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Stability of discontinuous diffusion coefficients for the heat equation on a star-shaped tree

Emmanuelle Crépeau*, Lionel Rosier † and Julie Valein ‡

February 5, 2025

Abstract

In this paper, we study the heat equation on a star-shaped tree network with a piecewise regular diffusion coefficient. By developing new Carleman estimates, we establish stability results for the identification of the diffusion coefficient. These stability estimates are derived using either internal measurements or boundary observations, offering robust insights into the inverse problem for this class of equations.

keywords network of partial differential equations, inverse problem, Carleman estimate, parabolic equation.

MSCcodes 35K05, 35R30, 93C20

1 Introduction

In this paper, we address inverse problems on a network formed by the edges of a tree. While our results are general and could apply to any tree structure, we focus on a star-shaped tree to simplify the presentation.

The inverse problem under consideration involves determining a discontinuous diffusion coefficient $c = c(x)$ in the heat equation

$$y_t - (c(x)y_x)_x = 0$$

using either internal or boundary measurements of the solution $y(x, t)$. This problem has been extensively studied in the literature (see e.g. [5, 6, 8, 9, 17, 18, 19]). The novelty of our approach lies in extending the analysis to cases where the spatial domain is defined on a tree network. Our objective is to derive stability estimates that provide "sharp" results with respect to the choice of measurements,

More specifically, we consider a star-shaped network \mathcal{R} consisting of $n + 1$ edges e_j , each of length $l_j > 0$, $j \in \llbracket 0, n \rrbracket$, all connected at a single vertex which we designate as the origin of all edges. For any function $f : \mathcal{R} \rightarrow \mathbb{R}$ we set

$f_j := f|_{e_j}$ the restriction of f to the edge e_j ,

$$[f]_0 := \sum_{j=0}^n f_j(0) \text{ the transmission bracket at the vertex } 0.$$

*Univ. Polytechnique Hauts-de-France, INSA Hauts-de-France, CERAMATHS - Laboratoire de Matériaux Céramiques et de Mathématiques, F-59313 Valenciennes, France. E-mail: emmanuelle.crepeau-jaisson@uphf.fr

†Université du Littoral Côte d'Opale, Laboratoire de Mathématiques Pures et Appliquées J. Liouville, B.P. 699, F-62228 Calais, France. E-mail: lionel.rosier@univ-littoral.fr

‡Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France. E-mail: julie.valein@univ-lorraine.fr

For any given $T > 0$, we define $Q := \mathcal{R} \times (0, T)$. On this planar 1-D network, we consider the heat equation with a variable diffusion coefficient, given by the following system

$$\begin{cases} y_{j,t}(x, t) - (c_j(x)y_{j,x}(x, t))_x = g_j(x, t), & j \in \llbracket 0, n \rrbracket, (x, t) \in (0, l_j) \times (0, T), \\ y_j(l_j, t) = h_j(t), & j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ y(x, 0) = y^0(x), & x \in \mathcal{R}, \end{cases} \quad (1)$$

together with the continuity condition and Kirchhoff law at the vertex 0, namely

$$y_j(0, t) = y_i(0, t) =: y(0, t), \quad \forall i, j \in \llbracket 0, n \rrbracket, t \in (0, T), \quad (2)$$

$$[cy_x(t)]_0 := \sum_{j=0}^n c_j(0)y_{j,x}(0, t) =: k(t), \quad t \in (0, T). \quad (3)$$

The diffusion coefficient c is assumed to be piecewise \mathcal{C}^1 , i.e. $c_i \in \mathcal{C}^1([0, l_i])$, $i \in \llbracket 0, n \rrbracket$, and

$$0 < c_{min} \leq c(x) \leq c_{max}, \quad x \in \mathcal{R}. \quad (4)$$

In the sequel, we shall use the following notations: $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$,

$$\begin{aligned} L^2(\mathcal{R}) &:= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in L^2(0, l_j), \forall j \in \llbracket 0, n \rrbracket\}, \\ H^1(\mathcal{R}) &:= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in H^1(0, l_j), f_j(0) = f_i(0), \forall i, j \in \llbracket 0, n \rrbracket\}, \\ H_0^1(\mathcal{R}) &:= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in H^1(0, l_j), f_j(l_j) = 0, f_j(0) = f_i(0), \forall i, j \in \llbracket 0, n \rrbracket\}. \end{aligned}$$

For the sake of shortness, for $f \in L^1(\mathcal{R}) := \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in L^1(0, l_j), \forall j \in \llbracket 0, n \rrbracket\}$, we write

$$\int_{\mathcal{R}} f dx := \sum_{j=0}^n \int_0^{l_j} f_j(x) dx.$$

The norms of the Hilbert spaces $L^2(\mathcal{R})$ and $H_0^1(\mathcal{R})$ are respectively defined by

$$\|f\|_{L^2(\mathcal{R})}^2 := \int_{\mathcal{R}} |f|^2 dx, \quad \|f\|_{H_0^1(\mathcal{R})}^2 := \int_{\mathcal{R}} |f_x|^2 dx.$$

We also denote by L the operator $L := \partial_t - \partial_x(c\partial_x)$ in Q and L^* its formal adjoint operator, namely $L^* := -\partial_t - \partial_x(c\partial_x)$ in Q .

Our first main result is concerned with stability estimates with internal measurements.

Theorem 1. *Let $T > 0$, $T' = T/2$, $\tau \in (0, T/2)$ and $\omega \subset \mathcal{R}$ be an open set with $\omega \cap (0, l_j) \neq \emptyset$ for $j \in \llbracket 1, n \rrbracket$. Let \tilde{c} be a piecewise \mathcal{C}^1 function satisfying (4) and let $h_j = 0$ for $j \in \llbracket 0, n \rrbracket$. Then for any $y^0 \in L^2(\mathcal{R})$, there exists $g \in C_c^\infty(\omega \times (0, T))$ such that for any piecewise \mathcal{C}^1 function c satisfying (4),*

$$c_i(0) = \tilde{c}_i(0) \quad \forall i \in \llbracket 0, n \rrbracket, \quad \text{and} \quad \max_{j \in \llbracket 0, n \rrbracket} \|c_j\|_{W^{1,\infty}(0, l_j)} \leq c_{max}^{1,\infty},$$

if y and \tilde{y} denote the solutions of (1), (2), and (3) associated with c and \tilde{c} respectively, with $g, k \equiv 0$ and $\tilde{y}_0 = y_0$, then we have

$$\begin{aligned} \|c - \tilde{c}\|_{L^2(\mathcal{R})} &\leq C (\|y_t - \tilde{y}_t\|_{L^2(\omega \times (\tau, T))} + \|y_{tt} - \tilde{y}_{tt}\|_{L^2(\omega \times (\tau, T))} \\ &\quad + \|y_t(T') - \tilde{y}_t(T')\|_{L^2(\omega)} + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})} + \|y(T') - \tilde{y}(T')\|_{H^2(\omega)}). \end{aligned} \quad (5)$$

Note that internal measurements are required on only n of the $n + 1$ edges in the network.

Our second main result is concerned with stability estimates derived from boundary measurements.

Theorem 2. Let $T > 0$, $T' = T/2$, $\tau \in (0, T/2)$ and \tilde{c} be a piecewise \mathcal{C}^1 function satisfying (4). Then for any $y^0 \in L^2(\mathcal{R})$, there exist some functions $h_j \in C_c^1(0, T)$, $j \in \llbracket 0, n \rrbracket$ with $h_0 \equiv 0$, such that for any c piecewise \mathcal{C}^1 function satisfying (4),

$$\begin{aligned} c_i(0) &= \tilde{c}_i(0) & \forall i \in \llbracket 0, n \rrbracket, \\ c_i(l_i) &= \tilde{c}_i(l_i) & \forall i \in \llbracket 1, n \rrbracket, \\ \max_{j \in \llbracket 0, n \rrbracket} \|c_j\|_{W^{1,\infty}(0, l_j)} &\leq c_{max}^{1,\infty}, \end{aligned}$$

if y and \tilde{y} denote the respective solutions of (1), (2), and (3) associated with c and \tilde{c} respectively, with $g \equiv 0$, $k \equiv 0$ and $\tilde{y}^0 = y^0$, we have

$$\begin{aligned} \|c - \tilde{c}\|_{L^2(\mathcal{R})} &\leq C \left(\sum_{j=1}^n \left[\|y_{tx}(l_j) - \tilde{y}_{tx}(l_j)\|_{L^2(\tau, T)} + \|y_{ttx}(l_j) - \tilde{y}_{ttx}(l_j)\|_{L^2(\tau, T)} \right. \right. \\ &\quad \left. \left. + |y_{j,x}(l_j, T') - \tilde{y}_{j,x}(l_j, T')| \right] + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})} \right). \end{aligned}$$

Note that only boundary measurements on n exterior nodes for $n + 1$ edges of the network are required.

Theorem 1 provides a stability estimate for the discontinuous diffusion coefficients using measurements of the solution's derivative on open sets of n edges along with measurements of the solution over the entire network of $n + 1$ edges at time $T/2$. That theorem holds under the technical assumption that the diffusion coefficients are equal at the central node. Theorem 2 establishes a similar result, based on n boundary measurements of the solution's derivative under the assumption of the equality of diffusion coefficients at the central node and at n exterior nodes where the measurements are taken.

The key to obtain both theorems is a global Carleman estimate. In the context of parabolic equation with discontinuous diffusion coefficient, a Carleman estimate was shown in [10] to prove the exact controllability to trajectories and in [6] in the context of inverse problem, under a condition on the monotonicity of the coefficient. Without imposing such monotonicity, a stability result for identifying the diffusion coefficient in the one-dimensional case was achieved in [5]. Moreover, [19] established the Lipschitz stability of principal coefficients for a parabolic system, assuming partial knowledge of the coefficients on an open subset of the boundary. For coupled systems, [8] addressed the inverse problem of simultaneously identifying two discontinuous diffusion coefficients in a one-dimensional parabolic system using observations of only one component. Additionally, [9] provided stability results for reaction-diffusion equations, focusing on the simultaneous determination of the diffusion coefficient and the potential using internal observations.

To bypass additional assumptions on the discontinuous diffusion coefficients, we derive a Carleman estimate in one less derivative. This approach builds on prior work: [15] introduced Carleman estimates for hyperbolic operators in H^{-1} spaces, yielding Hölder stability results for inverse problems involving coefficient determination in acoustic equations with a single measurement. This idea was further explored in [11] for the controllability of parabolic systems and in [18] for the stability of discontinuous diffusion coefficients in the heat equation.

Concerning the inverse problem on networks, the only works we know consist of determining a potential for the wave equation in [2, 3, 4], for the heat and Schrödinger equations in [13], and for parabolic equations on networks with loop in [1]. To the best of our knowledge, no stability results currently exist for the identification of discontinuous diffusion coefficients in the heat equation on star-shaped networks.

Our contribution addresses this gap. Both theorems presented in this work rely on global classical Carleman estimate in $L^2(\mathcal{R})$ for a tree-shaped network. The primary challenge lies in designing appropriate Carleman weights to handle terms at the central node arising from integrations by parts. By leveraging this $L^2(\mathcal{R})$ Carleman estimate, we derive estimates with one fewer derivative in the source term through a null controllability problem. Subsequently, we

establish stability results for stationary problems under specific assumptions. Our main theorems require repeated use of these refined Carleman estimates and stability results, necessitating the preliminary construction of a control ensuring the validity of assumptions about the stationary solutions.

The paper is organized as follows. In Section 2, we provide some Carleman estimates in $L^2(\mathcal{R})$ with internal measurements, and next we derive some Carleman estimates with one fewer derivative. Finally, Section 3 is dedicated to the establishment of the stability estimates.

2 Carleman estimates

In this section, we will establish three Carleman estimates for both operators L and L^* defined on a star-shaped network \mathcal{R} .

2.1 Carleman estimates in $L^2(\mathbb{R})$ with interior observations

Let $\lambda > 0$. We define the weight functions

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \quad \eta(x, t) = \frac{e^{\lambda\bar{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}, \quad (6)$$

for all $(x, t) \in \mathcal{R} \times (0, T)$, where β and $\bar{\beta}$ are defined as in the following lemma.

Lemma 3. *Let $\omega_i \subset (0, l_i)$, $i \in \llbracket 1, n \rrbracket$, be n (nonempty) open sets and let $\omega := \cup_{i \in \llbracket 1, n \rrbracket} \omega_i$. Then there exist $(a_0, \dots, a_n, b_1, \dots, b_n, d) \in (\mathbb{R}_-)^{n+1} \times (\mathbb{R}_+)^{n+1}$ and $\bar{\beta} > 0$ such that, defining the weight function β on \mathcal{R} as*

$$\beta_0(x) = a_0x + d, \quad \text{for } x \in [0, l_0], \quad (7)$$

$$\beta_j(x) = a_jx^2 + b_jx + d, \quad \text{for } x \in [0, l_j], \quad j \in \llbracket 1, n \rrbracket, \quad (8)$$

then for all $i, j \in \llbracket 0, n \rrbracket$ and all $x \in [0, l_j]$, we have

$$\beta_i(0) = \beta_j(0), \quad (9)$$

$$0 < \frac{\bar{\beta}}{2} < \beta_j(x) < \bar{\beta}, \quad (10)$$

$$\beta'_j(l_j) < 0, \quad (11)$$

$$\beta'(x) \neq 0 \text{ on } \mathcal{R} \setminus \omega, \quad (12)$$

and the following $(n+1) \times (n+1)$ symmetric matrix

$$A_\beta := \begin{pmatrix} \beta'_0(0) + \beta'_1(0) & \beta'_0(0) & \cdots & \beta'_0(0) & -\beta'_0(0)[c\beta']_0 \\ & \beta'_0(0) + \beta'_2(0) & \ddots & \vdots & -\beta'_0(0)[c\beta']_0 \\ & & \ddots & \beta'_0(0) & \vdots \\ & & & \beta'_0(0) + \beta'_n(0) & -\beta'_0(0)[c\beta']_0 \\ & & & & \beta'_0(0)([c\beta']_0)^2 + [c^2(\beta')^3]_0 \end{pmatrix} \quad (13)$$

is positive definite, i.e.

$$\exists \alpha > 0, \forall \xi \in \mathbb{R}^{n+1}, (A_\beta \xi, \xi) \geq \alpha |\xi|^2. \quad (14)$$

Proof. First we fix $a_0 < 0$. The continuity condition (9) follows directly from the choice of the weights in (7)-(8). Second, the matrix A_β is positive definite if and only if all of its leading principal minors are positive. Using the definition (7)-(8), we get $\beta'_0(0) = a_0$ and $\beta'_j(0) = b_j$,

$j \in \llbracket 1, n \rrbracket$. We choose $b_j = b_1 > 0$ for all $j \geq 1$. Thus, the symmetric matrix A_β becomes

$$A_\beta := \begin{pmatrix} a_0 + b_1 & a_0 & \cdots & a_0 & -a_0 \left(a_0 c_0(0) + b_1 \sum_{j=1}^n c_j(0) \right) \\ & a_0 + b_1 & \ddots & \vdots & -a_0 \left(a_0 c_0(0) + b_1 \sum_{j=1}^n c_j(0) \right) \\ & & \ddots & a_0 & \vdots \\ & & & a_0 + b_1 & -a_0 \left(a_0 c_0(0) + b_1 \sum_{j=1}^n c_j(0) \right) \\ & & & & a_0 \left(a_0 c_0(0) + b_1 \sum_{j=1}^n c_j(0) \right)^2 + a_0^3 c_0^2(0) + b_1^3 \sum_{j=1}^n c_j^2(0) \end{pmatrix},$$

and to get the positivity of leading principal minors, we actually need

$$(ja_0 + b_1)b_1^{j-1} > 0, \quad \forall j \in \llbracket 1, n \rrbracket \quad (15)$$

$$\text{Det}(A_\beta) > 0. \quad (16)$$

On the one hand, choosing $b_1 > -na_0$ yields (15), since $b_1 > 0$. On the other hand, the function $\text{Det}(A_\beta) = P(a_0, b_1, c_0(0), \dots, c_n(0))$ is a polynomial in b_1 of degree $n+3$. More precisely, the leading order term in b_1 of $\text{Det}(A_\beta)$ comes from the product of the diagonal terms and is given by

$$b_1^{n+3} \sum_{j=1}^n c_j^2(0)$$

and choosing b_1 sufficiently large (positive) gives $P(a_0, b_1, c_0(0), \dots, c_n(0)) > 0$, which gives (16).

