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What teaching practices should be used to introduce the limits of functions in the first year of university? A case study

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The introduction of the formal definition of the limit of a function during the first year of university is a source of many difficulties for students. In this exploratory research, we study the discourse of lecturers when they introduce this notion. To do so, we determine the ‘relief’ of the notion of limit, resulting from the combination of epistemological, curricular and cognitive studies. This relief enables us to envisage ‘proximity opportunities’, that is possible attempts to bring together students’ prior knowledge and the notion of the limit. We demonstrate the value of using these tools by studying the discourse of one particular teacher, which enables us to identify the discursive proximities that he actually attempted.

Keywords: Teachers’ and Students’ practices at university level, Teaching and learning of analysis and calculus, Limit of a function, Discursive proximities, Lectures.

CONTEXT OF THE RESEARCH

We present here an ongoing study of university teachers’ practices who teach the limits of functions. More specifically, we focus on the introduction of the formal definition of a limit, the first examples and the first results presented by the teachers. The choice of this topic is motivated in particular by the fact that limits of functions are taught in Calculus courses during the first year of university in many countries and that this content is often a source of difficulties for students (Oktaç & Vivier, 2016). Secondly, the limits of functions have been much less studied than the limits of sequences (see Chorlay (2019) for a state-of-the-art) and existing work does not place much emphasis on the study of teachers’ discourse. Finally, this problematic, linked to the study of teachers’ discourse, aims more globally at the study of the conduct of lectures and their impact on students’ learning. In a previous research, we have shown that certain practices cause a discrepancy between teachers’ objectives on the one hand and the way in which students receive the content delivered to them on the other: a main result attests that lectures are not necessarily a source of inactivity for students, and thus that this teaching space deserves to be studied (Bridoux et al., in press).

We begin by presenting our tools for analysing teachers’ discourse and formulating the research issues that follow. We then show how we used these tools to study a lecture dedicated to the introduction of the limit of a function. Finally, we present our first results and a few prospects for further work.

THEORETICAL TOOLS AND ISSUES

Our research is based on Activity Theory, adapted to the didactics of mathematics (Bridoux et al., 2016). It leads us to study students' learning through the prism of their mathematical activities organised by the teacher through a coherent scenario. However, during the lectures, these activities are difficult to observe. We are therefore led to study the teacher's discourse specifically. We hypothesise that, in order to advance the students' knowledge, the teacher tries to use a discourse that is close to the students' work in order to introduce new knowledge, for example by building on acquired knowledge. Within our framework, this theoretical hypothesis is related to the ZPD¹ model of Vygotsky: it asserts that lectures can contribute to the appropriation of knowledge by students and, ultimately, to the conceptualization of this knowledge (Bridoux et al., 2016). The connections between the teacher's discourse and students that we seek to study are called 'discursive proximities' (Robert & Vandebrouck, 2014). Three types of proximities are distinguished. *Bottom-up proximities* lie between what students have already done and the introduction of a new object or property. In this case, the teacher's discourse therefore aims to move from the contextualised to the decontextualised by generalising the particular case. *Top-down proximities* are situated between what has been explained and examples or exercises. The teacher can then explain how the particular case fits into the general case, moving from the decontextualised to the contextualised. Finally, *horizontal proximities* do not lead to any change between contextualised and decontextualised. They consist of reformulations, explanations of the links between concepts, comments on the structure of the course, etc. Examples of proximities will be given in the next section for limits of functions.

To prepare for the study of teachers' discourse, the researcher must have an *a priori* reference, which we call '*relief on the concepts to be taught*' (Bridoux et al., 2016). The relief² of a notion to be taught is a cross-study combining epistemological, curricular and cognitive analyses. Relief thus makes it possible to study the specific features of the concepts to be taught, taking account of the curricula, while being aware of students' difficulties already identified by research. These analyses then make it possible to identify, *a priori*, *opportunities for proximities*, which will then be compared to the proximities actually attempted by the teacher to see whether or not these opportunities are taken during the course.

