_6^{13} v_8^4}{v_5^3} & \frac{v_5^{30} v_8^{38}}{v_7^{39}} & \frac{v_6^{10} v_8^6}{v_7^3} & & & & \end{array} \right\}$$

Whereas, Algorithm 5.3 provides a polynomial generating set of cardinality 329*. They consists of*

- 18 *monomials of support of cardinality* 3*:*

$$
\begin{array}{cccccccccc}
v_1^2v_6^2v_7 & v_4^3v_6^6v_8^4 & v_3^3v_6^7v_7^3 & v_1^3v_6^8v_8^3 & v_2^3v_6^8v_8^5 & v_1v_2^9v_7^8 & v_1^2v_4^9v_7^7 & v_3^3v_6^{17}v_8^6 & v_4^{15}v_7^9v_8^2 \\
v_2^{15}v_7^{12}v_8 & v_2^{13}v_5^8v_8^{11} & v_3^2v_4^{21}v_7^{16} & v_3^{19}v_5^{21}v_7^5 & v_1^{38}v_5^{18}v_7^7 & v_1^{39}v_5^{24}v_8^7 & v_2^{42}v_3^2v_7^{37} & v_4^{39}v_5^{18}v_8^{28} & v_3^{39}v_5^{51}v_8^{10}
\end{array}
$$

