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Boundary value problems and Heisenberg uniqueness pairs

S. Rigat and F. Wielonsky

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Abstract

We describe a general method for constructing Heisenberg uniqueness pairs (Γ, Λ) in the euclidean space \mathbb{R}^n based on the study of boundary value problems for partial differential equations. As a result, we show, for instance, that any pair made of the boundary Γ of a bounded convex set Ω and a sphere Λ is an Heisenberg uniqueness pair if and only if the square of the radius of Λ is not an eigenvalue of the Laplacian on Ω . The main ingredients for the proofs are the Paley-Wiener theorem, the uniqueness of a solution to a homogeneous Dirichlet or initial boundary value problem, the continuity of single layer potentials, and some complex analysis in \mathbb{C}^n . Denjoy's theorem on topological conjugacy of circle diffeomorphisms with irrational rotation numbers is also useful.

1 Introduction

The interpretation of the uncertainty principle in mathematics is a classical theme, especially in harmonic analysis, see e.g. [14] for a thorough presentation of the subject. Heisenberg uniqueness pair is a related notion which was introduced in [15]. It consists in the following.

A pair (Γ, Λ) , where Γ is a hypersurface in \mathbb{R}^n , $n \geq 2$, and Λ is a set in \mathbb{R}^n , is an *Heisenberg uniqueness pair* (sometimes abbreviated HUP) if for any complex-valued function g integrable on Γ with respect to $d\sigma$, the surface measure on Γ , the following implication holds true,

$$\widehat{g}(\lambda) = \int_{\Gamma} e^{-i\lambda \cdot x} g(x) d\sigma(x) = 0 \text{ on } \Lambda \quad \implies \quad g = 0 \text{ a.e. on } \Gamma.$$

The function $\widehat{g}(\lambda)$ is the Fourier transform, in the sense of distribution, of the measure $g d\sigma$, also called the Fourier-Stieltjes transform of g . Note that, by the Hahn-Banach theorem, the above condition is equivalent to the fact that the vector space spanned by the family of exponentials $e^{i\lambda \cdot \xi}$, $\lambda \in \Lambda$, is weak-* dense in $L^\infty(\Gamma)$. Also, a similar notion is that of a mutually annihilating pair (S, Σ) of Borel subsets of \mathbb{R} of positive measure, satisfying

$$\forall \varphi \in L^2(S), \text{ supp}(\widehat{\varphi}) \subset \Sigma \quad \implies \quad \varphi = 0,$$

see [14].

It is easy to check that the notion of an Heisenberg uniqueness pair satisfies two elementary properties, namely,

- 1) it is invariant by translation: if (Γ, Λ) is an HUP then $(\Gamma + x_0, \Lambda + \lambda_0)$ with $x_0, \lambda_0 \in \mathbb{R}^n$, is also an HUP,
- 2) if T is an invertible linear transformation with adjoint T^* , then

$$(\Gamma, \Lambda) \text{ is an HUP} \implies (T(\Gamma), (T^{-1})^*(\Lambda)) \text{ is also an HUP.} \quad (1.1)$$

In [15], it was proven that the pair made of the hyperbola and a cross lattice,

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\}, \quad \Lambda_{\alpha, \beta} := (\alpha \mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta \mathbb{Z}), \quad \alpha, \beta > 0,$$

is a HUP if and only if $\alpha\beta \leq 1$. Since the publication of [15], many other examples have appeared in the literature. For instance, in [2, 3, 11, 17, 19, 23, 24], one may find examples of HUP's in the plane \mathbb{R}^2 involving sets like circles, union of lines, ellipses, parabola, polygons or very specific curves. In [12, 26], one may also find examples of HUP's in \mathbb{R}^n , $n > 2$, again involving spheres, paraboloids or certain cones.

The aim of the present paper is to obtain Heisenberg uniqueness pairs in \mathbb{R}^n , $n \geq 2$, under rather general assumptions on the set Γ since, in some of our examples, it is only assumed to be the boundary of a bounded convex set. Our method is inspired by the study in [1], which was concerned with providing rigorous foundations of the Fokas method for boundary value problems, showing, for linear elliptic pde's on bounded convex domains, that a solution to the global relation between the Dirichlet and Neumann boundary data implies the existence of a solution to the Dirichlet problem, and that the solution to the global relation is unique. The link between [1] and the problem of determining Heisenberg uniqueness pairs does not seem to have been previously noticed.

When convenient, we will denote by X a n -tuple of variables X_1, \dots, X_n and by D the tuple of partial derivations $\partial_{X_1}, \dots, \partial_{X_n}$. For a polynomial $P(X) = \sum_{|\alpha| \leq m} a_\alpha X^\alpha \in \mathbb{C}[X]$, the differential operator $P(-iD)$ is defined by

$$P(-iD) = \sum_{|\alpha| \leq m} a_\alpha (-i)^{|\alpha|} \partial_{X_1}^{\alpha_1} \dots \partial_{X_n}^{\alpha_n}.$$

We will also denote by $Z(P)$ the subset of \mathbb{C}^n of zeros of P , and by $Z_{\mathbb{R}}(P) = Z(P) \cap \mathbb{R}^n$ the subset of its real zeros.

Let $E_P(x)$ be a fundamental solution of the differential operator $P(-iD)$, that is $P(-iD)E_P = \delta$ in the sense of distribution, where δ denotes the Dirac delta distribution. We will make use of potentials of a single layer,

$$\Phi g(x) := \int_{\Gamma} g(y) E_P(x - y) d\sigma(y), \quad x \in \mathbb{R}^n, \quad (1.2)$$

where g is some function defined on Γ , see Section 2 for more details.

Let us now state our main result.

Theorem 1.1. *Let Ω be a bounded convex domain in \mathbb{R}^n with a piecewise C^1 boundary Γ , and let $d\sigma$ be the surface measure on Γ (the arc measure when $n = 2$). Let $P(X_1, \dots, X_n)$ be a polynomial with real coefficients such that*

(i) P is a square-free polynomial in $\mathbb{C}[X]$, i.e. each irreducible factor of P is of multiplicity 1, and for at least one variable, e.g. X_1 , one has $P(X) = a_1 X_1^d + \dots$, where the nonzero leading coefficient a_1 is a constant (i.e. independent from the other variables).

(ii) each irreducible component V_i of the variety $Z(P)$ is defined by a real polynomial, and contains a point in $V_{i,\mathbb{R}} = V_i \cap \mathbb{R}^n$, which is smooth as a point of V_i (i.e. the Jacobian has rank 1 at this point).

(iii) For some space $\mathcal{D}_P(\overline{\Omega})$ of functions defined on $\overline{\Omega}$, the homogeneous Dirichlet problem

$$\begin{aligned} P(-iD)u(x) &= 0, & x \in \Omega, \\ u(x) &= 0, & x \in \Gamma, \end{aligned}$$

has the unique solution $u = 0$ in $\mathcal{D}_P(\overline{\Omega})$.

(iv) For some space $\mathcal{G}(\Gamma) \subset L^1(\Gamma)$, of complex-valued functions g defined on Γ , the potential Φg , associated to $g \in \mathcal{G}(\Gamma)$, defined in (1.2), is a continuous function in \mathbb{R}^n , and

$$\forall g \in \mathcal{G}(\Gamma), \quad \Phi g|_{\overline{\Omega}} \in \mathcal{D}_P(\overline{\Omega}).$$

Then $(\Gamma, Z_{\mathbb{R}}(P))$ is a Heisenberg uniqueness pair for $\mathcal{G}(\Gamma)$, that is,

$$\forall g \in \mathcal{G}(\Gamma), \quad \hat{g} = 0 \text{ on } Z_{\mathbb{R}}(P) \implies g = 0 \text{ a.e. on } \Gamma.$$

The spaces of functions $\mathcal{D}_P(\overline{\Omega})$ and $\mathcal{G}(\Gamma)$ in the above statement will be chosen in accordance with the domain Ω and the polynomial P , see Section 4 for details.

