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# NEW INVARIANT SURFACE MEASURES FOR THE CUBIC SCHRÖDINGER EQUATION

JEAN-BAPTISTE CASTERAS, ANA BELA CRUZEIRO, AND ANNIE MILLET

ABSTRACT. We construct new invariant measures supported on mass level sets for the cubic defocusing nonlinear Schrödinger equation in dimensions 1 and 2.

## 1. INTRODUCTION

Consider the following equation for  $\lambda \in \{-1, +1\}$

$$(1) \quad i \frac{\partial v}{\partial t} = -\Delta v + \lambda |v|^2 v,$$

in the torus  $\mathbb{T}^d$  of dimension  $d = 1, 2$ , with initial conditions  $v_0 \in H^s$ , for some  $s \in \mathbb{R}$ . When  $\lambda = 1$  (resp.  $\lambda = -1$ ), the equation is defocusing (resp. focusing). Recall that (1) can be seen as a Hamiltonian PDE, i.e.,

$$\frac{du}{dt} = i \frac{\partial H}{\partial \bar{u}},$$

where

$$H(u) = \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 dx + \frac{\lambda}{4} \int_{\mathbb{T}^d} |u|^4 dx.$$

From this reformulation, it is easily seen that  $H(u)$  is an invariant quantity. One can also check that the mass

$$M(u) = \int_{\mathbb{T}^d} |u|^2 dx,$$

and the momentum

$$P(u) = i \int_{\mathbb{T}^d} u \bar{u}_x dx$$

are conserved. In general, these three conservation laws are the only known ones. But in some very special cases, others are available. For instance, if  $d = 1$ , the cubic Schrödinger equation possesses a Lax pair structure which allows to construct an infinity of conservation laws see [10, 29]. In this case, the equation is said to be completely integrable. Let us point out that very recently people were able to construct a continuum of conservation laws (whereas the earlier results only provided infinite but countable ones). We

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refer to [28] for results regarding the Benjamin-Ono equation, [15] for the cubic NLS in the whole line, [7] for the KdV and modified KdV equations (the non-linear term  $v\partial_x v$  is replaced by  $v^2\partial_x v$ ).

Our goal in this paper is to construct new invariant measures for the cubic NLS equation in dimension 1 and 2. First, let us describe in a few words the situation in the  $d$ -dimensional case. Let  $H(p, q) = H(p_1, \dots, p_d, q_1, \dots, q_d)$  be a Hamiltonian and let us consider its associated flow in  $\mathbb{R}^{2d}$  i.e.

$$(2) \quad \dot{p}_i = \frac{\partial H}{\partial q_j}, \quad \dot{q}_i = -\frac{\partial H}{\partial p_j}, \quad i, j = 1, \dots, d.$$

Then, using the conservation of  $H$ , Liouville's theorem states that the Gibbs measure  $e^{-H(p,q)} \prod_{j=1}^d dp_j dq_j$  is invariant under the flow generated by (2). Notice that, for any (reasonable) function  $F(p, q)$  which is conserved by the flow (2), the measure  $F(p, q) e^{-H(p,q)} \prod_{j=1}^d dp_j dq_j$  is also invariant. The situation is much more delicate in the infinite dimensional case.

In analogy with the Euclidean case  $\mathbb{R}^d$ , by a scaling argument, one expects (1) to be (at least locally) well-posed for initial data in  $H^s$ , for  $s > s_c = \frac{d}{2} - 1$ . The space  $H^{s_c}$  is called critical. In [12], it is proved that (1) for  $d = 1$  in the focusing and defocusing case is globally well-posed for initial data in  $H^s$ ,  $s > -\frac{1}{2}$ . The same result holds for the 1d modified KdV equation. When  $d = 2$ , Bourgain [2] obtained the local well-posedness of (1) for initial data in  $H^s$ ,  $s > 0$ . The global well-posedness in this exact situation was obtained in [13] for small initial data in  $H^s$ , for any  $s > 0$ . In dimensions 3 and 4, for energy-critical non-linearity, local well posedness was proved in  $H^1$ , while global well-posedness was shown for "small" initial data; see [14] and the references therein. For higher dimensions, we refer to the very recent reference [18] concerning local well-posedness results in critical spaces.

Starting from the work of Lebowitz, Rose and Speer [19], and later on from the paper of Bourgain [3], the statistical approach to Schrödinger equations has been considerably developed. It consists in constructing invariant probability (Gibbs) measures for the solutions of these equations, based on their invariant quantities. The solutions are proved to exist starting almost everywhere from the support of the measures and can be rather singular, typically living in Sobolev spaces of low regularity.

This approach uses techniques from probability theory, where transport properties of Gaussian measures under linear and nonlinear transformations have an old tradition. In the framework of Hamiltonian partial differential equations, of which Schrödinger systems are an example, invariant (or quasi-invariant) measures are associated to conservation laws and are weighted Gaussian measures. We refer to the works [23] and [25] where the authors prove the quasi-invariance of Gaussian measures under the flow generated by the quintic NLS in the defocusing case on  $\mathbb{T}^3$  (which is energy critical).

For the cubic non-linear Schrödinger equation (NLS), in the defocusing case and on the two-dimensional torus, a renormalisation must be introduced. This equation was studied in [4], with the construction of a Gibbs measure associated to the Wick ordered Hamiltonian and the existence of the corresponding flow defined on the support of the measure was proved. We also refer to [24] for related results.

In the recent paper [9] the authors generalize Bourgain's work and consider Wick ordered power nonlinearities in the Schrödinger equation which are of higher degree, proving the existence of strong solutions with respect to the associated Gibbs measure.

The Gibbs measures are absolutely continuous with respect to Gaussian ones. We consider the corresponding abstract Wiener spaces in the sense of Gross [11]. Then we use techniques from quasi-sure analysis on these spaces, as established by Malliavin [20]. For a non degenerate functional on an abstract Wiener space we can consider its level sets, which are (finite codimensional) submanifolds of that space. In [1] Airault and Malliavin defined a disintegration of the Gaussian measure into a family of conditional laws or surface measures and proved a geometric Federer-type coarea formula providing this disintegration. Such measures do not charge sets of capacity zero.

In this work we study the defocusing cubic renormalised NLS on the two dimensional torus (as in [4]) and construct surface measures for the mass, which is a conserved quantity of the equation. Then we prove existence of weak solutions living on the level sets of the mass and defined almost everywhere with respect to the surface measures. We therefore provide a more precise result than those obtained with respect to the Gibbs measures constructed so far.

We refer also to the work [21], where a different kind of conditioning and corresponding measures for NLS have been considered.

Let us describe more precisely the result of [21]. The authors construct a measure of the form

$$d\mu_0 = Z_0^{-1} e^{\frac{1}{4} \int_{\mathbb{T}} |u|^4} dP_0,$$

where  $Z_0$  is a normalisation constant and  $P_0$  is defined by

$$P_0(E) = \lim_{\varepsilon \rightarrow 0} P\left(E \mid \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a), i \int_{\mathbb{T}} u \overline{u_x} \in B_\varepsilon(b)\right),$$

where  $A_\varepsilon(a)$ ,  $B_\varepsilon(b)$  are neighborhoods shrinking nicely to  $a$  and  $b$  as  $\varepsilon \rightarrow 0$ . In fact, this measure is shown to exist for polynomial nonlinearities  $\pm |u|^{p-2}u$  with  $2 < p \leq 6$  (the mass needing to be small if  $p = 6$ ). We also refer to [5] for related results for the derivative NLS.

Let us also mention that an active line of work in the construction of Gibbs measure is to find optimal integrability thresholds for such a measure to exist; see for instance [22].

Our methods are similar to those in [8], where surface measures supported by level sets of the enstrophy were constructed for the Euler equation on the torus.

We are interested in the cubic defocusing Schrödinger equation in the one and two dimensional torus, namely

$$(3) \quad i \frac{\partial v}{\partial t} = -\Delta v + |v|^2 v$$

with initial conditions  $v_0$  of low regularity. If the solution is assumed to be at least  $L^2$ , we can use the following change of variables  $v = e^{it(1-2\|u\|_{L^2}^2)} u$ . In this case, (3) is equivalent to

$$(4) \quad i \frac{\partial u}{\partial t} = -\Delta u + (|u|^2 - 2\|u\|_{L^2}^2) u + u.$$

Notice that it is not anymore the case if the solution does not have a finite mass. Our aim is to take initial data in the support of a surface measure supported on mass level sets. Formally, let

$$d\mu_{2s} = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du,$$

where  $Z_s$  is a normalisation constant and  $du$  is formally the Lebesgue measure, which is well-known to not exist on infinite dimensional vector spaces. The Gaussian measure  $\mu_{2s}$  can also be seen as the law of the random variable

$$\omega \rightarrow \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{(1 + |n|^2)^{s/2}} e^{inx},$$

where  $\{g_n\}$  are independent standard complex-valued Gaussian variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . For  $\sigma \in \mathbb{R}$ , it is easy to check that

$$\mathbb{E} \left[ \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^\sigma \left| \frac{g_n}{(1 + |n|^2)^{s/2}} \right|^2 \right] < \infty \text{ iff } \sigma < s - \frac{d}{2}.$$

So this random series converges in  $\mathbb{L}^2(\Omega, H^\sigma(\mathbb{T}^d))$  if and only if  $\sigma < s - \frac{d}{2}$ . In what follows, we will always take  $s = 1$ . We see that, when  $d = 2$ , solutions distributed according to  $\mu_2$  do not have a finite mass, which implies that equation (3) and (4) are not equivalent. The previous remarks also show that the support of this Gibbs measure is necessarily rough. We point out that an alternative method to construct invariant measures is the so-called Inviscid-Infinite-dimensional limits method due to Kuksin and Shirikyan [16, 17]. We refer to [6, 26, 27] for applications related to NLS.