- 112 *monomials of support of cardinality* 4*:*

$$
\begin{array}{cccccccc}
v_3v_4v_5v_7 & v_2^2v_5v_6v_8^2 & v_2^2v_3v_5v_7^2 & v_2v_4^3v_5^2v_8^3 & v_1^4v_3v_5^3v_7 & v_1v_2v_4^2v_7^4 & v_3^3v_5^3v_6^4v_8^2 & v_1v_2^3v_4^3v_7^5 \\
v_3^4v_5^3v_6^3v_7^2 & v_4^6v_6^2v_7^3v_8^2 & v_3^2v_6^8v_7v_8^2 & v_3v_4^6v_6v_7^5 & v_3^2v_4^2v_6^4v_7^4 & v_1^6v_5^3v_6^3v_8^2 & v_2v_4^7v_7^5v_8 & v_1v_2^5v_4^2v_7^6 \\
v_2^7v_5^2v_7^3v_8^3 & v_3^5v_4v_5^2v_8^2 & v_2^3v_4^6v_7^6v_8 & v_1v_2^7v_4v_7^7 & v_1^2v_3v_6^{11}v_8^4 & v_2^5v_4^2v_7^7v_8 & v_2^6v_6^6v_7^3v_8^4 & v_1^{11}v_5^6v_7v_8 \\
v_2^6v_3^2v_6^4v_7^7 & v_3v_4^8v_5^5v_8^6 & v_2^5v_3^3v_7^5v_8^5 & v_1^4v_3^5v_5^9v_8^2 & v_2^7v_4^4v_7^8v_8 & v_4^{11}v_5^2v_7^4v_8^4 & v_1^5v_2^7v_5^2v_7^7 & v_1v_3^2v_6^{14}v_8^5 \\
v_2^9v_6^4v_7^6v_8^3 & v_2^9v_4^3v_7^9v_8 & v_1v_4^{12}v_7^8v_8 & v_1^6v_4^8v_5^2v_7^7 & v_3^9v_5^9v_6^2v_7^3 & v_1^{12}v_2v_5^8v_8^3 & v_2^{11}v_4^2v_7^{10}v_8 & v_1^9v_2^5v_5^4v_7^6 \\
v_4^9v_5^3v_6^5v_8^8 & v_1^3v_4^9v_5^6v_8^7 & v_2^{12}v_6^2v_7^9v_8^2 & v_2^{12}v_3v_6v_7^{11} & v_1^9v_2^4v_5^8v_8^5 & v_2^{13}v_4v_7^{11}v_8 & v_1^{13}v_2^3v_5^6v_7^5 & v_1^2v_3^{10}v_5^{12}v_7^3 \\
v_1^6v_2^7v_5^8v_8^7 & v_2^2v_3^9v_5^{13}v_8^4 & v_1^{15}v_5^9v_6v_8^3 & v_1^{10}v_4^7v_5^4v_7^7 & v_1^3v_2^{10}v_5^8v_8^9 & v_1^7v_4^8v_5^8v_8^7 & v_1v_3^{11}v_5^{15}v_8^3 & v_1^{11}v_3^4v_5^{12}v_8^3 \\
v_1^{17}v_2v_5^8v_7^4 & v_1^{14}v_4^6v_5^6v_7^7 & v_3^{12}v_5^{15}v_6^3v_8^4 & v_3^{14}v_5^{15}v_6v_7^4 & v_1^{20}v_5^9v_6v_7^4 & v_1^{11}v_4^7v_5^{10}v_8^7 & v_4^{15}v_5^6v_6^4v_8^{12} & v_4^{18}v_5^6v_7^3v_8^{10} \\
v_1^{18}v_4^5v_5^8v_7^7 & v_1^{15}v_4^6v_5^{12}v_8^7 & v_1^{18}v_3^3v_5^{15}v_8^4 & v_2^2v_3^2v_4^{20}v_7^{17} & v_2^4v_3^2v_4^{19}v_7^{18} & v_1^{22}v_4^4v_5^{10}v_7^7 & v_1^2v_4^{19}v_5^{10}v_8^{14} & v_1^{19}v_4^5v_5^{14}v_8^7 \\
v_2^6v_3^2v_4^{18}v_7^{19} & v_2^8v_3^2v_4^{17}v_7^{20} & v_1^{26}v_4^3v_5^{12}v_7^7 & v_4^{21}v_5^9v_6^3v_8^{16} & v_2^{10}v_3^2v_4^{16}v_7^{21} & v_1^{23}v_4^4v_5^{16}v_8^7 & v_1^{25}v_3^2v_5^{18}v_8^5 & v_2^{12}v_3^2v_4^{15}v_7^{22} \\
v_4^{25}v_5^{10}v_7^2v_8^{16} & v_2^{14}v_3^2v_4^{14}v_7^{23} & v_1^{30}v_4^2v_5^{14}v_7^7 & v_1^{27}v_4^3v_5^{18}v_8^7 & v_2^{16}v_3^2v_4^{13}v_7^{24} & v_3^{21}v_5^{27}v_6^2v_8^6 & v_3^{23}v_5^{27}v_7^4v_8^2 & v_2^{18}v_3^2v_4^{12}v_7^{25} \\
v_1^{34}v_4v_5^{16}v_7^7 & v_2^{20}v_3^2v_4^{11}v_7^{26} & v_1^{31}v_4^2v_5^{20}v_8^7 & v_1^{32}v_3v_5^{21}v_8^6 & v_4^{27}v_5^{12}v_6^2v_8^{20} & v_2^{22}v_3^2v_4^{10}v_7^{27} & v_2^{24}v_3^2v_4^9v_7^{28} & v_2v_3^{24}v_5^{32}v_8^7 \\
v_1v_4^{29}v_5^{14}v_8^{21} & v_1^{35}v_4v_5^{22}v_8^7 & v_2^{26}v_3^2v_4^8v_7^{29} & v_3^{27}v_5^{33}v_7^3v_8^4 & v_2^{28}v_3^2v_4^7v_7^{30} & v_4^{32}v_5^{14}v_7v_8^{22} & v_2^{30}v_3^2v_4^6v_7^{31} & v_2^{32}v_3^2v_4^5v_7^{32} \\
v_4^{33}v_5^{15}v_6v_8^{24} & v_2^{34}v_3^2v_4^4v_7^{33} & v_2^{36}v_3^2v_4^3v_7^{34} & v_2^{38}v_3^2v_4^2v_7^{35} & v_3^{30}v_5^{39}v_6v_8^8 & v_3^{31}v_5^{39}v_7^2v_8^6 & v_2^{40}v_3^2v_4v_7^{36} & v_3^{35}v_5^{45}v_7v_8^8
\end{array}
$$