Applying the Theorem 1.1 in specific cases, we derive general Heisenberg uniqueness pairs as described next.

Theorem 1.2. *The following holds true:*

i) Let Γ be the boundary of any bounded convex domain in \mathbb{R}^n of class C^1 . Let $S_{n-1}(c_1)$ be the sphere of dimension $n - 1$ and radius c_1 . Then $(\Gamma, S_{n-1}(c_1))$ is a HUP for $L^p(\Gamma)$, $p > n - 1$, if and only if c_1^2 is not an eigenvalue of $-\Delta$ on Ω .

ii) Let Γ be the rectangle with vertices $(0, 0), (T, 0), (T, L), (0, L)$, $T, L > 0$, and let \mathcal{P} be the parabola $X_1 + X_2^2 = 0$ in \mathbb{R}^2 . The pair (Γ, \mathcal{P}) is a HUP for $C^2(\Gamma)$.

iii) Let Γ be the rectangle as in ii), and assume that $T/L \notin \mathbb{Q}$. Let Δ_+ and Δ_- be the two lines $\{X_1 = X_2\}, \{X_1 = -X_2\}$ in \mathbb{R}^2 . Then the pair $(\Gamma, \Delta_+ \cup \Delta_-)$ is a HUP for $C^1(\Gamma)$.

iv) Let \mathbb{T} denote the unit circle, and let Δ_1, Δ_2 be two lines through the origin. Then the pair $(\mathbb{T}, \Delta_1 \cup \Delta_2)$ is a HUP for $L^1(\mathbb{T})$ if and only if Δ_1 and Δ_2 make an angle of $\pi\rho$, with ρ an irrational number.

We note that the HUP in item iv) was obtained previously, in [23], by a different method. Also, making use of the property (1.1), item iii) may be rephrased as the fact that the pair $(\mathcal{C}, \Delta_\rho \cup \Delta_{-\rho})$ is a HUP, where \mathcal{C} is the rectangle of sides π and 1, and $\Delta_{\pm\rho} = \{X_2 = \pm\pi\rho X_1\}$, with $\rho \notin \mathbb{Q}$. This shows some relation between iii) and iv).

Acually, item iv) can be extended in the following way.

Theorem 1.3. *Let Γ be any strictly convex planar curve of class C^2 . There exists an angle ρ between two lines Δ_1, Δ_2 through the origin, such that the pair $(\Gamma, \Delta_1 \cup \Delta_2)$ is a HUP for $L^1(\Gamma)$.*

Unfortunately, we were unable to characterize the angles ρ , in terms of Γ , for which the above pair is a HUP. It may possibly be an interesting problem.

In Section 2 we recall or state several preliminary results that will be useful for our study. In Section 3, we give the proof of Theorem 1.1. In Section 4, we apply our result in the cases of an elliptic equation (the Helmholtz equation), the Schrödinger equation, a hyperbolic equation (the wave equation), and finally, transport equations. For each of these equations, we derive the corresponding Heisenberg uniqueness pairs.

2 Preliminaries

We start with recalling a few details about the notion of a potential of a single layer, as defined in (1.2). The following can be found in [7, Chap. 2, §3], in the case of the Newton potential. In the sequel, φ denotes a test function. For $g \in L^1(\Gamma)$, the Radon measure $gd\sigma$ is the distribution, supported on Γ , defined by

$$\langle gd\sigma, \varphi \rangle = \int_{\Gamma} g(m)\varphi(m)d\sigma(m).$$

The potential of a single layer Φg , associated to g , is then defined as the convolution of distributions,

$$\Phi g = E * gd\sigma,$$

where E denotes a fundamental solution of the differential operator $\mathcal{P} = P(-iD)$. Note that the convolution is well defined as the measure $gd\sigma$ has compact support.

By the classical result of Malgrange and Ehrenpreis, a fundamental solution E of the linear differential operator $P(-iD)$ with constant coefficients, always exists, satisfying

$$P(-iD)E = \delta. \quad (2.1)$$

With \check{E} denoting the distribution symmetric to E , defined by $\langle \check{E}, \varphi \rangle = \langle E, \check{\varphi} \rangle$, $\check{\varphi}(x) = \varphi(-x)$, and from basic properties of the convolution, we get

$$\langle \Phi g, \varphi \rangle = \langle gd\sigma, \check{E} * \varphi \rangle = \int_{\Gamma} (\check{E} * \varphi)gd\sigma = \int_{\Gamma} \langle \check{E}(y), \varphi(x-y) \rangle g(x)d\sigma(x). \quad (2.2)$$

Now, with \mathcal{P}^* , the adjoint of the differential operator \mathcal{P} , we have

$$\begin{aligned} \langle \mathcal{P}(\Phi g), \varphi \rangle &= \langle \Phi g, \mathcal{P}^*\varphi \rangle = \int_{\Gamma} (\check{E} * \mathcal{P}^*\varphi)gd\sigma = \int_{\Gamma} (\mathcal{P}^*\check{E} * \varphi)gd\sigma \\ &= \int_{\Gamma} (\widetilde{\mathcal{P}E} * \varphi)gd\sigma = \int_{\Gamma} (\delta * \varphi)gd\sigma = \int_{\Gamma} \varphi gd\sigma = \langle gd\sigma, \varphi \rangle, \end{aligned}$$

and thus $\mathcal{P}(\Phi g) = gd\sigma$, supported on Γ . Hence

$$\mathcal{P}(\Phi g)|_{\mathbb{R}^n \setminus \Gamma} = 0. \quad (2.3)$$

In the case when the fundamental solution E is a locally integrable function, one gets, from (2.2),

$$\langle \Phi g, \varphi \rangle = \int_{x \in \Gamma} \int_{y \in \mathbb{R}^n} \check{E}(x-y)\varphi(y)dyg(x)d\sigma(x) = \int_{x \in \Gamma} E(y-x)g(x)d\sigma(x), \varphi(y) \rangle$$

i.e. Φg is just the function

$$\Phi g(y) = \int_{\Gamma} E(y-x)g(x)d\sigma(x).$$

Next, we recall a version of the Paley-Wiener theorem that will be useful to us, see [16, Theorem 7.3.1].

Theorem 2.1 (Paley-Wiener). *Let K be a convex compact subset of \mathbb{R}^n with supporting function H . Every entire function u in \mathbb{C}^n satisfying, for some positive integer N , an estimate*

$$|u(\zeta)| \leq C(1 + |\zeta|)^N e^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n,$$

is the Fourier transform of a distribution with support contained in K .

We will also make use of a result about analytic functions in \mathbb{C}^n vanishing on the real points of a variety, see e.g. [25, Theorem 5.1]. Recall that a point of an affine variety $V \subset \mathbb{C}^n$ of dimension d , defined by a family of polynomials, is smooth if the Jacobian of the defining family has rank $n-d$ at this point. We first consider the case of an irreducible variety V .