Third, we choose $a_j < 0$ such that (12) holds. Indeed, for all $j \in \llbracket 1, n \rrbracket$, $\beta'_j(x) = 2a_j x + b_j = 2a_j x + b_1$, and if $\omega_j \subset (p_j, q_j) \subset (0, l_j)$ we want that $-b_1/(2a_j) \in (p_j, q_j)$ which is equivalent to $-\frac{b_1}{2p_j} < a_j < -\frac{b_1}{2q_j} < -\frac{b_1}{2l_j}$. This directly implies (11).

Finally, it remains to verify (10). We clearly have $a_0 l_0 + d \leq \beta_0(x) \leq d$ and $a_j l_j^2 + b_1 l_j + d \leq \beta_j(x) \leq d - \frac{b_1^2}{4a_j}$ for $j \in \llbracket 1, n \rrbracket$. Therefore, (10) is equivalent to

$$\begin{cases} d < \bar{\beta} \\ d - \frac{b_1^2}{4a_j} < \bar{\beta}, & \forall j \in \llbracket 1, n \rrbracket \\ \bar{\beta} < 2(a_0 l_0 + d) \\ \bar{\beta} < 2d \\ \bar{\beta} < 2(a_j l_j^2 + b_1 l_j + d), & \forall j \in \llbracket 1, n \rrbracket. \end{cases}$$

Since $a_j < 0$ for $j \in \llbracket 0, n \rrbracket$, the conditions reduce to

$$\begin{cases} 1 - \frac{b_1^2}{4da_j} < \frac{\bar{\beta}}{d}, & \forall j \in \llbracket 1, n \rrbracket \\ \frac{\bar{\beta}}{d} < 2\left(\frac{a_0 l_0}{d} + 1\right) \\ \frac{\bar{\beta}}{d} < 2\left(\frac{a_j l_j^2 + b_1 l_j}{d} + 1\right), & \forall j \in \llbracket 1, n \rrbracket. \end{cases}$$

It is then sufficient to take d large enough so that

$$\max_{j \in \llbracket 1, n \rrbracket} \left(1 - \frac{b_1^2}{4da_j} \right) < \min \left\{ 2\left(\frac{a_0 l_0}{d} + 1\right), \min_{j \in \llbracket 1, n \rrbracket} 2\left(\frac{a_j l_j^2 + b_1 l_j}{d} + 1\right) \right\},$$

and next to pick any number

$$\nu \in \left(\max_{j \in \llbracket 1, n \rrbracket} \left(1 - \frac{b_1^2}{4da_j} \right), \min \left\{ 2\left(\frac{a_0 l_0}{d} + 1\right), \min_{j \in \llbracket 1, n \rrbracket} 2\left(\frac{a_j l_j^2 + b_1 l_j}{d} + 1\right) \right\} \right),$$

and to set $\bar{\beta} = \nu d$. \square

Theorem 4. Let $\tilde{\omega}_i \subset (0, l_i)$, $i \in \llbracket 1, n \rrbracket$, be n nonempty open sets and let $\tilde{\omega} := \cup_{i \in \llbracket 1, n \rrbracket} \tilde{\omega}_i$. Assume that φ and η are as in (6) with β as given by Lemma 3. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(T, \mathcal{R}, \tilde{\omega}, c_{\min}, c_{\max}, \beta)$ such that the following Carleman estimate holds

$$\begin{aligned} s\lambda^2 \iint_Q \varphi |u_x|^2 e^{-2s\eta} dxdt + s^3 \lambda^4 \iint_Q \varphi^3 |u|^2 e^{-2s\eta} dxdt + s^3 \lambda^3 \int_0^T |\varphi(0, t)|^3 |u(0, t)|^2 e^{-2s\eta(0, t)} dt \\ \leq C \left[s^3 \lambda^4 \int_0^T \int_{\tilde{\omega}} \varphi^3 |u|^2 e^{-2s\eta} dxdt + \iint_Q |u_t \pm (cu_x)_x|^2 e^{-2s\eta} dxdt \right. \\ \left. + s\lambda \int_0^T \varphi(0, t) |k(t)|^2 e^{-2s\eta(0, t)} dt + \frac{1}{s^2} \int_0^T |k_t(t)|^2 e^{-2s\eta(0, t)} dt \right], \quad (17) \end{aligned}$$

for $s \geq s_0$, $\lambda \geq \lambda_0$ and $u \in D(L) := \{u \in L^2(0, T; H_0^1(\mathcal{R})), Lu \in L^2(Q)\}$ if $\pm = -$, or $u \in D(L^*) := \{u \in L^2(0, T; H_0^1(\mathcal{R})), L^*u \in L^2(Q)\}$ if $\pm = +$, where $k \in H^1(0, T)$ is as in (3).

Proof. We only treat the case of the operator $L^* = -\partial_t - \partial_x(c\partial_x)$, the other case $(\partial_t - \partial_x(c\partial_x))$ being obtained as a direct consequence by performing the change of variable $t \rightarrow T - t$. Let $s > 0$ and $\lambda > 0$, and set $w = e^{-s\eta}u$ and $Pw = -e^{-s\eta}L^*(e^{s\eta}w)$. After computing Pw , we split the terms as follows:

$$Pw = P_1w + P_2w + Rw$$

where

$$P_1w := (cw_x)_x + s^2\lambda^2\varphi^2(\beta')^2cw + s\eta_t w, \quad (18)$$

$$P_2w := w_t - 2s\lambda\varphi c\beta'w_x - 2s\lambda^2\varphi c(\beta')^2w, \quad (19)$$

$$Rw := -s\lambda\varphi(c\beta')_x w + s\lambda^2\varphi c(\beta')^2w. \quad (20)$$

Therefore, $P_1w + P_2w = Pw - Rw$ and

$$\begin{aligned} \iint_Q |Pw - Rw|^2 dxdt &= \iint_Q |P_1w|^2 dxdt + \iint_Q |P_2w|^2 dxdt \\ &\quad + 2 \iint_Q P_1w P_2w dxdt. \end{aligned} \quad (21)$$

The main task in the proof will be to minimize the cross-terms in $P_1w P_2w$ by positive and dominant terms looking similar to the ones in the left-hand-side of (17) and negative boundary terms that will be moved to the right-hand-side of the estimate. For the sake of clarity, we divide the proof in several steps.

Step 1. Main calculations

We set $\langle P_1w, P_2w \rangle_{L^2(Q)} = \sum_{i,k=1}^3 I_{i,k}$ where $I_{i,k}$ is the integral over Q of the product of the i th-term

in P_1w and the k th-term in P_2w . Then after some integrations by parts, we obtain

$$I_{1,1} = \iint_Q (cw_x)_x w_t dxdt = - \int_0^T w_t(0, t) [cw_x]_0 dt,$$

since $w(x, 0) = w(x, T) = 0$ and $w_{j,t}(l_j, t) = 0$;

$$\begin{aligned} I_{1,2} &= s\lambda^2 \iint_Q |cw_x|^2 (\beta')^2 \varphi dxdt + s\lambda \iint_Q |cw_x|^2 \beta'' \varphi dxdt \\ &\quad - s\lambda \sum_{j=0}^n \int_0^T |c_j(l_j) w_{j,x}(l_j, t)|^2 \varphi_j(l_j, t) \beta'_j(l_j) dt + s\lambda \int_0^T [|cw_x|^2 \beta']_0 \varphi(0, t) dt; \end{aligned}$$

$$\begin{aligned}
I_{1,3} &= 2s\lambda^2 \iint_Q |cw_x|^2 (\beta')^2 \varphi \, dxdt + 2s\lambda^3 \iint_Q c^2 w w_x (\beta')^3 \varphi \, dxdt \\
&\quad + 2s\lambda^2 \iint_Q c w w_x \varphi (c(\beta')^2)_x \, dxdt + 2s\lambda^2 \int_0^T w(0,t) [c^2 w_x (\beta')^2]_0 \varphi(0,t) \, dt; \\
I_{2,1} &= -s^2 \lambda^2 \iint_Q c |w|^2 (\beta')^2 \varphi \varphi_t \, dxdt; \\
I_{2,2} &= 3s^3 \lambda^4 \iint_Q |cw|^2 (\beta')^4 \varphi^3 \, dxdt + s^3 \lambda^3 \iint_Q |w|^2 (c^2 (\beta')^3)_x \varphi^3 \, dxdt \\
&\quad + s^3 \lambda^3 \int_0^T [c^2 (\beta')^3]_0 w^2(0,t) \varphi^3(0,t) \, dt; \\
I_{2,3} &= -2s^3 \lambda^4 \iint_Q |cw|^2 \varphi^3 (\beta')^4 \, dxdt; \\
I_{3,1} &= -\frac{s}{2} \iint_Q |w|^2 \eta_{tt} \, dxdt; \\
I_{3,2} &= s^2 \lambda^2 \iint_Q c |w|^2 (\beta')^2 \eta_t \varphi \, dxdt - s^2 \lambda^2 \iint_Q c |w|^2 (\beta')^2 \varphi \varphi_t \, dxdt \\
&\quad + s^2 \lambda \iint_Q |w|^2 (c\beta')_x \varphi \eta_t \, dxdt + s^2 \lambda \int_0^T [c\beta']_0 w^2(0,t) \varphi(0,t) \eta_t(0,t) \, dt; \\
I_{3,3} &= -2s^2 \lambda^2 \iint_Q c |w|^2 \varphi (\beta')^2 \eta_t \, dxdt.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\langle P_1(w), P_2(w) \rangle_{L^2(Q)} &= 3s\lambda^2 \iint_Q |cw_x|^2 \varphi (\beta')^2 \, dxdt + s^3 \lambda^4 \iint_Q |cw|^2 \varphi^3 (\beta')^4 \, dxdt \\
&\quad - s\lambda \sum_{j=0}^n \int_0^T |c_j(l_j) w_{j,x}(l_j, t)|^2 \varphi_j(l_j, t) \beta'_j(l_j) \, dt + X + B \quad (22)
\end{aligned}$$

where X is the sum of the remaining interior terms and B is the sum of the trace terms at the central node.

From now on, $M > 0$ will be a generic constant depending only on the network \mathcal{R} , the time T , c_{\min} and c_{\max} and on β_0, \dots, β_n , but independent of s and λ .

Step 2. Boundary terms at the central node.

Lemma 5. *Under the hypothesis of Lemma 3 for the weight function β , the sum of the trace terms at the central node 0, denoted B in (22) satisfies*

$$\begin{aligned}
B &\geq \frac{\alpha}{2} s^3 \lambda^3 \int_0^T |\varphi(0,t)|^3 |w(0,t)|^2 \, dt - Ms\lambda \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} |k(t)|^2 \, dt \\
&\quad - \frac{M}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 \, dt \quad (23)
\end{aligned}$$

for s sufficiently large and any $\lambda \geq 1$.

Proof. Gathering all the terms at the central node 0, we get

$$B = B_{1,1} + B_{1,2} + B_{1,3} + B_{2,2} + B_{3,2}$$

where $B_{i,j}$ comes from $I_{i,j}$.

We denote by $B_{12,2}$ the sum of $B_{1,2}$ and $B_{2,2}$, i.e.

$$B_{12,2} = s\lambda \int_0^T [|cw_x|^2 \beta']_0 \varphi(0,t) dt + s^3 \lambda^3 [|c|^2 (\beta')^3]_0 \int_0^T |w(0,t)|^2 \varphi^3(0,t) dt.$$

Since $w = e^{-s\eta}u$ gives $w_x = s\lambda\varphi\beta'w + e^{-s\eta}u_x$, then

$$[cw_x]_0 = \sum_{j=0}^n c_j(0)w_{j,x}(0,t) = s\lambda\varphi(0)w(0)[c\beta']_0 + e^{-s\eta(0)}k(t), \quad (24)$$

where we wrote $\varphi(0)$, $w(0)$ and $\eta(0)$ instead of $\varphi(0,t)$, $w(0,t)$ and $\eta(0,t)$ for the sake of shortness. We can write

$$c_0(0)w_{0,x}(0) = s\lambda\varphi(0)w(0)[c\beta']_0 + e^{-s\eta(0)}k(t) - \sum_{j=1}^n c_j(0)w_{j,x}(0).$$

Using this expression, we thus obtain

$$\begin{aligned} [\beta' |cw_x|^2]_0 &= \beta'_0(0) \left(s\lambda\varphi(0)w(0)[c\beta']_0 + e^{-s\eta(0)}k(t) - \sum_{j=1}^n c_j(0)w_{j,x}(0) \right)^2 \\ &\quad + \sum_{j=1}^n \beta'_j(0) |c_j(0)w_{j,x}(0)|^2 \\ &= s^2\lambda^2\varphi(0)^2\beta'_0(0)w(0)^2([c\beta']_0)^2 \\ &\quad + \beta'_0(0)e^{-2s\eta(0)}|k(t)|^2 + \sum_{j=1}^n (\beta'_0(0) + \beta'_j(0)) |c_j(0)w_{j,x}(0)|^2 \\ &\quad + 2\beta'_0(0) \sum_{i,j=1\dots n, i<j} c_i(0)c_j(0)w_{i,x}(0)w_{j,x}(0) \\ &\quad - 2s\lambda\beta'_0(0)w(0)\varphi(0)[c\beta']_0 \sum_{j=1}^n c_j(0)w_{j,x}(0) \\ &\quad + 2s\lambda\beta'_0(0)w(0)\varphi(0)[c\beta']_0 e^{-s\eta(0)}k(t) - 2\beta'_0(0) \sum_{j=1}^n e^{-s\eta(0)}k(t)c_j(0)w_{j,x}(0). \end{aligned}$$

Therefore,

$$\begin{aligned} B_{12,2} &= s\lambda \int_0^T \varphi(0,t) (A_\beta W(t), W(t)) dt + s\lambda \int_0^T \varphi(0,t) \beta'_0(0) e^{-2s\eta(0,t)} |k(t)|^2 dt \\ &\quad + 2s^2\lambda^2 \int_0^T \beta'_0(0) w(0,t) \varphi^2(0,t) [c\beta']_0 e^{-s\eta(0,t)} k(t) dt \\ &\quad - 2s\lambda \sum_{j=1}^n \int_0^T \beta'_0(0) \varphi(0,t) e^{-s\eta(0,t)} k(t) c_j(0) w_{j,x}(0,t) dt, \end{aligned}$$

with $W(t) = ((c_j(0)w_{j,x}(0,t))_{j=1\dots n}, s\lambda\varphi(0,t)w(0,t))$ and A_β defined by (13).

