

Improved S-variable results applied to the analysis of time-varying uncertain systems

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Abstract

Finite horizon performance analysis is addressed for time-varying uncertain linear systems. Results apply to state-space systems with matrices rational in both time and a scalar uncertainty. Performances include stability and L_2 induced gain like criteria over a given time interval and extend to usual stability and L_2 induced norm as the upper bound goes to infinity. Results are formalized in terms of linear matrix inequalities by applying the S-variable approach. Two improvements to this approach are proposed to reduce conservatism and deal with positive unbounded indeterminates such as time. Descriptor modeling is adopted and a lifting based methodology allows to build tractable results aiming at decreasing conservatism. *Keywords:* Robustness, LMIs, time varying-system, S-variables

1 Introduction

To address robustness with respect to parametric uncertainties the core issue usually boils down to proving that some function of indeterminates is negative (or positive) for all indeterminates constrained to lie in given bounded or unbounded sets. Solutions to these core problems rely on (a) modeling tools to manipulate the intricate functions of indeterminates and (b) relaxations that, often with some conservatism, reformulate the problem into a search for variables (certificates) that serve as solutions to numerically tractable optimization problems. [16] provides an overview of most of these relaxations leading to semi-definite programming (SDP) problems formalized as Linear Matrix Inequalities (LMIs). In this paper we consider an alternative approach which relies on differential-algebraic (descriptor) representations for the modeling and on the S-variable approach for SDP relaxation.

Descriptor modeling which combines differential and algebraic equations is a very flexible tool for modeling but also raises many specific questions for the existence and analysis of solutions, see [7] for the relevant literature. In the paper we consider a subclass of these descriptor models where existence and unicity of solutions is guaranteed by the fact that the models are built from ordinary state-space differential equations, but their flexibility is used to transform complicated rational dependent formulas into affine equations in the indeterminates. The approach is inspired by [2, 20] and offers an alternative to Linear-Fractional Transform (LFT) modeling [3, 5]. As illustrated in [12] it is a lossless alternative and provides models of smaller or equal dimensions. Size reduction occurs, in part, when considering multi-affine descriptor models inspired by [9], but as shown in [11] size reduction also leads to an increased conservatism of the relaxations. In the present paper we propose to slightly enlarge the size of the descriptor representations to separate indeterminates in independent rows. It does not augment the size of models compared to LFTs but provides even less conservative results than the previous affine non-decoupled modeling.

The S-variable approach [4] is applied for SDP relaxation. As demonstrated in [14] it is an alternative to the Sum-of-Squares approach [10, 8, 17]. We do not claim it to be better or even different in terms of the resulting SDPs, but we believe it brings some simplicity in the mathematical manipulations. In most of literature on S-variables the relaxations are built for indeterminates lying in convex bounded sets such as polytopes. In [14] we also considered

the case of indeterminates in the whole real axis. In the present paper we propose a new result for indeterminates constrained to be positive.

These descriptor modeling and S-variable relaxation tools are classically applied to systems with parametric uncertainties [4] but they are also applicable to non-linear systems [2, 20, 13] and time delay systems [6]. In the present paper they are applied for the first time to time-varying linear systems. This goes beyond the case of parameter-varying systems as in [15] where the time-varying feature lies in bounded parameters with bounded derivatives. In our case we exploit directly the time-dependency of the coefficients assumed rational in time t and prove finite-horizon quadratic properties as in [18]. In that last paper the modeling tool is LFT based and SDP relaxations rely on sum-of-squares. We propose here a descriptor modeling and S-variable relaxation variant.

The outline of the paper is as follows. A first section is dedicated to reminding some essentials of the S-variable approach and presents the two proposed enhancements concerning less conservative relaxations thanks to row decoupling of indeterminates and treatment of positive unbounded indeterminates. Section III is then dedicated to descriptor modeling of time-varying uncertain systems and includes a new time and uncertainty dependent lifting. Section IV provides the central result for the analysis of these time-varying systems in both the finite or infinite horizon case. Section V gives some illustrative academic examples. A final section concludes the paper and outlines some future perspectives.

Notation: $\mathbb{R}^{m \times p}$ denotes the set of $m \times p$ real matrices. I_n stands for the identity matrix of size n . $0_{m,p}$ is the zero matrix with m rows and p columns. When the size can be deduced from the context the subscripts are avoided. A^T is the transpose of the matrix A . $\{A\}^{\mathcal{H}}$ stands for the symmetric (Hermitian) matrix $\{A\}^{\mathcal{H}} = A + A^T$. For a matrix $M \in \mathbb{R}^{m \times p}$ of rank r , $M^\perp \in \mathbb{R}^{p \times (p-r)}$ stands for the matrix of maximal rank such that $MM^\perp = 0$, i.e. M^\perp spans the null space of M . A^\dagger stands for the Moore-Penrose of A . For full-column rank matrices $A^\dagger A = I$. $M(\succeq) \succ N$ is the matrix inequality stating that $M - N$ is symmetric positive (semi-)definite. A matrix inequality of the type $N(X) \succeq 0$ is said to be a linear matrix inequality (LMI for short), if $N(X)$ is affine in the decision variables X . Decision variables are highlighted using the blue color. Uncertainties δ are denoted in red color. In this paper only scalar real bounded $\delta \in [\underline{\delta} \ \bar{\delta}]$ and positive real unbounded $\delta \in [0 \ \infty[$ are considered. Solutions η of an uncertainty dependent constraint $M(\delta)\eta = 0$ are denoted in red color as they are δ -dependent.