Proposition 2.2. *Let $V \subset \mathbb{C}^n$ be an irreducible affine variety of dimension d , defined by a family of $n-d$ polynomials $P_1, \dots, P_{n-d} \in \mathbb{R}[X]$. Assume V has a smooth real point $a \in V_{\mathbb{R}}$. Then $V_{\mathbb{R}}$ is Zariski dense in V , that is any function f analytic in V , which vanishes on $V_{\mathbb{R}}$, must vanish on V .*

Proof. Consider the smooth point $a \in V$, and assume, for simplicity, that the first minor $(\partial_{z_j} P_i(a))_{i,j=1,\dots,n-d}$ of the Jacobian is non singular. By the implicit function theorem, there exists a small ball \mathcal{B}_a in \mathbb{C}^d , centered at (a_{n-d+1}, \dots, a_n) , a neighborhood W of (a_1, \dots, a_n) , and $n-d$ analytic functions

$$h_j(z_{n-d+1}, \dots, z_n), \quad j = 1, \dots, n-d,$$

defined in \mathcal{B}_a such that,

$$\begin{aligned} (z_1, \dots, z_n) \in W \quad \text{and} \quad P_i(z_1, \dots, z_n) = 0, \quad i = 1, \dots, n-d &\iff \\ (z_{n-d+1}, \dots, z_n) \in \mathcal{B}_a \quad \text{and} \quad z_j = h_j(z_{n-d+1}, \dots, z_n), \quad j = 1, \dots, n-d. &\quad (2.4) \end{aligned}$$

Since the P_i 's have real coefficients, a is also a regular point of $V_{\mathbb{R}}$. Applying the implicit function theorem, this time in \mathbb{R}^n , we get,

$$\begin{aligned} (x_1, \dots, x_n) \in W \cap \mathbb{R}^n \quad \text{and} \quad P_i(x_1, \dots, x_n) = 0, \quad i = 1, \dots, n-d &\iff \\ (x_{n-d+1}, \dots, x_n) \in \mathcal{B}_a \cap \mathbb{R}^d \quad \text{and} \quad x_j = h_j(x_{n-d+1}, \dots, x_n), \quad j = 1, \dots, n-d. &\quad (2.5) \end{aligned}$$

where the functions h_j in (2.4) and in (2.5) are actually the same (and are real-valued when restricted to \mathbb{R}^d). Now, for f analytic on V and vanishing on $V_{\mathbb{R}}$, the function

$$g(z_{n-d+1}, \dots, z_n) = f(h_1(z_{n-d+1}, \dots, z_n), \dots, h_{n-d}(z_{n-d+1}, \dots, z_n), z_{n-d+1}, \dots, z_n)$$

is analytic in \mathcal{B}_a and, because of (2.5), vanishes on $\mathcal{B}_{a,\mathbb{R}}$. Hence, g vanishes on the whole of \mathcal{B}_a . Equivalently, f vanishes in a neighborhood (in the topology of V) of a . Finally, applying the identity principle for holomorphic functions on the irreducible algebraic set V , see [5, Chapter 1, §5.3], f must vanish on all of V . \square

The above result extends easily to the case of a non irreducible variety.

Corollary 2.3. *Let $V \subset \mathbb{C}^n$ be an affine variety of dimension d , and let $V = \cup V_i$ its decomposition into a finite number of irreducible components V_i . Assume each V_i is defined by real polynomials, and contains a smooth real point $a_i \in V_{i,\mathbb{R}}$. Then, as in Proposition 2.2, $V_{\mathbb{R}}$ is Zariski dense in V .*

Proof. It suffices to notice that $V_{\mathbb{R}} = \cup V_{i,\mathbb{R}}$ and to apply Proposition 2.2 to each of the components V_i . \square

Another result about analytic functions in \mathbb{C}^n , useful to us, is the following lemma.

Lemma 2.4. *Let Ω be an open connected subset of \mathbb{C}^n and let f be analytic in Ω . Assume $P \in \mathbb{C}[z_1, \dots, z_n]$ is a square-free polynomial. If f vanishes on the zero set $Z(P)$ of P , then f/P is analytic in Ω .*

Proof. By the local analytic nullstellensatz, see [13, Chapter 3], for each $z \in \Omega$, there exists a neighborhood U_z of z and a function g_z analytic in U_z such that $f = g_z P$. By the identity principle applied in Ω , we may glue together the functions g_z to get a function g analytic in Ω such that $f = gP$ there. \square

Finally, for one of our applications, we will need some results about diffeomorphisms of the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The rotation number of an orientation-preserving homeomorphism $f : \mathbb{T} \rightarrow \mathbb{T}$ is defined as the limit

$$\tau(f) = \lim_{n \rightarrow \infty} \frac{\tilde{f}^n(x) - x}{2\pi n}, \quad \text{mod } 1,$$

independent of x , where \tilde{f}^n denotes the n -th iterate of a lift \tilde{f} to \mathbb{R} of f , satisfying $\Pi \circ \tilde{f} = f \circ \Pi$, with Π the projection $\Pi : \mathbb{R} \rightarrow \mathbb{T}$. Intuitively, the rotation number represents the average portion of the circle that a point is moved by f .

Lemma 2.5 ([18, Proposition 11.1.6]). *The map $f \mapsto \tau(f)$ is continuous in the uniform topology of $C(\mathbb{T})$, the space of continuous maps from \mathbb{T} to \mathbb{T} .*

A basic result concerning the diffeomorphisms of the circle is the theorem of Denjoy.

Theorem 2.6 (Denjoy, [18, Theorem 12.1.1]). *A C^1 diffeomorphism $f : \mathbb{T} \rightarrow \mathbb{T}$ with irrational rotation number $\tau(f)$ and derivative of bounded variation (e.g. a C^2 diffeomorphism) is topologically conjugate to the rotation $R_{\tau(f)}$ of angle $2\pi\tau(f)$. Namely, there exists a homeomorphism h of the circle such that*

$$f = h^{-1} \circ R_{\tau(f)} \circ h.$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $g \in \mathcal{G}(\Gamma)$, the function space introduced in assumption (iv), such that its Fourier-Stieltjes transform \hat{g} satisfies

$$\hat{g}(\lambda) = \int_{\Gamma} e^{-i\lambda \cdot x} g(x) d\sigma(x) = 0, \quad \lambda \in Z_{\mathbb{R}}(P),$$

with $d\sigma$ the surface measure on Γ . Taking the Fourier transform \mathcal{F} in \mathbb{R}^n on both sides of (2.1), we get $P(\lambda)\mathcal{F}(E) = 1$. Moreover, The function \hat{g} is an entire function in \mathbb{C}^n which, by assumption, vanishes on $Z_{\mathbb{R}}(P)$. Hence, by Proposition 2.2 and its Corollary 2.3, it vanishes on $Z(P)$, which implies, together with Lemma 2.4, that the quotient \hat{g}/P is also an entire function (recall assumption (i) that P is a square-free polynomial in $\mathbb{C}[X]$). Hence, for the Fourier transform of Φg , we have

$$\mathcal{F}(\Phi g) = \mathcal{F}(E * gd\sigma) = \mathcal{F}(E)\mathcal{F}(gd\sigma) = \hat{g}\mathcal{F}(E) = (\hat{g}/P)P\mathcal{F}(E) = \hat{g}/P. \quad (3.1)$$

Next, we use the elementary fact that, if $h(z)$ is an analytic function in a neighborhood of the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{C}$ and $p(z)$ is a polynomial with leading coefficient α , one has

$$|\alpha h(0)| \leq \sup_{|z|=1} |h(z)p(z)|.$$

Thus, applying this inequality with the variable z_1 , $z = (z_1, 0, \dots, 0)$, $p(z_1) = P(z + \lambda)$, $h(z_1) = \mathcal{F}(\Phi g)(z + \lambda)$, and recalling the second part of assumption (i), we get

$$\begin{aligned} |\mathcal{F}(\Phi g)(\lambda)| &\leq a_1^{-1} \sup_{|z_1|=1} |\mathcal{F}(\Phi g)(z + \lambda)P(z + \lambda)| \leq a_1^{-1} \sup_{|z_1|=1} \left| \int_{\Gamma} e^{-i(z+\lambda)\cdot y} g(y) d\sigma(y) \right| \\ &\leq C \sup_{y \in \Gamma} e^{\text{Im}(\lambda)\cdot y} = C e^{H_{\overline{\Omega}}(\text{Im}(\lambda))}, \end{aligned}$$