Using Lemma 3 (see (14)), we get

$$\begin{aligned} B_{12,2} &\geq \alpha s^3 \lambda^3 \int_0^T |\varphi(0,t)|^3 |w(0,t)|^2 dt + \alpha s \lambda \sum_{j=1}^n \int_0^T \varphi(0,t) |c_j(0)w_{j,x}(0,t)|^2 dt \\ &\quad + s\lambda \int_0^T \varphi(0,t) \beta'_0(0) e^{-2s\eta(0,t)} |k(t)|^2 dt + 2s^2\lambda^2 \int_0^T \beta'_0(0) w(0,t) \varphi^2(0,t) [c\beta']_0 e^{-s\eta(0,t)} k(t) dt \\ &\quad - 2s\lambda \sum_{j=1}^n \int_0^T \beta'_0(0) \varphi(0,t) e^{-s\eta(0,t)} k(t) c_j(0) w_{j,x}(0,t) dt. \end{aligned}$$

By Young's inequality,

$$\begin{aligned} & \left| s^2 \lambda^2 \int_0^T \beta'_0(0) w(0, t) \varphi^2(0, t) [c\beta']_0 e^{-s\eta(0, t)} k(t) dt \right| \\ & \leq \epsilon s^3 \lambda^3 \int_0^T |w(0, t)|^2 |\varphi(0, t)|^3 dt + \frac{M}{\epsilon} s \lambda \int_0^T \varphi(0, t) e^{-2s\eta(0, t)} |k(t)|^2 dt \end{aligned}$$

and

$$\begin{aligned} & \left| s \lambda \sum_{j=1}^n \int_0^T \beta'_0(0) \varphi(0, t) e^{-s\eta(0, t)} k(t) c_j(0) w_{j,x}(0, t) dt \right| \\ & \leq \epsilon s \lambda \sum_{j=1}^n \int_0^T \varphi(0, t) |c_j(0) w_{j,x}(0, t)|^2 dt + \frac{M}{\epsilon} s \lambda \int_0^T \varphi(0, t) e^{-2s\eta(0, t)} |k(t)|^2 dt, \end{aligned}$$

for any $\epsilon > 0$.

Moreover, by (24) and the fact that $\varphi(x, 0)w(x, 0) = \varphi(x, T)w(x, T) = 0$, we have by integration by parts

$$\begin{aligned} |B_{1,1}| &= \left| s \lambda \int_0^T \varphi(0, t) w(0, t) w_t(0, t) [c\beta']_0 dt + \int_0^T e^{-s\eta(0, t)} w_t(0, t) k(t) dt \right| \\ &\leq \frac{s\lambda}{2} \left| \int_0^T \varphi_t(0, t) [c\beta']_0 w(0, t)^2 dt \right| \\ &\quad + \left| - \int_0^T w(0, t) e^{-s\eta(0, t)} k_t(t) dt + s \int_0^T w(0, t) k(t) \eta_t(0, t) e^{-s\eta(0, t)} dt \right|. \end{aligned}$$

Using Young's inequality and the facts that for $t \in (0, T)$,

$$T^2 |\varphi(0, t)| \geq 1, \quad (25)$$

$$|\varphi_t(0, t)| \leq T |\varphi(0, t)|^2 \leq T^3 |\varphi(0, t)|^3, \quad (26)$$

(since $\beta(0) \geq 0$) and

$$|\eta_t(0, t)| \leq \frac{T}{t^2(T-t)^2} e^{\lambda\bar{\beta}} \leq \frac{T}{t^2(T-t)^2} e^{2\lambda\beta(0)} = T |\varphi(0, t)|^2 = T |\varphi(0, t)|^{3/2} |\varphi(0, t)|^{1/2} \quad (27)$$

(see (10)), we obtain

$$\begin{aligned} |B_{1,1}| &\leq M \left((s\lambda T^3 + sT^2 + s^2 T^6) \int_0^T |\varphi(0, t)|^3 |w(0, t)|^2 dt \right. \\ &\quad \left. + \frac{1}{s^2} \int_0^T e^{-2s\eta(0, t)} |k_t(t)|^2 dt + s \int_0^T e^{-2s\eta(0, t)} |\varphi(0, t)| |k(t)|^2 dt \right). \end{aligned}$$

Using again (24), we have

$$\begin{aligned} B_{1,3} &= 2s^2 \lambda^3 \int_0^T w(0, t)^2 \varphi(0, t)^2 c_0(0) (\beta'_0(0))^2 [c\beta']_0 dt \\ &\quad - 2s\lambda^2 \sum_{j=1}^n \int_0^T c_0(0) c_j(0) (\beta'_0(0))^2 w(0, t) \varphi(0, t) w_{j,x}(0, t) dt \\ &\quad + 2s\lambda^2 \sum_{j=1}^n \int_0^T (\beta'_j(0))^2 c_j(0)^2 w_{j,x}(0, t) w(0, t) \varphi(0, t) dt \\ &\quad + 2s\lambda^2 \int_0^T (\beta'_0(0))^2 c_0(0) w(0, t) \varphi(0, t) e^{-s\eta(0, t)} k(t) dt, \end{aligned}$$

and then, using Young's inequality and (25), we obtain

$$|B_{1,3}| \leq M \left(s^2 \lambda^3 (T^2 + T^4) \int_0^T |w(0,t)|^2 |\varphi(0,t)|^3 dt + \lambda \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} |k(t)|^2 dt \right. \\ \left. + \lambda \sum_{j=1}^n \int_0^T |c_j(0) w_{j,x}(0,t)|^2 \varphi(0,t) dt \right).$$

Finally, by (27), we can prove that

$$|B_{3,2}| \leq MT s^2 \lambda \int_0^T |w(0,t)|^2 |\varphi(0,t)|^3 dt.$$

As $B = B_{12,2} + B_{1,1} + B_{1,3} + B_{3,2} \geq B_{12,2} - |B_{1,1}| - |B_{1,3}| - |B_{3,2}|$, we then obtain for $s \geq 1$, $\lambda \geq 1$ and $\epsilon > 0$,

$$B \geq ((\alpha - \epsilon) s^3 \lambda^3 - M(s\lambda T^3 + sT^2 + s^2 T^6 + s^2 \lambda^3 (T^2 + T^4) + T s^2 \lambda)) \int_0^T |w(0,t)|^2 |\varphi(0,t)|^3 dt \\ + (\alpha s \lambda - M \lambda - \epsilon s \lambda) \sum_{j=1}^n \int_0^T \varphi(0,t) |c_j(0) w_{j,x}(0,t)|^2 dt + s \lambda \int_0^T \varphi(0,t) \beta'_0(0) e^{-2s\eta(0,t)} |k(t)|^2 dt \\ - \left(\frac{M}{\epsilon} s \lambda + M s + M \lambda \right) \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} |k(t)|^2 dt - \frac{M}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 dt. \quad (28)$$

Consequently, by taking s sufficiently large and ϵ small enough, we get (23). \square

Step 3. Interior terms.

Lemma 6. *The sum X of the remaining interior terms of (22) satisfies*

$$|X| \leq M \left(\left(s^3 \lambda^3 + \frac{1}{\epsilon} s \lambda^4 \right) \iint_Q |w|^2 \varphi^3 dx dt + (s \lambda + \epsilon s \lambda^2) \iint_Q |w_x|^2 \varphi dx dt \right)$$

for some suitable constant $M = M(T) > 0$ and all $\epsilon > 0$, $\lambda \geq 1$ and $s \geq 1$.

Proof. Gathering all remaining interior terms of (22), we get

$$X = X_{1,2} + X_{1,3} + \tilde{X}_{2,1} + X_{2,2} + X_{3,1} + \tilde{X}_{3,2},$$

where

$$|X_{1,2}| = \left| s \lambda \iint_Q |c w_x|^2 \beta'' \varphi dx dt \right| \leq M s \lambda \iint_Q |w_x|^2 \varphi dx dt,$$

$$|\tilde{X}_{2,1}| = \left| -2s^2 \lambda^2 \iint_Q c |w|^2 (\beta')^2 \varphi \varphi_t dx dt \right| \leq M T s^2 \lambda^2 \iint_Q |w|^2 |\varphi|^3 dx dt,$$

(by an estimation similar to (26)),

$$|X_{2,2}| = \left| s^3 \lambda^3 \iint_Q |w|^2 (c^2 (\beta')^3)_x \varphi^3 dx dt \right| \leq M s^3 \lambda^3 \iint_Q |w|^2 |\varphi|^3 dx dt,$$

$$|X_{3,1}| = \left| \frac{s}{2} \iint_Q |w|^2 \eta_{tt} dx dt \right| \leq M T^2 s \iint_Q |w|^2 |\varphi|^3 dx dt,$$

and

$$\begin{aligned} |\tilde{X}_{3,2}| &= \left| s^2 \lambda^2 \iint_Q c |w|^2 (\beta')^2 \varphi \eta_t \, dx dt + s^2 \lambda \iint_Q |w|^2 (c \beta')_x \varphi \eta_t \, dx dt \right| \\ &\leq (MT s^2 \lambda^2 + MT s^2 \lambda) \iint_Q |w|^2 |\varphi|^3 \, dx dt \end{aligned}$$

since

$$|\eta_{tt}(x, t)| \leq MT^2 |\varphi(x, t)|^3$$

and

$$|\eta_t(x, t)| \leq MT |\varphi(x, t)|^2. \quad (29)$$

Moreover, using Young's inequality, the fact that

$$T^2 |\varphi(x, t)| \geq 1 \quad \forall (x, t) \in \mathcal{R} \times (0, T), \quad (30)$$

and $\lambda > 1$, we obtain

$$\begin{aligned} |X_{1,3}| &= \left| 2s \lambda^3 \iint_Q c^2 w w_x (\beta')^3 \varphi \, dx dt + 2s \lambda^2 \iint_Q c w w_x \varphi (c (\beta')^2)_x \, dx dt \right| \\ &\leq \frac{MT^4}{\epsilon} s \lambda^4 \iint_Q |w|^2 |\varphi|^3 \, dx dt + M \epsilon s \lambda^2 \iint_Q |w_x|^2 \varphi \, dx dt \end{aligned}$$

for any $\epsilon > 0$, which concludes the proof of Lemma 6. \square

Combining (21), (22), Lemma 5 and Lemma 6, we obtain

$$\begin{aligned} &\iint_Q |P_1 w|^2 \, dx dt + \iint_Q |P_2 w|^2 \, dx dt + 6s \lambda^2 \iint_Q |c w_x|^2 (\beta')^2 \varphi \, dx dt \\ &+ 2s^3 \lambda^4 \iint_Q |c w|^2 \varphi^3 (\beta')^4 \, dx dt - 2s \lambda \sum_{j=0}^n \int_0^T \varphi_j(l_j, t) |c_j(l_j) w_{j,x}(l_j, t)|^2 \beta'_j(l_j) \, dt \\ &+ \frac{\alpha}{2} s^3 \lambda^3 \int_0^T |\varphi(0, t)|^3 |w(0, t)|^2 \, dt \leq \iint_Q |Pw - Rw|^2 \, dx dt \\ &+ M \left(\left(s^3 \lambda^3 + \frac{1}{\epsilon} s \lambda^4 \right) \iint_Q |w|^2 \varphi^3 \, dx dt + (s \lambda + \epsilon s \lambda^2) \iint_Q |w_x|^2 \varphi \, dx dt \right) \\ &+ Ms \lambda \int_0^T \varphi(0, t) e^{-2s\eta(0,t)} |k(t)|^2 \, dt + \frac{M}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 \, dt. \quad (31) \end{aligned}$$

Using (4), (11) and (12), we then obtain

$$\begin{aligned} &\iint_Q |P_1 w|^2 \, dx dt + \iint_Q |P_2 w|^2 \, dx dt + s \lambda^2 \int_0^T \int_{\mathcal{R} \setminus \omega} |w_x|^2 \varphi \, dx dt + s^3 \lambda^4 \int_0^T \int_{\mathcal{R} \setminus \omega} |w|^2 \varphi^3 \, dx dt \\ &+ s^3 \lambda^3 \int_0^T |\varphi(0, t)|^3 |w(0, t)|^2 \, dt \\ &\leq M \left(\iint_Q |Pw - Rw|^2 \, dx dt + \left(s^3 \lambda^3 + \frac{1}{\epsilon} s \lambda^4 \right) \iint_Q |w|^2 \varphi^3 \, dx dt + (s \lambda + \epsilon s \lambda^2) \iint_Q |w_x|^2 \varphi \, dx dt \right. \\ &\quad \left. + s \lambda \int_0^T \varphi(0, t) e^{-2s\eta(0,t)} |k(t)|^2 \, dt + \frac{1}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 \, dt \right). \end{aligned}$$

By (20), we have

$$\begin{aligned} \iint_Q |Pw - Rw|^2 dxdt &\leq 2 \iint_Q |Pw|^2 dxdt + 2 \iint_Q |Rw|^2 dxdt \\ &\leq 2 \iint_Q |Pw|^2 dxdt + Ms^2\lambda^4 \iint_Q |w|^2 \varphi^3 dxdt. \end{aligned} \quad (32)$$

Consequently, we have

$$\begin{aligned} &\iint_Q |P_1w|^2 dxdt + \iint_Q |P_2w|^2 dxdt + s\lambda^2 \iint_Q |w_x|^2 \varphi dxdt + s^3\lambda^4 \iint_Q |w|^2 \varphi^3 dxdt \\ &\quad + s^3\lambda^3 \int_0^T |\varphi(0,t)|^3 |w(0,t)|^2 dt \\ &\leq M \left(\iint_Q |Pw|^2 dxdt + \left(s^3\lambda^3 + s^2\lambda^4 + \frac{1}{\epsilon}s\lambda^4 \right) \iint_Q |w|^2 \varphi^3 dxdt \right. \\ &+ (s\lambda + \epsilon s\lambda^2) \iint_Q |w_x|^2 \varphi dxdt + s\lambda \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} |k(t)|^2 dt + \frac{1}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 dt \\ &\quad \left. + s\lambda^2 \int_0^T \int_\omega |w_x|^2 \varphi dxdt + s^3\lambda^4 \int_0^T \int_\omega |w|^2 \varphi^3 dxdt \right). \end{aligned}$$

Taking ϵ small enough, λ sufficiently large and then s sufficiently large (which depends to λ and ϵ), we arrive to

$$\begin{aligned} &\iint_Q |P_1w|^2 dxdt + \iint_Q |P_2w|^2 dxdt + s\lambda^2 \iint_Q |w_x|^2 \varphi dxdt + s^3\lambda^4 \iint_Q |w|^2 \varphi^3 dxdt \\ &\quad + s^3\lambda^3 \int_0^T |\varphi(0,t)|^3 |w(0,t)|^2 dt \\ &\leq M \left(\iint_Q |Pw|^2 dxdt + s\lambda \int_0^T \varphi(0,t) e^{-2s\eta(0,t)} |k(t)|^2 dt + \frac{1}{s^2} \int_0^T e^{-2s\eta(0,t)} |k_t(t)|^2 dt \right. \\ &\quad \left. + s\lambda^2 \int_0^T \int_\omega |w_x|^2 \varphi dxdt + s^3\lambda^4 \int_0^T \int_\omega |w|^2 \varphi^3 dxdt \right). \end{aligned} \quad (33)$$

Step 4. Back to the variable u .