We set  $V_r = \{\varphi \in L^2 | E(\varphi) = r\}$ , for  $r > 0$  where

$$(5) \quad E(\varphi) = \int_{\mathbb{T}^d} |\varphi|^2 dx - 1_{\{d=2\}} \sum_{k \neq 0} \frac{2}{|k|^2}.$$

In section 4, we prove that this quantity is well-defined in dimension 2. Our main result reads as follows :

**Theorem 1.1.** *Let  $d = 1$  or  $d = 2$ . Then, for all  $r > 0$ , there exists a probability measure  $\sigma_r$  defined on  $L^2$  with support on  $V_r$ , a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a  $C(\mathbb{R}, V_r)$ -valued stochastic process  $u$  such that*

- $u = u^\omega$  is a global weak solution to (1) if  $d = 1$  or (4) if  $d = 2$  for  $\mathbb{P}$  a.e.  $\omega \in \Omega$ .
- $\sigma_r$  is invariant in the sense that the law of  $u(t)$  is  $\sigma_r$  for all  $t \in \mathbb{R}$ .

Let us also point out that our result can be easily generalised to more general dispersions namely in (1), we could have taken any power of the Laplace operator  $(-\Delta)^s$ ,  $s > 0$  as well as any odd power nonlinearities. In the case of a more general dispersion of the form  $(-\Delta)^s$ , the difference would have been the value of the parameter  $a = 2$  in the above renormalisation to define  $E(\varphi)$  (see (5)). Instead of taking it equal to 2, it would have been chosen as  $a = 2s$ .

In section 2, we describe the Gaussian measures we will use, define the Sobolev and Wiener spaces. Finally, we state a result by Airault and Malliavin [1] which constructs a surface measure supported by a level set of some functional defined on a Wiener space; this will be the main ingredient in the construction of our invariant measure. In section 3, we prove Sobolev estimates on the non linear term projected on low frequencies; in dimension 2 this will require a renormalisation of this cubic term. A similar renormalisation was also considered in [4]. In Section 4, we prove several estimates for a renormalisation of the mass. Finally, we prove Theorem 1.1 in Section 5. Some technical lemma is proved in the appendix.

## 2. PRELIMINARIES

Let us first state several basic facts about Gaussian measures. Next, we rewrite the cubic Schrödinger equation into frequency.

**2.1. The Gaussian measures.** Given a function  $\varphi \in L^2(\mathbb{T}^d)$ , set  $\varphi = \sum_{k \in \mathbb{Z}^d} \varphi_k e_k$ , where  $e_k(x) = \frac{1}{2} e^{ikx}$ ,  $k \in \mathbb{Z}^d$ , with  $kx = k_1 x_1 + \dots + k_d x_d$ .

For  $a > 0$  let us consider the Gaussian measures on  $L^2(\mathbb{T}^d)$

$$d\mu_a = \prod_{k \in \mathbb{Z}^d} d\mu_a^k, \quad d\mu_a^k(z) = \frac{|k|^a}{2\pi} e^{-\frac{1}{2}|k|^a |z|^2} dx dy,$$

with  $z = x + iy \in \mathbb{C}$ ,  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$  and  $|k|^2 = k_1^2 + \dots + k_d^2$ .

Suppose that  $\{\varphi_k, k \in \mathbb{Z}^d\}$  are independent complex-valued Gaussian random variables with distribution  $\mu_a^k$ . For later purpose, let us notice that

$$E_{\mu_a}(\varphi_k) = 0, \quad E_{\mu_a}(\varphi_k \varphi_{k'}) = 0, \quad E_{\mu_a}(\varphi_k \overline{\varphi_{k'}}) = \delta_{k,k'} \frac{2}{|k|^a},$$

and, for any  $p \geq 1$ ,

$$(6) \quad \mathbb{E}_{\mu_a}(|\varphi_k|^{2p}) = \frac{c_p}{|k|^{ap}}, \quad \text{where } c_p = 2^p p!.$$

Indeed, letting  $x = \operatorname{Re}(\varphi_k)$ ,  $y = \operatorname{Im}(\varphi_k)$ , we have

$$\begin{aligned} \mathbb{E}_{\mu_a}(|\varphi_k|^{2p}) &= \frac{|k|^a}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (x^2 + y^2)^p e^{-|k|^a \frac{x^2+y^2}{2}} dx dy \\ &= |k|^a \int_{\mathbb{R}^+} r^{2p} e^{-|k|^a \frac{r^2}{2}} r dr \\ &= 2^p p! |k|^{-ap}. \end{aligned}$$

**2.2. The Sobolev spaces.** We next recall some definitions of functional spaces such as Sobolev spaces and Wiener spaces. The Sobolev spaces of order  $\beta$  on the torus are defined by

$$H^\beta = \left\{ \varphi \in L^2 : \sum_{k \neq 0} |k|^{2\beta} |\varphi_k|^2 < +\infty \right\}$$

and are endowed with the corresponding Hilbert scalar product.

We will work with the Gaussian measure  $\mu_2$ . This measure is supported in the space  $H^\beta$  with  $\beta < 0$ . This is a consequence of the fact that  $\int \sum_{k \neq 0} |k|^{2\beta} |\varphi_k|^2 d\mu_2 = 2 \sum_{k \neq 0} \frac{|k|^{2\beta}}{|k|^2}$  converges for  $\beta < 0$ ; it is also a consequence of  $\mu_2$  being the measure induced by the process  $\sum_{|n|>0} \frac{1}{n} G_n e_n(x)$ , where  $\{G_n\}_n$  are independent standard complex Gaussian random variables.

For  $\beta < 0$ , the triple  $(H^\beta, H^1, \mu_2)$  is an abstract Wiener space in the sense of Gross (see [11]).

Given a Hilbert space  $G$ , we consider the Malliavin derivative of a functional  $F : H^\beta \rightarrow G$ , in the Cameron-Martin direction  $v \in H^1$ , namely the (a.e.) limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(\varphi + \varepsilon v) - F(\varphi)), \quad x \in H^1.$$

By the Riesz theorem this derivative gives rise to a linear gradient operator  $\nabla$  such that the limit above coincides with  $\langle \nabla F(\varphi), v \rangle$ . One can iterate this procedure and obtain higher order gradients  $\nabla^r$ . Let us for instance consider the second derivative of  $F$ . For  $\varphi \in H^\beta$ , and  $v, w \in H^1$ , iterating the first derivative, we obtain

$$\nabla^2 F(\varphi)(v, w) = D_v D_w F(\varphi).$$

We define the Hilbert-Schmidt norm

$$\|\nabla^2 F(\varphi)\|_{H.S.(H^1 \otimes H^1, H^\beta)}^2 = \sum_{j,k} \|D_{\hat{e}_j} D_{\hat{e}_k} F(\varphi)\|_{H^\beta}^2,$$

where  $\{\hat{e}_j\}_j$  is an orthonormal basis of  $H^1$ . The corresponding Sobolev spaces are  $W_r^p(H^\beta; G)$ , that is the space of maps  $F : H^\beta \rightarrow G$  such that  $F \in L_{\mu_2}^p(H^\beta; G)$ , the gradients  $\nabla^s F : H^\beta \rightarrow H.S.(\otimes_s H^1, G)$  are defined for every  $1 \leq s \leq r$  and belong to  $L_{\mu_2}^p(H^\beta; H.S.(\otimes_s H^1, G))$ . We equip this space with the norm

$$\|f\|_{W_r^p(H^\beta; G)} = \|f\|_{L_{\mu_2}^p(H^\beta; G)} + \sum_{1 \leq s \leq r} \|f\|_{L_{\mu_2}^p(H^\beta; H.S.(\otimes_s H^1, G))}.$$

We also denote  $W_\infty = \bigcap_{r,p} W_r^p(H^\beta; \mathbb{R})$  and  $W_\infty(H^\beta) = \bigcap_{r,p} W_r^p(H^\beta; H^\beta)$ .