- 139 *monomials of support of cardinality* 5*:*

$$
\begin{array}{ccccccc}
v_3v_4^2v_5^2v_6v_8^2 & v_1v_3^2v_5^3v_6v_8 & v_1^2v_4v_5v_6^3v_8^2 & v_1v_3^4v_6^2v_7^2v_8 & v_1v_3v_6^5v_7v_8 & v_1^3v_4^2v_5^2v_7v_8 & v_1^3v_2v_5^2v_6^2v_8^2 \\
v_2^3v_4^2v_5^2v_7v_8^3 & v_1v_2^3v_6^4v_7^2v_8^2 & v_1v_2^3v_3v_6^3v_7^4 & v_2^5v_4v_5^2v_7^2v_8^3 & v_1^3v_4^3v_5^3v_6v_8^3 & v_3v_4^3v_6^5v_7^2v_8^2 & v_1^2v_4^5v_5^2v_7^2v_8^2 \\
v_1^3v_2^4v_5^2v_7^3v_8 & v_2v_4^4v_6^4v_7^2v_8^3 & v_1^4v_2v_4^2v_5^4v_8^3 & v_1^5v_2v_3v_5^5v_8^2 & v_1^7v_4v_5^4v_7v_8 & v_1^4v_4^4v_5v_6v_7^4 & v_2^2v_4^2v_6^6v_7v_8^4 \\
v_2^2v_4^5v_6^2v_7^4v_8^2 & v_1v_2^6v_6^2v_7^5v_8 & v_2^2v_3v_5^4v_6v_7^6 & v_2^2v_3^2v_4^2v_6^4v_7^5 & v_1^5v_2v_4^3v_5^2v_7^4 & v_1v_2^4v_4^2v_5^4v_8^5 & v_1^2v_4^2v_3v_5^5v_8^4 \\
v_2^3v_4^3v_6^4v_7^3v_8^3 & v_2^3v_3v_6^7v_7^2v_8^3 & v_2^3v_3^2v_6^6v_7^4v_8 & v_1^7v_2^2v_5^4v_7^2v_8 & v_1^2v_3^5v_5^6v_6v_7^2 & v_1v_4^5v_5^2v_6^4v_8^5 & v_2^4v_4v_6^6v_7^2v_8^4 \\
v_7^4v_5v_6^3v_7^2v_8^4 & v_3^2v_4v_5v_6^9v_8^4 & v_1v_4^8v_5^2v_7^3v_8^3 & v_2^4v_4^2v_6^2v_7^5v_8^2 & v_2^2v_3^5v_5^7v_7v_8^2 & v_2^4v_3v_4^4v_6v_7^7 & v_2^4v_3^2v_4^4v_6^7v_8^4 \\
v_1^5v_2^3v_4^2v_5^2v_7^5 & v_2^5v_2^4v_6^4v_7^4v_8^3 & v_1^4v_2^5v_5^4v_7^2v_8^3 & v_1^5v_3v_5^3v_6^6v_8^3 & v_1^7v_4^2v_5^5v_6v_8^3 & v_1^3v_2^2v_3^3v_5^7v_8^3 & v_1^8v_3v_5^6v_6v_8^2 \\
v_1^2v_2v_4v_6^{10}v_8^5 & v_1^8v_2v_4v_5^6v_8^3 & v_2^6v_4^3v_6^2v_7^6v_8^2 & v_2^6v_3v_5^5v_6^5v_7^2v_8^2 & v_1v_3^7v_5^9v_7v_8 & v_2^6v_3v_4^3v_6v_7^8 & v_1^5v_2^5v_4v_5^2v_7^6 \\
v_1^8v_4^3v_5^3v_6v_7^4 & v_1v_2^8v_5^4v_7^2v_8^5 & v_1v_2^3v_4^6v_5^6v_8^5 & v_1^2v_3^3v_4^4v_5^7v_8^4 & v_2^7v_4v_6^4v_7^5v_8^3 & v_2v_3^6v_5^8v_6^2v_8^3 & v_1^3v_3^4v_4^2v_5^8v_8^3 \\
v_1^9v_2v_4^2v_5^4v_7^4 & v_4^8v_5^2v_6^4v_7v_8^6 & v_1^2v_4^7v_5^4v_6^2v_8^6 & v_1^5v_2^4v_4v_5^6v_8^5 & v_1^8v_2^3v_5^6v_7v_8^3 & v_2^8v_4^2v_6^2v_7^2v_8^2 & v_2^8v_3v_4^2v_6v_7^9 \\