The facts that $u = e^{s\eta}w$ and $\eta_x = -\lambda\beta'\varphi$ give that

$$s\lambda^2\varphi e^{-2s\eta} |u_x|^2 \leq C \left(s\lambda^2\varphi |w_x|^2 + s^3\lambda^4 |\varphi|^3 |w|^2 \right)$$

and

$$s\lambda^2\varphi |w_x|^2 \leq C \left(s\lambda^2\varphi e^{-2s\eta} |u_x|^2 + s^3\lambda^4 |\varphi|^3 e^{-2s\eta} |u|^2 \right). \quad (34)$$

These estimates combined with (33) leads to

$$\begin{aligned}
& \iint_Q |P_1 w|^2 dxdt + \iint_Q |P_2 w|^2 dxdt + s\lambda^2 \iint_Q e^{-2s\eta} |u_x|^2 \varphi dxdt \\
& \quad + s^3 \lambda^4 \iint_Q e^{-2s\eta} |u|^2 \varphi^3 dxdt + s^3 \lambda^3 \int_0^T |\varphi(0, t)|^3 |u(0, t)|^2 e^{-2s\eta(0, t)} dt \\
& \leq M \left(\iint_Q e^{-2s\eta} |u_t + (cu_x)_x|^2 dxdt + s\lambda \int_0^T \varphi(0, t) e^{-2s\eta(0, t)} |k(t)|^2 dt \right. \\
& \quad \left. + \frac{1}{s^2} \int_0^T e^{-2s\eta(0, t)} |k_t(t)|^2 dt + s^3 \lambda^4 \int_0^T \int_\omega e^{-2s\eta} |u|^2 \varphi^3 dxdt \right. \\
& \quad \left. + s\lambda^2 \int_0^T \int_\omega e^{-2s\eta} |u_x|^2 \varphi dxdt \right).
\end{aligned}$$

To prove the Carleman estimate (17), it remains to absorb the last term in the previous inequality. To do that, following [10, Proof of (100)], we can show that, for $\omega \subset \tilde{\omega}$,

$$s\lambda^2 \int_0^T \int_\omega \varphi |u_x|^2 e^{-2s\eta} dxdt \leq M \left(s^3 \lambda^4 \int_0^T \int_{\tilde{\omega}} \varphi^3 |u|^2 e^{-2s\eta} dxdt + \iint_Q |u_t + (cu_x)_x|^2 e^{-2s\eta} dxdt \right),$$

which leads to the Carleman estimate (17).

2.2 Carleman estimates with one less derivative, with interior observations

Theorem 7. *Let us consider the following problem:*

$$\begin{cases} q_t - (cq_x)_x = f_x, & \text{in } Q, \\ q_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ q_i(0, t) = q_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [cq_x(t)]_0 = 0, & \forall t \in (0, T), \\ q(x, 0) = q_0(x), & \forall x \in \mathcal{R}. \end{cases} \quad (35)$$

Let $\omega_i \subset (0, l_i)$, $i \in \llbracket 1, n \rrbracket$, be n nonempty open sets, and let $\omega := \cup_{i \in \llbracket 1, n \rrbracket} \omega_i$. Assume that φ and η are as in (6), and that β is as given by Lemma 3. Assume that $q_0 \in L^2(\mathcal{R})$, $f \in L^2(0, T, H^1(\mathcal{R}))$ with $f(0, t) = 0$ for a.e. $t \in (0, T)$. Let s_0, λ_0 be as in Theorem 4, pick $\lambda \geq \lambda_0$, then there exist $\bar{s} \geq s_0 > 0$ and a positive constant $C = C(T, \mathcal{R}, \omega, c_{\min}, c_{\max})$ such that the following Carleman estimate holds

$$\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \leq C \left(\iint_Q s^2 \lambda^2 \varphi^2 e^{-2s\eta} |f|^2 dxdt + \int_0^T \int_\omega s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \right), \quad (36)$$

for $s \geq \bar{s}$ and q satisfying (35).

The proof relies on the original proof of Imanuvilov and Yamamoto [14], see also [15], to obtain a Carleman estimate for a parabolic operator with one less derivative in the source term. For simplicity, we follow the proofs given in [11] and [18].

Proof. Using integrations by parts and the fact that $f(0, t) = 0$, we see that the function q is solution of (35) in $L^2(Q)$ if and only if it satisfies for any $g \in L^2(Q)$

$$\iint_Q qg dxdt = - \iint_Q f z_x dxdt + \int_{\mathcal{R}} q_0 z(0) dx, \quad (37)$$

where $z \in D(L^*)$ is the solution of

$$\begin{cases} -z_t - (cz_x)_x = g, & \text{in } Q, \\ z_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \\ z_i(0, t) = z_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [cz_x(t)]_0 = 0, & \forall t \in (0, T), \\ z(x, T) = 0, & \forall x \in \mathcal{R}. \end{cases}$$

Let s and λ be as in Theorem 4 and let us introduce the following fourth order problem with unknown

$$p \in P_0 := \{u \in C^2(\bar{Q}); \quad u_j(l_j, t) = 0, \quad u_i(0, t) = u_j(0, t), \quad \forall i, j \in \llbracket 0, n \rrbracket, \quad [cu_x(t)]_0 = 0 \quad \forall t \in [0, T]\},$$

$$\begin{cases} L^*(e^{-2s\eta}Lp) + s^3\lambda^4\varphi^3e^{-2s\eta}p\mathbf{1}_\omega = s^3\lambda^4\varphi^3e^{-2s\eta}q, & \text{in } Q, \\ (e^{-2s\eta}Lp)_j(l_j, t) = 0 & \forall j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ (e^{-2s\eta}Lp)_i(0, t) = (e^{-2s\eta}Lp)_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [c(e^{-2s\eta}Lp)_x(t)]_0 = 0, & \forall t \in (0, T), \\ (e^{-2s\eta}Lp)(x, 0) = (e^{-2s\eta}Lp)(x, T) = 0, & \forall x \in \mathcal{R}, \end{cases} \quad (38)$$

where $q \in L^2(Q)$ is solution of (35).

We define on P_0^2 the bilinear form a and on P_0 the linear form b_q as follows,

$$\begin{cases} a(p, p') & := \iint_Q e^{-2s\eta}LpLp'dxdt + \int_0^T \int_\omega s^3\lambda^4\varphi^3e^{-2s\eta}pp'dxdt, \\ b_q(p') & := \iint_Q s^3\lambda^4\varphi^3e^{-2s\eta}qp'dxdt. \end{cases}$$

It is clear that a is a positive bilinear symmetric form on P_0^2 . As $P_0 \subset D(L)$, we can apply the Carleman estimate (17) to p to obtain

$$\iint_Q s^3\lambda^4\varphi^3e^{-2s\eta}|p|^2dxdt \leq Ca(p, p), \quad \text{for all } p \in P_0. \quad (39)$$

Thus $(P_0, \|\cdot\|_a)$, where $\|\cdot\|_a := \sqrt{a(\cdot, \cdot)}$, is a normed space.

Let $P := \bar{P}_0^{\|\cdot\|_a}$ be the completion of P_0 for the norm $\|\cdot\|_a$. Then P is a Hilbert space for the scalar product $a(\cdot, \cdot)$ and all functions in P satisfy inequality (39). Furthermore, thanks to Cauchy-Schwarz inequality and (39) we obtain,

$$|b_q(p)| \leq C^{1/2} \left(\iint_Q s^3\lambda^4\varphi^3e^{-2s\eta}|q|^2dxdt \right)^{1/2} \|p\|_a. \quad (40)$$

Thus b_q is a continuous linear form on P .

Then we can apply the Lax-Milgram Theorem and the following variational problem possesses one and only one solution $p \in P$:

$$a(p, p') = b_q(p'), \quad \forall p' \in P. \quad (41)$$

Using some test functions $p' \in P_0$, we obtain after integrations by parts that the solution p also solves the first equation in (38) in the distributional sense.

Let us set $\hat{u} := -s^3\lambda^4\varphi^3e^{-2s\eta}\mathbf{1}_\omega p$, $\hat{z} = e^{-2s\eta}Lp$. Using appropriate test functions in (41), we see that $\hat{z} \in D(L^*)$ and \hat{z} is solution of the null controllability problem,

$$\begin{cases} -\hat{z}_t - (c\hat{z}_x)_x = \hat{u} + s^3\lambda^4\varphi^3e^{-2s\eta}q, & \text{in } Q, \\ \hat{z}_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \\ \hat{z}_i(0, t) = \hat{z}_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [c\hat{z}_x(t)]_0 = 0, & \forall t \in (0, T), \\ \hat{z}(x, T) = 0, \quad \hat{z}(x, 0) = 0, & \forall x \in \mathcal{R}, \end{cases}$$

and using (37) applied to \hat{z} we have,

$$\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt = - \iint_Q f \hat{z}_x dxdt - \int_0^T \int_\omega q \hat{u} dxdt. \quad (42)$$

With (40) we have

$$a(p, p) = b_q(p) \leq C^{1/2} \left(\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \right)^{1/2} a(p, p)^{1/2}.$$

Thus

$$a(p, p) = \iint_Q e^{2s\eta} |\hat{z}|^2 dxdt + \int_0^T \int_\omega s^{-3} \lambda^{-4} \varphi^{-3} e^{2s\eta} |\hat{u}|^2 dxdt \leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \quad (43)$$

We will now give an estimate of the integral term $\iint_Q s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} c |\hat{z}_x|^2 dxdt$. Let $Q_n := \mathcal{R} \times (t_n, T_n)$ where the numbers $0 < t_n < T_n < T$, $n \in \mathbb{N}$, will be defined later. Let

$$I_1 := \iint_{Q_n} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} c |\hat{z}_x|^2 dxdt.$$

First, we make an integration by parts to obtain

$$I_1 = \underbrace{\iint_{Q_n} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} [-(c\hat{z}_x)_x] \hat{z} dxdt}_{I_2} + \underbrace{\iint_{Q_n} s^{-2} \lambda^{-2} (\varphi^{-2} e^{2s\eta})_x (-c\hat{z}_x) \hat{z} dxdt}_{I_3}.$$

As $-(c\hat{z}_x)_x = \hat{u} + s^3 \lambda^4 \varphi^3 e^{-2s\eta} q + \hat{z}_t$ we have

$$I_2 = \underbrace{\iint_{Q_n} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} \hat{z} \hat{u} dxdt}_{I_4} + \underbrace{\iint_{Q_n} s \lambda^2 \varphi \hat{z} q dxdt}_{I_5} + \underbrace{\iint_{Q_n} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} \hat{z} \hat{z}_t dxdt}_{I_6}.$$

First of all, we have $(\varphi^{-2} e^{2s\eta})_x = -2\lambda \varphi^{-2} (1 + s\varphi) \beta' e^{2s\eta}$, thus for s sufficiently large we have,

$$|(\varphi^{-2} e^{2s\eta})_x| \leq C \lambda s \varphi^{-1} e^{2s\eta}.$$

Thus we get by using (43)

$$\begin{aligned} |I_3| &\leq C \iint_{Q_n} s^{-1} \lambda^{-1} \varphi^{-1} e^{2s\eta} c |\hat{z}| |\hat{z}_x| dxdt \\ &\leq \frac{1}{2} \iint_{Q_n} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} c |\hat{z}_x|^2 dxdt + C \iint_{Q_n} e^{2s\eta} |\hat{z}|^2 dxdt \\ &\leq \frac{1}{2} I_1 + C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \end{aligned} \quad (44)$$

By using Minkowski estimate and (43), we obtain

$$\begin{aligned} |I_4| &\leq C \iint_{Q_n} e^{2s\eta} |\hat{z}|^2 dxdt + C \iint_{Q_n} s^{-4} \lambda^{-4} \varphi^{-4} e^{2s\eta} |\hat{u}|^2 dxdt \\ &\leq C \iint_{Q_n} e^{2s\eta} |\hat{z}|^2 dxdt + C \int_0^T \int_\omega s^{-3} \lambda^{-4} \varphi^{-3} e^{2s\eta} |\hat{u}|^2 dxdt \\ &\leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \end{aligned} \quad (45)$$

By using Minkowski estimate and (43), we obtain

$$\begin{aligned} |I_5| &\leq C \iint_{Q_n} e^{2s\eta} |\hat{z}|^2 dxdt + C \iint_{Q_n} s^2 \lambda^4 \varphi^2 e^{-2s\eta} |q|^2 dxdt \\ &\leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \end{aligned} \quad (46)$$

First of all, we make an integration by parts in time:

$$I_6 = -\frac{1}{2} \iint_{Q_n} s^{-2} \lambda^{-2} (\varphi^{-2} e^{2s\eta})_t |\hat{z}|^2 dxdt + \left[\frac{1}{2} \int_{\mathcal{R}} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} |\hat{z}|^2 dx \right]_{t_n}^{T_n}.$$

But we have

$$\begin{aligned} |(\varphi^{-2} e^{2s\eta})_t| &= |(-2\varphi^{-3} \varphi_t + 2s\varphi^{-2} \eta_t) e^{2s\eta}| \\ &\leq C\varphi^{-3} (|\eta_t| + |\varphi_t|) (1 + s\varphi) e^{2s\eta} \leq Cse^{2s\eta}, \end{aligned}$$

since $|\varphi_t| + |\eta_t| \leq C_T \varphi^2$. Then, we get with (43),

$$\begin{aligned} \left| \frac{1}{2} \iint_{Q_n} s^{-2} \lambda^{-2} (\varphi^{-2} e^{2s\eta})_t |\hat{z}|^2 dxdt \right| &\leq C \iint_Q s^{-1} \lambda^{-2} e^{2s\eta} |\hat{z}|^2 dxdt \\ &\leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \end{aligned} \quad (47)$$

With estimates (44), (45), (46) and (47), we infer that

$$|I_1| \leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt + \left| \left[\frac{1}{2} \int_{\mathcal{R}} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} |\hat{z}|^2 dx \right]_{t_n}^{T_n} \right|. \quad (48)$$

To choose the times t_n and T_n , we need the following lemma.