2 S-variables

In this section we recall a result from the S-variable approach described in details in [4]. The result provides conservative LMI formulations to prove robust properties of the following type:

$$\eta^T \Upsilon(\delta) \eta \leq 0 \quad , \quad \forall \begin{cases} M(\delta) \eta = 0 \\ \delta \in \Delta \end{cases} \quad (1)$$

that is, a quadratic form defined by $\Upsilon(\delta)$ is negative for all vectors in the null space of the linear application $M(\delta)$ and the properties hold for all parameters δ in a given set Δ .

For this paper we shall consider the case of two scalar independent uncertain parameters $\delta = (\delta_1, \delta_2)$. δ_2 is assumed to be bounded $\delta_2 \in [\underline{\delta}_2 \ \bar{\delta}_2]$, while two cases are assumed for δ_1 : positive real $\delta_1 \in [0 \ \infty[$ and positive bounded $\delta_1 \in [0 \ \bar{\delta}_1]$. Cases with more than two uncertainties and other bounded/unbounded cases can be derived by easy extensions, at the expense of an increased complexity in notations.

$\Upsilon(\delta_1, \delta_2)$ is assumed to be a multi-affine matrix of the two scalar independent parameters δ_1, δ_2 . $M(\delta_1, \delta_2)$ is assumed to have the following structure

$$M(\delta_1, \delta_2) = \begin{bmatrix} M_0 \\ M_1(\delta_1) \\ M_2(\delta_2) \end{bmatrix}$$

where $M_1(\delta_1) = N_{01} + \delta_1 N_{11}$ and $M_2(\delta_2) = N_{02} + \delta_2 N_{12}$ are affine matrices of each parameter respectively. If there are no parameter independent rows then M_0 is an empty matrix and $M_0^\perp = I$.

Lemma 1 *If there exists four matrices $\underline{S}_1, \bar{S}_1, \underline{S}_2, \bar{S}_2$ such that the following four LMIs hold*

$$\begin{aligned}
M_0^{\perp T} \Upsilon(0, \underline{\delta}_2) M_0^{\perp} + \{(\underline{S}_1 M_1(0) + \underline{S}_2 M_2(\underline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \\
M_0^{\perp T} \Upsilon(\bar{\delta}_1, \underline{\delta}_2) M_0^{\perp} + \{(\underline{S}_1 M_1(\bar{\delta}_1) + \underline{S}_2 M_2(\underline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \\
M_0^{\perp T} \Upsilon(0, \bar{\delta}_2) M_0^{\perp} + \{(\bar{S}_1 M_1(0) + \bar{S}_2 M_2(\bar{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \\
M_0^{\perp T} \Upsilon(\bar{\delta}_1, \bar{\delta}_2) M_0^{\perp} + \{(\bar{S}_1 M_1(\bar{\delta}_1) + \bar{S}_2 M_2(\bar{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0
\end{aligned} \tag{2}$$

then the robust constraint (1) holds for all $\delta_1 \in [0 \bar{\delta}_1]$ and all $\delta_2 \in [\underline{\delta}_2 \bar{\delta}_2]$.

Proof Let $S_2(\delta_1) = (1 - \frac{\delta_1}{\bar{\delta}_1}) \underline{S}_2 + \frac{\delta_1}{\bar{\delta}_1} \bar{S}_2$. It satisfies $S_2(0) = \underline{S}_2$ and $S_2(\bar{\delta}_1) = \bar{S}_2$. By convexity of LMI conditions and since all matrices involved in the sum are affine in δ_1 , the two first inequalities in (2) imply for all $\delta_1 \in [0 \bar{\delta}_1]$:

$$\begin{aligned}
M_0^{\perp T} \Upsilon(\delta_1, \underline{\delta}_2) M_0^{\perp} \\
+ \{(\underline{S}_1 M_1(\delta_1) + S_2(\delta_1) M_2(\underline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0.
\end{aligned} \tag{3}$$

For the same reason, the last two inequalities in (2) imply

$$\begin{aligned}
M_0^{\perp T} \Upsilon(\delta_1, \bar{\delta}_2) M_0^{\perp} \\
+ \{(\bar{S}_1 M_1(\delta_1) + S_2(\delta_1) M_2(\bar{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0.
\end{aligned} \tag{4}$$

Let $S_1(\delta_2)$ be affine in δ_2 such that $S_1(\underline{\delta}_2) = \underline{S}_1$ and $S_1(\bar{\delta}_2) = \bar{S}_1$. By convexity of LMIs and since all matrices involved in the sum are affine in δ_2 , the inequalities (3) and (4) imply that

$$\begin{aligned}
M_0^{\perp T} \Upsilon(\delta_1, \delta_2) M_0^{\perp} \\
+ \{(S_1(\delta_2) M_1(\delta_1) + S_2(\delta_1) M_2(\delta_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0
\end{aligned} \tag{5}$$

for all parameters $\delta_1 \in [0 \bar{\delta}_1]$ and all $\delta_2 \in [\underline{\delta}_2 \bar{\delta}_2]$. As M_0^{\perp} is full column rank: $M_0^{\perp T} M_0^{\perp \dagger T} = I$. It implies for $\tilde{S}_1(\delta_2) = M_0^{\perp \dagger T} S_1(\delta_2)$ and $\tilde{S}_2(\delta_1) = M_0^{\perp \dagger T} S_2(\delta_1)$

$$M_0^{\perp T} \left(\begin{array}{c} \Upsilon(\delta_1, \delta_2) \\ + \{\tilde{S}_1(\delta_2) M_1(\delta_1) + \tilde{S}_2(\delta_1) M_2(\delta_2)\}^{\mathcal{H}} \end{array} \right) M_0^{\perp} \preceq 0$$