**2.3. The nonlinear Schrödinger equation.** Using the decomposition of the solution  $\varphi(x, t) = \sum_k \varphi_k(t) e_k(x)$  into frequencies, the cubic Schrödinger equation (3) for  $\varphi$  can be rewritten as the family of equations

$$\frac{d}{dt} \varphi_k = -i|k|^2 \varphi_k - iB_k(\varphi), \quad |k| > 0,$$

where

$$B_k(\varphi) = \sum_{l,m} \varphi_{k-m} \varphi_{l+m} \bar{\varphi}_l.$$

The functions  $\{e_k\}_k$  are eigenfunctions for the operator  $-\Delta$  with eigenvalues  $|k|^2 = k_1^2 + \dots + k_d^2$ ; we set  $A_k(\varphi) = -|k|^2 \varphi_k$ , corresponding to the Laplacian  $A = \Delta$ .

For any  $n \in \mathbb{N}$ , we define

$$B^n(\varphi) = \Pi_n(B(\Pi_n(\varphi))),$$

where

$$\Pi_n \left( \sum_k \varphi_k e_k \right) = \sum_{k: 1 \leq |k|^2 \leq n} \varphi_k e_k.$$

**2.4. Surface measures.** When  $g \in W_\infty$ , it is possible to consider a redefinition of  $g$ , namely a function  $g^*$  that coincides with  $g$  almost surely, and is defined and continuous outside sets of capacity zero (c.f [20]). Using such redefinitions a construction of surface measures on Wiener spaces was made in [1]. These measures are defined on level sets of  $W_\infty$  functionals, and do not charge zero capacity sets.

Let  $E$  be a real-valued functional defined on  $H^\beta$  and consider the surfaces  $V_r = \{\varphi | E(\varphi) = r\}$  for  $r > 0$ . Assume that  $E$  and  $\|\nabla E\|_{H^1}^{-1}$  belong to the corresponding spaces  $W_\infty$ .

Denote the  $C^\infty$  densities of  $d(E * \mu_2)$  (resp.  $d(E * g\mu_2)$ ) with respect to the Lebesgue measure by  $\rho(r) = \frac{d(E * \mu_2)}{dr}$  (resp.  $\rho_g(r) = \frac{d(E * g\mu_2)}{dr}$ ); they are well defined and smooth, cf. [1]. We have

**Theorem 2.1.** [1] *Let  $r > 0$  be such that  $\rho(r) > 0$ . Then there exists a Borel probability measure defined on  $H^\beta$ , denoted by  $\nu^r$ , with support on  $V_r$  and such that*

$$\int_{V_r} g^*(\varphi) d\nu^r = \frac{\rho_g(r)}{\rho(r)},$$

for any  $g^*$  redefinition of  $g$ .

This will be a crucial ingredient to construct our invariant measures.



3. NORM ESTIMATES FOR  $B$ 

Our goal in this section is to derive Sobolev estimates for the vector field  $B$ . We will see that we need to renormalize  $B$  when the dimension  $d = 2$ . We proceed as follows: for every  $N \in \mathbb{N}$  and  $k \in \mathbb{Z}^2$ , we let

$$(7) \quad (: B :)_k^N = B_k - 4 \left( \sum_{m: 1 \leq |m| \leq |N|} \frac{1}{|m|^a} \right) \varphi_k,$$

with  $a = 2$ . Notice that this sum diverges when  $|N| \rightarrow \infty$  if  $d = 2$ . In dimension 1, a renormalisation is not needed (since the sum is converging). More precisely our aim in this section is to show that in dimension 2  $(: B :)^N$  belongs to  $L_{\mu_a}^{2p}(H^\beta, H^\beta)$  and  $\nabla^s (: B :)^N$  belongs to  $L_{\mu_a}^{2p}(H^\beta, H.S.(\otimes^s H^1, H^\beta))$ , for any  $p \geq 1$ ,  $s \in \mathbb{N}^*$  and  $\beta < 0$ , uniformly in  $N$ . Thanks to our estimate, we will be able to prove that  $(: B :)^N$  converges in  $L_{\mu_a}^{2p}(H^\beta, H^\beta)$ . We will denote its limit by  $: B :$ . We begin by estimating  $(: B :)^N$ .

**Proposition 3.1.** *Let  $a = 2$  and  $\beta < 0$  and  $p \geq 1$  an integer.*

*In dimension 1,  $B$  is  $L^p$ -integrable with respect to the measure  $\mu_a$  and to the norm  $H^\beta$ .*

*In dimension 2,  $(: B :)^N$  is  $L^p$ -integrable with respect to the measure  $\mu_a$  and to the norm  $H^\beta$ , uniformly in  $N$ . We denote by  $: B :$  its limit as  $N \rightarrow \infty$ .*

Notice that our choice of parameters is related to the fact that we want to work on the abstract Wiener space  $(H^\beta, H^1, \mu_2)$ , with  $\beta < 0$ .

*Proof.* By abuse of notation, in this proof, we will drop the subscript  $N$  in  $: B :^N$  in dimension 2. We will work at fixed  $N$  and will notice at the end that, thanks to the normalisation, our estimates do not depend on  $N$ .

Let  $p$  be odd; we are going to show that  $: B : \in L_{\mu_a}^{2p}(H^\beta, H^\beta)$ .

In a first step, we compute  $E_{\mu_a}(|B|^{2p})$ .

For every  $k \in \mathbb{Z}$  with  $k \neq 0$  we have

$$|B_k(\varphi)|^2 = \sum_{l, l' \neq 0, m, m', |l| \vee |m| \vee |l'| \vee |m'| \leq N} T_k(l, m, l', m'),$$

where, for  $m \notin \{k, -l\}$  and  $m' \notin \{k, -l'\}$ ,

$$T_k(l, m, l', m') := \varphi_{k-m} \varphi_{l+m} \overline{\varphi_{l'} \varphi_{k-m'} \varphi_{l'+m'}}.$$

Note that  $E_{\mu_a}[T_k(l, m, l', m')^p] = 0$  except in three cases:  $k - m = l$ ,  $k - m = k - m'$  and  $k - m = l' + m'$ .

*Case 1* Let  $k - m = l$ ; then we have to impose that either  $l + m = k - m'$  or  $l + m = l' + m'$ .

*Case 1.1* If  $l + m = k - m'$ , we deduce that, since  $k - l = m = m + m'$ , we have  $m' = 0$ . Therefore,  $T_k(l, k - l, l', 0) = |\varphi_k|^2 |\varphi_l|^2 |\varphi_{l'}|^2$ . Thus,

- if  $m \neq 0$ ,  $k \neq l$  and  $k \neq l'$ , the indices  $k, l, l'$  are pairwise different;
- if  $m \neq 0$  and  $l = l'$ , then  $T_k(l, k - l, l, 0) = |\varphi_k|^2 |\varphi_l|^4$ ;

- if  $m \neq 0$  and  $l' = k$ , then  $T_k(l, k-l, k, 0) = |\varphi_k|^4 |\varphi_l|^2$ ;
- if  $m = m' = 0$  and  $l' \neq k$ , then  $T_k(k, 0, l', 0) = |\varphi_k|^4 |\varphi_{l'}|^2$ ;
- if  $m = m' \neq 0$  and  $l = l' \neq k$ , then  $T_k(l, k-l, l, k-l) = |\varphi_k|^2 |\varphi_l|^4$ ;
- finally, if  $m = m' = 0$  and  $l' = k$ , then  $T_k(k, 0, k, 0) = |\varphi_k|^6$ .

*Case 1.2* If  $l+m = l'+m'$ , we deduce that  $k = l+m = l'+m'$  and hence  $k-m' = l'$ . Therefore,  $T_k(l, k-l, l', k-l') = |\varphi_k|^2 |\varphi_l|^2 |\varphi_{l'}|^2$ . Thus,

- if  $m \neq 0$ ,  $m' \neq 0$  and  $l \neq l'$ , the indices  $k, l, l'$  are pairwise different;
- if  $m = 0$  and  $m' \neq 0$ , then  $T_k(k, 0, l', k-l') = |\varphi_k|^4 |\varphi_{l'}|^2$ .

In the other cases, that is  $m \neq 0$  and  $m' = 0$  (resp.  $m = m' = 0$ ), we have again  $T_k(l, k-l, k, 0)$  (resp.  $T_k(k, 0, k, 0)$ ).

*Case 2* Let  $k-m = k-m'$ , which implies  $m = m'$ . Then we have to impose either  $l+m = l$  or  $l+m = l'+m'$ .

*Case 2.1* If  $l+m = l$ , we deduce  $m = m' = 0$ , so that  $l' = l'+m'$ . Therefore,  $T_k(l, 0, l', 0) = |\varphi_k|^2 |\varphi_l|^2 |\varphi_{l'}|^2$ . Thus

- the indices  $k, l, l'$  can be pairwise distinct;
- if  $k = l' \neq l$ , then  $T_k(l, 0, k, 0) = |\varphi_k|^4 |\varphi_l|^2$ ;

In the other cases  $k = l \neq l'$  (resp.  $k = l = l'$ ), we have again  $T_k(k, 0, l', 0)$  (resp.  $T_k(k, 0, l, 0)$ ).