In this context, the "relief" helps the researcher to analyse the content taught by considering possible ways of introducing it and to study the distance between previous students' knowledge and the new notion. Thus the "relief" allows the researcher to describe the attempted conceptualization by taking into account students' difficulties.

¹ Zone of Proximal Development.

² The word 'relief' is a French word which is a metaphor for 'relief map'.

In the context of teaching the limits of functions and on the basis of the tools just presented, our research questions are formulated as follows: What are the relief elements that give rise to proximities in teachers' discourse during the course? What are the links between the content organisation choices made by teachers and the proximities attempted in their discourse?

ANALYSIS OF A LECTURE

In order to provide some answers to our questions, we present some curricular information about the limits of functions and then analyse the discourse of a teacher in a lecture by means of the tools described above.

In France, the limits of functions are intuitively introduced in *Première* (grade 11, students aged 16-17) on the basis of examples and without formalisation. They are taken up again in *Terminale* (grade 12). The objectives described in the syllabi are aimed at practising the operative aspects. At university, the definition frequently given is as follows: f has a finite limit l at a if $\forall \varepsilon > 0 \exists \eta > 0 \forall x \in D_f (|x - a| \leq \eta \Rightarrow |f(x) - l| \leq \varepsilon)$. As students have already calculated the limits of functions at the end of their secondary education without using this definition, one of the challenges of university teaching is to make them feel the need for introducing this definition. It also requires knowledge with respect to logic, absolute values, real numbers and inequalities. However, this knowledge is not widely used in high school and is therefore probably not available to a large number of students. It is therefore difficult for the teacher to find an initial problem where the notion of limit would be the optimal tool for solving and where the students could construct the new notion independently. Thus, it is unlikely to find any bottom-up proximities in the teacher's discourse introducing this definition. To introduce the definition, teaching sequences developed by researchers are often based on the articulation between several semiotic representation registers (in the sense of Duval, 2006): natural language, graphs and algebraic (for example Bloch, 2003). This articulation seems to be an effective lever to give sense to the new notion which, in our view, would imply to find horizontal proximities within the teacher's discourse. However, our experience show that these sequences are rarely used in classic lectures. Instead, teachers often give a first intuitive formulation like " $f(x)$ approaches l if x approaches a " and then formalize these words to build the definition. Sometimes they also use graphics.

It has been shown that students often develop a dynamic conception with respect to limits (Robert, 1983) in which the notion of limit is described as 'getting closer to', which can give rise to conceptions such as 'the limit is a number that the function cannot reach' (Mamona Downs, 2001). In contrast, a static conception in the sense of Robert (ibid.), in which the limit is associated with expressions such as 'as close as you want', allows students to give more meaning to the notion. The vocabulary used by the teacher can therefore have an impact on the students' conceptions. The teacher could also associate a graph with the definition (before or after its introduction). This link between different semiotic representation registers can lead the teacher to

attempt horizontal proximities, for example via reformulations to interpret the inequalities present in the definition in terms of intervals or distances.

Once the definition has been introduced, the teacher often gives examples to show how to manipulate the formalism it contains. This type of task can lead to top-down proximities in the teacher's discourse, in particular to show the logical organisation required to manipulate the definition as an object or to make explicit the prior knowledge that students need, which could also be a source of horizontal proximities. But the first manipulative tasks, such as showing that $\lim_{x \rightarrow 2} (3x - 5) = 1$ or $\lim_{x \rightarrow a} x^2 = a^2$ by means of the formal definition, are already complex for many students. It is also not uncommon for the teacher to use the definition as a tool for proving results such as the uniqueness of the limit, calculation rules, etc., thus leading the teacher to attempt other top-down or horizontal proximities.

In this context, it is difficult for the teacher to introduce the definition of limit because of the distance between intuitive high school conceptions and the needed formalism at university. Furthermore, the skills needed to write the definition are not taught at high school level. That is why students are not able to build the definition by themselves or to solve first tasks where the notion is worked in its double dimension of object and tool. The proximities attempted by the teacher are thus crucial to show students feature of the notion of limit.