By definition of M_0^{\perp} , this inequality reads as

$$\eta^T (\Upsilon(\delta_1, \delta_2) + \{S_1(\delta_2) M_1(\delta_1) + S_2(\delta_1) M_2(\delta_2)\}^{\mathcal{H}}) \eta \leq 0$$

for all vectors in the null space of M_0 . As η is additionally constrained to lie in the null spaces of $M_1(\delta_1)$ and $M_2(\delta_2)$, that is for all $M(\delta_1, \delta_2)\eta = 0$, one gets $\eta^T \Upsilon(\delta_1, \delta_2)\eta \leq 0$. ■

Compared to existing literature on S-variables, the lemma has the following characteristics:

- It borrows from [4] the strategy of exploiting the case when M contains parameter independent rows, these rows do not need not generate S variables thus keeping the LMIs of reasonable dimensions. If not exploiting this feature (proved to be lossless in [4]), the inequality (5) would read as

$$\begin{aligned}
\Upsilon(\delta_1, \delta_2) + \\
\{S_0(\delta_1, \delta_2) M_0 + \tilde{S}_1(\delta_2) M_1(\delta_1) + \tilde{S}_2(\delta_1) M_2(\delta_2)\}^{\mathcal{H}} &\preceq 0
\end{aligned}$$

with many more decisions variables needed to define the three S-variables and with LMIs of larger size.

- It exploits, which is at our knowledge new, the assumption on the parameter-dependent rows of M to be functions of independent parameters. It allows to build parameter-dependent S variables, thus reducing the conservatism of the conditions. In classical S-variable approach the terms $S_1(\delta_2) M_1(\delta_1) + S_2(\delta_1) M_2(\delta_2)$ would be written as $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} M_1(\delta_1) \\ M_2(\delta_2) \end{bmatrix}$.

The new result allows to consider parameter-dependent S-variables.

Lemma 2 *If there exists five matrices $\underline{D} \succeq 0$, $\overline{D} \succeq 0$, $\underline{G} = -\underline{G}^T$, $\overline{G} = -\overline{G}^T$, S_2 such that the following two LMIs hold*

$$\begin{aligned} M_0^{\perp T} \Upsilon(\underline{\delta}_2) M_0^{\perp} + \{(\underline{S}_1 N_{01} + S_2 M_2(\underline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \\ M_0^{\perp T} \Upsilon(\overline{\delta}_2) M_0^{\perp} + \{(\overline{S}_1 N_{01} + S_2 M_2(\overline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \end{aligned} \quad (6)$$

where $\underline{S}_1 = M_0^{\perp T} N_{11}^T (\underline{G} - \underline{D})$ and $\overline{S}_1 = M_0^{\perp T} N_{11}^T (\overline{G} - \overline{D})$, then the robust constraint (1) holds for all $\delta_1 \in [0 \infty[$ and all $\delta_2 \in [\underline{\delta}_2 \overline{\delta}_2]$.

Proof First, notice that the unbounded case assumes that the $\Upsilon(\delta_2)$ matrix is independent of δ_1 . Second, recall that $N_{01} = M_1(\delta_1) - \delta_1 N_{11}$. With this in mind one gets

$$\begin{aligned} \{\overline{S}_1 N_{01} M_0^{\perp}\}^{\mathcal{H}} \\ = \{\overline{S}_1 M_1(\delta_1) M_0^{\perp}\}^{\mathcal{H}} - \{M_0^{\perp T} N_{11}^T (\overline{G} - \overline{D})(\delta_1 N_{11}) M_0^{\perp}\}^{\mathcal{H}} \end{aligned}$$

Because of the choice of matrices $\underline{D} \succeq 0$, $\underline{G} = -\underline{G}^T$ the last term is such that

$$\begin{aligned} \{M_0^{\perp T} N_{11}^T (\overline{G} - \overline{D})(\delta_1 N_{11}) M_0^{\perp}\}^{\mathcal{H}} \\ = M_0^{\perp T} N_{11}^T \{(\overline{G} - \overline{D}) \delta_1\}^{\mathcal{H}} N_{11} M_0^{\perp} \\ = -2\delta_1 M_0^{\perp T} N_{11}^T \overline{D} N_{11} M_0^{\perp} \preceq 0 \end{aligned}$$

for all positive $\delta_1 \geq 0$. Hence the equations (6) imply

$$\begin{aligned} M_0^{\perp T} \Upsilon(\underline{\delta}_2) M_0^{\perp} + \{(\overline{S}_1 M_1(\delta_1) + S_2 M_2(\underline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \\ M_0^{\perp T} \Upsilon(\overline{\delta}_2) M_0^{\perp} + \{(\underline{S}_1 M_1(\delta_1) + S_2 M_2(\overline{\delta}_2)) M_0^{\perp}\}^{\mathcal{H}} &\preceq 0 \end{aligned}$$

hold for all $\delta_1 \in [0 \infty[$. The remaining of the proof follows the same lines as for lemma 1. \blacksquare

Compared to existing literature on S-variables, the lemma is new. It provides a S-variable method to deal with unbounded positive uncertainties. It generalizes the result from [14] that considered the case of unbounded real parameters $\delta_1 \in \mathbb{R}$.