*Case 2.2* If  $l+m = l'+m'$ , since  $m = m'$  we conclude that  $l = l'$ . Therefore,  $T_k(l, m, l, m) = |\varphi_{k-m}|^2 |\varphi_{l+m}|^2 |\varphi_l|^2$ . Thus,

- if  $m \neq 0$ ,  $l \neq k-m$  and  $l \neq l-2m$ , the indices  $k-m, l, l+m$  are pairwise distinct;
- if  $m \neq 0$  and  $l+m = k-m$ , then  $l \neq k-m$  and  $T_k(k-2m, m, k-2m, m) = |\varphi_{k-m}|^4 |\varphi_{k-2m}|^2$ ;
- if  $m = 0$  and  $l \neq k$ , then  $T_k(l, 0, l, 0) = |\varphi_k|^2 |\varphi_l|^4$ ;

Finally, when  $m = 0$  and  $k = l$  (resp.  $m \neq 0$  and  $l = k-m$ ) we have again  $T_k(k, 0, k, 0)$  (resp.  $T_k(l, k-l, l, k-l)$ ).

*Case 3* Let  $k-m = l'+m'$ ; then we have to impose either  $l+m = l$  or  $l+m = k-m'$ .

*Case 3.1* If  $l+m = l$ , that is  $m = 0$ , we have  $k = l'+m'$ . Therefore,  $T_k(l, 0, l', k-l') = |\varphi_k|^2 |\varphi_l|^2 |\varphi_{l'}|^2$ . Thus,

- the indices  $k, l, l'$  can be pairwise distinct;
- if  $k \neq l = l'$ , then  $T_k(l, 0, l, k-l) = |\varphi_k|^2 |\varphi_l|^4$ .

When  $k = l \neq l'$  (resp.  $k = l' \neq l$ ), we have again  $T_k(k, 0, l', k-l')$  (resp.  $T_k(l, 0, k, 0)$ ).

Finally, if  $k = l = l'$ , we have again  $T_k(k, 0, k, 0)$ .

*Case 3.2* If  $l+m = k-m'$ , we deduce that  $m+m' = k-l = k-l'$ , so that  $l' = l$ . Therefore,  $T_k(l, m, l, k-m-l) = |\varphi_{k-m}|^2 |\varphi_{l+m}|^2 |\varphi_l|^2$ . Thus

- if  $m \neq 0$ ,  $l \neq k-m$  and  $l \neq k-2m$ , the indices  $k-m, l, l+m$  are pairwise distinct;
- if  $m \neq 0$  and  $l = k-m$ , then  $T_k(l, k-l, l, 0) = |\varphi_l|^4 |\varphi_k|^2$ .

The other cases  $m \neq 0$  and  $l + m = k - m$  (resp.  $m = 0$  and  $l \neq k$ ) give again  $T_k(k - 2m, m, k - 2m, m)$  (resp.  $T_k(l, 0, l, k - l)$ ). Finally, the case  $m = 0$  and  $l = k$  gives again  $T_k(k, 0, k, 0)$ .

As a summary, we have proven that

$$\begin{aligned}
E_{\mu_a}(|B_k|^{2p}) &= E[|\varphi_k|^{6p}] + 4 \sum_{|l| \leq N, |l'| \leq N, l \neq k, l' \neq k, l \neq l'} E[|\varphi_k|^{2p}] E[|\varphi_l|^{2p}] E[|\varphi_{l'}|^{2p}] \\
&+ 2 \sum_{|l| \neq N, l \neq k, 1 \leq |m| \leq N, l+m \neq k-m} E[|\varphi_{k-m}|^{2p}] E[|\varphi_{l+m}|^{2p}] E[|\varphi_l|^{2p}] \\
&+ 4 \sum_{|l| < N, l \neq k} E[|\varphi_k|^{4p}] E[|\varphi_l|^{2p}] + 4 \sum_{|l| \leq N, l \neq k} E[|\varphi_k|^{2p}] E[|\varphi_l|^{4p}] \\
(8) \quad &+ \sum_{1 \leq |m| \leq N, m \neq k, 2m \neq k} E[|\varphi_{k-m}|^{2p}] E[|\varphi_{k-2m}|^{4p}].
\end{aligned}$$

We first deal with  $p = 1$ . Set

$$D_N := \sum_{l: 0 < |l| \leq N} \int |\varphi_l|^2 d\mu_a.$$

The above discussion implies that

$$(9) \quad E_{\mu_a}[|B_k|^2] = 4D_N^2 E_{\mu_a}[|\varphi_k|^2] + 4D_N E_{\mu_a}[|\varphi_k|^4] + \mathcal{I}_{N,1}(k) + \mathcal{I}_{N,2}(k),$$

where

$$\begin{aligned}
\mathcal{I}_{N,1}(k) &= E_{\mu_a}[|\varphi_k|^6] + 4E_{\mu_a}[|\varphi_k|^2] \sum_{0 < |l| \leq N, l \neq k} E_{\mu_a}[|\varphi_l|^4] \\
&+ \sum_{0 < |m| \leq N} E_{\mu_a}[|\varphi_{k-m}|^4] E[|\varphi_{k-2m}|^2], \\
\mathcal{I}_{N,2}(k) &= 2 \sum_{0 < |m| \leq N, 0 < |l| \leq N, l+m \neq k, l+2m \neq k} E_{\mu_a}[|\varphi_{k-m}|^2] E_{\mu_a}[|\varphi_l|^2] E_{\mu_a}[|\varphi_{l+m}|^2].
\end{aligned}$$

In dimension 1, it is easy to see that  $\sup_N [D_N + \mathcal{I}_{N,1}(k) + \mathcal{I}_{N,2}(k)] < \infty$ .

In dimension 2, the identity (6) and the upper estimate (12) in Lemma 6.1 imply that for  $\beta < 0$  we have

$$\sup_N \sum_{0 < |k| \leq N} |k|^{2\beta} \mathcal{I}_{N,1}(k) < \infty,$$

and the upper estimate (11) implies

$$\sup_N \sum_{0 < |k| \leq N} |k|^{2\beta} \mathcal{I}_{N,2}(k) < \infty.$$

A similar (easier) computation, based on  $m = 0$  and  $l = k - m \neq k$ , implies that for  $\tilde{T}_k(l, m) = \varphi_{k-m} \varphi_{l+m} \overline{\varphi_l \varphi_k}$ , we have

$$(10) \quad \sum_{|l|, |m| \leq N} E_{\mu_a}[\operatorname{Re}(\tilde{T}_k(l, m))] = E[|\varphi_k|^4] + 2 \sum_{1 \leq |l| \leq N, l \neq k} E[|\varphi_k|^2] E[|\varphi_l|^2].$$

Using (9) and (10) we deduce that

$$\begin{aligned} E_{\mu_a}[|B_k(\varphi) - C_N \varphi_k|^2] &= \mathcal{I}_{N,1}(k) + 4D_N^2 E[|\varphi_k|^2] + 4D_N E[|\varphi_k|^4] \\ &\quad - 2C_N E[|\varphi_k|^4] - 2C_N E[|\varphi_k|^2] 2D_N + \mathcal{I}_{N,2}(k) + C_N^2 E[|\varphi_k|^2]. \end{aligned}$$

Therefore, if  $C_N = 2D_N$ , we have

$$E_{\mu_a}[|B_k(\varphi) - C_N \varphi_k|^2] = \mathcal{I}_{N,1}(k) + \mathcal{I}_{N,2}(k),$$

and, for  $\beta < 0$ ,

$$\sup_N \sum_{0 < |k| \leq N} |k|^{-2\beta} E_{\mu_a}[|B_k(\varphi) - C_N \varphi_k|^2] < \infty.$$

For more general integers  $p$ , we proceed recursively. Indeed, we have formally (to simplify computations, we use erroneously the independence of  $|B_k(\varphi) - C_N \varphi_k|^{2p}$  and  $|B_k|^2 - C_N^2 |\varphi_k|^2$ , but the argument can be made rigorous)

$$\begin{aligned} &E_{\mu_a}(|B_k(\varphi) - C_N \varphi_k|^{2p+2}) \\ &= E_{\mu_a}(|B_k(\varphi) - C_N \varphi_k|^{2p} [|B_k(\varphi)|^2 - 2C_N \operatorname{Re}(\varphi_k \overline{B_k}) + C_N^2 |\varphi_k|^2]) \\ &= E_{\mu_a}(|B_k(\varphi) - C_N \varphi_k|^{2p} [|B_k(\varphi)|^2 - C_N^2 |\varphi_k|^2]) \\ &= E_{\mu_a}(|B_k(\varphi) - C_N \varphi_k|^{2p} [\mathcal{I}_{N,1}(k) + \mathcal{I}_{N,2}(k)]) \\ &\leq C E_{\mu_a}(|B_k(\varphi) - C_N \varphi_k|^{2p}). \end{aligned}$$

This concludes the proof.  $\square$

Next, we turn to estimate the first derivative of  $:B:$ . We have

**Proposition 3.2.** *Let  $d = 1, 2$ . For  $a = 2$  and  $\beta < 0$ , the gradient of the vector field  $B$  for  $d = 1$  (resp.  $:B:$  for  $d = 2$ ) belongs to  $L_{\mu_a}^{2p}(H^\beta; H.S.(H^1, H^\beta))$  for any integer  $p \geq 1$ .*