We now show how these elements of relief help us to study a teacher's discourse. We focus on a one-hour lecture given in the second semester to 200 first-year university students. To analyse the teacher's discourse, we compared what the teacher wrote on the blackboard with what he said orally. Tableau 1 in Bridoux et al. (2015, p. 48) gives an overview of the different phases organised by the teacher, showing their duration and content.

We have chosen to look specifically at the emergence of the formal definition and its first use as a tool to prove a property.

First, the teacher chooses to introduce the notion of limit intuitively by saying: *'How would you define an intuitive notion of limit?'* A student replies: *'f(x) gets as close as you want to l'*, then adds *'when x gets close to x₀'*. The teacher then shows a continuous function on the board and comments on the graph: *'We're trying to look at a diagram to explore this concept. So x is approaching x₀ the point M is approaching the point M₀, f(x) is the ordinate of M is approaching l there, OK?'* Then he writes at the same time as saying: *'f(x) is as close as we want to l if x is close enough to x₀'* (reformulation 1). The teacher thus reformulates the intuitive student's definition by combining the graphic and natural language registers and proposes a definition that can be associated with a static conception of the notion of limit, as we mentioned in the relief elements presented earlier.

The teacher continues: *'Here we're using sentences, what we'd like is to have a mathematical reformulation. Because these sentences leave a lot of room for ambiguity'*. He illustrates his point by giving the following example: *'We define a*

function of \mathbb{R} in \mathbb{R} by $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}^*$, $f(x) = \frac{1}{q}$ if $x = \frac{p}{q} \in \mathbb{Q}$ where $\frac{p}{q}$ is irreducible'. The teacher points out:

'Here, the intuitive notion becomes complicated, because you're going to have trouble tracing this curve. So we're not going to be able to use a geometric notion of the limit. So, to solve a certain number of problems, we need a more mathematical, more rigorous definition'.

Here, the teacher uses an example to demonstrate the need for a definition that he would like to write in the algebraic register. However, the students did not have to work with this kind of function at high school. This example does not probably show the need of a formal definition for most students.

The teacher builds on reformulation 1 to construct the formal definition of limit step by step, as shown in Table 1.

What is written on the board	What the teacher says (extracts)
$ f(x) - l < \varepsilon$	<i>What is the distance from $f(x)$ to l? Yes, it's the absolute value of $f(x) - l$ OK. So we want $f(x) - l$ to be as small as we want, $f(x)$ is going to be as close as we want to l. What does that mean? It means that the absolute value of $f(x) - l$ is less than epsilon, for epsilon to be as small as we want, we agree.</i>
$\forall \varepsilon > 0 \quad f(x) - l < \varepsilon$	<i>We want epsilon to be as small as we want, which means that it will be true for any positive epsilon, so we'll have:</i>
$\forall \varepsilon > 0 \quad x - x_0 < \alpha \quad f(x) - l < \varepsilon$	<i>And thus we have that if x is close enough to x_0 that is, if the distance from x to x_0 is less than a certain alpha value.</i>
$\forall \varepsilon > 0 \quad x - x_0 < \alpha \Rightarrow f(x) - l < \varepsilon$	<i>So the if-then implication is that however small epsilon is, if $x - x_0$ is less than alpha, we must have $f(x) - l$ is less than epsilon, it's starting to look like the definition your friend gave.</i>

$\forall \varepsilon > 0, \exists \alpha > 0, \quad x - x_0 < \alpha$ $\Rightarrow f(x) - l < \varepsilon$	<p><i>So alpha, how do we introduce it, because here we're introducing a notation, which means that alpha has to be in which set? It has to increase a distance so... [student answer] positive, that's it. Whatever epsilon is, as soon as x is close enough to x₀ i.e. if the distance from x to x₀ is less than alpha, so behind this is the notion that 'there exists alpha such that'. So our definition, if we want to write it in a rigorous way, looks something like this, OK?</i></p>
$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in Df, x - x_0 < \alpha$ $\Rightarrow f(x) - l < \varepsilon$	<p><i>So there's still something missing, and that's x belongs to which set, for what x we have this involvement. It's for the x belonging to the definition set, so I must have...</i></p>