v_1^3v_2v_3^6v_5^8v_7^3 & v_2v_3^3v_5^2v_6^{11}v_8^5 & v_2^3v_3^4v_4^2v_5^8v_8^5 & v_1v_2^3v_5^3v_5^9v_8^4 & v_2^9v_3v_6^3v_7^8v_8 & v_1^4v_2^8v_5v_6v_7^8 & v_1^9v_2^3v_4v_5^4v_7^5 \\
v_1^2v_2^7v_4v_5^6v_8^7 & v_1^5v_2^6v_5^6v_7v_8^5 & v_1^{11}v_4v_5^7v_6v_8^3 & v_2^{10}v_4v_6^2v_7^8v_8^2 & v_1^2v_2^{10}v_5^2v_7^7v_8^2 & v_3^8v_5^9v_6^3v_7v_8^2 & v_2^{10}v_3v_4v_6v_7^{10} \\
v_1v_2^2v_4^2v_6^{12}v_8^7 & v_1^2v_2v_3^7v_5^{11}v_8^3 & v_1^{12}v_4^2v_5^2v_6v_7^4 & v_1^2v_2^9v_5^6v_7v_8^7 & v_4^{12}v_5^3v_6v_7^3v_8^6 & v_4^4v_3v_7^4v_5^7v_8^6 & v_1^5v_3^2v_4^5v_5^8v_8^5 \\
v_1^6v_3^3v_4^3v_5^9v_8^4 & v_1^7v_3^4v_4v_5^{10}v_8^3 & v_1^8v_2^6v_5^3v_6v_7^7 & v_1^{13}v_2v_4v_5^6v_7^4 & v_2^8v_3^2v_4v_5^8v_8^4 & v_2v_3^2v_4v_6^{16}v_8^7 & v_2^2v_3v_4^2v_6^{15}v_8^8 \\
v_1^{12}v_2^4v_5^5v_6v_7^6 & v_1v_4^{11}v_5^5v_6^3v_8^9 & v_4^{13}v_5^4v_6^2v_7^2v_8^8 & v_2^{10}v_3^2v_5^8v_7v_8^8 & v_1^2v_4^{12}v_5^6v_7v_8^8 & v_1^{16}v_4v_5^7v_6v_7^4 & v_1^8v_3v_4^6v_5^9v_8^6 \\
v_1^9v_3^2v_4^4v_5^{10}v_8^5 & v_4^{16}v_5v_6v_7^8v_8^4 & v_1^{10}v_3^3v_4^2v_5^{11}v_8^4 & v_2v_3^{12}v_5^{14}v_7^3v_8 & v_1^{16}v_2^2v_5^7v_6v_7^5 & v_4^{14}v_5^5v_6^3v_7v_8^{10} & v_1^2v_4^{13}v_5^7v_6v_8^{10} \\
v_1v_4^{15}v_5^6v_7^2v_8^9 & v_2^{11}v_3v_4^2v_5^9v_8^{11} & v_1v_2^{11}v_3^2v_5^{10}v_8^{10} & v_3^{13}v_5^{15}v_6^2v_7^2v_8^2 & v_1^{12}v_3v_4^5v_5^{11}v_8^6 & v_1^{13}v_2^3v_4^3v_5^{12}v_8^5 & v_1^{14}v_3^3v_4v_5^{13}v_8^4 \\
v_1v_2v_3^{15}v_5^{17}v_7^5 & v_1^{16}v_3v_4^4v_5^{13}v_8^6 & v_1^{17}v_3^2v_4^2v_5^{14}v_8^5 & v_1v_4^{17}v_5^8v_6^2v_8^{13} & v_4^{19}v_5^7v_6v_7^2v_8^{12} & v_2v_3^{15}v_5^{20}v_6v_8^5 & v_2v_3^{16}v_5^{20}v_7^2v_8^3 \\
v_4^{20}v_5^8v_6^2v_7v_8^{14} & v_1^{20}v_3v_4^3v_5^{15}v_8^6 & v_1^{21}v_3^2v_4v_5^{16}v_8^5 & v_3^{17}v_5^{21}v_6^2v_7v_8^4 & v_3^{18}v_5^{21}v_6v_7^3v_8^2 & v_1v_4^{22}v_5^{10}v_7v_8^{15} & v_1^{24}v_3v_4^2v_5^{17}v_8^6 \\
v_1v_4^{23}v_5^{11}v_6v_8^{17} & v_2v_3^{20}v_5^{26}v_7v_8^5 & v_1^{28}v_3v_4v_5^{19}v_8^6 & v_3^{22}v_5^{27}v_6v_7^2v_8^4 & v_4^{26}v_5^{11}v_6v_7v_8^{18} & v_3^{26}v_5^{33}v_6v_7v_8^6 &
\end{array}
$$