where C is some constant independent of λ , and $H_{\overline{\Omega}}$ is the supporting function of $\overline{\Omega}$, the closure of Ω . Now, applying the Paley-Wiener theorem, see Theorem 2.1, one derives that Φg vanishes outside $\overline{\Omega}$. On the other hand, from the continuity of Φg in \mathbb{R}^n , recall assumption (iv), Φg also vanishes on Γ . From assumption (iii), the homogeneous Dirichlet problem for the operator $P(-iD)$ has the unique zero solution in $\mathcal{D}_P(\overline{\Omega})$. Hence Φg , which lies in $\mathcal{D}_P(\overline{\Omega})$ by assumption (iv), and satisfies, see (2.3),

$$P(-iD)(\Phi g)(x) = 0, \quad x \in \mathbb{R}^n \setminus \Gamma,$$

must vanish inside Ω and thus everywhere in \mathbb{R}^n . Consequently, $\mathcal{F}(\Phi g) = 0$ i.e. $\hat{g} = \mathcal{F}(gd\sigma) = 0$. The measure $gd\sigma$ is a distribution with compact support, hence a tempered distribution. From the fact that the Fourier transform is an isomorphism from the space of tempered distributions to itself, we thus obtain that $g = 0$. We have thus proved that $(\Gamma, Z_{\mathbb{R}}(P))$ is a Heisenberg pair for the space $\mathcal{G}(\Gamma)$. \square

4 Applications

We shall restrict ourselves to quadratic polynomial $P(X_1, \dots, X_n) \in \mathbb{R}[X]$ of the form

$$P(X_1, \dots, X_n) = - \sum_{i,j=1}^n a_{ij} X_i X_j + \sum_{i=1}^n b_i X_i + c, \quad a_{i,j}, b_i, c \in \mathbb{R}, \quad (4.1)$$

with $(a_{ij})_{i,j}$, a symmetric matrix.

4.1 Elliptic equations

In this section, we consider elliptic operators, that is, we assume that the matrix $(a_{ij})_{i,j}$ in (4.1) is positive. By a linear change of variables with real coefficients (so that real points of $Z(P)$ correspond by the change of variables), we may restrict ourselves to operators of the form

$$P(-iD) = \Delta - i \sum_{i=1}^n b_i \partial_{x_i} + c \quad \text{where} \quad P(X) = - \sum_{i=1}^n X_i^2 + \sum_{i=1}^n b_i X_i + c, \quad b_i, c \in \mathbb{R}.$$

Assume that the b_i 's are zero. Then, $Z(P)$ is defined by the equation

$$\sum_{i=1}^n X_i^2 - c = 0. \quad (4.2)$$

- If $c < 0$, $Z_{\mathbb{R}}(P)$ is the empty set so that Theorem 1.1 does not apply.
- If $c = 0$ then $Z_{\mathbb{R}}(P)$ reduces to the singleton 0 , which is a singular point of $Z(P)$.
- If $c = c_1^2 > 0$, with $c_1 > 0$, then $Z_{\mathbb{R}}(P)$ is the sphere $S_{n-1}(c_1)$ of dimension $n - 1$ and radius c_1 .

Before we elaborate on the third case, and prove the first item in Theorem 1.2, let us mention a result about continuity of potentials of a single layer of the form

$$(\Phi g)(x) = \int_{\Gamma} g(y) \varphi(x - y) d\sigma(y), \quad x \in \mathbb{R}^n. \quad (4.3)$$

Theorem 4.1. *Assume the following holds :*

- 1) *the hypersurface Γ is of class C^1 ,*
- 2) *The function $\varphi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ is continuous,*
- 3) *The function φ is weakly singular at 0 , namely there exists constants $0 < \nu < n - 1$ and $C > 0$ such that*

$$|\varphi(x)| \leq \frac{C}{|x|^\nu} \quad \text{as } x \rightarrow 0.$$

- 4) *$g \in L_{d\sigma}^p(\Gamma)$ with $p > 1 + \nu/(n - 1 - \nu)$.*

Then the potential Φg in (4.3) is continuous on \mathbb{R}^n . In particular, its limits on Γ from the inside and the outside coincide,

$$(\Phi g)_+(x) = (\Phi g)_-(x), \quad x \in \Gamma.$$

Since we were unable to find a convenient reference for this result, we provide a proof of Theorem 4.1 in the Appendix.

Proof of i) in Theorem 1.2. We assume $c = c_1^2 > 0$. When $n \geq 2$ and $c \neq 0$, the polynomial in (4.2) is irreducible in $\mathbb{C}[X]$, and all points of $Z(P)$ are non singular. Moreover, the homogeneous Dirichlet problem for the Helmholtz operator $\Delta + c_1^2$ on Ω has non trivial solutions in $C^0(\overline{\Omega}) \cap C^2(\Omega)$ only when c_1^2 is an eigenvalue of the positive operator $-\Delta$ on Ω . Finally, a fundamental solution of the Helmholtz operator is given (see e.g. [22, p.40]), up to a multiplicative constant, by

$$E_n(x) = |x|^{1-n/2} Y_{n/2-1}(c_1|x|),$$

for $x \neq 0$, where Y_α is the Bessel function of the second kind of order α . Near 0, the following estimates hold,

$$Y_\alpha(z) \sim \begin{cases} (2/\pi) \log(z), & \alpha = 0, \\ -(\Gamma(\alpha)/\pi)(2/z)^\alpha, & \alpha > 0. \end{cases}$$

and thus

$$E_n(x) \sim \begin{cases} (2/\pi) \log(|x|), & n = 2, \\ -(\Gamma(n/2 - 1)/\pi)(2/c_1)^{n/2-1}|x|^{2-n}, & n > 2. \end{cases} \quad (4.4)$$

In view of (4.4), Theorem 4.1 applies with $\varphi = E_n$ and $p > n - 1$. Together with Theorem 1.1, we get that $(\Gamma, S_{n-1}(c_1))$ is a HUP for $L^p(\Gamma)$, $p > n - 1$.

Assume now that c_1^2 is an eigenvalue of $-\Delta$ on Ω . Then, there exists some nonzero solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the homogeneous Dirichlet problem in Ω . Moreover, it is known, see e.g. Theorems 3.27 and 3.1 of [6] for the case $n = 3$, that u actually lies in $C^1(\overline{\Omega})$, and that

$$\int_\Gamma \frac{\partial u}{\partial \nu}(y) E_n(x - y) d\sigma(y) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

where ν denotes the outward normal to the surface Γ . Hence, with $v = \partial u / \partial \nu \in C^1(\Gamma)$, one obtains $u = \Phi v$. First, notice that $v \neq 0$ since $u \neq 0$. Second, similarly to (3.1), one has

$$P(\lambda) \mathcal{F}(u)(\lambda) = \hat{v}(\lambda),$$

and $\mathcal{F}(u)$ is an entire function since u is continuous, with compact support, hence in $L^1(\mathbb{R}^n)$. Hence, \hat{v} vanishes on $Z_{\mathbb{R}}(P)$ and $(\Gamma, S_{n-1}(c_1))$ is not a HUP. This finishes the proof of item i) of Theorem 1.2. \square

Remark 4.2. When Ω is a ball, it is known that the eigenvalues of Δ are the zeros of Bessel functions of the first kind.