*Proof.* As in the previous proof, we work at fixed  $N$  but we drop the subscript  $N$  in  $(:B:)^N$ . A straight-forward computation gives

$$\begin{aligned} D_{e_j} B_k(\varphi) &= \sum_{l,m} (\varphi_{l+m} \overline{\varphi_l} \delta_{j,k-m} + \varphi_{k-m} \overline{\varphi_l} \delta_{j,l+m} + \varphi_{l+m} \varphi_{k-m} \delta_{-j,l}) \\ &= \sum_{l \notin \{0, j-k\}} \varphi_{l+k-j} (2\overline{\varphi_l} + \varphi_{-l}). \end{aligned}$$

Thus, we have

$$|D_{e_j} B_k(\varphi)|^2 = \sum_{l, l' \notin \{0, j-k\}} \varphi_{l+k-j} (2\overline{\varphi_l} + \varphi_{-l}) \overline{\varphi_{l'+k-j}} (2\varphi_{l'} + \overline{\varphi_{l'}}).$$

Let us first consider the 1-dimensional case. Since

$$\|\nabla B(\varphi)(\hat{e}_j)\|_{H^\beta}^2 \leq c \sum_k \frac{|k|^{2\beta}}{|j|^2} \left| \sum_{l \notin \{0, j-k\}} \varphi_{l+k-j} (2\overline{\varphi_l} + \varphi_{-l}) \right|^2,$$

we obtain that

$$E_{\mu_a}[\|\nabla B(\varphi)\|_{H.S.(H^b, H^\beta)}^2] \leq c \sum_{k,j} \frac{|k|^{2\beta}}{|j|^2} E_{\mu_a} \left[ \sum_{l, l' \notin \{0, l+k-j\}} T_{j,k}(l, l') \right],$$

where  $T_{j,k}(l, l') = \varphi_{l+k-j}(2\overline{\varphi_l} + \varphi_{-l})\overline{\varphi_{l'+k-j}}(2\varphi_{l'} + \overline{\varphi_{-l'}})$ .

*Case 1:  $l = l'$ ,  $T_{j,k}(l, l) = |\varphi_{l+k-j}|^2(4|\varphi_l|^2 + 2\overline{\varphi_l}\varphi_{-l} + 2\varphi_{-l}\varphi_l + |\varphi_{-l}|^2)$ .*

- If  $k = j$ ,  $E_{\mu_a}[T_{j,j}(l, l)] = 4E_{\mu_a}[|\varphi_l|^4] + E_{\mu_a}[|\varphi_l|^2]E_{\mu_a}[|\varphi_{-l}|^2]$ .
- If  $k \neq j$  and  $2l+k-j \neq 0$ ,  $E_{\mu_a}[T_{j,k}(l, l)] = 4E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_l|^2] + E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_{-l}|^2]$ .
- If  $k \neq j$  and  $2l+k-j = 0$ ,  $E_{\mu_a}[T_{j,k}(l, l)] = 4E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_l|^2] + E_{\mu_a}[|\varphi_{l+k-j}|^4]$ .

*Case 2:  $l \neq l'$ .*

- If  $k = j$ ,  $T_{j,j}(l, l') = 4|\varphi_l|^2|\varphi_{l'}|^2 + 2|\varphi_{l'}|^2\varphi_l\varphi_{-l} + 2|\varphi_l|^2\overline{\varphi_{l'}\varphi_{-l'}} + \overline{\varphi_{l'}\varphi_{-l'}}$ . Therefore,  $E_{\mu_a}[T_{j,j}(l, l')] = 4E_{\mu_a}[|\varphi_l|^2]E_{\mu_a}[|\varphi_{l'}|^2]$ .
- If  $k \neq j$ ,  $E_{\mu_a}[T_{j,k}(l, l')] = 0$ , except if  $l+k-j = -l'$ . In that case we have  $E_{\mu_a}[T_{j,k}(l, l')] = E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_{-l}|^2]$  if  $l+k-j \neq -l$ , and  $E_{\mu_a}[|\varphi_{l+k-j}|^4]$  if  $l+k-j = -l$ .

For  $a = 2$  and  $\beta < 0$  we have

$$\sum_{k,j \neq 0} \frac{|k|^{2\beta}}{|j|^2} \left[ \sum_{l \neq 0} |l|^{-2a} + \sum_{l \notin \{0, j-k\}} |l|^{-a}|l+k-j|^{-a} + \sum_{l, l' \neq 0, l \neq l'} |l|^{-a}|l'|^{-a} \right] < \infty.$$

Next, we deal with the 2-dimensional case. We need to be more careful in this case and we begin by computing  $\|D_{e_j}B_k(\varphi)\|_{L^2_{\mu_a}}$ . As in the one-dimensional case, we have

$$\begin{aligned} \int |D_{e_j}B_k(\varphi)|^2 d\mu_a &= \delta_{j,k} \left( 4 \sum_{l \neq 0} E_{\mu_a}[|\varphi_l|^4] + \sum_{l \neq 0} E_{\mu_a}[|\varphi_l|^2]E_{\mu_a}[|\varphi_{-l}|^2] \right. \\ &\quad \left. + 4 \sum_{l, l' \neq 0, l \neq l'} E_{\mu_a}[|\varphi_l|^2]E_{\mu_a}[|\varphi_{l'}|^2] \right) \\ &+ 1_{\{j \neq k\}} \left( 4 \sum_{l \notin \{0, j-k\}} E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_l|^2] + 21_{\{j-k \text{ even}\}} E_{\mu_a}[|\varphi_{(k-j)/2}|^4] \right. \\ &\quad \left. + 2 \sum_{2l \notin \{0, j-k\}} E_{\mu_a}[|\varphi_{l+k-j}|^2]E_{\mu_a}[|\varphi_{-l}|^2] \right). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{J}_N(j, k) &= \delta_{j,k} \left( 4 \sum_{l \neq 0} E_{\mu_a} [|\varphi_l|^4] + \sum_{l \neq 0} E_{\mu_a} [|\varphi_l|^2] E_{\mu_a} [|\varphi_{-l}|^2] \right) \\ &\quad + 1_{\{j \neq k\}} \left( 4 \sum_{l \notin \{0, j-k\}} E_{\mu_a} [|\varphi_{l+k-j}|^2] E_{\mu_a} [|\varphi_l|^2] + 2 1_{\{j-k \text{ even}\}} E_{\mu_a} [|\varphi_{(k-j)/2}|^4] \right. \\ &\quad \left. + 2 \sum_{2l \notin \{0, j-k\}} E_{\mu_a} [|\varphi_{l+k-j}|^2] E_{\mu_a} [|\varphi_{-l}|^2] \right). \end{aligned}$$

Using (7), we deduce that

$$D_{e_j} (: B :)_k(\varphi) = D_{e_j} B_k(\varphi) - C_N \delta_{j,k}.$$

Therefore, since  $C_N = 2D_N$ ,

$$\int |D_{e_j} B_k(\varphi)|^2 d\mu_a = \mathcal{J}_N(j, k) + \delta_{j,k} (4D_N^2 - 2C_N 2D_N + C_N^2) = \mathcal{J}_N(j, k).$$

Using (6) and (11) in Lemma 6.1, we deduce that

$$\sup_N \sum_{j,k} \frac{|k|^{2\beta}}{|j|^2} \mathcal{J}_N(j, k) < \infty.$$

For more general integers  $p$ , we proceed recursively. Indeed, we have formally (as previously, we use erroneously the independence of  $|D_{e_j} B_k(\varphi)|^2$  and  $|D_{e_j} B_k(\varphi)|^{2p}$ , but the argument can be made rigorous)

$$\begin{aligned} E_{\mu_a} (|D_{e_j} B_k(\varphi)|^{2p+2}) &= E_{\mu_a} (|D_{e_j} B_k(\varphi)|^{2p} |D_{e_j} B_k(\varphi)|^2) \\ &= E_{\mu_a} (|D_{e_j} B_k(\varphi)|^{2p} \mathcal{J}_N(j, k)) \\ &\leq C E_{\mu_a} (|D_{e_j} B_k(\varphi)|^{2p}). \end{aligned}$$

This concludes the proof.  $\square$

Using the definition of the second derivatives and corresponding norms presented in section 2, we have the following result:

**Proposition 3.3.** *Let  $a = 2$ ,  $\beta < 0$ ; the second derivative of  $B$  belongs to  $L_{\mu_a}^{2p}(H^\beta, H.S.(H^1 \otimes H^1, H^\beta))$ , for any integer  $p \geq 1$ .*

*Proof.* A straight-forward computation gives

$$\|D_{\hat{e}_l} D_{\hat{e}_m} B(\varphi)\|_{H^\beta}^2 \lesssim \frac{1}{(|l||m|)^2} \sum_k |k|^{2\beta} (|\varphi_{l+m-k}|^2 + |\varphi_{l-m+k}|^2).$$