3 Time-varying uncertain systems

The S-variable approach as recalled in the previous section is applied for the analysis of time-varying uncertain systems $\dot{x} = A(t, \delta)x + B(t, \delta)w$ where t is time and δ is a scalar constant uncertain parameter. The matrix $A(t, \delta)$ is assumed to be rational of the two parameters t and δ . The system admits an equivalent affine representation:

$$\begin{aligned} M_{\uparrow}(t, \delta) \eta_{\uparrow} &= \begin{bmatrix} M_{\uparrow 0} \\ M_{\uparrow 1}(t) \\ M_{\uparrow 2}(\delta) \end{bmatrix} \eta_{\uparrow} \\ &= \begin{bmatrix} M_{\uparrow 01} & M_{\uparrow 02} & M_{\uparrow 03} \\ M_{\uparrow 11}(t) & M_{\uparrow 12}(t) & M_{\uparrow 13}(t) \\ M_{\uparrow 21}(\delta) & M_{\uparrow 22}(\delta) & M_{\uparrow 23}(\delta) \end{bmatrix} \begin{pmatrix} \pi_{\uparrow} \\ \dot{x} \\ x \end{pmatrix} = 0 \end{aligned} \quad (7)$$

where $\pi_{\uparrow} = (\pi^T \ w^T)^T$ contains the inputs w and some expanding vector π allowing to transform the rational representation $\dot{x} = A(t, \delta)x + B(t, \delta)w$ into the algebraic-differential affine descriptor model $M_{\uparrow}(t, \delta)\eta_{\uparrow} = 0$. The transformation is always possible for example by using the LFT representation as an intermediate step as shown in [11]. The representation is not unique. We assume it is well posed: π_{\uparrow} and \dot{x} are uniquely defined for given $M_{\uparrow}(t, \delta)$, x and w .

For the analysis of this system, we shall also consider a lifted state $x_L = L(t, \delta)x$ where $L(t, \delta)$ is a tall matrix rational in the parameters t and δ . For the same reason as upper, this rationally-dependent definition of x_L admits a descriptor affine representation

$$\begin{aligned} M_L(t, \delta) \eta_L &= \begin{bmatrix} M_{L0} \\ M_{L1}(t) \\ M_{L2}(\delta) \end{bmatrix} \eta_L \\ &= \begin{bmatrix} M_{L01} & M_{L02} & M_{L03} \\ M_{L11}(t) & M_{L12}(t) & M_{L13}(t) \\ M_{L21}(\delta) & M_{L22}(\delta) & M_{L23}(\delta) \end{bmatrix} \begin{pmatrix} \pi_L \\ x_L \\ x \end{pmatrix} = 0 \end{aligned} \quad (8)$$

and the derivative of the lifted state satisfies

$$M_L(t, \delta)\dot{\eta}_L + \dot{M}_L\eta_L = 0 \quad (9)$$

where \dot{M}_L is by construction a constant matrix. Combining the last three equations provides a descriptor representation of the lifted dynamics:

$$\hat{M}_\uparrow(t, \delta)\hat{\eta}_\uparrow = \begin{bmatrix} \hat{M}_{\uparrow 0} \\ \hat{M}_{\uparrow 1}(t) \\ \hat{M}_{\uparrow 2}(\delta) \end{bmatrix} \begin{pmatrix} \hat{\pi}_\uparrow \\ \hat{x} \\ \hat{x} \end{pmatrix} = 0 \quad (10)$$

where $\hat{\pi}_\uparrow = (\hat{\pi}_L^T \ \pi_L^T \ \pi_\uparrow^T)^T$, $\hat{x} = (x_L^T \ x^T)^T$ and the matrices defining \hat{M}_\uparrow have the following structure (t and δ dependence of the M_{1i} and M_{2i} sub-matrices are omitted for space saving reasons):

$$\begin{aligned} \hat{M}_{\uparrow 0} &= \begin{bmatrix} 0 & 0 & M_{\uparrow 01} & | & 0 & M_{\uparrow 02} & | & 0 & M_{\uparrow 03} \\ 0 & M_{L01} & 0 & | & 0 & 0 & | & M_{L02} & M_{L03} \\ M_{L01} & 0 & 0 & | & M_{L02} & M_{L03} & | & 0 & 0 \end{bmatrix} \\ \hat{M}_{\uparrow 1}(t) &= \begin{bmatrix} 0 & 0 & M_{\uparrow 11} & | & 0 & M_{\uparrow 12} & | & 0 & M_{\uparrow 13} \\ 0 & M_{L11} & 0 & | & 0 & 0 & | & M_{L12} & M_{L13} \\ M_{L11} & \dot{M}_{L11} & 0 & | & M_{L12} & M_{L13} & | & \dot{M}_{L12} & \dot{M}_{L13} \end{bmatrix} \\ \hat{M}_{\uparrow 2}(\delta) &= \begin{bmatrix} 0 & 0 & M_{\uparrow 21} & | & 0 & M_{\uparrow 22} & | & 0 & M_{\uparrow 23} \\ 0 & M_{L21} & 0 & | & 0 & 0 & | & M_{L22} & M_{L23} \\ M_{L21} & 0 & 0 & | & M_{L22} & M_{L23} & | & 0 & 0 \end{bmatrix} \end{aligned}$$