Table 1: Teacher's discourse during the emergence of formal definition

Here, the teacher builds up the definition step by step in the algebraic register, following the order of the quantifiers. With reformulation 1, he remains in the natural language register to reformulate the idea of closeness by introducing an inequality containing an absolute value and then in terms of distance. Our relief study enabled us to anticipate the presence of such horizontal proximities during the definition construction phase. The teacher's discourse does indeed contain several of these, in this case reformulations linked to the formalism and logical structure of the definition, but there is nevertheless a discrepancy between what is said orally and what the teacher writes in the algebraic register. Finally, the teacher writes the following definition: 'Let $x_0 \in \mathbb{R}$ and f be a function defined on a neighbourhood of x_0 . f has a limit $l \in \mathbb{R}$ in x_0 ($\lim_{x \rightarrow x_0} f(x) = l$) if and only if $\forall \varepsilon > 0 \exists \alpha > 0$ tel que $\forall x \in D_f (|x - x_0| < \alpha \Rightarrow |f(x) - l| < \varepsilon)$.

While writing this definition, the teacher reads out the absolute values in terms of distances and then writes on the board: 'In other words, however small ε is, we can find a sufficiently small interval around x_0 over which the distance from $f(x)$ to l is less than ε . The teacher continues orally: 'OK, the 'whatever' is 'as small as ε is', " " $\exists \alpha > 0$ such that $\forall x \in D_f |x - x_0| < \alpha$ ' this is an interval, so there is a sufficiently small interval around x over which the distance from $f(x)$ to l is less than ε '.

During this phase, the teacher does not make links with the previous graphic. Horizontal proximities are thus not attempted in this regard.

We now analyse the first result proved by the teacher: ‘If f has a limit l at x_0 and is defined in x_0 then $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ ’. The use of the formal definition is compulsory for writing the demonstration. It is used here as a tool, as we had anticipated in the study of relief. To start the proof, the teacher draws a real number line and says: ‘We are going to use reduction ad absurdum. l less than $f(x_0)$. What does the definition say?’ He writes: ‘The definition of $\lim_{x \rightarrow x_0} f(x) = l$ is

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in Df, |x - x_0| < \alpha \Rightarrow |f(x) - l| < \varepsilon$$

After recalling the negation of an implication $A \Rightarrow B$ the teacher writes on the board:

$$\exists \varepsilon > 0, \forall \alpha > 0, \exists x \in Df \text{ such that } |x - x_0| < \alpha \text{ and } |f(x) - l| \geq \varepsilon$$

Then:

‘So how can we prove an existence? What are the possibilities? There are cases where you can’t produce any. The simplest way is to give an example: we show that for a certain number, it works. That’s what we’re going to do here, we’re going to show that there’s an epsilon for which we have the property’

The teacher makes a methodological comment here, which we interpret as an attempt to relate it to the students' knowledge of logic.

The teacher then takes over the proof:

‘I’m going to take epsilon equal to half the distance. What is this distance? $f(x_0) - l$! This distance is not zero because we said that these are two different numbers. It’s the absolute value of $f(x_0) - l$, here it is $f(x_0) - l$ because I considered that l is smaller but it can be larger [...] I take half of this absolute value, so epsilon is this distance.’

After choosing ε , the teacher writes on the board $\forall \alpha > 0 |x - x_0| < \alpha$ and says: ‘So what could we say to show that for any alpha, there is a x in the definition set such that we have the following property? What do you think the x that will cause the problem?’ A student gives the right answer but the teacher continues to comment: ‘So what does it mean that $\forall \alpha > 0 |x - x_0| < \alpha$, what numbers verify this? The distance from x to x_0 is as small as I want it to be, that’s what it means [...] If we want to find an x that works, it’s bound to be [...], x_0 , yes?’ He then points to the logical sentence whose negation he has considered: ‘So what happens with the two propositions for x_0 ? The distance from x_0 to x_0 is zero, so less than any alpha, it works’. He writes on the board $|x_0 - x_0| = 0 < \alpha$ then $|f(x_0) - l| < \varepsilon = \frac{|f(x_0) - l|}{2}$ and comments: ‘So can we have the second property? Epsilon, we defined it as being... So we still have this property, so we do have the two properties on the right’. Here, the teacher’s discourse contains reformulations based on the notion of distance which can be more intuitive for students and linked to the graphic register. However, this notion is lost in a very complex reasoning, to which students are not used at this level of teaching.