By definition of the Hilbert-Schmidt norm, this leads to

$$\|\nabla^2 B(\varphi)\|_{H.S.(H^1 \times H^1, H^\beta)}^2 \lesssim \sum_{l,m} \frac{1}{(|l||m|)^2} \sum_k |k|^{2\beta} (|\varphi_{l+m-k}|^2 + |\varphi_{l-m+k}|^2).$$

Using once more (6), we deduce that

$$E_{\mu_a} \|\nabla^2 B(\varphi)\|_{H.S.(H^1 \times H^1, H^\beta)}^2 \lesssim \sum_{l,m} \frac{1}{(|l||m|)^2} \sum_{k \neq l+m} |k|^{2\beta} \frac{1}{|l+m-k|^a}.$$

This last term converges when  $\beta < 0$  (see (11) in Lemma 6.1). As previously, the case for more general integers  $p$  can be made by induction.  $\square$

More generally, the following holds:

**Proposition 3.4.** *For any  $s \in \mathbb{N}$  and any integer  $p \geq 1$ ,  $\nabla^s : B$  : belong to  $L^p_{\mu_a}(H^\beta; H.S.(\otimes^s H^1, H^\beta))$  when  $a = 2$  and  $\beta < 0$ .*

#### 4. THE RENORMALISED MASS

We will consider level surfaces for the renormalised mass  $E$ , where

$$E(\varphi) = \sum_k (|\varphi_k|^2 - Z_k).$$

Let us recall that the mass is an invariant of the NLS flow. We choose  $Z_k$  such that  $E \in L^1_{\mu_a}$ , i.e.,  $Z_k = c_1 |k|^{-a}$  where  $c_p$  was defined in (6). As for  $B$ , this renormalisation is not needed in dimension 1 since the serie is convergent. We recall that  $W_\infty = \bigcap_{r,p} W_r^p(H^\beta, \mathbb{R})$ .

Our goal in this section is to show that  $E$  and  $\|\nabla E\|_{H.S.(H^l, \mathbb{R})}^{-1}$  belong to the space  $W_\infty$ , as it was defined before, namely for  $a = 2$  and  $l = 1$  (let us recall that the norm of this space depends on  $\mu_a$ ). As previously, our choice of parameters are linked with the fact that we want to work on the abstract Wiener space  $(H^\beta, H^1, \mu_2)$ ,  $\beta < 0$ . We begin by estimating  $E$ .

**Proposition 4.1.** *The mass  $E$  belongs to  $W_\infty$ .*

*Proof.* A straight-forward computation gives

$$\begin{aligned} & \int |E(\varphi)|^2 d\mu_a \\ &= \int \sum_k (|\varphi_k|^2 - Z_k)^2 d\mu_a + \sum_{k \neq l} \int |\varphi_k|^2 |\varphi_l|^2 d\mu_a - 2 \sum_{k \neq l} \int Z_l |\varphi_k|^2 d\mu_a + \sum_{k \neq l} Z_k Z_l \\ &= \int \sum_k |\varphi_k|^4 d\mu_a - 2 \int \sum_k Z_k |\varphi_k|^2 d\mu_a + \sum_k Z_k^2 \\ & \quad + \sum_{k \neq l} \int |\varphi_k|^2 |\varphi_l|^2 d\mu_a - 2 \sum_{k \neq l} \int Z_l |\varphi_k|^2 d\mu_a + \sum_{k \neq l} Z_k Z_l \\ &= \sum_k c_2 |k|^{-2a} - 2 \sum_k Z_k c_1 |k|^{-a} + \sum_k Z_k^2 \\ & \quad + \sum_{k \neq l} c_1 |k|^{-a} c_1 |l|^{-a} - 2 \sum_{k \neq l} Z_l c_1 |k|^{-a} + \sum_{k \neq l} Z_k Z_l \\ &= \sum (c_2 |k|^{-2a} - Z_k^2) \lesssim |k|^{-2a}. \end{aligned}$$

To deal with a general integer  $p \geq 2$ , using the independence of the  $\varphi_{k_j}, j = 1, \dots, p$  and an induction argument, we have

$$\begin{aligned}
\int |E(\varphi)|^p d\mu_a &= \int \sum_k (|\varphi_k|^2 - Z_k)^p d\mu_a + \sum_{k_1 \neq k_j} \int (|\varphi_{k_1}|^2 - Z_{k_1}) \Pi_{k_j} (|\varphi_{k_j}|^2 - Z_{k_j}) d\mu_a \\
&= \int \sum_k (|\varphi_k|^2 - Z_k)^p d\mu_a + \sum_{k_1 \neq k_j} \int |\varphi_{k_1}|^2 \Pi_{k_j} (|\varphi_{k_j}|^2 - Z_{k_j}) d\mu_a \\
&\quad - \sum_{k_1 \neq k_j} \int Z_{k_1} \Pi_{k_j} (|\varphi_{k_j}|^2 - Z_{k_j}) d\mu_a \\
&\lesssim \sum_k |k|^{-ap} + \sum_{k_1 \neq k_j} \int Z_{k_1} \Pi_{k_j} (|\varphi_{k_j}|^2 - Z_{k_j}) d\mu_a \\
&\quad - \sum_{k_1 \neq k_j} \Pi_{k_j} \int Z_{k_1} (|\varphi_{k_j}|^2 - Z_{k_j}) d\mu_a \\
&\lesssim \sum_k |k|^{-ap}.
\end{aligned}$$

Next, we have

$$\nabla E(\varphi)(e_k) = 2\text{Re}(\varphi_k).$$

Therefore, we deduce

$$\|\nabla E(\varphi)\|_{L_{\mu_a}^{2m}(H.S.(H^1, \mathbb{R}))}^{1/m} \leq c \sum_k (E_{\mu_a}[|k|^{-2m} |\varphi_k|^{2m}])^{1/m} \leq c(p) \sum_k \frac{1}{|k|^{a+2}} < \infty.$$

Finally, we also have

$$\|\nabla^2 E(\varphi)\|_{L_{\mu_2}^{2m}(H.S.(H^1 \otimes H^1, \mathbb{R}))}^{1/m} \leq C \sum_k |k|^{-4} < \infty.$$

□

We now turn to the estimate of  $\|\nabla E\|_{H.S.(H^1, \mathbb{R})}^{-1}$ .

**Proposition 4.2.** *The random variable  $\|\nabla E\|_{H.S.(H^1, \mathbb{R})}^{-1}$  belongs to the space  $W_\infty$ .*

*Proof.* By Chebycheff's inequality, we have, for all  $t > 0$  and  $\varepsilon > 0$ ,

$$\begin{aligned}
\mu_a\{\|\nabla E(\varphi)\|_{H.S.(H^1, \mathbb{R})}^2 \leq \varepsilon\} &\leq e^{t\varepsilon} E_{\mu_a} [e^{-t\|\nabla E(\varphi)\|_{H.S.(H^1, \mathbb{R})}^2}] \\
&\leq C e^{t\varepsilon} \prod_{k \neq 0} |k|^a \int_0^\infty e^{-4t|k|^{-2}r^2 - \frac{|k|^a}{2}r^2} r dr \\
&\leq C e^{t\varepsilon} \prod_{k \neq 0} \frac{1}{1 + 8t/|k|^{a+2}}.
\end{aligned}$$



On the other hand, we have, for some  $c > 0$ ,

$$\begin{aligned} \prod_{k \neq 0} \left(1 + \frac{8t}{|k|^{a+2}}\right) &\geq \prod_{k \neq 0, |k| \leq (8t)^{1/(a+2)}} \left(1 + \frac{8t}{|k|^{a+2}}\right) \\ &\geq 2^{(8t)^{\frac{1}{a+2}}} = e^{ct^{\frac{1}{a+2}}}. \end{aligned}$$

Therefore, we deduce that for all  $\varepsilon > 0$ ,

$$\mu_a \{ \|\nabla E(\varphi)\|_{H.S.(H^1, \mathbb{R})}^2 \leq \varepsilon \} \leq \inf_t e^{t\varepsilon - ct^{\frac{1}{a+2}}} = e^{-c\varepsilon^{\frac{1}{1-(a+2)}}}.$$

Thus, letting  $\|\nabla E(\varphi)\|_{H.S.(H^1, \mathbb{R})}^2 = X$  and using the equality  $\mu_a(1/|X| \geq \varepsilon) = \mu_a(|X| \leq 1/\varepsilon)$ , we have

$$\begin{aligned} E_{\mu_a} \left( \frac{1}{|X|} \right)^p &= \int_0^\infty p\varepsilon^{p-1} \mu_a \left( \frac{1}{|X|} \geq \varepsilon \right) d\varepsilon \\ &\leq \int_0^\infty p\varepsilon^{p-1} e^{-c\varepsilon^{\frac{1}{(a+2)-1}}} d\varepsilon < \infty \end{aligned}$$

when  $1 < a + 2$  and, in particular, for  $a = 2$ .  $\square$

## 5. THE NLS EQUATION AND INVARIANT MEASURES

Following [4], if we consider the renormalised NLS equation and its corresponding renormalised Hamiltonian, which is the limit in  $N$  of

$$H_N(u) = \int |\nabla u|^2 + \frac{1}{2} \int |u|^4 - 2a_N \int |u|^2 + a_N^2,$$

with  $a_N = 2 \sum_{k: |k| < N, k \neq 0} \frac{1}{|k|^2}$ . The corresponding Gibbs measure

$$d\mu(\varphi) = \lim_N e^{-\frac{1}{2} \int |u|^4 - 2a_N \int |u|^2 + a_N^2} d\mu_2(\varphi)$$

is invariant for the renormalised flow (4), and the measures  $e^{-H_N(\varphi)} \Pi d\varphi$  are invariant for the corresponding truncated equations. Moreover  $\mu$  has a Radon-Nikodym derivative w.r.t.  $\mu_2$  which belongs to all  $L^p$  spaces (all these statements were shown in [4]).