The solutions of the time-varying system are functions of the initial conditions $x(0)$ and the properties may be studied in terms of constraints on the final condition $x(T)$. We shall assume that these initial/final conditions are constrained by some affine descriptor form

$$\begin{aligned} M.(\delta)\eta &= \begin{bmatrix} M_{.0} \\ M_{.2}(\delta) \end{bmatrix} \eta \\ &= \begin{bmatrix} M_{.01} & M_{.02} & M_{.03} \\ M_{.21}(\delta) & M_{.22}(\delta) & M_{.23}(\delta) \end{bmatrix} \begin{pmatrix} \pi. \\ x(0) \\ x(T) \end{pmatrix} = 0 \end{aligned} \quad (11)$$

If no constraints are formulated $M. = M_{.0}$ is an empty matrix such that $M_{.0}^\perp = I$. The lifted dynamics are therefore constrained by descriptor boundary constraints

$$\hat{M}.\!(\delta)\hat{\eta}. = \begin{bmatrix} \hat{M}_{.0} \\ \hat{M}_{.2}(\delta) \end{bmatrix} \begin{pmatrix} \hat{\pi}. \\ \hat{x}(0) \\ \hat{x}(T) \end{pmatrix} = 0 \quad (12)$$

where $\hat{\pi}. = (\pi_L^T(0) \ \pi_L^T(T) \ \pi^T)^T$ and the matrices have the following structure $\hat{M}_{.0} = [\hat{M}_{.01} \ \hat{M}_{.02} \ \hat{M}_{.03}]$ where

$$\begin{aligned} \hat{M}_{.01} &= \begin{bmatrix} 0 & 0 & M_{.01} \\ M_{L01} & 0 & 0 \\ M_{L11}(0) & 0 & 0 \\ 0 & M_{L01} & 0 \\ 0 & M_{L11}(T) & 0 \end{bmatrix} \\ \hat{M}_{.02} &= \begin{bmatrix} 0 & M_{.02} \\ M_{L02} & M_{L03} \\ M_{L12}(0) & M_{L13}(0) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \hat{M}_{.03} &= \begin{bmatrix} 0 & M_{.03} \\ 0 & 0 \\ 0 & 0 \\ M_{L02} & M_{L03} \\ M_{L12}(T) & M_{L13}(T) \end{bmatrix} \end{aligned}$$

and $\hat{M}_{.2}(\delta) =$

$$\begin{bmatrix} 0 & 0 & M_{.21} & | & 0 & M_{.22} & | & 0 & M_{.23} \\ M_{L21} & 0 & 0 & | & M_{L22} & M_{L23} & | & 0 & 0 \\ 0 & M_{L21} & 0 & | & 0 & 0 & | & M_{L22} & M_{L23} \end{bmatrix}.$$

Notice that the lifting procedure has no influence neither on the dynamics and nor on the boundary constraints of the original system. The lifted state x_L is driven by x but there is no influence of x_L on x . A property, such as stability, proved for the lifted system would hence be inherited by the original system.

4 Finite horizon integral quadratic performances

The aim is to prove finite-horizon integral quadratic performances defined by

$$\langle \eta_{\uparrow} | \Psi_{\uparrow} \eta_{\uparrow} \rangle + \eta^* \Psi \cdot \eta \leq 0 \quad (13)$$

where the notation defines a finite-horizon integral

$$\langle \eta_{\uparrow} | \Psi_{\uparrow} \eta_{\uparrow} \rangle = \int_0^T \eta_{\uparrow}^*(t) \Psi_{\uparrow}(t) \eta_{\uparrow}(t) dt.$$

Examples of such performances are

- Bounded final state / bounded initial state (also called finite-time stability in [1])

$$\Psi_{\uparrow} = 0, \quad \Psi \cdot = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & I \end{bmatrix}$$

For this choice the performance reads as $\|x(T)\|^2 \leq \alpha \|x(0)\|^2$ where $\|x(0)\|$ stands for the euclidian norm of the initial conditions.

- Bounded induced L_2 norm

$$\Psi_{\uparrow} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -\alpha I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C^T C \end{array} \right], \quad \Psi \cdot = 0$$

For this choice the performance reads as $\|z\|^2 \leq \alpha \|w\|^2$ where $\|w\|^2 = \langle w|w \rangle$ is the finite-horizon L_2 norm of the input w and $\|z\|^2 = \langle z|z \rangle$ is the L_2 norm of the output $z = Cx$. Such performance is usually studied assuming zero initial conditions (modeled as a constraint on the vector η at $t = 0$).

- Bounded L_2 norm of the output / bounded initial conditions:

$$\Psi_{\uparrow} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^T C \end{bmatrix}, \quad \Psi \cdot = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With this choice the performance is $\|z\|^2 \leq \alpha \|x(0)\|^2$.

Many other combinations of performances combining initial, final conditions as well as L_2 norms can be readily derived from these examples.