Lemma 8. *Let $T > 0$, $r > 1$ and $\zeta : (0, T) \rightarrow \mathbb{R}$ be a measurable function with $\zeta \in L^1(0, T)$. Then there exists a decreasing sequence $(t_n)_{n \geq 0}$ in $(0, T)$ with $t_n \rightarrow 0$, such that $t_n^r |\zeta(t_n)| \rightarrow 0$.*

Proof. To prove Lemma 8, we have to show the following statement:

$$\forall C > 0, \forall \epsilon > 0, \exists t \in (0, \epsilon) \quad t^r |\zeta(t)| < C.$$

If the statement is false, we may pick some numbers $C > 0$ and $\epsilon \in (0, T)$ such for all $t \in (0, \epsilon)$, we have $t^r |\zeta(t)| \geq C$. This yields $\int_0^\epsilon |\zeta(t)| dt \geq C \int_0^\epsilon t^{-r} dt = \infty$, contradicting the assumption $\zeta \in L^1(0, T)$. \square

We apply Lemma 8 twice to the function $\zeta(t) := \int_{\mathcal{R}} e^{2s\eta} |\hat{z}|^2 dx$ with $r = 2$. Note that $\zeta \in L^1(0, T)$ by (43). We may pick two sequences $(t_n)_{n \geq 0}$, $(T_n)_{n \geq 0}$, with $0 < t_n < T_n < T$, $t_n \searrow 0$, $T_n \nearrow T$ and

$$t_n^2 |\zeta(t_n)| + (T - T_n)^2 |\zeta(T_n)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $\varphi(x, t_n)^{-2} \leq Ct_n^2$ and $\varphi(x, T_n)^{-2} \leq C(T - T_n)^2$, we infer that as $n \rightarrow +\infty$

$$\left| \left[\frac{1}{2} \int_{\mathcal{R}} s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} |\hat{z}|^2 dx \right]_{t_n}^{T_n} \right| \rightarrow 0.$$

Letting $n \rightarrow \infty$ in (48), we conclude that

$$\iint_Q s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} c |\hat{z}_x|^2 dxdt \leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt. \quad (49)$$

With (42), (43) and (49), we have

$$\begin{aligned}
\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt &= - \iint_Q f \hat{z}_x dxdt - \int_0^T \int_\omega q \hat{u} dxdt \\
&\leq \left(\iint_Q s^{-2} \lambda^{-2} \varphi^{-2} e^{2s\eta} |\hat{z}_x|^2 dxdt \right)^{1/2} \left(\iint_Q s^2 \lambda^2 \varphi^2 e^{-2s\eta} |f|^2 dxdt \right)^{1/2} \\
&\quad + \left(\int_0^T \int_\omega s^{-3} \lambda^{-4} \varphi^{-3} e^{2s\eta} |\hat{u}|^2 dxdt \right)^{1/2} \left(\int_0^T \int_\omega s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \right)^{1/2} \\
&\leq C \left(\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \right)^{1/2} \left(\left(\iint_Q s^2 \lambda^2 \varphi^2 e^{-2s\eta} |f|^2 dxdt \right)^{1/2} \right. \\
&\quad \left. + \left(\int_0^T \int_\omega s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt \right)^{1/2} \right).
\end{aligned}$$

Thus we obtain the desired Carleman inequality (36). \square

2.3 Carleman estimates with one less derivative, with boundary observations

By following the same lines as in [18], we shall obtain a new Carleman inequality for boundary observations. We will use again the Carleman estimate with distributed observation in $L^2(\mathcal{R})$.

Theorem 9. *Let us consider the following problem:*

$$\begin{cases} q_t - (cq_x)_x = f_x, & \text{in } Q, \\ q_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \\ q_i(0, t) = q_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [cq_x(t)]_0 = 0, & \forall t \in (0, T), \\ q(x, 0) = q_0(x), & \forall x \in \mathcal{R}. \end{cases} \quad (50)$$

Extend the network \mathcal{R} in another network $\hat{\mathcal{R}}$ with the edges $[0, \hat{l}_j]$, $j \in \llbracket 0, n \rrbracket$, where $0 < l_j < \hat{l}_j$ for $j \in \llbracket 1, n \rrbracket$ and $\hat{l}_0 = l_0$, and fix an open subset $\omega \subset \hat{\mathcal{R}} \setminus \mathcal{R}$ intersecting each interval (l_j, \hat{l}_j) for $1 \leq j \leq n$. Assume that φ and η are as in (6), and that β is as given by Lemma 3 for $\hat{\mathcal{R}}$ and ω . Assume that $q_0 \in L^2(\mathcal{R})$, $f \in L^2(0, T, H^1(\mathcal{R}))$ with $f(0, t) = 0$ for a.e. $t \in (0, T)$. Let s_0, λ_0 be as in Theorem 4. Pick $\lambda \geq \lambda_0$. Let $V_\gamma := \cup_{1 \leq i \leq n} (l_i - \epsilon, l_i)$ for some $\epsilon \in (0, \inf_{1 \leq i \leq n} l_i)$. Then there exist $\bar{s} \geq s_0 > 0$ and a positive constant $C = C(T, \mathcal{R}, c_{\min}, c_{\max})$ such that if $q \in L^2(Q) \cap L^2(0, T, H^2(V_\gamma))$ solves (50), then the following Carleman estimate holds for $s \geq \bar{s}$

$$\begin{aligned}
\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dxdt &\leq C \left(\iint_Q s^2 \lambda^2 \varphi^2 e^{-2s\eta} |f|^2 dxdt \right. \\
&\quad \left. + \int_0^T \sum_{j=1}^n s \lambda \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |f_j(l_j, t) + c_j(l_j) q_{j,x}(l_j, t)|^2 dt \right).
\end{aligned}$$

Proof. We still follow the proof given in [18, A.2.3]. Let $\hat{Q} = (0, T) \times \hat{\mathcal{R}}$. We extend q and f on \hat{Q} by setting $f = 0$ and $q = 0$ in $\hat{Q} \setminus Q$. We set $q_0(x) = 0$ for $x \in \hat{\mathcal{R}} \setminus \mathcal{R}$. We also extend c smoothly such that $0 < c_{\min} \leq c(x) \leq c_{\max} < \infty$ in $\hat{\mathcal{R}}$. We are then in the same situation as in

the proof of Theorem 7 with \mathcal{R} replaced by $\hat{\mathcal{R}}$. The function q satisfies the system

$$\begin{cases} \hat{L}q := q_t - (cq_x)_x = (f_x)\mathbf{1}_{\mathcal{R}} + \sum_{j=1}^n (c_j q_{j,x})(l_j)\delta_{l_j}, & \text{in } \hat{Q}, \\ q_j(\hat{l}_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \\ q_i(0, t) = q_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [cq_x(t)]_0 = 0, & \forall t \in (0, T), \\ q(x, 0) = q_0(x), & \forall x \in \hat{\mathcal{R}}. \end{cases} \quad (51)$$

Then (37) is replaced by

$$\iint_{\hat{Q}} qq dx dt = - \iint_{\hat{Q}} f z_x dx dt + \sum_{j=1}^n \int_0^T (f_j(l_j, t) + c_j(l_j)q_{j,x}(l_j, t))z_j(l_j, t) dt + \int_{\hat{\mathcal{R}}} q_0 z(0) dx \quad (52)$$

where $z \in D(\hat{L}^*)$ is a solution to

$$\begin{cases} -z_t - (cz_x)_x = g, & \text{in } \hat{Q}, \\ z_j(\hat{l}_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, \\ z_i(0, t) = z_j(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, \forall t \in (0, T), \\ [cz_x(t)]_0 = 0, & \forall t \in (0, T), \\ z(x, T) = 0, & \forall x \in \hat{\mathcal{R}}. \end{cases}$$

We modify the calculations in the proof of Theorem 7 by taking into account the boundary terms. Since $q = 0$ and $f = 0$ in $\hat{Q} \setminus Q$, then (42) is replaced by

$$\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dx dt = - \iint_Q f \hat{z}_x dx dt + \sum_{j=1}^n \int_0^T (f_j(l_j, t) + c_j(l_j)q_{j,x}(l_j, t)) \hat{z}_j(l_j, t) dt. \quad (53)$$

Then we get (43) and (49) with \hat{Q} instead of Q . To obtain the desired Carleman inequality, we just need to prove that

$$\begin{aligned} & \left| \sum_{j=1}^n \int_0^T (f_j(l_j, t) + c_j(l_j)q_{j,x}(l_j, t)) \hat{z}_j(l_j, t) dt \right| \\ & \leq C \left(\iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dx dt \right)^{1/2} \left(\sum_{j=1}^n \int_0^T s \lambda \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} (f_j(l_j, t) + c_j(l_j)q_{j,x}(l_j, t))^2 dt \right)^{1/2}. \quad (54) \end{aligned}$$

As for any $j \in \llbracket 1, n \rrbracket$ and for any $t \in (0, T)$, we have $(\varphi^{-1} e^{2s\eta} \hat{z}^2)(\cdot, t)|_{(0, l_j)} \in W^{1,1}(0, l_j)$, and as for any function $\mu \in W^{1,1}(0, l_j)$, $|\mu(l_j)| \leq C(\int_0^{l_j} |\mu_x| dx + \int_0^{l_j} |\mu| dx)$, we have by using (43) and (49)

$$\begin{aligned} & \int_0^T \sum_{j=1}^n s^{-1} \lambda^{-1} \varphi_j^{-1}(l_j, t) e^{2s\eta_j(l_j, t)} \hat{z}_j^2(l_j, t) dt \\ & \leq C \left(\iint_Q s^{-1} \lambda^{-1} \varphi^{-1} e^{2s\eta} \hat{z}^2 dx dt + \iint_Q s^{-1} \lambda^{-1} |(\varphi^{-1} e^{2s\eta} \hat{z}^2)_x| dx dt \right) \\ & \leq C \iint_Q s^{-1} \lambda^{-1} \varphi^{-1} e^{2s\eta} |\hat{z}|^2 dx dt + C s^{-1} \lambda^{-1} \iint_Q |(\varphi^{-1} e^{2s\eta})_x| \hat{z}^2 dx dt \\ & \quad + C s^{-1} \lambda^{-1} \iint_Q \varphi^{-1} e^{2s\eta} |\hat{z} \hat{z}_x| dx dt \\ & \leq C \iint_Q e^{2s\eta} |\hat{z}|^2 dx dt + C s^{-1} \lambda^{-1} \iint_Q \varphi^{-1} e^{2s\eta} |\hat{z} \hat{z}_x| dx dt \\ & \leq C \iint_Q e^{2s\eta} |\hat{z}|^2 dx dt + C s^{-2} \lambda^{-2} \iint_Q \varphi^{-2} e^{2s\eta} |\hat{z}_x|^2 dx dt \\ & \leq C \iint_Q s^3 \lambda^4 \varphi^3 e^{-2s\eta} |q|^2 dx dt \end{aligned}$$

which gives (54) and completes the proof of the Carleman inequality. \square

3 Stability results

Let y and \tilde{y} be the solutions of

$$\begin{cases} y_t - (cy_x)_x = g\mathbb{1}_\omega, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ y_j(l_j, t) = h_j(t), & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ y_j(0, t) = y_i(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ [cy_x(t)]_0 = 0, & \forall t \in (0, T), \\ y(x, 0) = y^0(x), & \forall x \in \mathcal{R}, \end{cases} \quad (55)$$

and

$$\begin{cases} \tilde{y}_t - (\tilde{c}\tilde{y}_x)_x = g\mathbb{1}_\omega, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ \tilde{y}_j(l_j, t) = h_j(t), & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ \tilde{y}_j(0, t) = \tilde{y}_i(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ [\tilde{c}\tilde{y}_x(t)]_0 = 0, & \forall t \in (0, T), \\ \tilde{y}(x, 0) = \tilde{y}^0(x), & \forall x \in \mathcal{R}, \end{cases} \quad (56)$$

respectively. Let

$$\xi := c - \tilde{c}, \text{ and } v := (y - \tilde{y})_t.$$

Then v satisfies the following problem

$$\begin{cases} v_t - (cv_x)_x = (\xi\tilde{y}_{tx})_x, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ v_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ v_j(0, t) = v_i(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, 0 < t < T, \\ [cv_x(t)]_0 = -[\xi\tilde{y}_{tx}(t)]_0, & \forall t \in (0, T), \\ v(x, 0) = (cy_x^0)_x - (\tilde{c}\tilde{y}_x^0)_x, & \forall x \in \mathcal{R}. \end{cases} \quad (57)$$

3.1 Stability results with interior observations

In this subsection we derive stability theorem 1 with interior observations. Our method is based on ideas borrowed from [8, 18].

3.1.1 Stability for a stationary problem

We first investigate the $L^2(\mathcal{R})$ -stability of the solution $c(x)$ of the following stationary problem

$$\begin{cases} -(cu_x)_x = f, & \text{in } \mathcal{R}, \\ u_j(0) = u_i(0), & \forall i, j \in \llbracket 0, n \rrbracket, \\ [cu_x]_0 = 0, \end{cases} \quad (58)$$

where $u \in H^1(\mathcal{R})$ and $f \in L^2(\mathcal{R})$.

Let $\tilde{c}(x)$ solve

$$\begin{cases} -(\tilde{c}\tilde{u}_x)_x = \tilde{f}, & \text{in } \mathcal{R}, \\ \tilde{u}_j(0) = \tilde{u}_i(0), & \forall i, j \in \llbracket 0, n \rrbracket, \\ [\tilde{c}\tilde{u}_x]_0 = 0, \end{cases} \quad (59)$$

where $\tilde{u} \in H^1(\mathcal{R})$ and $\tilde{f} \in L^2(\mathcal{R})$, and let $\xi := c - \tilde{c}$.

For any function $q : \mathcal{R} \rightarrow \mathbb{R}$, we set $q_s(x) := q(x)e^{-s\eta(x, T')}$ where η is as in (6) and $T' \in (0, T)$. At several places, to simplify the notation, we shall write $\eta(T')$ instead of $\eta(x, T')$. Moreover in this section η (resp. φ) denotes $\eta(T')$ (resp. $\varphi(T')$).

We begin with the following stability result:

Lemma 10. *Let $\delta > 0$ and $T' \in (0, T)$. Assume that the functions c and \tilde{c} are piecewise \mathcal{C}^1 , that they satisfy (4), and that they are such that*

$$(H_c) \quad c_i(0) = \tilde{c}_i(0) \quad \forall i \in \llbracket 0, n \rrbracket.$$

Let β be as in Lemma 3. Assume further that

$$\begin{aligned} (H_1) \quad & |\beta' \tilde{u}_x| \geq \delta > 0 \quad \text{in } \mathcal{R} \setminus \omega^0, \\ (H_2) \quad & \forall j \in \llbracket 0, n \rrbracket, u_j \in \mathcal{C}^2([0, l_j]) \text{ and } \tilde{u}_j \in \mathcal{C}^2([0, l_j]), \\ (H_3) \quad & \|\tilde{u}\|_{L^\infty(\mathcal{R})} + \|\tilde{u}_x\|_{L^\infty(\mathcal{R})} \leq \delta^{-1}, \end{aligned}$$

for some open set ω^0 with $\omega^0 \subset \omega$.