Remark 4.3. By the remark in (1.1) and the fact that convexity is preserved by linear transformation, one can make use of a linear diagonal transformation

$$T : (x_1, \dots, x_n) \mapsto (a_1 x_1, \dots, a_n x_n), \quad a_i > 0, \quad i = 1, \dots, n,$$

normalized by $\sum_{i=1}^n a_i^2 = 1$, so that the statement i) in Theorem 1.2 becomes the following one. Let $\mathcal{E}_{n-1}(c_1)$ be the ellipsoid

$$\sum_{i=1}^n \left(\frac{X_i}{a_i} \right)^2 = c_1^2,$$

whose vector of semi-axes $c_1(a_1, \dots, a_n)$ has norm c_1 . Let Γ be the boundary of a bounded convex domain in \mathbb{R}^n . Then, $(\Gamma, \mathcal{E}_{n-1}(c_1))$ is a HUP for $L^p(\Gamma)$, $p > n - 1$ if c_1^2 is different from an eigenvalue of the operator

$$-\sum_{i=1}^n \frac{1}{a_i^2} \frac{\partial^2}{\partial x_i^2} \quad \text{on } \Omega.$$

4.2 The Schrödinger equation

In this section, we prove item ii) of Theorem 1.2 by considering the 1-dimensional Schrödinger equation, corresponding to the operator

$$P(-iD) = i\partial_t + \partial_x^2 \quad \text{where} \quad P(X) = -X_1 - X_2^2.$$

Then $Z(P) = \{(X_1, X_2) \in \mathbb{C}^2, X_1 + X_2^2 = 0\}$ and $Z_{\mathbb{R}}(P)$ is a parabola in \mathbb{R}^2 .

In the literature, a fundamental solution of the Schrödinger equation is usually given by

$$H(t)t^{-1/2} \exp(ix^2/(4t)), \quad t \geq 0,$$

see e.g. [27, Sect. 6.2], where $H(t)$ denotes the Heaviside function

$$H(t) = 0 \text{ for } t \leq 0 \text{ and } H(t) = 1 \text{ for } t > 0.$$

Actually, one can also take as a fundamental solution,

$$E_1(t, x) = t^{-1/2} \exp(ix^2/(4t)), \quad t \neq 0,$$

without the factor $H(t)$, which allows one to reconstruct solutions to the Schrödinger equation for $t \in (-\infty, \infty)$ from an initial data at $t = 0$, see [10, Section 4.3.1.b, Example 3].

For some $T, L > 0$, and $I = (0, L)$, we consider the rectangular domain

$$\Omega := (0, T) \times I \subset \mathbb{R}^2, \quad (4.5)$$

with vertices

$$V_1 = (0, 0), \quad V_2 = (T, 0), \quad V_3 = (T, L), \quad V_4 = (0, L).$$

Let

$$\Phi g(t, x) = \int_{\Gamma} g(u, y) \frac{e^{i(x-y)^2/(4(t-u))}}{\sqrt{t-u}} d\sigma(u, y), \quad (4.6)$$

where g is a function defined on the boundary Γ of Ω , and $d\sigma$ is the arc measure on Γ , which we assume to be positively oriented. Also, we assume that the square root in (4.6) satisfies

$$\sqrt{t-u} = -i\sqrt{|t-u|} \quad \text{when } t < u.$$

Our aim is to show the following result.

Proposition 4.4. *Assume the function g to be of class C^2 on Γ_1 and Γ_3 . Then the function $\Phi g(t, x)$ is of class C^2 on $\mathbb{R}^2 \setminus (\Delta_0 \cup \Delta_T)$, where Δ_0, Δ_T are the vertical lines $t = 0$ and $t = L$ respectively. Moreover, $\Phi g(t, x)$ admits left and right limits at each point of Δ_0 , resp. Δ_L , and these limits coincide.*

Proof. We decompose the integral according to the four sides $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, of the rectangle, starting with the horizontal side Γ_1 from $(0, 0)$ to $(T, 0)$, and get

$$\Phi g(t, x) = \Phi_1 g(t, x) + \Phi_2 g(t, x) - \Phi_3 g(t, x) - \Phi_4 g(t, x),$$

with

$$\begin{aligned} \Phi_1 g(t, x) &:= \int_{u=0}^T g(u, 0) \frac{e^{ix^2/(4(t-u))}}{\sqrt{t-u}} du, & \Phi_2 g(t, x) &:= \int_{y=0}^L g(T, y) \frac{e^{i(x-y)^2/(4(t-T))}}{\sqrt{t-T}} dy, \\ \Phi_3 g(t, x) &:= \int_{u=0}^T g(u, L) \frac{e^{i(x-L)^2/(4(t-u))}}{\sqrt{t-u}} du, & \Phi_4 g(t, x) &:= \int_{y=0}^L g(0, y) \frac{e^{i(x-y)^2/(4t)}}{\sqrt{t}} dy. \end{aligned}$$

It is clear that the first and third integrals are well-defined for any $(t, x) \in \mathbb{R}^2$ since the integrands are absolutely integrable. The second one is well-defined when $t \neq T$, and the fourth one is well defined when $t \neq 0$. Moreover, the four integrals define C^∞ functions with respect to the variable $x \in \mathbb{R}$. The second and fourth ones define C^∞ functions with respect to the variable t when $t \notin \{0, T\}$. The first and third ones are also C^∞ functions of t when $t \notin [0, T]$. When $0 \leq t \leq T$, we may write, with a change of variable $v = t - u$,

$$\Phi_1 g(t, x) = \int_{v=t-T}^t g(t-v, 0) \frac{e^{ix^2/(4v)}}{\sqrt{v}} dv, \quad \Phi_3 g(t, x) = \int_{v=t}^{t-T} g(t-v, L) \frac{e^{i(x-L)^2/(4v)}}{\sqrt{v}} dv,$$

which shows, together with the Leibniz integral rule, that the two integrals are C^2 functions of t (recall that we assumed $g(u, y)$ to be a C^2 function on Γ_1 and Γ_3).

Finally, it remains to consider $\Phi_2 g(t, x)$ and $\Phi_4 g(t, x)$, respectively when $t = T$ and $t = 0$. Obviously, the integral expressions for $\Phi_2 g(T, x)$ and $\Phi_4 g(0, x)$ are not defined, we thus compute the left and right limits as $t \rightarrow T_\pm$ or $t \rightarrow 0_\pm$. Recall from the method of stationary phase that, for $\alpha < 0 < \beta$,

$$\frac{1}{\sqrt{\pi\epsilon}} \int_\alpha^\beta e^{iy^2/\epsilon} a(y) dy = e^{i\pi/4} a(0) + O(\epsilon) \quad \text{as } \epsilon \rightarrow 0_+,$$

where $a(y)$ is a C^1 function, see [20, Theorem 13.1] or [10, Section 4.5.3]. Hence, for $0 < x < L$,

$$\lim_{t \rightarrow T_+} \Phi_2 g(t, x) = \lim_{t \rightarrow T_+} \int_{y=0}^L g(T, y) \frac{e^{i(x-y)^2/(4(t-T))}}{\sqrt{t-T}} dy = \frac{\sqrt{\pi}}{2} e^{i\pi/4} g(T, x),$$

and

$$\begin{aligned} \lim_{t \rightarrow T_-} \Phi_2 g(t, x) &= \lim_{t \rightarrow T_-} \int_{y=0}^L g(T, y) \frac{e^{-i(x-y)^2/(4(T-t))}}{-i\sqrt{T-t}} dy \\ &= i \lim_{t \rightarrow T_-} \int_{y=0}^L g(T, y) \frac{e^{i(x-y)^2/(4(T-t))}}{\sqrt{T-t}} dy = i \frac{\sqrt{\pi}}{2} e^{-i\pi/4} g(T, x) = \lim_{t \rightarrow T_+} \Phi_2 g(t, x), \end{aligned}$$

which shows, together with the assumption that $x \in [0, L] \mapsto g(T, x)$ is continuous, that $\Phi_2 g$ can be extended to the right vertical side Γ_2 as a continuous function. With help of [20, Theorem 13.2], one could also check that, for $x \notin (0, L)$, the left and right limits of $\Phi_2 g(t, x)$ coincide at T , and are equal to 0 when $x \notin [0, L]$. Though this will be not important for our analysis, we note that $\Phi_2 g$ has a discontinuity at the right vertices $(T, 0)$ and (T, L) of the rectangle if the function $g(t, x)$ does not vanish at these points.