- 56 *monomials of support of cardinality* 6*:*

$$
\begin{array}{ccccccc}
v_1v_2v_3v_4v_5^3v_8^2 & v_2v_3^2v_5^2v_6^2v_7v_8 & v_1v_2v_4v_6^4v_7v_8^2 & v_1v_2v_3v_4v_6^3v_7^3 & v_1^3v_2v_3v_5^2v_6v_7^2 & v_1v_2^3v_3v_5^3v_7v_8^2 \\
v_1v_2^2v_4^2v_6^2v_7^3v_8 & v_1^3v_2^2v_4v_5^2v_7^2v_8 & v_1v_4^4v_5v_6^3v_7v_8^3 & v_1v_3v_4v_5v_6^6v_8^3 & v_2v_3^2v_4v_5^3v_6^3v_8^3 & v_1^4v_3v_4v_5^4v_6v_8^2 \\
v_1v_2^4v_4v_6^2v_7^4v_8 & v_1^2v_2v_3^3v_5^3v_7v_8 & v_1v_2v_4^2v_5v_6^5v_8^4 & v_2v_3v_4v_6^7v_7v_8^3 & v_1^2v_2v_3v_5^2v_6^5v_8^3 & v_2v_3v_4^4v_6^3v_7^4v_8 \\
v_1^2v_2^5v_5v_6v_7^4v_8 & v_2v_2^2v_3^2v_4v_6^3v_7^3v_8 & v_2^2v_3v_4^2v_6^3v_7^3v_8^2 & v_1^2v_2^2v_4^4v_5^2v_7^3v_8^2 & v_1^4v_2^3v_4v_5^4v_7v_8^3 & v_2^3v_3v_4^3v_6^3v_7^2v_8 \\
v_1^4v_2^2v_4^3v_5v_6v_7^5 & v_2^2v_1^6v_5^3v_6v_7v_8^4 & v_2^4v_3v_4v_6^5v_7^4v_8^2 & v_2^2v_1^2v_4^3v_5^2v_7^4v_8^2 & v_1^6v_2^3v_5^3v_6v_7^3v_8 & v_1v_2v_3^5v_5^5v_6^2v_7^3 \\
v_2v_4^5v_5v_6^5v_7v_8^5 & v_1v_2^6v_4v_5^4v_7v_8^5 & v_2v_3v_4^2v_5v_6^8v_8^5 & v_1v_2v_3^2v_5^2v_6^8v_8^4 & v_2v_4^8v_5v_6v_7^4v_8^3 & v_5^2v_3v_4^2v_6^3v_7^6v_8 \\
v_1^3v_4^7v_5v_6v_7^5v_8 & v_1^4v_2^4v_4^2v_5v_6v_7^6 & v_1v_2v_3^4v_5^2v_6^6v_7^4 & v_1^2v_2^6v_4^2v_5^2v_7^5v_8^2 & v_2^7v_3v_4v_6^3v_7^3v_8 & v_2v_3^7v_5^8v_6v_7^2v_8 \\
v_1^{10}v_2v_5^5v_6v_7^2v_8 & v_1^4v_2^6v_4v_5v_6v_7^7 & v_1v_4^9v_5^3v_6v_7^2v_8^5 & v_1^2v_2^8v_4^2v_5^2v_7^6v_8^2 & v_1^8v_2^2v_4^3v_5^3v_6v_7^5 & v_1^2v_4^{10}v_5v_6v_7^6v_8^2 \\
v_1v_2v_3v_4v_6^{13}v_8^6 & v_1^8v_2^4v_4v_5^3v_6v_7^6 & v_1v_4^{10}v_5^4v_6^2v_7v_8^7 & v_1v_4^{13}v_5v_6v_7^3v_8^3 & v_1^{12}v_2^2v_4v_5^5v_6v_7^5 & v_1v_2v_3^{10}v_5^{11}v_6v_7^4 \\
v_2v_3^{11}v_5^{14}v_6v_7v_8^3 & v_1v_4^{16}v_5^7v_6v_7v_8^{11} &
\end{array}
$$