Let s_0, λ_0 be associated with the set ω^0 as in Theorem 4. Pick any $\lambda \geq \lambda_0$. Then there exists $s_1 \geq s_0$ such that for any $s \geq s_1$ we have the following estimate (with $\xi = c - \tilde{c}$)

$$s^2 \delta^2 \int_{\mathcal{R} \setminus \omega^0} |\xi|^2 e^{-2s\eta} dx \leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx + C s \int_{\omega^0} |\xi|^2 e^{-2s\eta} dx + C(s, c_{max}) \|u - \tilde{u}\|_{H^1(\mathcal{R})}^2. \quad (60)$$

Proof. By (58)-(59), we have

$$\begin{cases} -(c_s u_x)_x - s\eta_x c_s u_x = f_s \\ -(\tilde{c}_s \tilde{u}_x)_x - s\eta_x \tilde{c}_s \tilde{u}_x = \tilde{f}_s. \end{cases}$$

Thus

$$\|f_s - \tilde{f}_s\|_{L^2(\mathcal{R})}^2 = \int_{\mathcal{R}} [(c_s u_x - \tilde{c}_s \tilde{u}_x)_x + s\eta_x (c_s u_x - \tilde{c}_s \tilde{u}_x)]^2 dx.$$

As $(a + b)^2 \geq a^2 + 2ab$ we obtain,

$$\|f_s - \tilde{f}_s\|_{L^2(\mathcal{R})}^2 \geq s^2 \int_{\mathcal{R}} |\eta_x (c_s u_x - \tilde{c}_s \tilde{u}_x)|^2 dx + 2s \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)_x (\eta_x (c_s u_x - \tilde{c}_s \tilde{u}_x)) dx.$$

We estimate each term in the right hand side of the previous inequality. We first have a look at

$$B_1 := \int_{\mathcal{R}} |\eta_x (c_s u_x - \tilde{c}_s \tilde{u}_x)|^2 dx = \int_{\mathcal{R}} \lambda^2 \varphi^2(\beta')^2 (c_s u_x - \tilde{c}_s \tilde{u}_x)^2 dx.$$

As $c_s u_x - \tilde{c}_s \tilde{u}_x = \xi_s \tilde{u}_x + c_s (u_x - \tilde{u}_x)$ and $(a + b)^2 \geq \frac{a^2}{2} - b^2$, we have

$$\begin{aligned} B_1 &= \lambda^2 \int_{\mathcal{R}} \varphi^2(\beta')^2 (\xi_s \tilde{u}_x + c_s (u_x - \tilde{u}_x))^2 dx \\ &\geq \lambda^2 \int_{\mathcal{R}} \varphi^2(\beta')^2 \left(\frac{1}{2} \xi_s^2 \tilde{u}_x^2 - c_s^2 (u_x - \tilde{u}_x)^2 \right) dx. \end{aligned}$$

With Hypothesis (H_1) , $|\beta_x \tilde{u}_x| \geq \delta$ on $\mathcal{R} \setminus \omega^0$, we infer that

$$\begin{aligned} B_1 &\geq \frac{\lambda^2 \delta^2}{2} \int_{\mathcal{R} \setminus \omega^0} \varphi^2 \xi_s^2 dx - C(\beta, \lambda, c_{max}) \int_{\mathcal{R}} \varphi^2 e^{-2s\eta} |u_x - \tilde{u}_x|^2 dx \\ &\geq \frac{\lambda^2 \delta^2 e^{\lambda \bar{\beta}}}{2(T'(T - T'))^2} \int_{\mathcal{R} \setminus \omega^0} \xi_s^2 dx - C(\beta, c_{max}, s, \lambda, T') \int_{\mathcal{R}} |u_x - \tilde{u}_x|^2 dx. \end{aligned} \quad (61)$$

On the other hand,

$$\begin{aligned} B_2 &:= \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)_x (\eta_x (c_s u_x - \tilde{c}_s \tilde{u}_x)) dx \\ &= \frac{1}{2} \sum_{j=0}^n [(c_{j,s} u_{j,x} - \tilde{c}_{j,s} \tilde{u}_{j,x})^2 \eta_{j,x}]_0^{l_j} - \frac{1}{2} \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)^2 \eta_{xx} dx \\ &= -\frac{\lambda}{2} \sum_{j=0}^n (c_{j,s} u_{j,x} - \tilde{c}_{j,s} \tilde{u}_{j,x})^2 (l_j) \beta'_j(l_j) \varphi_j(l_j) + \frac{\lambda}{2} \sum_{j=0}^n (c_{j,s} u_{j,x} - \tilde{c}_{j,s} \tilde{u}_{j,x})^2 (0) \beta'_j(0) \varphi(0) \\ &\quad - \frac{1}{2} \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)^2 \eta_{xx} dx. \end{aligned}$$

First of all, we have $\beta'_j(l_j) < 0$ for all $j \in \llbracket 0, n \rrbracket$, by (11). Thus the first term in the right hand side of the previous equation is nonnegative. We now prove that the second term in this expression is also nonnegative.

Let $B_3 := \sum_{j=0}^n (c_{j,s}u_{j,x} - \tilde{c}_{j,s}\tilde{u}_{j,x})^2(0)\beta'_j(0)\varphi(0)$. We have that $c_{j,s}(0) = \tilde{c}_{j,s}(0)$ for all $j \in \llbracket 0, n \rrbracket$ (see (H_c)), and hence $[c_s(u_x - \tilde{u}_x)]_0 = [c_s u_x - \tilde{c}_s \tilde{u}_x]_0 = 0$, so that

$$c_{0,s}(0)(u_{0,x} - \tilde{u}_{0,x})(0) = -\sum_{j=1}^n c_{j,s}(0)(u_{j,x} - \tilde{u}_{j,x})(0).$$

It follows that

$$\begin{aligned} B_3 &= \varphi(0) \left(\beta'_0(0) \left(\sum_{j=1}^n c_{j,s}(0)(u_{j,x} - \tilde{u}_{j,x})(0) \right)^2 + \sum_{j=1}^n (c_{j,s})^2(0)(u_{j,x} - \tilde{u}_{j,x})^2(0)\beta'_j(0) \right) \\ &= \varphi(0) \left(\sum_{j=1}^n (\beta'_0(0) + \beta'_j(0))(c_{j,s}(0)(u_{j,x} - \tilde{u}_{j,x})(0))^2 \right. \\ &\quad \left. + 2\beta'_0(0) \sum_{i,j=1, i < j}^n c_{i,s}(0)(u_{i,x} - \tilde{u}_{i,x})(0)c_{j,s}(0)(u_{j,x} - \tilde{u}_{j,x})(0) \right) \\ &= \varphi(0)(A_\beta W, W), \end{aligned}$$

with A_β denoting the matrix defined in Lemma 3, and $W = ((c_{j,s}(0)(u_{j,x} - \tilde{u}_{j,x})(0))_{j=1 \dots n}, 0)$. As the matrix A_β is positive definite, we infer that $B_3 \geq 0$. It follows that, by (H_2) ,

$$\begin{aligned} B_2 &\geq -\frac{1}{2} \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)^2 \eta_{xx} dx \\ &\geq -C(\lambda, \beta, T') \int_{\mathcal{R}} (c_s u_x - \tilde{c}_s \tilde{u}_x)^2 dx \\ &\geq -C \int_{\mathcal{R}} (c_s(u_x - \tilde{u}_x) + \xi_s \tilde{u}_x)^2 dx \\ &\geq -C \|\tilde{u}_x\|_\infty^2 \int_{\mathcal{R}} \xi_s^2 dx - C(s, c_{max}) \int_{\mathcal{R}} (u_x - \tilde{u}_x)^2 dx. \end{aligned} \quad (62)$$

With (61), (62) and (H_3) we obtain

$$\begin{aligned} &\|f_s - \tilde{f}_s\|_{L^2(\mathcal{R})}^2 \\ &\geq s^2 \frac{\lambda^2 \delta^2 e^{\lambda \bar{\beta}}}{2(T'(T - T'))^2} \int_{\mathcal{R} \setminus \omega^0} |\xi_s|^2 dx - C(s, \beta, \lambda, T', c_{max}) \int_{\mathcal{R}} (u_x - \tilde{u}_x)^2 dx - \frac{Cs}{\delta^2} \int_{\mathcal{R}} \xi_s^2 dx. \end{aligned} \quad (63)$$

Thus we obtain (60) for s large enough. \square

The following lemma gives an L^2 -estimate of ξ on a neighborhood of ω^0 .

Lemma 11. *Let $c, u, f, \tilde{c}, \tilde{u}, \tilde{f}$ be as in Lemma 10. Assume further that*

(H_4) *There exist some open intervals $\omega_i^1 \subset (0, l_i)$, $i \in \llbracket 1, n \rrbracket$, such that:*

1. $\omega^0 \subset \omega^1 := \cup_{i \in \llbracket 1, n \rrbracket} \omega_i^1 \subset \omega$,
2. $\tilde{u}|_{\partial \omega^1} = 0$,
3. $\tilde{u} \in \mathcal{C}^2(\omega^1)$ and $|\tilde{u}_x|^2 - \tilde{u}\tilde{u}_{xx} \geq \delta > 0$ in ω^1 .

Assume in addition that $\|c\|_{W^{1,\infty}(\omega_1)} \leq c_{max}^{1,\infty}$. Then there exist $C > 0$ and $C(c_{max}^{1,\infty}) > 0$ such that

$$\int_{\omega^1} |\xi|^2 dx \leq C \|f - \tilde{f}\|_{L^2(\omega^1)}^2 + C(c_{max}^{1,\infty}) \|u - \tilde{u}\|_{H^2(\omega^1)}^2. \quad (64)$$

Proof. In ω^1 , we have

$$-(\xi \tilde{u}_x)_x = -(c \tilde{u}_x - \tilde{c} \tilde{u}_x)_x = f - \tilde{f} + (c(u - \tilde{u}))_x. \quad (65)$$

Thus

$$\int_{\omega^1} -\xi \tilde{u} (\xi \tilde{u}_x)_x dx = \int_{\omega^1} (f - \tilde{f} + (c(u - \tilde{u}))_x) \xi \tilde{u} dx. \quad (66)$$

On the other hand, since $\tilde{u}|_{\partial\omega^1} = 0$, we obtain by an integration by parts that

$$\int_{\omega^1} -\xi \tilde{u} (\xi \tilde{u}_x)_x dx = \int_{\omega^1} \xi^2 \left((\tilde{u}_x)^2 - \frac{1}{4} (\tilde{u}^2)_{xx} \right) dx.$$

Thanks to (H_4) 3., we have $(\tilde{u}_x)^2 - \frac{1}{4} (\tilde{u}^2)_{xx} \geq \frac{\delta}{2}$. This yields

$$\int_{\omega^1} \xi^2 dx \leq C(\delta) \int_{\omega^1} (f - \tilde{f} + (c(u - \tilde{u}))_x) \xi \tilde{u} dx,$$

and hence, using Cauchy-Schwarz inequality and (H_3) ,

$$\begin{aligned} \int_{\omega^1} \xi^2 dx &\leq C(\delta) \int_{\omega^1} (f - \tilde{f} + (c(u - \tilde{u}))_x)^2 dx \|\tilde{u}\|_{L^\infty(\omega^1)}^2 \\ &\leq C(\delta) \|f - \tilde{f}\|_{L^2(\omega^1)}^2 + C(c_{max}^{1,\infty}, \delta) \|u - \tilde{u}\|_{H^2(\omega^1)}^2, \end{aligned}$$

which ends the proof of Lemma 11. \square

We are in a position to state a stationary stability result:

Proposition 12. *Let c, \tilde{c} piecewise \mathcal{C}^1 satisfying (4) and $\max_{j \in [0, n]} \|c_j\|_{W^{1,\infty}(0, l_j)} \leq c_{max}^{1,\infty}$. Under hypothesis (H_c) , (H_1) , (H_2) , (H_3) and (H_4) , there exist $C > 0$, $s_0 > 0$ such that for all $s \geq s_0$, there exist $C(s) > 0$ and $C(s, c_{max}, c_{max}^{1,\infty}) > 0$ with*

$$\begin{aligned} s^2 \int_{\mathcal{R}} |\xi|^2 e^{-2s\eta} dx &\leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx + C(s) \|f - \tilde{f}\|_{L^2(\omega^1)}^2 \\ &\quad + C(s, c_{max}, c_{max}^{1,\infty}) \left(\|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 + \|u - \tilde{u}\|_{H^2(\omega^1)}^2 \right). \end{aligned}$$

Proof. We have by using Lemma 10 and Lemma 11

$$\begin{aligned} s^2 \int_{\mathcal{R}} |\xi|^2 e^{-2s\eta} dx &\leq s^2 \int_{\mathcal{R} \setminus \omega^0} |\xi|^2 e^{-2s\eta} dx + s^2 \int_{\omega^1} |\xi|^2 e^{-2s\eta} dx \\ &\leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx + Cs \int_{\omega^0} |\xi|^2 e^{-2s\eta} dx + C(s, c_{max}) \|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 \\ &\quad + C(s) \int_{\omega^1} |\xi|^2 dx \\ &\leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx + C(s, c_{max}) \|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 + C(s) \int_{\omega^1} |\xi|^2 dx \\ &\leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx + C(s, c_{max}) \|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 \\ &\quad + C(s) \|f - \tilde{f}\|_{L^2(\omega^1)}^2 + C(s, c_{max}^{1,\infty}) \|u - \tilde{u}\|_{H^2(\omega^1)}^2. \end{aligned}$$

\square

3.1.2 Stability for the evolution problem

Assume given $y^0 \in L^2(\mathcal{R})$ and $g \in C_c^\infty(\omega \times (0, T))$ (conveniently chosen). Let y (resp. \tilde{y}) denote the solution of (55) (resp. (56)) with $h_j(t) = 0$ for $j \in \llbracket 0, n \rrbracket$ and $t \in (0, T)$, and $\tilde{y}^0(x) = y^0(x)$ for $x \in \mathcal{R}$. We aim at deriving stability estimates for $c - \tilde{c}$ from measurements of derivatives of $y - \tilde{y}$ for $(x, t) \in \omega \times (0, T)$ and for $(x, t) \in \mathcal{R} \times \{T/2\}$.

Notice first that there is no loss of generality in assuming that $\omega = \cup_{j=1}^n (p_j, q_j)$ with $0 < p_j < q_j < l_j$ for $1 \leq j \leq n$ by shrinking ω if needed. In what follows, the function β is those given in Lemma 3 for the open set $\omega = \cup_{j=1}^n (p_j, q_j)$.

As we shall use the stability estimates in the stationary case, we need first to prove the existence of some control input g such that the assumptions $(H_1), (H_2), (H_3)$ and (H_4) are satisfied for $\tilde{u} = \tilde{y}(\cdot, T/2)$. This is done in the following proposition.

Proposition 13. *Let \tilde{c} be a piecewise \mathcal{C}^1 function satisfying (4) and (H_c) . Let $T > 0$ and $\tilde{y}^0 \in L^2(\mathcal{R})$. Pick $T' = T/2$. Then there exist two open sets $\omega^0 \subset \omega^1 \subset \omega$, a function $g \in C_c^\infty(\omega \times (0, T))$ such that the solution \tilde{y} of (56) associated with $\tilde{y}_2^0(x) := \tilde{y}^0(x) - 2$ and $h_j(t) = -2$ for $j \in \llbracket 0, n \rrbracket$ and $t \in (0, T)$ satisfies for some $\delta > 0$*

$$|\beta' \tilde{y}_x(\cdot, T')| \geq \delta > 0 \quad \text{in } \mathcal{R} \setminus \omega^0; \quad (67)$$

$$\tilde{y}_j(\cdot, T') \in \mathcal{C}^2([0, l_j]) \quad \forall j \in \llbracket 0, n \rrbracket; \quad (68)$$

$$\|\tilde{y}(\cdot, T')\|_{L^\infty(\mathcal{R})} + \|\tilde{y}_x(\cdot, T')\|_{L^\infty(\mathcal{R})} \leq \delta^{-1}; \quad (69)$$

$$\tilde{y}(\cdot, T')|_{\partial\omega^1} = 0; \quad (70)$$

$$|\tilde{y}_x(\cdot, T')|^2 - \tilde{y}(\cdot, T')\tilde{y}_{xx}(\cdot, T') \geq \delta > 0 \quad \text{in } \omega^1. \quad (71)$$

Proof. We proceed as in [8]. In a first step, we construct a function $\rho(x)$ satisfying conditions very similar to (67)-(71). In a second step, we show that a control input g can be designed so that the solution \tilde{y} of (56) is close enough to ρ for the norm $\|u\|_{L^\infty(\mathcal{R})} + \|u_x\|_{L^\infty(\mathcal{R})} + \|u_{xx}\|_{L^\infty(\mathcal{R})}$ at time $t = T'$ (approximate controllability).