One may check that the function $\Phi_4 g$ exhibits similar behavior on the line $t = 0$, in particular, it can be continuously extended on the left vertical side Γ_4 of the rectangle. \square

Next, in order to apply Theorem 1.1, we need an uniqueness result for the initial boundary value problem related to the Schrödinger equation, see [4, Lemma 7.1.1, Theorem 7.2.1],

Theorem 4.5. *The initial boundary value problem*

$$\begin{cases} iu_t(t, x) + \Delta u(t, x) = 0, & x \in I, t > 0, \\ u(0, x) = \varphi(x) \in H_0^1(I), & x \in \bar{I} = [0, L], \\ u(t, 0) = 0, u(t, L) = 0, & t \geq 0, \end{cases}$$

has a unique solution u in the space $C([0, \infty), L^2(I))$ of continuous functions from $[0, \infty)$ to $L^2(I)$ (actually, the solution is in

$$C([0, \infty), H_0^1(I)) \cap C^1([0, \infty), H^{-1}(I)),$$

but this result will not be needed to us). Here, $H_0^1(I)$ and $H^{-1}(I)$ denote the usual Sobolev spaces of the segment I .

Since $\Phi g(t, x)$ is a C^2 function when $t > 0$, that is on the right of the vertical line Δ_0 , in order to prove that $t \mapsto \Phi g(t, x)$ is a continuous function from $[0, T)$ to $L^2(I)$, it is sufficient to check that

$$\|\Phi g(t, x) - \Phi g(0, x)\|_{L^2(I)} \rightarrow 0 \quad \text{as } t \rightarrow 0_+,$$

and, more precisely, that

$$\|\Phi_4 g(t, x) - \Phi_4 g(0, x)\|_{L^2(I)} \rightarrow 0 \quad \text{as } t \rightarrow 0_+, \quad (4.7)$$

since $\Phi_i g$, $i = 1, 2, 3$, are of class C^2 in a neighborhood of the segment $\{0\} \times I$. The fact that (4.7) holds true can be derived from uniform error estimates for the stationary phase approximations, see [21]. We let the details to the reader.

Putting together Proposition 4.4 and Theorem 4.5 applied with the initial data $\varphi(x) = 0$ on $\bar{I} = [0, L]$, we see that the method from Theorem 1.1 applies, leading to item ii) of Theorem 1.2.

4.3 Hyperbolic equations

Here we prove item iii) of Theorem 1.2 by considering the 1-dimensional wave equation, that is, the operator

$$P(-iD) = \partial_t^2 - \partial_x^2 \quad \text{where} \quad P(X) = -X_1^2 + X_2^2 = (X_2 - X_1)(X_2 + X_1).$$

Then $Z(P)$ decomposes into the union of two irreducible varieties, namely the two lines $X_1 = \pm X_2$ in \mathbb{C}^2 , and

$$Z_{\mathbb{R}}(P) = \Delta_+ \cup \Delta_- \subset \mathbb{R}^2, \quad \Delta_+ : X_1 = X_2, \quad \Delta_- : X_1 = -X_2.$$

A fundamental (distributional) solution of the wave equation is known to be, up to a multiplicative constant,

$$E_1(t, x) = H(t)H(t-x)H(t+x),$$

see [27, Sect. 7], where H still denotes the Heaviside function.

As in the case of the Schrödinger equation, we consider the rectangular domain $\Omega \subset \mathbb{R}^2$ with vertices $(0, 0)$, $(T, 0)$, (T, L) , $(0, L)$, with $T, L > 0$. Let g be a C^1 function defined on the boundary Γ of Ω . Then,

$$\Phi g(t, x) = \int_{\Gamma} g(u, y) H(t - u) H((t - u) - (x - y)) H((t - u) + (x - y)) d\sigma(u, y),$$

where $d\sigma$ is the arc measure on Γ , which we assume to be positively oriented. In view of the definition of H , the function Φg rewrites as

$$\Phi g(t, x) = \int_{\Gamma_{t,x}} g(u, y) d\sigma(u, y), \quad (4.8)$$

where $\Gamma_{t,x}$ is the subarc (possibly empty) of Γ , which intersects the sector

$$\{(u, y) \in \mathbb{R}^2, u < t, \quad u - y < t - x, \quad u + y < t + x\}.$$

Depending on the location of the point $(t, x) \in \mathbb{R}^2$, it is readily checked that the endpoints of $\Gamma_{t,x}$ are two of the points

$$(0, x-t), (0, t+x), (t-x, 0), (t+x, 0), (T, T-t+x), (T, t+x-T), (L+t-x, L), (t+x-L, L).$$

Because g has been assumed to be a C^1 function, and all coordinates of the above points are just affine functions of t and x , it follows from (4.8) that the function Φg is of class C^2 in \mathbb{R}^2 .

Assuming $\hat{g} = 0$ on the two lines $X_1 = \pm X_2$ in \mathbb{R}^2 , we know from the proof of Theorem 1.1 that the function Φg vanishes outside of Ω , and by continuity also on Γ . Hence, Φg , which belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of the wave equation, and solves the homogeneous Dirichlet problem in $\overline{\Omega}$. Let us now recall the following result, see [8, Theorem 1],

Theorem 4.6. *The homogeneous Dirichlet problem*

$$(\partial_t^2 - \partial_x^2)f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \Gamma,$$

has the unique solution $f = 0$ in $C^2(\Omega) \cap C^1(\overline{\Omega})$ if and only if T/L is an irrational number.

Actually, [8] considers the $(n+1)$ -dimensional wave equation in a rectangular domain, and [9] considers more general hyperbolic equations in cylindrical domains.

From what precedes, we see that Theorem 1.1 applies, leading to item iii) of Theorem 1.2. More generally, making use of (1.1), one gets that, for $c > 0$, $(\Gamma, \Delta_{c,+} \cup \Delta_{c,-})$ is an Heisenberg uniqueness pair for $C^1(\Gamma)$ if $T/(cL) \notin \mathbb{Q}$, where $\Delta_{c,+}$ and $\Delta_{c,-}$ are the lines with slopes c and $-c$.

4.4 A simple first order equation

The goal of this section is to prove item iv) of Theorem 1.2 and Theorem 1.3. We consider the simple differential equation of first order $u_y = 0$ in \mathbb{R}^2 , associated to the operator

$$P(-iD) = i\partial_y \quad \text{where} \quad P(X) = -X_2,$$

in a convex domain Σ with boundary Γ . The set Λ is the horizontal line $\Delta_0 := \{X_2 = 0\}$.