- *And* 4 *monomials of support of cardinality* 7*:*

$$v_1^2v_2v_4^2v_5v_6v_7^2v_8 \quad v_1^2v_2^3v_4v_5v_6v_7^3v_8 \quad v_1v_2v_4^5v_5v_6v_7^3v_8^2 \quad v_1^6v_2v_4v_5^3v_6v_7^2v_8$$

Example 5.5. *Consider the representation of* $(\mathbb{G}_\times)^3$ *associated to*

$$M_3 = \begin{pmatrix} 4 & -3 & -1 & 4 & 4 & 5 & 2 & -3 \\ -5 & 3 & 6 & -5 & -4 & 6 & -4 & -3 \\ -3 & -1 & 0 & -2 & -5 & 3 & -2 & 6 \end{pmatrix}$$

- *Thanks to Smith Normal Forms, Theorem 3.4 provides a rational generating set of cardinality* 5*:*

$$\mathcal{E}^{(3)}_{gen} = \left\{ \frac{v_2^{19}v_3^3v_4^{64}}{v_1^{49}} \quad \frac{v_2^{14}v_3^2v_4^{49}v_5}{v_1^{39}} \quad \frac{v_2^{10}v_4^{25}v_6}{v_1^{19}v_3} \quad \frac{v_2^{17}v_3^3v_4^{58}v_7}{v_1^{45}} \quad \frac{v_2^{10}v_3^3v_4^{31}v_8}{v_1^{22}} \right\}$$

- *According to Theorem 4.1,* $\#\gamma(M_3)$ *is bounded by* $\binom{8}{4} = 70$*. But since the columns* $3, 5$ *and* 7 *are linearly dependent, Theorem 3.13 provides a rational separating set consisting of only* 66 *monomials. These are of support of cardinality* 4*, except from* $P_{\{3,5,7\}}$ *whose support is of size* 3*:*