Therefore, by Theorem 2.1 and thanks to Propositions 4.1 and 4.2, we can repeat the construction of the surface measures on level sets  $(V_r)$  of the mass  $E$ , starting from this weighted Wiener measure  $\mu$  instead of  $\mu_2$ . Let us denote by  $\sigma^r$  this surface measure on the level set of renormalised mass equal to  $r$ .

In this section we prove the existence of a global solution of the renormalised cubic nonlinear Schrödinger equation in the two-dimensional torus in a weak sense and living in the level sets of the renormalised mass. More precisely, we have

**Theorem 5.1.** *For each  $r$  such that  $\rho(r) = \frac{d(E*\mu)}{dr} > 0$  there exists a probability space  $(\Omega, \mathbb{F}, P)$  and a random process  $u_\omega \in C(\mathbb{R}, V_r)$ ,  $\omega \in \Omega$ , such that for every  $t$*

$$(i) \quad u_t(\omega) = e^{itA}u_0(\omega) + \int_0^t e^{-i(t-s)A} : B :^* (u_s(\omega)) ds \quad P - a.s.$$

where  $: B :^*$  is a redefinition of  $: B :$  as defined in Section 2.4. Moreover the measure  $\sigma^r$  is invariant for the process, namely

$$(ii) \quad \int_{\Omega} f(u_t(\omega)) dP(\omega) = \int_{V_r} f d\sigma^r$$

for every cylindrical  $f$  functional and every  $t$ .

*Proof.* The proof is similar to the one of [8, Theorem 4.1] for the Euler equation.

First notice that, given the estimates of all Sobolev norms of the field  $: B :$  which were previously derived, one can consider a redefinition  $: B :^*$  on each level set  $V_r$ . By the definition of the surface measure,  $: B :^*$  belongs also to all Sobolev spaces with respect to this measure.

From the regularity of the finite dimensional approximations  $: B :^n$  for every  $n$ , we know that there exists a global solution  $v_t^n(\varphi)$  of the equation, written here in its integral form,

$$v_t^n(\varphi) = e^{itA}\varphi + \int_0^t e^{-i(t-s)A} : B :^n (v_s^n(\varphi)) ds, \varphi \in H^\beta.$$

For every cylindrical functional  $f$  and every  $h \in C^\infty(\mathbb{R})$  with compact support we have,

$$\begin{aligned} \int_{\mathbb{R}} h(r)\rho(r) \int_{V_r} (A- : B :^n)(\varphi), \nabla f(\varphi) >_{H^1} d\sigma^r(\varphi) dr \\ = \int_{H^\beta} < h((E(\varphi))(A- : B :^n)(\varphi), \nabla f(\varphi)) >_{H^1} d\mu(\varphi) \\ = - \int_{H^\beta} [h'(E(\varphi))(A- : B :^n), \nabla E >_{H^1}(\varphi)] f(\varphi) d\mu(\varphi) \\ = 0, \end{aligned}$$

where the last equality is a consequence of the invariance of the mass and, in particular,

$$\int_{H^\beta} < (A- : B :^n), \nabla(h(E)f) >_{H^1}(\varphi) d\mu(\varphi) = 0.$$

This shows the invariance of the surface measure with respect to the flows  $v_t^n$ .

Denote by  $Q_n^k$  the law of  $[v_t^n]^k$  in the space  $C(\mathbb{R}^+; \mathbb{R})$  endowed with the supremum norm, namely

$$Q_n^k(\Gamma) = \sigma^r(\{[v^n(\varphi)]^k \in \Gamma, \quad \Gamma \subset C(\mathbb{R}^+; \mathbb{R})\})$$

The index  $k$  stands here for component in the  $H^\beta$  basis  $\{\frac{1}{k^\beta} e_k\}$ .

On the space of probability measures over the real valued continuous path space we consider the weak topology. We show that, for  $w \in C(\mathbb{R}^+; \mathbb{R})$ ,

$$\lim_{R \rightarrow \infty} \sup_n Q_n^k(|w(0)| > R) = 0,$$

and that for every  $R > 0$  and  $T > 0$

$$\lim_{\delta \rightarrow 0} \sup_n Q_n^k\left(\sup_{0 \leq t \leq t' \leq T, t' - t \leq \delta} |w(t) - w(t')| \geq R\right) = 0.$$

For negative times the proof is analogous.

The first statement is simply due to the estimate

$$\begin{aligned} Q_n^k(|w(0)| > R) &\leq \frac{1}{R^2} \int |w(0)|^2 dQ_n^k \\ &= \frac{1}{R^2} \int |\varphi^k|^2 d\sigma^r(\varphi), \end{aligned}$$

where in the first inequality we used Chebycheff's inequality. Concerning the second one, we have

$$\begin{aligned} &Q_n^k\left(\sup_{0 \leq t \leq t' \leq T, t' - t \leq \delta} |w(t) - w(t')| \geq R\right) \\ &\leq \frac{1}{R} \int_{V_r} \sup | [v_t^n(\varphi) - v_{t'}^n(\varphi)]^k | d\sigma^r(\varphi) \\ &\leq \frac{1}{R} \sup \left( \int_{V_r} \left( |[e^{it'A}\varphi - e^{it'A}\varphi]^k| + \left| \left[ \int_0^t (e^{-i(t-s)A}) : B :^n (v_s^n(\varphi)) ] ds \right. \right. \right. \right. \\ &\quad \left. \left. \left. - \int_0^{t'} (e^{-i(t'-s)A}) : B :^n (v_s^n(\varphi)) ]^k \right| \right) d\sigma^r(\varphi) \right) \end{aligned}$$

Using equalities  $\frac{d}{dt} e^{itA}\varphi = itAe^{itA}\varphi$  and

$$\begin{aligned} &\frac{d}{dt} \int_0^t (e^{-i(t-s)A}) : B :^n (v_s^n(\varphi)) ds \\ &= -iAe^{-itA} \int_0^t e^{isA} : B :^n (v_s^n(\varphi)) ds + : B :^n (v_t^n(\varphi)), \end{aligned}$$

we derive

$$\begin{aligned}
& Q_n^k \left( \sup_{0 \leq t \leq t' \leq T, t' - t \leq \delta} |w(t) - w(t')| \geq R \right) \\
& \leq \frac{\delta^2}{R^2} c(k, T) \left( \int_0^T \int [ |e^{isA} \varphi|^k ]^2 ds d\sigma^r(\varphi) \right)^{\frac{1}{2}} \\
& \quad + \frac{\delta^2}{R^2} c(k, T) \left( \int_0^T \int_0^T [ |e^{-isA} e^{irA} \varphi|^k ]^2 ds dr d\sigma^r(\varphi) \right)^{\frac{1}{2}} \\
& \quad + \frac{\delta^2}{R^2} c(k, T) \left( \int_0^T \int |[: B^n : (\varphi)]^k|^2 dt d\sigma^r(\varphi) \right)^{\frac{1}{2}} \\
& \leq \frac{\delta c_r}{R^2} \int_{V_r} \int_0^T \left( \|\varphi\|_{H^\beta} + \|\ : B :^n (\varphi) \|_{H^\beta} \right) ds d\sigma^r(\varphi),
\end{aligned}$$

where we have denoted  $c_r$  a generic constant depending on  $r, k$  and  $T$ , and the last step is a consequence of the invariance of the process with respect to the surface measure.

The limit follows from the fact that, since the sequence  $\{ : B :^n \}_n$ , as well as that of their gradients converges to  $: B :$  in all  $L_\mu^p$  spaces, it also converges (to  $: B :^*$ ) in  $L_{\sigma^r}^p$ . Hence

$$Q_n^k \left( \sup_{0 \leq t \leq t' \leq T, t' - t \leq \delta} |w(t) - w(t')| \geq R \right) \leq \frac{\delta}{R^2} T c$$

for some constant  $c$ , independent of  $n$ . We conclude that the sequence of probability laws  $Q_n^k$  is tight on  $C(R^+; R)$ , and therefore that we can extract a subsequence converging weakly to a probability measure  $Q$ . By Skorohod's theorem there exists a probability space  $(\Omega, \mathbb{F}, P)$  and processes  $u_t^n(\omega) = \sum_k [u_t^n(\omega)]^k e_k, u_t(\omega)$  with laws  $Q_n^k$  and  $Q^k$  resp., such that  $u^n \rightarrow u$ .  $P$ -almost surely.