As is easily checked, a fundamental solution is given by the distribution

$$E : \varphi \mapsto \int_0^\infty \varphi(0, s) ds.$$

In view of (2.2), we have, for g a function on Γ , integrable with respect to $d\sigma$, φ a test function, and $x = (x_1, x_2)$,

$$\langle \Phi g, \varphi \rangle = \int_\Gamma \langle E(y), \varphi(x + y) \rangle g(x) d\sigma(x) = \int_{x \in \Gamma} \int_{y=0}^\infty \varphi(x_1, x_2 + y) g(x) dy d\sigma(x).$$

We choose as a domain the unit disk $\Sigma = \mathbb{D}$ with boundary the unit circle $\Gamma = \mathbb{T}$. We set

$$\mathbb{T}_+ = \{(x_1, x_2) \in \mathbb{T}, x_2 \geq 0\}, \quad \mathbb{T}_- = \{(x_1, x_2) \in \mathbb{T}, x_2 < 0\},$$

and we parameterize these two arcs by the functions $s \in [-1, 1] \rightarrow (s, \pm\sqrt{1-s^2})$, so that

$$d\sigma(x) = \frac{|s| ds}{\sqrt{1-s^2}}.$$

Setting $g(x) = g_+(s)$ for $x \in \mathbb{T}_+$ and $g(x) = g_-(s)$ for $x \in \mathbb{T}_-$, we get,

$$\begin{aligned} \langle \Phi g, \varphi \rangle &= \int_{s=-1}^1 \int_{y=0}^\infty \varphi(s, \sqrt{1-s^2} + y) |s| g_+(s) \frac{dy ds}{\sqrt{1-s^2}} \\ &\quad + \int_{s=-1}^1 \int_{y=0}^\infty \varphi(s, -\sqrt{1-s^2} + y) |s| g_-(s) \frac{dy ds}{\sqrt{1-s^2}}. \end{aligned}$$

Letting $u = y + \sqrt{1-s^2}$ in the first double integral and $u = y - \sqrt{1-s^2}$ in the second one, the above expression becomes

$$\langle \Phi g, \varphi \rangle = \int_{(s,u) \in D_+} \varphi(s, u) |s| g_+(s) \frac{d\lambda(s, u)}{\sqrt{1-s^2}} + \int_{(s,u) \in D_-} \varphi(s, u) |s| g_-(s) \frac{d\lambda(s, u)}{\sqrt{1-s^2}},$$

where D_+ (resp. D_-) denotes the subset of \mathbb{R}^2 which lie above \mathbb{T}_+ (resp. \mathbb{T}_-), and $d\lambda$ denotes the planar Lebesgue measure. Thus, the distribution Φg is actually a locally integrable function in $L^1_{loc}(\mathbb{R}^2)$, namely

$$\begin{aligned} \Phi g(s, u) &= \chi_{D_+}(s, u) \frac{g_+(s)|s|}{\sqrt{1-s^2}} + \chi_{D_-}(s, u) \frac{g_-(s)|s|}{\sqrt{1-s^2}} \\ &= \chi_{\mathbb{D}}(s, u) \frac{g_-(s)|s|}{\sqrt{1-s^2}} + \chi_{D_+}(s, u) (g_+(s) + g_-(s)) \frac{|s|}{\sqrt{1-s^2}}, \end{aligned}$$

where χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^2$.

Now, assume that $\hat{g} = 0$ on Λ . Here we can not apply directly Theorem 1.1 since, a priori, Φg is not a continuous function. Nevertheless, the first part of the proof of Theorem 1.1 tells us that $\Phi g = 0$ on D_+ , that is

$$g_+(s) = -g_-(s), \quad a.e. s \in [-1, 1] \quad \iff \quad g(e^{i\theta}) = -g(e^{-i\theta}), \quad a.e. \theta \in [0, 2\pi]. \quad (4.9)$$

Now, if we assume that, for some given angle $\rho \in (0, \pi)$, one has $\hat{g} = 0$ on the line $\Delta_\rho := \{X_1 = (\cot \rho)X_2\}$, and consider, instead of $u_y = 0$, the differential equation

$$u_x - (\cot \rho)u_y = 0,$$

we derive, in a completely similar fashion, that

$$g(e^{i(\rho+\theta)}) + g(e^{i(\rho-\theta)}) = 0, \quad a.e. \theta \in [0, 2\pi]. \quad (4.10)$$

Combining (4.9) and (4.10), we obtain

$$g(e^{i(\theta+\rho)}) = g(e^{i(\theta-\rho)}), \quad a.e. \theta \in [0, 2\pi],$$

that is, 2ρ is a period of the function $\theta \mapsto g(e^{i\theta})$. By uniqueness of the Fourier coefficients of a function in $L^1(\mathbb{T})$, we get

$$a_n = a_n e^{i2n\rho}, \quad n \in \mathbb{Z}.$$

Assume $a_n \neq 0$ for some $n \neq 0$. Then $2n\rho = 2k\pi$ for some $k \in \mathbb{Z}$ which implies that ρ/π is a rational number. Hence, if $\rho/\pi \notin \mathbb{Q}$, g is a constant function, and by (4.9), it has to vanish almost everywhere. Finally, making use of the property (1.1) with T a rotation, we obtain the HUP stated in item iv) of Theorem 1.2.

Conversely, assume that $\rho/\pi = k/n \in \mathbb{Q}$. Then, the nonzero function $g(e^{i\theta}) = e^{in\theta} - e^{-in\theta}$ satisfies (4.9) and (4.10). Hence, with $\Phi_1 g$ and $\Phi_2 g$ the potentials associated to the fundamental solutions of ∂_y and $\partial_x - (\cot \rho)\partial_y$, one gets that $\Phi_1 g$ and $\Phi_2 g$ have compact support since they vanish outside \mathbb{D} . Consequently, their Fourier transforms are entire functions in \mathbb{C}^n , which, in view of (3.1), implies that \hat{g} vanishes on $\Delta_0 \cup \Delta_\rho$. Hence, $(\mathbb{T}, \Delta_0 \cup \Delta_\rho)$ cannot be a HUP, and item iv) of Theorem 1.2 is completely proved.

The above result for the circle \mathbb{T} can be extended, in some sense, to any strictly convex curve Γ of class C^2 , see Theorem 1.3. A proof of this theorem is the goal of the remaining part of this section. Thus we consider a function $g \in L^1(\Gamma)$ such that \hat{g} vanishes on the union $\Delta_0 \cup \Delta_\rho$ of two lines. We introduce two diffeomorphisms s_0 and s_ρ on Γ defined by the condition that, for $x \in \Gamma$, the segment $(x, s_0(x))$ is vertical (resp. $(x, s_\rho(x))$ makes an angle $\rho + \pi/2$ with the oriented horizontal axis). Note that, by the assumption of strict convexity of Γ , s_0 and s_ρ are well-defined. They are involutive maps, and s_0 has exactly two fixed points M_1, M_2 which are the points where the tangents to Γ are vertical. Setting $t_\rho = s_\rho \circ s_0$, a reasoning similar to the one we just made for the circle leads to the conclusion that

$$g(s_0(x)) = -g(x), \quad g(s_\rho(x)) = -g(x), \quad a.e. x \in \Gamma, \quad (4.11)$$

and thus,

$$g(t_\rho(x)) = g(x), \quad a.e. x \in \Gamma. \quad (4.12)$$

Next, pick any point inside Γ . By translation, we may assume it is the origin O of the plane, and by dilation, we may also assume that the unit circle \mathbb{T} lies inside Γ . We denote by p the diffeomorphism from Γ to \mathbb{T} ,

$$p : \Gamma \rightarrow \mathbb{T}, \quad x \mapsto x/\|x\|,$$

where $\|x\|$ denotes the euclidean norm in \mathbb{R}^2 .

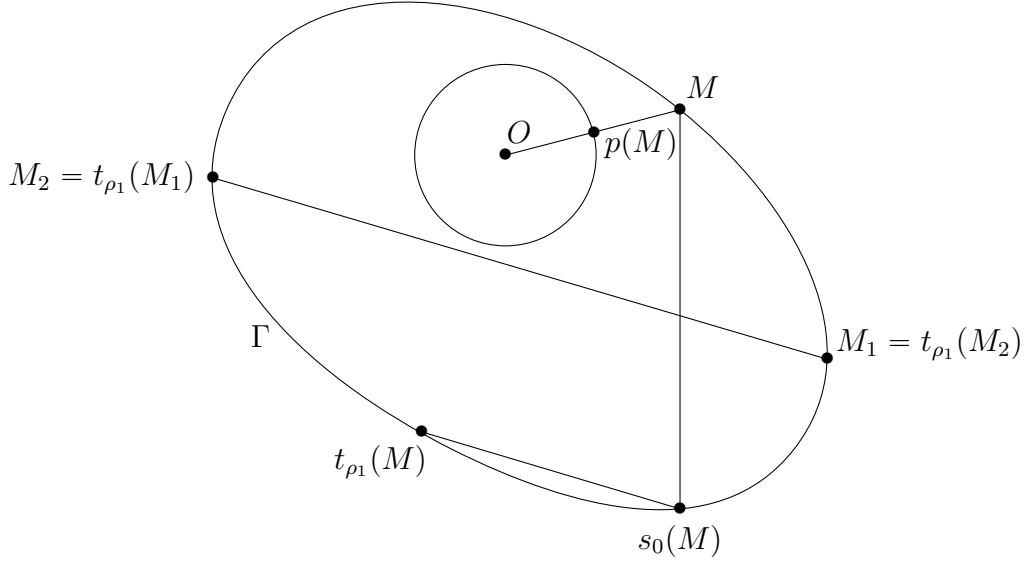


Figure 1 : The convex curve Γ , the circle \mathbb{T} , the points M, M_1, M_2 on Γ , and some of their transforms, as defined in the text.