$$\mathcal{E}^{(3)}_{sep} = \left\{ \begin{array}{cccccc} \frac{v_3^2v_7^5}{v_5^2} & \frac{v_2^{19}v_3^3v_4^{64}}{v_1^{49}} & \frac{v_5^{64}}{v_1^{95}v_2^{35}v_3^{19}} & \frac{v_1^9v_2^{165}v_6^{64}}{v_3^{139}} & \frac{v_3^9v_7^{32}}{v_1^{19}v_2^7} & v_1^{111}v_2^{51}v_3^{99}v_8^{64} \\ \frac{v_2^4v_4^{19}v_5^3}{v_1^{19}} & \frac{v_2^{49}v_4^{139}v_6^3}{v_1^{106}} & \frac{v_1^4v_7}{v_2^2v_4^6} & \frac{v_1^{27}v_8}{v_3^9v_4^{33}} & \frac{v_1^{209}v_2^{125}v_6^{19}}{v_3^{139}} & \frac{v_5^{18}v_7^{19}}{v_2^{38}v_5^{14}} \\ \frac{v_5^{99}v_8^{19}}{v_1^{114}v_3^{39}} & \frac{v_2^{16}v_6^{18}v_7^{139}}{v_1^{80}} & v_1^{255}v_2^{366}v_6^{99}v_8^{139} & \frac{v_1^{16}v_4^4v_2^8}{v_1^{11}} & \frac{v_4^{35}v_5^{19}}{v_5^{55}v_4^4} & \frac{v_1^{129}v_6^{19}}{v_1^{49}v_4^{165}} \\ \frac{v_3^6v_4^{14}v_7^{19}}{v_1^{22}} & \frac{v_1^{72}v_3^{27}v_8^{19}}{v_4^{51}} & \frac{v_5^{33}v_6^7}{v_1^{48}v_3^{25}} & \frac{v_3^{39}v_5^{51}v_8^{35}}{v_1^{15}} & \frac{v_3^{16}v_6^{14}v_7^{165}}{v_1^{96}} & \frac{v_1^{93}v_3^{122}v_8^{55}}{v_6^{17}} \\ \frac{v_3^{36}v_7^{51}v_8^{14}}{v_1^6} & \frac{v_1^{169}v_6^4}{v_4^{125}v_5^{49}} & \frac{v_4^7v_5^3v_7^2}{v_1^{11}} & \frac{v_4^{39}v_5^{27}v_8^4}{v_1^{63}} & \frac{v_6^6v_7^{49}}{v_1^{16}v_4^{16}} & \frac{v_1^{369}v_6^{27}v_8^{49}}{v_4^{366}} \\ \frac{v_1^{18}v_2^2}{v_4^{12}v_7^9} & \frac{v_5^{16}v_6^{14}v_7^{125}}{v_1^{96}} & \frac{v_5^{366}v_6^{39}v_8^{125}}{v_1^{321}} & \frac{v_5^{36}v_7^{14}}{v_5^6v_7^{39}} & \frac{v_1^{120}v_8^8}{v_6^{18}v_7^{183}} & \frac{v_5^{49}}{v_2^{55}v_3^{19}v_4^{95}} \\ \frac{v_2^{129}v_4^9v_6^{49}}{v_1^{106}} & \frac{v_3^{12}v_7^{49}}{v_2^{22}v_4^{38}} & v_2^{72}v_3^{81}v_4^{111}v_8^{49} & \frac{v_2^{240}v_5^9v_6^{95}}{v_3^{209}} & v_2^{15}v_3^{114}v_5^{111}v_8^{95} & \frac{v_2^{96}v_6^{38}v_7^9}{v_3^{80}} \\ \frac{v_3^{85}v_8^3}{v_2^{93}v_6^{37}} & v_2^6v_3^{90}v_7^{111}v_8^{38} & \frac{v_2^{169}v_4^{209}v_6^{19}}{v_5^{106}} & \frac{v_5^{12}v_7^{19}}{v_2^{22}v_4^{38}} & \frac{v_5^{81}v_8^{19}}{v_6^{63}v_7^{114}} & \frac{v_6^3v_7^{53}}{v_8^8v_2^{40}} \\ v_2^{369}v_4^{255}v_6^{81}v_8^{106} & \frac{v_2^{18}v_4^{30}v_8^4}{v_7^{27}} & \frac{v_2^{96}v_6^{38}v_7^{209}}{v_5^{80}} & v_2^{321}v_5^{255}v_6^{114}v_8^{209} & \frac{v_2^3v_5^{45}v_8^{19}}{v_7^{57}} & v_2^{48}v_6^{18}v_7^{51}v_8^{16} \\ \frac{v_5^{129}v_6^{55}}{v_1^{169}v_2^{240}} & \frac{v_3^{63}v_5^{72}v_8^{55}}{v_1^{15}} & \frac{v_6^{22}v_7^{129}}{v_1^{16}v_3^{96}} & \frac{v_3^{123}v_4^{93}v_8^{43}}{v_6^{24}} & \frac{v_3^{27}v_7^{36}v_8^{11}}{v_4^3} & \frac{v_3^{107}v_5^{93}v_8^{80}}{v_6^5} \\ \frac{v_3^{80}v_7^{93}v_8^{32}}{v_6^2} & \frac{v_6^{22}v_7^{169}}{v_4^{96}v_5^{16}} & \frac{v_5^{369}v_6^{63}v_8^{169}}{v_4^{321}} & \frac{v_5^{54}v_8^{22}}{v_4^6v_7^{63}} & \frac{v_6^{54}v_7^{369}v_8^{16}}{v_4^{240}} & \frac{v_5^{80}v_7^{32}}{v_6^2v_7^{107}} \end{array} \right\}$$

Whereas, Algorithm 5.3 provides a polynomial generating set of cardinality 541*, that I do not exhibit here. Let me just detail the cardinality of their supports. They are*

- 8 *of support of cardinality* 4*,*
- 118 *of support of cardinality* 5*,*
- 299 *of support of cardinality* 6*,*
- 109 *of support of cardinality* 7*,*
- *and* 7 *of support of cardinality* 8*.*

In this example, the advantage to restrict to low support appears particularly clearly. Indeed, the explosion of the generating set cardinality is due to monomials of bigger supports of size 5*,* 6 *and* 7*.*

Beyond the cardinality, the complexity of the computation of these three sets is very different. To compute the rational generating set, we need the Smith Normal form of M, whose computation has a polynomial cost in d, n [KB79]. We have already noticed in Remark 3.15 that this cost is largely avoided in our method. The true difficulty that we meet to compute our separating set is finally to compute the set of optimal supports $\gamma(M)$, also in polyn(d, n) time. Algorithm 5.3 to compute polynomial generating set is clearly the heaviest. My computer being apparently too weak for it as soon as $d \geq 4$.