Remark that, the performance trivially reads as follows for the lifted system

$$\langle \hat{\eta}_{\uparrow} | \hat{\Psi}_{\uparrow} \hat{\eta}_{\uparrow} \rangle + \hat{\eta}^* \hat{\Psi} \cdot \hat{\eta} \leq 0 \quad (14)$$

where the $\hat{\Psi}$ matrices are obtained from Ψ by adding rows and columns of zeros.

The following result is formalized for the lifted system but applies readily to the original system when removing the $\hat{\cdot}$ notation.

Define the following matrices which are functions of a time and uncertainty dependent symmetric matrix $\hat{P}(t, \delta)$:

$$\hat{\Theta}_\uparrow(t, \delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{P} \\ 0 & \hat{P} & \dot{\hat{P}} \end{bmatrix}, \quad \hat{\Theta}_\cdot(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{P}(0, \delta) & 0 \\ 0 & 0 & -\hat{P}(T, \delta) \end{bmatrix}.$$

Theorem 1 *If there exists $\hat{P}(t, \delta)$ such that the following two robust conditions hold*

$$\hat{\eta}_\uparrow^T (\hat{\Psi}_\uparrow + \hat{\Theta}_\uparrow(t, \delta)) \hat{\eta}_\uparrow \leq 0, \quad \forall \begin{cases} \hat{M}_\uparrow(t, \delta) \hat{\eta}_\uparrow = 0 \\ t \in [0, T] \\ \delta \in [\underline{\delta}, \bar{\delta}] \end{cases} \quad (15)$$

$$\hat{\eta}_\cdot^T (\hat{\Psi}_\cdot + \hat{\Theta}_\cdot(\delta)) \hat{\eta}_\cdot \leq 0, \quad \forall \begin{cases} \hat{M}_\cdot(\delta) \hat{\eta}_\cdot = 0 \\ \delta \in [\underline{\delta}, \bar{\delta}] \end{cases} \quad (16)$$

then the performance (13) holds along the trajectories of the system described by (7) and (11).

Proof of Theorem 1

First let us prove the performance (14) holds for the lifted system defined by (10) and (12). Let $V = \hat{x}^T \hat{P} \hat{x}$. It is such that

$$\langle \hat{\eta}_\uparrow | \hat{\Theta}_\uparrow \hat{\eta}_\uparrow \rangle = \int_0^T \dot{V} dt = -\hat{\eta}_\cdot^T \hat{\Theta}_\cdot \hat{\eta}_\cdot.$$

therefore one has

$$\langle \hat{\eta}_\uparrow | (\hat{\Psi}_\uparrow + \hat{\Theta}_\uparrow) \hat{\eta}_\uparrow \rangle + \hat{\eta}_\cdot^T (\hat{\Psi}_\cdot + \hat{\Theta}_\cdot) \hat{\eta}_\cdot = \langle \hat{\eta}_\uparrow | \hat{\Psi}_\uparrow \hat{\eta}_\uparrow \rangle + \hat{\eta}_\cdot^T \hat{\Psi}_\cdot \hat{\eta}_\cdot.$$

As conditions (15), (16) hold, the two terms of the sum in the left hand side of this inequality are negative along the trajectories of the lifted system. Thus proving that the performance (14) holds subject to (10) and (12).

Now, consider the original system. First, notice that

$$V = x^T P x \text{ where } P = \begin{bmatrix} L \\ I \end{bmatrix}^T \hat{P} \begin{bmatrix} L \\ I \end{bmatrix}; \text{ and denoting}$$

$$\Theta_\uparrow(t, \delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & P \\ 0 & P & \dot{P} \end{bmatrix}, \quad \Theta_\cdot(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & P(0, \delta) & 0 \\ 0 & 0 & -P(T, \delta) \end{bmatrix},$$

simple manipulations give that (15) reads also as

$$\eta_\uparrow^T (\Psi_\uparrow + \Theta_\uparrow(t, \delta)) \eta_\uparrow \leq 0, \quad \forall \begin{cases} M_\uparrow(t, \delta) \eta_\uparrow = 0 \\ t \in [0, T] \\ \delta \in [\underline{\delta}, \bar{\delta}] \end{cases} \quad (17)$$

and (16) reads also as

$$\eta_\cdot^T (\Psi_\cdot + \Theta_\cdot(\delta)) \eta_\cdot \leq 0, \quad \forall \begin{cases} M_\cdot(\delta) \eta_\cdot = 0 \\ \delta \in [\underline{\delta}, \bar{\delta}] \end{cases} \quad (18)$$

i.e. the same conditions but for the original system and with a matrix P that is quadratic in the lifting L . This proves the performance for the original system by the following the same lines as above. ■

The robust conditions of the Theorem 1 are exactly in the form of conditions in Section 2 with $\Upsilon = \Psi + \Theta$. When T is bounded, choosing \hat{P} to be multi-affine in t and δ , the robust conditions of Theorem 1 can be proved feasible using LMIs from Lemma 1. When $T = \infty$, choosing \hat{P} to be independent of t and affine in δ , the robust conditions of Theorem 1 can be proved feasible using LMIs from Lemma 2. These two lemmas allow hence to solve conditions of Theorem 1 for the case of affine dependence of \hat{P} . The LMIs are conservative since the two lemmas are conservative.