We now prove (ii).

For every cylindrical functional  $f$ , we have

$$\begin{aligned}
\int f(u_t(\omega)) dP(\omega) &= \lim_n \int f(u_t^n(\omega)) dP(\omega) \\
&= \lim_n \int_{V_r} f(v_t^n(\varphi)) d\sigma^r(\varphi) \\
&= \int f d\sigma^r.
\end{aligned}$$

For the last equality we have used the invariance of the measure  $\sigma^r$  for the flow  $v_t^n$ .

Recalling that  $\rho(r) = \frac{d(E * \mu_2)}{dr} > 0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^+} \psi(r) \rho(r) \int_{V_r} f(v_t^n(\varphi)) d\sigma^r(\varphi) dr &= \int \psi(E(\varphi)) f(v_t^n(\varphi)) d\mu(\varphi) \\ &= \int_{V_r} \psi(E(v_{-t}^n(\varphi))) f(\varphi) d\mu(\varphi) \\ &= \int_{V_r} \psi(E(\varphi)) f(\varphi) d\mu(\varphi) \\ &= \int_{\mathbb{R}^+} \psi(r) \rho(r) \int_{V_r} f(\varphi) d\sigma^r(\varphi) dr \end{aligned}$$

for every smooth function  $\psi$ . In particular the process  $u_t$  takes values in  $V_r$ . It remains to show (i). For every  $k$ , we get for  $\beta < 0$

$$\begin{aligned} &\int \int_0^t \left| \left[ e^{-i(t-s)A} : B :^n (u_t^n(\omega)) - \int_0^t e^{-i(t-s)A} : B :^* (u_t(\omega)) \right]^k \right| ds dP(\omega) \\ &\leq \int \int_0^T \left| \left[ e^{-i(t-s)A} \left( : B :^n (u_t^n) - : B :^* (u_t^n) \right) \right]^k \right| ds dP \\ &+ \int \int_0^T \left| \left[ e^{-i(t-s)A} \left( : B :^* (u_t^n) - : B :^* (u_t) \right) \right]^k \right| ds dP \\ &\leq c(k) \left( \int \int_0^T \left\| \left( : B :^n (u_t^n) - : B :^* (u_t^n) \right) \right\|_{H^\beta} ds dP \right. \\ &\left. + \int \int_0^T \left\| \left( : B :^* (u_t^n) - : B :^* (u_t) \right) \right\|_{H^\beta} ds dP \right) \end{aligned}$$

The first term converges to zero due to the invariance of the measure  $P$  and the  $(L^1)$  convergence of  $: B :^n$  to  $: B :^*$ . Concerning the second term, let  $\Lambda \subset H_\beta$  be a set of  $\sigma^r$  measure equal to zero such that  $: B :^*$  is continuous in  $\Lambda^c$ . Define

$$\Lambda_n = \{(t, \omega) \in [0, T] \times \Omega : u_t^n(t, \omega) \in C\},$$

and

$$\Lambda_\infty = \{(t, \omega) \in [0, T] \times \Omega : u_t(t, \omega) \in C\}.$$

We have,

$$\int_0^T \int \chi_\Lambda(u_t^n(\omega)) dP(\omega) dt = \int_0^T \int \chi_\Lambda(\varphi) d\sigma^r(\varphi) dt = 0$$

i.e, the measure of  $\Lambda_n$  (and, analogously, of  $\Lambda$ ), considering the Lebesgue measure in the time coordinate, is equal to zero.

On the other hand, since  $u^n$  converges to  $u$  almost surely, consider a set  $D \subset \Omega$  such that  $P(D^c) = 0$  where this convergence holds, in the space  $H^\beta$  and for every  $t \in [0, T]$ . Then the measure of  $[0, T] \times D^c$  is zero.

Using Egorov's theorem, we conclude that, for  $(t, \omega)$  in  $\Lambda_\infty^c \cap_n \Lambda_n^c \cap ([0, t] \times D^c)$ , and  $\beta < 0$

$$\int \int_0^T \left\| \left( : B :^* (u_t^n) - : B :^* (u_t) \right) \right\|_{H^\beta} ds dP \rightarrow 0,$$

which concludes the proof.  $\square$

## 6. APPENDIX

In this appendix, we show a convergence result for series that were used quite often in section 3.

**Lemma 6.1.** *Let  $d = 1, 2$ . For any  $\beta < 0$ , it holds that*

$$(11) \quad \sum_{0 \neq l, m \in \mathbb{Z}^2} \frac{1}{|l|^2} \frac{1}{|m|^2} \sum_{k \notin \{0, l+m\}} \frac{|k|^\beta}{|l+m-k|^2} < \infty,$$

and

$$(12) \quad \sum_{k \neq 0, m \notin \{k, k/2\}} \frac{|k|^\beta}{|k-m|^4 |k-2m|^2} < \infty.$$

*Proof.* It is immediate to see that the above series converge in dimension 1.

Let  $d = 2$ . We first prove (11). Using a change of variable, the convergence of the serie is equivalent to the one of

$$\sum_{0 \neq l, m \in \mathbb{Z}^2} \frac{1}{|l|^2} \frac{1}{|m|^2} \sum_{k \notin \{0, -(l+m)\}} \frac{|k+l+m|^\beta}{|k|^2}.$$

Let  $\lambda \in (0, 1)$ . We are going to split the sum in  $k$  into three parts.

- First, we take  $|k| \leq (1-\lambda)|l+m|$ ; this yields

$$\sum_{0 \neq k \in \mathbb{Z}^2, |k| \leq (1-\lambda)|l+m|} \frac{|k+l+m|^\beta}{|k|^2} \leq \lambda^\beta |l+m|^\beta \sum_{1 \leq n \leq (1-\lambda)|l+m|} \frac{2}{n} \leq C_\lambda |l+m|^{\beta/2}.$$

- Then, we consider  $|k| \geq (1+\lambda)|l+m|$ . In this case, we get

$$\begin{aligned} \sum_{0 \neq k \in \mathbb{Z}^2, |k| \geq (1+\lambda)|l+m|} \frac{|k+l+m|^\beta}{|k|^2} &\leq C_\lambda \sum_{0 \neq k \in \mathbb{Z}^2, |k| \geq (1+\lambda)|l+m|} |k|^{\beta-2} \\ &\leq C_\lambda |l+m|^\beta. \end{aligned}$$

- Finally, when  $(1-\lambda)|l+m| \leq |k| \leq (1+\lambda)|l+m|$ , i.e.,  $|k| \approx |l+m|$ , we have

$$\begin{aligned} \sum_{0 \neq k \in \mathbb{Z}^2, |k| \approx |l+m|} \frac{|k+l+m|^\beta}{|k|^2} &\leq C_\lambda |l+m|^{\beta/2} \sum_{0 \neq k \in \mathbb{Z}^2, |k| \approx |l+m|} |k|^{\beta/2-2} \\ &\leq C_\lambda |l+m|^{\beta/2}. \end{aligned}$$

Therefore, we have to upper estimate

$$\sum_{0 \neq l, m \in \mathbb{Z}^2, l+m \neq 0} \frac{1}{|l|^2} \frac{|l+m|^{\beta/2}}{|m|^2}.$$

A similar argument implies that

$$\sum_{0 \neq l, m \in \mathbb{Z}^2, l+m \neq 0} \frac{1}{|l|^2} \frac{|l+m|^{\beta/2}}{|m|^2} \leq C \sum_{0 \neq l \in \mathbb{Z}^2} \frac{|l|^{\beta/4}}{|l|^2} < \infty,$$

which completes the proof of (11).

A change of variables implies that

$$\sum_{k \notin \{0, m, 2m\}} \frac{|k|^\beta}{|k-m|^4 |k-2m|^2} = \sum_{k \notin \{0, -m, m\}} \frac{|k+m|^\beta}{|k|^4 |k-m|^2}.$$

A decomposition similar to the above one implies (12).  $\square$

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CEMSUL, FACULDADE DE CIÊNCIAS DA UNIVERSIDADE DE LISBOA, EDIFÍCIO C6,  
 PISO 1, CAMPO GRANDE 1749-016 LISBOA, PORTUGAL  
*Email address:* `jeanbaptiste.casteras@gmail.com`

GFM, DEP. DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, LISBOA, PORTUGAL  
*Email address:* `ana.cruzeiro@tecnico.ulisboa.pt`

SAMM, EA 4543, UNIVERSITÉ PARIS 1 PANTHÉON SORBONNE, 90 RUE DE TOLBIAC,  
 75634 PARIS CEDEX FRANCE AND LABORATOIRE DE PROBABILITÉS, STATISTIQUE  
 ET MODÉLISATION, UMR 8001, SORBONNE UNIVERSITÉ - UNIVERSITÉ PARIS CITÉ.  
*Email address:* `annie.millet@univ-paris1.fr`