Let us finish the comparisons from a separation point of view. Rosenlicht Theorem [PV94] claims that a rational generating set separates orbits almost everywhere. In particular, the set provided by the Smith Normal form separates any orbits of full support $\mathrm{supp}(v) = [\![1, n]\!]$ [Bür+21]. As $(\mathbb{G}_\times)^d$ is a reductive group, the invariant algebra separate any closed orbits. We have the following equivalence:

Closed orbits exist $\Leftrightarrow$ *Orbits of full support are closed.*

Thus a polynomial generating set is interestingly more accurate as soon as closed orbits exists, which is quite frequent when n grows. But the most accurate set is clearly our rational separating set, as previously highlighted in Remark 3.9. In a synthetic way we have the following qualitative hierarchy:

$$\underbrace{\left(\ \textit{rational separating set}\ \right)}_{\textit{Orbits of support in } \Gamma(M)} \quad >> \quad \underbrace{\left(\ \textit{generating } \mathbb{C}[\mathcal{V}]^{\left(\mathbb{G}_\times\right)^d}\right)}_{\textit{closed orbits}} \quad >> \quad \underbrace{\left(\ \textit{generating } \mathbb{C}(\mathcal{V})^{\left(\mathbb{G}_\times\right)^d}\right)}_{\textit{orbits of full support}}.$$

Acknowledgements

Martin Jalard receives funding from INRIA Sophia Antipolis.

References

[Azz+22] P. Azzi, R. Desmorat, B. Kolev, and F. Priziac. "Distance to a constitutive tensor isotropy stratum by Lasserre polynomial optimization method". In: *arXiv preprint arXiv:2207.04729* (2022).

[BV23] B Blum-Smith and S. Villar. "Machine learning and invariant theory". In: *Notices of the American Mathematical Society* (2023).

[Bür+21] P. Bürgisser, M. Doğan, V. Makam, M. Walter, and A. Wigderson. "Polynomial time algorithms in invariant theory for torus actions". In: *arXiv preprint arXiv:2102.07727* (2021).

[Che+19] Y. Chen, Z. Ming, L. Qi, and W. Zou. "A Polynomially Irreducible Functional Basis of Hemitropic Invariants of Piezoelectric Tensors". In: *arXiv preprint arXiv:1901.01701* (2019).

[Duf09] Emilie Dufresne. "Separating invariants and finite reflection groups". In: *Advances in Mathematics* 221.6 (2009), pp. 1979–1989.

[DG24] Nadav Dym and Steven J Gortler. "Low-Dimensional Invariant Embeddings for Universal Geometric Learning". In: *Foundations of Computational Mathematics* (2024), pp. 1–41.

[HL13] E. Hubert and G. Labahn. "Scaling invariants and symmetry reduction of dynamical systems". In: *Found. Comput. Math.* 13.4 (2013), pp. 479–516.

[Jal24] M Jalard. "Separation of the orbits in representations of SO2 and O2 over R and C". working paper or preprint. June 2024. URL: `https://hal.science/hal-04656738`.

[KB79] Ravindran Kannan and Achim Bachem. "Polynomial algorithms for computing the Smith and Hermite normal forms of an integer matrix". In: *siam Journal on Computing* 8.4 (1979), pp. 499–507.

[Kem07] G Kemper. "The computation of invariant fields and a constructive version of a theorem by Rosenlicht". In: *Transformation Groups* 12 (2007), pp. 657–670.

[Kem24] Gregor Kemper. "Invariant Theory: a Third Lease of Life". In: *arXiv preprint arXiv:2403.12709* (2024).

[PV94] V. Popov and E. Vinberg. "Invariant theory". In: *Algebraic geometry. IV*. Springer, 1994, pp. 122–278.

[PS85] Claudio Procesi and Gerald Schwarz. "Inequalities defining orbit spaces". In: *Inventiones mathematicae* 81.3 (1985), pp. 539–554.

[Smi61] Henry John Stephen Smith. "Xv. on systems of linear indeterminate equations and congruences". In: *Philosophical transactions of the royal society of london* 151 (1861), pp. 293–326.

[Stu08] Bernd Sturmfels. *Algorithms in invariant theory*. Springer Science & Business Media, 2008.