Yet, let us remind that even if \hat{P} is a constant matrix, P is a rationally time and uncertainty dependent matrix. By lifting, the approach allows to build LMI conditions where the Lyapunov-like matrix P for the original system is rationally dependent in both t and δ . The choice of a basis in which to express P is equivalent to the choice of the lifting L . The conditions of Theorem 1 applied to the original system with rationally dependent P defined by some basis function is equivalent to searching for constant \hat{P} for the lifted system where L contains these basis functions.

5 Examples

5.1 LTI systems - stability

For a smooth start, consider an LTI system $\dot{x} = Ax$. Adopting the notations of the paper the system is modeled by $M_{\uparrow} = M_{\uparrow 0} = \begin{bmatrix} I & -A \end{bmatrix}$ for which $M_{\uparrow 0}^{\perp} = \begin{bmatrix} A \\ I \end{bmatrix}$. The performance to be achieved is bounded final state for bounded initial state. No constraints is set on these final and initial states, $M_{\cdot} = M_{\cdot 0}$ is empty and $M_{\cdot 0}^{\perp} = I$. The LMIs of Lemma 1 applied to conditions of Theorem 1 with a constant $P = \hat{P}$ read as:

$$\begin{bmatrix} A \\ I \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} = A^T P + P A \preceq 0$$

$$\begin{bmatrix} -\alpha I + P & 0 \\ 0 & I - P \end{bmatrix} \preceq 0 \Rightarrow I \preceq P \preceq \alpha I$$

which is fully coherent with Lyapunov theory.

5.2 LTI systems - L_2 induced norm

Consider now an LTI system with inputs $\dot{x} = Ax + Bw$. Adopting the notations of the paper the system is modeled by $M_{\uparrow} = M_{\uparrow 0} = \begin{bmatrix} -B & I & -A \end{bmatrix}$ for which $M_{\uparrow 0}^{\perp} = \begin{bmatrix} 0 & I \\ A & B \\ I & 0 \end{bmatrix}$. The performance to be achieved is in terms of induced L_2 norm for zero initial conditions defined by $M_{\cdot} = M_{\cdot 0} = \begin{bmatrix} I & 0 \end{bmatrix}$ for which $M_{\cdot 0}^{\perp} = \begin{bmatrix} 0 \\ I \end{bmatrix}$. The LMIs of Lemma 1 applied to conditions of Theorem 1 with a constant $P = \hat{P}$ read as:

$$\begin{bmatrix} 0 & I \\ A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & P \\ 0 & P & C^T C \end{bmatrix} \begin{bmatrix} 0 & I \\ A & B \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A^T P + P A + C^T C & P B \\ B^T P & -\alpha I \end{bmatrix} \preceq 0$$

$$\begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = -P \preceq 0$$

which is fully coherent with the H_{∞} norm LMIs.

5.3 Infinite horizon

Consider the following example from [19]

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k_0 & -c(t) \end{bmatrix} x = A(t)x$$

which is proved to be stable if $c(t) > 0$ and $\dot{c}(t) < 2k_0$. We consider the case $c(t) = 1 + \frac{1}{t+1}$ and $k_0 = 2$ which satisfies these conditions. The model (7) reads as

$$M_{\uparrow} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & -2 \\ \hline -t-1 & 0 & 0 & 1 \end{array} \right].$$

for the choice of expanding signal $\pi_{\uparrow} = \begin{bmatrix} 0 & \frac{1}{t+1} \end{bmatrix} x$. The performance to be achieved is bounded final state for bounded initial state for all final times T (*i.e.* stability) with a value $\alpha = 2^2$. No constraint is set on these final and initial states, $M_{\cdot} = M_{\cdot 0}$ is empty and $M_{\cdot 0}^{\perp} = I$. The LMIs of Lemma 2 applied to conditions of Theorem 1 with a constant $P = \hat{P}$ are found feasible. Stability is hence indeed proved.

5.4 Finite horizon

Consider the following academic example

$$\dot{x} = \begin{bmatrix} 1-t & 1 \\ \delta & t-2 \end{bmatrix} x = A(t, \delta)x$$

with $\delta \in [\underline{\delta} - 0.01]$. Bounded final state / bounded initial state performance is considered $\|x(T)\|^2 \leq \alpha \|x(0)\|^2$ with $\alpha = 2^2$. The matrix $A(0, -0.01)$ is not Hurwitz stable, neither is $A(t, -0.01)$ for $t \geq 2$. It is hence expected that only finite-horizon performances can be achieved. Meanwhile, $A(t, -0.01)$ is Hurwitz stable for some frozen values of t in $[1 \ 2]$ which indicates that after a diverging behavior at times close to zero, the trajectories may temporarily be converging to zero after $T = 1$. The bounded final state / bounded initial property is hence expected to be non monotone in T .

The model (7) is given by

$$M_{\uparrow} = \left[\begin{array}{c|cc|cc} 0 & -1 & 0 & 1-t & 1 \\ 1 & 0 & -1 & 0 & t-2 \\ \hline -1 & 0 & 0 & \delta & 0 \end{array} \right].$$

The system is from the start affine in both t and δ . The expanding signal $\pi_{\uparrow} = [\delta \ 0] x$ is used to achieve decoupling of the time and uncertainty dependent rows in M_{\uparrow} . As $M_{\uparrow 0}$ and $M_{\cdot} = M_{\cdot 0}$ are empty, $M_{\uparrow 0}^{\perp} = I$ and $M_{\cdot 0}^{\perp} = I$.