Step 1. We need the following lemma:

Lemma 14. *Let \tilde{c} be a piecewise \mathcal{C}^1 function satisfying (4) and (H_c) . Then there exist two open sets $\omega^0 \subset \omega^2 \subset \omega$ and a function $\rho \in H^1(\mathcal{R})$ such that for some $\hat{\delta} > 0$*

$$|\beta' \rho_x| \geq \hat{\delta} > 0 \quad \text{in } \mathcal{R} \setminus \omega^0; \quad (72) \quad \rho_j(l_j) = -2 \quad \forall j \in \llbracket 0, n \rrbracket; \quad (77)$$

$$\rho_j \in \mathcal{C}^2([0, l_j]) \quad \forall j \in \llbracket 0, n \rrbracket; \quad (73) \quad (\tilde{c}_j \rho_{j,x})_x(l_j) = 0 \quad \forall j \in \llbracket 0, n \rrbracket; \quad (78)$$

$$\|\rho\|_{L^\infty(\mathcal{R})} + \|\rho_x\|_{L^\infty(\mathcal{R})} \leq \hat{\delta}^{-1}; \quad (74) \quad (\tilde{c}_j \rho_{j,x})_x(0) = 0 \quad \forall j \in \llbracket 0, n \rrbracket; \quad (79)$$

$$\rho|_{\partial\omega^2} = 0; \quad (75) \quad \rho(0) = -1; \quad (80)$$

$$|\rho_x|^2 - \rho \rho_{xx} \geq \hat{\delta} > 0 \quad \text{in } \omega^2; \quad (76) \quad [\tilde{c} \rho_x]_0 = 0. \quad (81)$$

Proof of Lemma 14: We first construct $\rho_j = \rho|_{[0, l_j]}$ for any given $j \in \llbracket 1, n \rrbracket$. Since $\beta'_j(x) \neq 0$ for $x \in [0, l_j] \setminus (p_j, q_j)$ and since $[0, l_j] \setminus (p_j, q_j)$ is compact, we can find two numbers p_j^0, q_j^0 with $0 < p_j < p_j^0 < q_j^0 < q_j < l_j$ such that

$$|p_j^0 - p_j| + |q_j^0 - q_j| < \frac{1}{10}(q_j - p_j) \quad (82)$$

and

$$\beta'_j(x) \neq 0 \quad \forall x \in [0, l_j] \setminus (p_j^0, q_j^0).$$

Pick some numbers $p_j^2 \in (p_j, p_j^0)$ and $q_j^2 \in (q_j^0, q_j)$. Set

$$\rho_j(x) := r_j(x - p_j^2)(q_j^2 - x) \quad \forall x \in [p_j, q_j]$$

where $r_j > 0$ is chosen so that $\rho_j(p_j) > -1$ and $\rho_j(q_j) > -2$. We note that we have $\rho_{j,x}(x) > 0$ for $x \in [p_j, \frac{p_j^2+q_j^2}{2})$ and $\rho_{j,x}(x) < 0$ for $x \in (\frac{p_j^2+q_j^2}{2}, q_j]$. Since $p_j^0 < \frac{p_j^2+q_j^2}{2} < q_j^0$ by (82), we infer that $\rho_{j,x}(x) > 0$ for $x \in [p_j, p_j^0]$ and $\rho_{j,x}(x) < 0$ for $x \in [q_j^0, q_j]$.

Pick $\varepsilon_j \in (0, \min(p_j, l_j - q_j))$. We set

$$\begin{aligned}\rho_j(x) &:= -1 + k_j \int_0^x \frac{ds}{\tilde{c}_j(s)} \quad \text{for } 0 \leq x \leq \varepsilon_j, \\ \rho_j(x) &:= -2 + k_j \int_x^{l_j} \frac{ds}{\tilde{c}_j(s)} \quad \text{for } l_j - \varepsilon_j \leq x \leq l_j,\end{aligned}$$

where $k_j > 0$ is chosen (small enough) so that $\rho_j(\varepsilon_j) < \rho_j(p_j)$ and $\rho_j(l_j - \varepsilon_j) < \rho_j(q_j)$. Then $\rho_j(l_j) = -2$, $\rho_j(0) = -1$, and $(\tilde{c}_j \rho_{j,x})_x(l_j) = (\tilde{c}_j \rho_{j,x})_x(0) = 0$. Next, we can extend ρ_j as a function in $\mathcal{C}^2([0, l_j])$ so that $|\beta'_j \rho_{j,x}| > 0$ in $[0, l_j] \setminus (p_j^0, q_j^0)$. On the other hand, $\rho_j(p_j^2) = \rho_j(q_j^2) = 0$, and noticing that $\rho_{j,xx} < 0$ in (p_j, q_j) and that $\rho_{j,x}(p_j^2) \neq 0$, $\rho_{j,x}(q_j^2) \neq 0$, we see that $|\rho_{j,x}|^2 - \rho_j \rho_{j,xx} > 0$ in $[p_j^2, q_j^2]$.

For ρ_0 , we set $k_0 := -\sum_{j=1}^n k_j < 0$ and

$$\begin{aligned}\rho_0(x) &:= -1 + k_0 \int_0^x \frac{ds}{\tilde{c}_0(s)} \quad \text{for } 0 \leq x \leq \varepsilon_0, \\ \rho_0(x) &:= -2 - k_0 \int_x^{l_0} \frac{ds}{\tilde{c}_0(s)} \quad \text{for } l_0 - \varepsilon_0 \leq x \leq l_0,\end{aligned}$$

where $\varepsilon_0 \in (0, l_0/2)$ is chosen (small enough) so that $\rho_0(\varepsilon_0) > \rho_0(l_0 - \varepsilon_0)$. Next we extend ρ_0 as a function in $\mathcal{C}^2([0, l_0])$ so that $\rho_{0,x} < 0$ in $[0, l_0]$.

Setting

$$\omega^0 := \cup_{j=1}^n (p_j^0, q_j^0) \quad \text{and} \quad \omega^2 := \cup_{i=1}^n (p_i^2, q_i^2),$$

we see that (72)-(81) hold. We note that $\rho_{0,x}(0) < 0$ while $\rho_{j,x}(0) > 0$ for $j \in \llbracket 1, n \rrbracket$. This completes the proof of Lemma 14.

Step 2. It follows from Theorem 4 that the following *unique continuation property*

$$\left. \begin{aligned} z &\in L^2(0, T, H_0^1(\mathcal{R})) \\ z_t + (\tilde{c}z_x)_x &= 0 \quad \text{in } \mathcal{R} \times (0, T) \\ [\tilde{c}z_x(t)]_0 &= 0 \quad \forall t \in (0, T), \\ z &= 0 \quad \text{in } \omega \times (0, T) \end{aligned} \right\} \Rightarrow z = 0 \quad \text{in } \mathcal{R} \times (0, T)$$

holds for any $T > 0$. By duality, this yields the approximate controllability in $L^2(\mathcal{R})$ and for any $T > 0$ of the system

$$\begin{cases} u_t - (\tilde{c}u_x)_x = g\chi_\omega, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ u_j(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ u_j(0, t) = u_i(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, 0 < t < T, \\ [\tilde{c}u_x(t)]_0 = 0, & \forall t \in (0, T), \\ u(x, 0) = 0, & \forall x \in \mathcal{R}, \end{cases} \quad (83)$$

with control input $g \in C_c^\infty(\omega \times (0, T))$.

We denote by $\bar{U}(u^0, g, h)$ the solution of the system

$$\begin{cases} u_t - (\tilde{c}u_x)_x = g\chi_\omega, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ u_j(l_j, t) = h_j(t), & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ u_j(0, t) = u_i(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ [\tilde{c}u_x(t)]_0 = 0, & \forall t \in (0, T), \\ u(x, 0) = u^0(x), & \forall x \in \mathcal{R}, \end{cases} \quad (84)$$

where $h = (h_j)_{j \in \llbracket 0, n \rrbracket}$.

Let \tilde{A} denote the operator $\tilde{A}u := -\partial_x(\tilde{c}\partial_x u)$ with domain $D(\tilde{A}) := \{u \in H_0^1(\mathcal{R}), \partial_x(\tilde{c}\partial_x u) \in L^2(\mathcal{R}), [\tilde{c}u_x]_0 = 0\} \subset L^2(\mathcal{R})$. It is easily seen that the operator \tilde{A} is selfadjoint, nonnegative, and sectorial (see [12]), and hence $-\tilde{A}$ generates a semigroup of contractions $(e^{-t\tilde{A}})_{t \geq 0}$ which is analytic.

Let

$$C(\mathcal{R}) := \{u : \mathcal{R} \rightarrow \mathbb{R}; u_j \in C([0, l_j]) \forall j \in \llbracket 0, n \rrbracket, u_j(0) = u_i(0) \forall i, j \in \llbracket 0, n \rrbracket\}$$

and

$$C_0(\mathcal{R}) := \{u \in C(\mathcal{R}); u_j(l_j) = 0 \text{ for } j \in \llbracket 0, n \rrbracket\}$$

be endowed with the norm $\|\cdot\|_{L^\infty(\mathcal{R})}$. We note that $H_0^1(\mathcal{R}) \subset C_0(\mathcal{R})$.

Let $\tilde{A}u := -\partial_x(\tilde{c}\partial_x u)$ denote the operator with domain

$$D(\tilde{A}) := \{u \in H_0^1(\mathcal{R}); \partial_x(\tilde{c}\partial_x u) \in C_0(\mathcal{R}), [\tilde{c}u_x]_0 = 0\} \subset C_0(\mathcal{R}).$$

It is easily seen (see e.g. [7]) that $-\tilde{A}$ generates a semigroup of contractions in $C_0(\mathcal{R})$.

Note that $u \in D(\tilde{A})$ implies that $u_j \in \mathcal{C}^2([0, l_j])$ for all $j \in \llbracket 0, n \rrbracket$, and that

$$\|u^0\|_{D(\tilde{A})} := \|u^0\|_{L^\infty(\mathcal{R})} + \|\tilde{A}u^0\|_{L^\infty(\mathcal{R})} \sim \|u^0\|_{L^\infty(\mathcal{R})} + \|u_x^0\|_{L^\infty(\mathcal{R})} + \|u_{xx}^0\|_{L^\infty(\mathcal{R})}. \quad (85)$$

From classical semigroup theory, there exists $C = C(T) > 0$ such that for all $u^0 \in D(\tilde{A})$, we have

$$\|\tilde{U}(u^0, 0, 0)(t)\|_{D(\tilde{A})} \leq C\|u^0\|_{D(\tilde{A})} \quad \forall t \in [0, T].$$

Let $\tilde{y}^0 \in L^2(\mathcal{R})$ and $T' = T/2$. We note that $e^{-T'\tilde{A}}\tilde{y}^0 \in D(\tilde{A}^2) \subset D(\tilde{A})$. We set

$$\rho_2 := \rho + 2 - \tilde{U}(\tilde{y}^0, 0, 0)(T') = \rho + 2 - e^{-T'\tilde{A}}\tilde{y}^0.$$

Then $\rho_2 \in D(\tilde{A})$, by (73) and (77)-(81). Let $\varepsilon > 0$ be given. We pick $\eta \in (0, T')$ such that

$$\|\tilde{U}(\rho_2, 0, 0)(\eta) - \rho_2\|_{D(\tilde{A})} \leq \varepsilon. \quad (86)$$

By approximate controllability of (83) in time $T' - \eta$, we may pick a control input $g \in C_c^\infty(\omega \times (0, T' - \eta))$, that we may extend by 0 for $t \geq T' - \eta$, such that

$$\|\tilde{U}(0, g, 0)(T' - \eta) - \rho_2\|_{L^2(\mathcal{R})} \leq \varepsilon.$$

On the other hand, we notice that for some constant $C' > 0$ and all $v^0 \in L^2(\mathcal{R})$

$$\|\tilde{U}(v^0, 0, 0)(\eta)\|_{D(\tilde{A})} \leq C'\|v^0\|_{L^2(\mathcal{R})}. \quad (87)$$

Indeed, both the map $v^0 \in L^2(\mathcal{R}) \rightarrow \tilde{U}(v^0, 0, 0)(\eta) = e^{-\eta\tilde{A}}v^0 \in D(\tilde{A}^2)$ and the embedding $D(\tilde{A}^2) \subset D(\tilde{A})$ are continuous. Using (87) with $v^0 = \tilde{U}(0, g, 0)(T' - \eta) - \rho_2$ and the semigroup property yield

$$\|\tilde{U}(0, g, 0)(T') - \tilde{U}(\rho_2, 0, 0)(\eta)\|_{D(\tilde{A})} \leq C'\varepsilon,$$

and hence, with (86),

$$\|\tilde{U}(0, g, 0)(T') - \tilde{\rho}\|_{D(\tilde{A})} \leq (C' + 1)\varepsilon. \quad (88)$$

Let

$$\tilde{y} := \tilde{U}(\tilde{y}^0 - 2, g, -2) = \tilde{U}(\tilde{y}^0, 0, 0) + \tilde{U}(0, g, 0) - 2$$

for $\tilde{U}(-2, 0, -2) = -2$. Then

$$\|\tilde{y}(T') - \rho\|_{D(\tilde{A})} = \|\tilde{U}(0, g, 0)(T') - \rho_2\|_{D(\tilde{A})} \leq (C' + 1)\varepsilon. \quad (89)$$

For any $j \in \llbracket 1, n \rrbracket$, we have $\rho_j(p_j) < 0$, $\rho_j(q_j) < 0$ and $\min_{x \in [p_j^0, q_j^0]} \rho_j(x) > 0$. Using (89), we see that for ε small enough we still have $\tilde{y}(p_j, T') < 0$, $\tilde{y}(q_j, T') < 0$ and $\min_{x \in [p_j^0, q_j^0]} \tilde{y}(x, T') > 0$. Therefore we can pick $p_j^1 \in (p_j, p_j^0)$ and $q_j^1 \in (q_j^0, q_j)$ such that $\tilde{y}(p_j^1, T') = \tilde{y}(q_j^1, T') = 0$ and $\tilde{y}(x, T') > 0$ for $x \in (p_j^1, q_j^1)$. Set $\omega^1 = \cup_{j=1}^n (p_j^1, q_j^1)$. Then (67)-(71) hold true for ε and δ small enough. The proof of Proposition 13 is complete. \square

We are ready to prove the main stability result for the evolution problem with interior observations, namely Theorem 1.