Moreover, let

$$\tilde{t}_\rho := p \circ t_\rho \circ p^{-1} : \mathbb{T} \rightarrow \mathbb{T}, \quad \tilde{g} = g \circ p^{-1} \in L^1(\mathbb{T}),$$

where we notice that \tilde{t}_ρ is a C^2 diffeomorphism of \mathbb{T} . Indeed, Γ being a curve of class C^2 , all of the bijective maps s_0, s_ρ, t_ρ, p are diffeomorphisms of class C^2 .

First, it is clear that, when $\rho = 0$, t_0 is the identity map of Γ , and \tilde{t}_0 the identity map of \mathbb{T} , with rotation number 0. Second, consider the unique points M_1, M_2 alluded to above, e.g. M_1 with the largest x_1 -coordinate on Γ , and let choose $\rho = \rho_1$ such that the segment (M_1, M_2) makes an angle $\rho_1 + \pi/2$ with the oriented horizontal axis. Then,

$$t_{\rho_1}(M_1) = s_{\rho_1}(M_1) = M_2 \quad \text{and} \quad t_{\rho_1}(M_2) = s_{\rho_1}(M_2) = M_1.$$

Hence, M_1 is a 2-periodic point of t_{ρ_1} , and, similarly, $p(M_1)$ is a 2-periodic point of \tilde{t}_{ρ_1} , which implies that the rotation number of \tilde{t}_{ρ_1} is $1/2$. From the continuity of the map $\rho \mapsto \tau(\tilde{t}_\rho)$, recall Lemma 2.5, we derive that there exists some angle $0 < \rho_2 < \rho_1$ such that $\tau_2 := \tau(\tilde{t}_{\rho_2}) \notin \mathbb{Q}$. From Denjoy's theorem 2.6, there exists a homeomorphism $h : \mathbb{T} \rightarrow \mathbb{T}$ such that $h \circ \tilde{t}_{\rho_2} \circ h^{-1}$ is the rotation R_{τ_2} of angle $2\pi\tau_2$. It then follows, with (4.12), that

$$(\tilde{g} \circ h^{-1})(x) = (\tilde{g} \circ h^{-1})(R_{\tau_2}(x)), \quad x \in \mathbb{T}.$$

By the fact that τ_2 is irrational, and the uniqueness of the Fourier coefficients of $\tilde{g} \circ h^{-1}$, we obtain as above that the function $\tilde{g} \circ h^{-1}$ is constant on \mathbb{T} , hence also the function g on Γ . From the first equality in (4.11), we finally conclude that $g = 0$ almost everywhere on Γ . Applying property (1.1) with a rotation finishes the proof of Theorem 1.3.

5 Appendix

We give a proof of Theorem 4.1 about the continuity of single layer potential with weakly singular kernels.

Proof of Theorem 4.1. First, when $x_0 \notin \Gamma$, Φg is continuous at x_0 since

– the map $x \mapsto \varphi(x - y)g(y)$ is continuous at x_0 , for all $y \in \Gamma$,

– $|\varphi(x - y)g(y)| \leq Mg(y) \in L^p(\Gamma) \subset L^1(\Gamma)$, where M is the sup of $|\varphi(x - y)|$ when x is in a neighborhood of x_0 that does not intersect Γ , and $y \in \Gamma$.

Second, assume $x_0 \in \Gamma$. Let TT_{x_0} and n_{x_0} be the $(n - 1)$ -dimensional tangeant space to Γ at x_0 , and the normal unit vector to Γ at x_0 . We write $y = x_0 + \tau_y + \eta_y n_{x_0} \in \Gamma$ with $\tau \in \text{TT}_{x_0}$, and we let $\eta = \gamma(\tau)$ be the equation of Γ in a neighborhood of x_0 . For some small enough $\delta > 0$, let

$$\Gamma(x_0, \delta) = \{y \in \Gamma, \tau_y \in B(x_0, \delta)\},$$

where $B(x_0, \delta)$ is the ball of radius δ in TT_{x_0} . Finally, for x sufficiently close to x_0 , we write

$$x = x_0 + \tau_x + \eta_x n_{x_0}.$$

Then, we have

$$\begin{aligned} \Phi g(x) - \Phi g(x_0) &= \int_{y \in \Gamma(x_0, \delta)} \varphi(x - y)g(y)d\sigma(y) - \int_{y \in \Gamma(x_0, \delta)} \varphi(x_0 - y)g(y)d\sigma(y) \\ &\quad + \int_{y \in \Gamma \setminus \Gamma(x_0, \delta)} (\varphi(x - y) - \varphi(x_0 - y))g(y)d\sigma(y). \end{aligned} \quad (5.1)$$

For the first two integrals, for x possibly equal to x_0 , we have, by Hölder inequality, with q the exponent conjugate to p ,

$$\begin{aligned} \left| \int_{y \in \Gamma(x_0, \delta)} \varphi(x - y)g(y)d\sigma(y) \right| \\ \leq \left(\int_{y \in \Gamma(x_0, \delta)} |\varphi(x - y)|^q d\sigma(y) \right)^{1/q} \left(\int_{y \in \Gamma(x_0, \delta)} |g(y)|^p d\sigma(y) \right)^{1/p}. \end{aligned}$$

The last integral is finite since $g \in L^p_{d\sigma}(\Gamma)$. For the second one, note that

$$|\varphi(x - y)| \leq \frac{C}{|x - y|^\nu} \leq \frac{C}{|\tau_x - \tau_y|^\nu}.$$

Moreover,

$$d\sigma(y)^2 = d\tau^2 + d\eta^2 = (1 + \gamma'(t)^2)d\tau^2 \leq 4d\tau^2$$

for δ small (recall that γ' is continuous and $\gamma'(0) = 0$). Hence

$$\left| \int_{y \in \Gamma(x_0, \delta)} |\varphi(x - y)|^q d\sigma(y) \right| \leq 2C \int_{B(x_0, \delta)} \frac{d\tau}{|\tau_x - \tau|^\nu},$$

(with $\tau_x = 0$ if $x = x_0$) where the last integral over the $(n - 1)$ -dimensional ball $B(x_0, \delta)$ is convergent since

$$1 + \nu/(n - 1 - \nu) < p \quad \iff \quad \nu q < n - 1.$$

Hence, by taking δ small, it can be made as small as we wish (uniformly in τ_x).

It remains to show that the third integral in (5.1) can be made small. For x close to x_0 so that $x \notin \Gamma \setminus \Gamma(x_0, \delta)$, the function

$$x \mapsto \int_{y \in \Gamma \setminus \Gamma(x_0, \delta)} \varphi(x - y)g(y)d\sigma(y)$$

is continuous at x_0 (same argument as in the first case), and thus, the third integral is small when x is sufficiently close to x_0 . \square

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