For fixed value of T the finite-horizon conditions are tested with respect to different liftings. By bisection we seek for the lowest value $\underline{\delta}$ such that the bounded final state / bounded initial state performance is attested robustly for all $\delta \in [\underline{\delta} \ 0]$. The liftings are

- Lifting=0, is the case without lifting
- Lifting=1, is for $L = tI$
- Lifting=2, is for $L = [tI \ t^2I]^T$
- Lifting=3, is for $L = [tI \ \frac{1}{t+1}I]^T$
- Lifting=4, is for $L = [tI \ \frac{1}{t+1}I \ t^2I]^T$

Notice that case 2 includes case 1 and is therefore expected to produce less conservative results. Case 3 includes case 1 and is hence also expected to be less conservative. Case 4 includes both case 2 and 3 and is hence expected to be the less conservative of all.

Results are plotted in Figure 1 and indeed show all the expected properties. Without lifting (case 0) the LMI conditions are feasible only up to $T = 0.35$. Case 1 not only reduces conservatism by proving properties for smaller values of $\underline{\delta}$ but also gives feasible LMIs up to $T = 0.65$. Further liftings do not reduce conservatism for values of $T \leq 0.8$. For final times $T \geq 0.8$ the conservatism reduction is seen by the gap between the curves. It also reads in the fact that for case 2 the LMIs become unfeasible after $T = 3.55$, for case 3 this occurs at $T = 3.7$ and case 4 gives feasible LMIs up to $T = 4s$.

Figures 2 and 3 plot the state trajectories for two choices of T and δ . The plot in Figure 2 is for $\delta = -19$, the trajectories starting on the unit circle are indeed all inside the disc of radius 2 at the final time $T = 1.8$ (bold stars) although the property was temporarily violated. The proof that the property holds is obtained by Lifting=4 conditions (stared point in Figure 1).

The plot in Figure 3 is for $\delta = -0.01$, the trajectories starting on the unit circle are indeed all inside the disc of radius 2 up to the final time $T = 4$ (bold stars). For final times T slightly greater than 4s some trajectories exit the disc of radius 2 thus indicating that the conservatims of lifting case 4 is close to exactness.

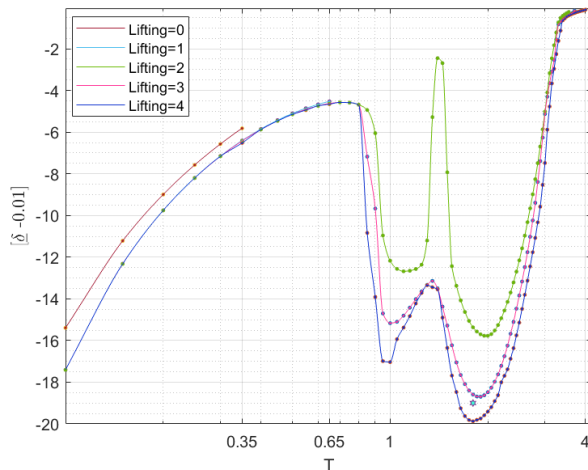


Figure 1: Minimal value of δ versus the final time T

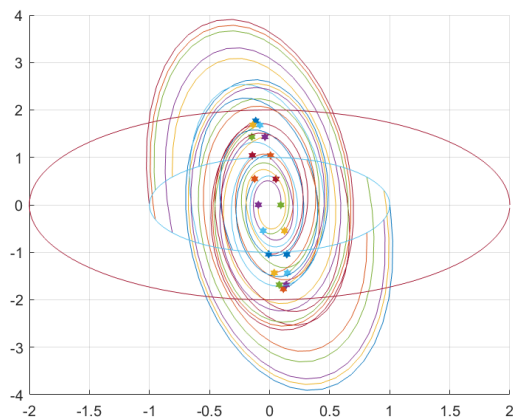


Figure 2: Trajectories for $\delta = -19$ and final time $T = 1.8$

6 Conclusions

The paper shows that the S-variable approach is also well suited to solve finite and infinite horizon performance problems for time-varying systems. A system lifting strategy allows to build sequences of semi-definite programs and results are tested on illustrative academic examples. Future work will study whether the series of lifted conditions can form hierarchies with decreasing conservatism with hopefully a proof of exactness at finite order of the lifting. Duality of SDPs is expected to contribute to this topic and at least provide worst case configurations. More involved examples shall be considered as well to test further the results.

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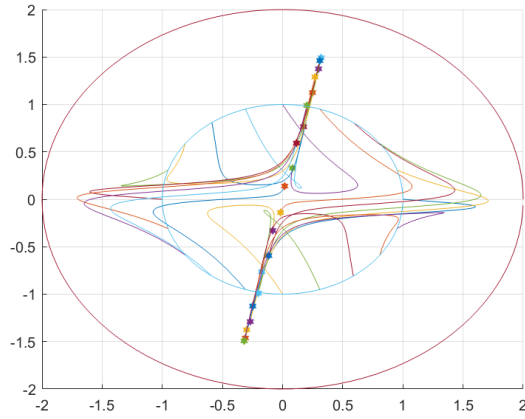


Figure 3: Trajectories for $\delta = -0.1$ and final time $T = 4$

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