Our interpretation of this episode is that horizontal proximity opportunities are not attempted by the teacher during this phase.

Finally, the teacher says ‘*Our demonstration is as follows*’ and then writes on the board:

We pose $\varepsilon = \frac{|f(x_0) - l|}{2}$. For all $\alpha > 0$. We have $|x_0 - x_0| = 0 < \alpha$ and $|f(x_0) - l| = 2\varepsilon > \varepsilon$. Conclusion: $l \neq f(x_0)$ is contradictory to $\lim_{x \rightarrow x_0} f(x) = l$.

After that, he says: ‘*OK, so why does it work, where does f defined in x_0 came in, it's here [He shows $|x_0 - x_0| = 0 < \alpha$] in the choice of x belonging to the definition set equal to x_0 , it's only possible if x_0 belongs to the definition set*’. The written trace on the blackboard does not make explicit what has been presented orally by the teacher on the structure of the demonstration and contains no trace of the different reformulations given orally by the teacher. In our opinion, there is therefore a lack of bottom-up proximities here.

DISCUSSION

Our relief study showed that the articulation of different semiotic registers associated with horizontal proximities in the teacher’s discourse helps students to give sense to the new notion. Top-down proximities allow the teacher to show how the definition is manipulated like an object or a tool. Horizontal proximities also lead students to understand how the formalism is used in the first tasks (examples of proofs of properties).

First of all, this relief study helped us to identify horizontal proximities in the construction phase of the definition formulated in the algebraic register. These proximities take the form of reformulations supported by work in different registers of writing (words, graphs, symbols) and revolve around the notions of absolute value, distance and interval. This choice is perhaps linked to the fact that the teacher imagines that this is old knowledge that has been stabilised among the students, whereas the links between these concepts are very little explained at secondary school.

We also hypothesised that the first examples could lead to attempts of top-down proximities. However, the teacher does not give an example to illustrate the definition, as is often done in a textbook (see for example Ramis and Warusfel, 2022). Instead, he chooses to create a gap associated with the use of the natural language register, which makes it impossible to deal with the example. The teacher will therefore not mobilise the definition as an object, thus causing the absence of top-down proximities in his discourse. What's more, the example chosen requires knowledge about numbers that is probably not readily available to a majority of students.

Finally, we have seen that the use of the definition as a demonstration tool is based on knowledge of logic which the teacher tries to take into account in his discourse, but we think that these attempts are once again only accessible to a small number of students.

As we had anticipated in the relief, we did not identify any bottom-up proximities in the teacher's discourse. In our opinion, this absence is linked to the choice of introducing the formal definition by attempting to (re)formulate the inequalities in terms of distance and with the idea of closeness. Another choice of introduction is to use sequences, which are often studied before the limits of functions, to construct an initial definition using the limits of sequences (see, for example, Ramis & Warusfel, 2022). The question then arises as to how to go about constructing the $\varepsilon - \alpha$ definition. Didactic engineering has also been developed to link the notion of the limit of a function to other knowledge (e.g. in topology, Branchetti et al., 2020), or to get students to interact more (in the form of a debate, for example, Lecorre, 2016) or to articulate different registers while leaving students more autonomy to construct the notion (Bloch, 2003).

Thus, the teacher's organisational choices mainly lead to attempts at horizontal proximity, but in our opinion these connections will have little impact on the students' conceptualization and thus on their learning, given the unavailability of the concepts on which these proximities are based.

This work, which remains exploratory at this stage, nevertheless shows how the tools presented make it possible to apprehend the teacher's discourse and to formulate hypotheses about the way in which students may receive this discourse. The aim now is to extend this work by studying more lectures and to compare the different proximities contained in the teachers' discourse in relation to their choice of introduction.

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