Proof of Theorem 1. Let $y^0 \in L^2(\mathcal{R})$, and let $\tilde{y}^0 = y^0$. Let $g \in C_c^\infty(\omega \times (0, T))$ be as given in Proposition 13. Pick $h_j(t) = 0$ for $j \in \llbracket 0, n \rrbracket$ and $t \in (0, T)$. Let y (resp. \tilde{y}) denote the solution to (55) (resp. (56)). Pick $T' = \frac{T}{2}$ and $\tau \in (0, T')$ with $g \equiv 0$ on $\omega \times (0, \tau)$. Let A and \mathcal{A} be the same operators as previously but corresponding to the coefficient c . Since $-A$ is the infinitesimal generator of an analytic semigroup with $0 \in \rho(A)$, we have by [16, Theorem 6.13] that for all $t > 0$, $e^{-tA} : L^2(\mathcal{R}) \rightarrow D(A^4)$ continuously. In particular, setting $Y(t) := e^{-tA}y^0$, we have

$$Y \in C([\tau, T], D(A^4)), \quad Y_t \in C([\tau, T], D(A^3)), \quad \text{and} \quad Y_{tt} \in C([\tau, T], D(A^2)).$$

Recall that $D(A^2) \subset D(\mathcal{A})$. Let $\xi := c - \tilde{c}$. Note that $\xi(0) = 0$. It follows from (85) and the fact that $\xi(0) = 0$ that the maps $u^0 \in D(\mathcal{A}) \rightarrow u_x^0 \in L^\infty(\mathcal{R})$ and $u^0 \in D(\mathcal{A}) \rightarrow \xi u_x^0 \in H^1(\mathcal{R})$ are continuous.

We infer that $Y_{tx}, Y_{ttx} \in C([\tau, T], L^\infty(\mathcal{R}))$, and that $\xi Y_{ttx} \in C([\tau, T], H^1(\mathcal{R}))$. Note that $y(t) = Y(t) + U(0, g, 0)(t)$, where U is defined as above with \tilde{c} replaced by c . Note that $C_c^2(\omega) \subset D(\mathcal{A})$, so that $g \in C_c^2((0, T), D(\mathcal{A}))$ and $U(0, g, 0) \in C^2([0, T], D(\mathcal{A}))$. It follows that $y_{tx}, y_{ttx} \in C([\tau, T], L^\infty(\mathcal{R}))$, and that $\xi y_{ttx} \in C([\tau, T], H^1(\mathcal{R}))$. Similarly, we have that $\tilde{y}_{tx}, \tilde{y}_{ttx} \in C([\tau, T], L^\infty(\mathcal{R}))$, and that $\xi \tilde{y}_{ttx} \in C([\tau, T], H^1(\mathcal{R}))$.

Replacing y^0 (resp. \tilde{y}^0) by $y(\tau)$ (resp. $\tilde{y}(\tau)$), we can assume that

$$y_{tx}, y_{ttx}, \tilde{y}_{tx}, \tilde{y}_{ttx} \in C([0, T], L^\infty(\mathcal{R})), \quad \xi \tilde{y}_{ttx} \in C([0, T], H^1(\mathcal{R})).$$

Note that it could now happen that $y^0 \neq \tilde{y}^0$. As (5) is invariant by uniform translation of both y and \tilde{y} by -2 , we can pick $h_j(t) = -2$ (instead of 0) for all $t \in (0, T)$ and all $j \in \llbracket 0, n \rrbracket$, so that the function \tilde{y} is now the one given by Proposition 13, while y is now the solution to (55) with initial data $y^0 - 2$, with boundary data $h_j(t) = -2$ for $t \in (0, T)$ and $j \in \llbracket 0, n \rrbracket$, and still with the same function $g \in C_c^\infty(\omega \times (0, T))$ as above. Let $v = (y - \tilde{y})_t$. Then v satisfies (57), and we have

$$\begin{aligned} \int_{\mathcal{R}} |v(T')|^2 e^{-2s\eta(T')} dx &= \int_0^{T'} \int_{\mathcal{R}} 2vv_t e^{-2s\eta} dx dt + \int_0^{T'} \int_{\mathcal{R}} |v|^2 \frac{\partial}{\partial t} (e^{-2s\eta}) dx dt \\ &= \int_0^{T'} \int_{\mathcal{R}} 2vv_t e^{-2s\eta} dx dt - 2s \int_0^{T'} \int_{\mathcal{R}} |v|^2 \eta_t e^{-2s\eta} dx dt. \end{aligned} \quad (90)$$

Let us denote by A_1 the term at the left hand side and by A_2 and A_3 the two right members of the previous equation.

1. Upper bounds for A_1 .

We know that $|\eta_t| \leq C\varphi^2$ (see (29)), thus

$$|A_3| \leq Cs \int_0^{T'} \int_{\mathcal{R}} |v|^2 \varphi^2 e^{-2s\eta} dx dt \leq Cs \iint_Q |v|^2 \varphi^3 e^{-2s\eta} dx dt \quad (91)$$

where we used (30). Furthermore, we note that $\xi_i(0) = 0$ for all $i \in \llbracket 0, n \rrbracket$ (due to (H_c)), so that $k(t) = -[\xi \tilde{y}_{tx}(t)]_0 = 0$ and $(\xi \tilde{y}_{tx})(0, t) = 0$ so we can apply the Carleman inequality given in Theorem 7 and get

$$s \iint_Q |v|^2 \varphi^3 e^{-2s\eta} dx dt \leq C \left(s \int_0^T \int_\omega |v|^2 \varphi^3 e^{-2s\eta} dx dt + \iint_Q |\xi \tilde{y}_{tx}|^2 \varphi^2 e^{-2s\eta} dx dt \right). \quad (92)$$

First of all, we note that (omitting variables x, t)

$$\varphi^2 e^{-2s\eta} = \varphi^2(T') e^{-2s\eta(T')} \left(\frac{T'(T-T')}{t(T-t)} \right)^2 e^{-2s\eta(T') \left(\frac{T'(T-T')}{t(T-t)} - 1 \right)}. \quad (93)$$

Picking $T' = T/2$, we easily obtain that, for s, λ large enough such that $s\eta(T') > 1$, we have $\left(\frac{T'(T-T')}{t(T-t)} \right)^2 e^{-2s\eta(T') \left(\frac{T'(T-T')}{t(T-t)} - 1 \right)} \leq 1$. Thus with (93) we have the following inequality:

$$\varphi^2 e^{-2s\eta} \leq \varphi^2(T') e^{-2s\eta(T')}. \quad (94)$$

Then we get, since $\tilde{y}_{tx} \in C([0, T], L^\infty(\mathcal{R}))$,

$$\iint_Q |\xi \tilde{y}_{tx}|^2 \varphi^2 e^{-2s\eta} dx dt \leq C \int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx.$$

Then, with (91) and (92), we obtain

$$|A_3| \leq C \left[\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx + s \int_0^T \int_\omega |v|^2 \varphi^3 e^{-2s\eta} dx dt \right]. \quad (95)$$

Using Cauchy-Schwarz inequality and (30), we have

$$|A_2| \leq C \left(\iint_Q s |v|^2 \varphi^3 e^{-2s\eta} dx dt + \iint_Q s^{-1} \varphi^{-1} e^{-2s\eta} |v_t|^2 dx dt \right). \quad (96)$$

Observe that v_t satisfies

$$\begin{cases} (v_t)_t - (cv_{xt})_x = (\xi \tilde{y}_{ttx})_x, & \forall (x, t) \in \mathcal{R} \times (0, T), \\ v_{j,t}(l_j, t) = 0, & \forall j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ v_{j,t}(0, t) = v_{i,t}(0, t), & \forall i, j \in \llbracket 0, n \rrbracket, t \in (0, T), \\ [cv_{tx}(t)]_0 = -[\xi \tilde{y}_{ttx}(t)]_0 = 0, & \forall t \in (0, T). \end{cases} \quad (97)$$

Applying Carleman estimate given in Theorem 7 to v_t , we have by using (94), (30) and the fact that $\tilde{y}_{ttx} \in C([0, T], L^\infty(\mathcal{R}))$

$$\begin{aligned} & \iint_Q s^{-1} \varphi^{-1} e^{-2s\eta} |v_t|^2 dx dt \\ & \leq C s^{-4} \iint_Q (s\varphi)^3 e^{-2s\eta} |v_t|^2 dx dt \\ & \leq C s^{-4} \left[\int_0^T \int_\omega s^3 |v_t|^2 \varphi^3 e^{-2s\eta} dx dt + \iint_Q |\xi \tilde{y}_{ttx}|^2 (s\varphi)^2 e^{-2s\eta} dx dt \right] \\ & \leq C \left[\int_0^T \int_\omega s^{-1} |v_t|^2 \varphi^3 e^{-2s\eta} dx dt + \int_{\mathcal{R}} |\xi|^2 s^{-2} \varphi^2(T') e^{-2s\eta(T')} dx \right]. \end{aligned} \quad (98)$$

Thus, with (92), (95)-(96) and (98), we obtain the following upper bounds for A_1 for s sufficiently large:

$$A_1 \leq C \left[\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx + \int_0^T \int_\omega (s|v|^2 + s^{-1}|v_t|^2) \varphi^3 e^{-2s\eta} dx dt \right]. \quad (99)$$

2. Lower bounds for A_1 .

First we have

$$v(T') = (cy_x - \tilde{c}\tilde{y}_x)_x(T'). \quad (100)$$

Thanks to the study of the stationary case and Proposition 12, we can write

$$\begin{aligned} A_1 &= \int_{\mathcal{R}} |(cy_x - \tilde{c}\tilde{y}_x)_x(T')|^2 e^{-2s\eta(T')} dx \\ &\geq Cs^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx - C(s) \left(\|y_t(T') - \tilde{y}_t(T')\|_{L^2(\omega^1)}^2 \right. \\ &\quad \left. + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})}^2 + \|y(T') - \tilde{y}(T')\|_{H^2(\omega^1)}^2 \right). \end{aligned} \quad (101)$$

Thus, combining (101) with (99), we obtain

$$\begin{aligned} s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx - C(s) (\|y_t(T') - \tilde{y}_t(T')\|_{L^2(\omega^1)}^2 \\ + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})}^2 + \|y(T') - \tilde{y}(T')\|_{H^2(\omega^1)}^2) \\ \leq C \left[\int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx + \int_0^T \int_{\omega} (s|v|^2 + s^{-1}|v_t|^2) \varphi^3 e^{-2s\eta} dx dt \right]. \end{aligned} \quad (102)$$

Thus, for s sufficiently large, we have the stability estimate

$$\begin{aligned} s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx \leq C \int_0^T \int_{\omega} \varphi^3 (s|y_t - \tilde{y}_t|^2 + s^{-1}|y_{tt} - \tilde{y}_{tt}|^2) e^{-2s\eta} dx dt \\ + C(s) (\|y_t(T') - \tilde{y}_t(T')\|_{L^2(\omega^1)}^2 + \|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})}^2 + \|y(T') - \tilde{y}(T')\|_{H^2(\omega^1)}^2), \end{aligned} \quad (103)$$

which completes the proof of Theorem 1. \square

3.2 Stability results with boundary observations

We derive, in the case of boundary measurements, results similar to those obtained in the previous section. In a first time, as for the proof of the Carleman result with one less derivative, we extend the network \mathcal{R} in another network $\hat{\mathcal{R}}$ and we follow the previous case. We make the following assumptions, (H_c) and

$$(H_{Ext}) \quad c_i(l_i) = \tilde{c}_i(l_i), \quad \forall i \in \llbracket 1, n \rrbracket.$$

3.2.1 Stability for the stationary problem

We easily prove the following result, with the same notations as in Section 3.1.1.

Proposition 15. *Let c, \tilde{c} piecewise \mathcal{C}^1 satisfying (4) and $\max_{j \in \llbracket 0, n \rrbracket} \|c_j\|_{W^{1,\infty}(0,l_j)} \leq c_{max}^{1,\infty}$. Under hypotheses (H_1) (with $\omega^0 = \emptyset$), (H_2) , (H_3) , (H_c) and (H_{Ext}) , there exist $C > 0$, $s_0 > 0$ such that for all $s \geq s_0$, there exist $C(s)$, $C(s, c_{max}) > 0$,*

$$\begin{aligned} s^2 \int_{\mathcal{R}} \varphi^2 |\xi|^2 e^{-2s\eta} dx \leq C \int_{\mathcal{R}} |f - \tilde{f}|^2 e^{-2s\eta} dx \\ + C(s, c_{max}) \left(\|u - \tilde{u}\|_{H^1(\mathcal{R})}^2 + \sum_{j=1}^n e^{-2s\eta(l_j, T')} |u_{j,x}(l_j) - \tilde{u}_{j,x}(l_j)|^2 \right). \end{aligned}$$

We only need to take care of the boundary term appearing in B_2 in Lemma 10. Using the fact that $\beta'_0(l_0) < 0$ and that $\beta'_j(l_j) > 0$ for $j \in \llbracket 1, n \rrbracket$ (since β is as given by Lemma 3 for $\hat{\mathcal{R}}$ and $\omega \subset \hat{\mathcal{R}} \setminus \mathcal{R}$), we can obtain the result.

3.2.2 Stability for the evolution problem

We also prove the main stability result for the evolution problem with boundary measurements, namely Theorem 2.

Proof of Theorem 2. Let $y^0 \in L^2(\mathcal{R})$. First, we pick $\tau \in (0, T')$ where $T' = T/2$. We take $h_j(t) = 0$ for $0 \leq t \leq \tau$, so that $y(\tau) \in D(A^4)$. As in the proof of Theorem 9, we extend the network \mathcal{R} in another network $\hat{\mathcal{R}}$, and we fix an open set $\omega \subset \cup_{j=1}^n (0, \hat{l}_j)$ which intersects each interval (l_j, \hat{l}_j) for $j \in \llbracket 1, n \rrbracket$. We also extend c and \tilde{c} smoothly such that $0 < c_{min} \leq c(x) \leq c_{max} < \infty$ in $\hat{\mathcal{R}}$. We can extend $y(\tau)$ (resp. $\tilde{y}(\tau)$) as a function in $D(A^4)$ (resp. $D(\tilde{A}^4)$) for c and $\hat{\mathcal{R}}$ (resp. for \tilde{c} and $\hat{\mathcal{R}}$). Applying Proposition 13 to $\hat{\mathcal{R}}$ and $\tilde{y}(\tau)$, we can find an input $g \in C_c^\infty(\omega \times (\tau, T))$ such that $|\tilde{y}_x(\cdot, T)| \geq \delta > 0$ in \mathcal{R} and $\|\tilde{y}(\cdot, T')\|_{L^\infty(\mathcal{R})} + \|\tilde{y}_x(\cdot, T')\|_{L^\infty(\mathcal{R})} \leq \delta^{-1}$. We pick a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t \leq T'$ and $\psi(t) = 0$ for $t \geq 3T/4$, and we set $h_j(t) = \psi(t)\tilde{y}_j(l_j, t)$ for $t \in [\tau, T]$ and $j \in \llbracket 1, n \rrbracket$, and $h_0(t) = 0$ for $t \in [0, T]$. We claim that $h_j \in C_c^1([0, T])$ for $j \in \llbracket 1, n \rrbracket$. Indeed, $0 = h_j'(\tau^-) = (\tilde{c}_j \tilde{y}_{j,x})_x(l_j, \tau) = h_j'(\tau^+)$ thanks to (56) applied with $x = l_j$ and $t = \tau^-$, $t = \tau^+$. On the other hand, $h_j(t) = 0$ for $t \in [0, T] \setminus (\tau, 3T/4)$ and $j \in \llbracket 1, n \rrbracket$.

Next, we use the same type of proof as previously for the case of interior observation. We have three terms A_1 , A_2 and A_3 to estimate as in (90).

By using the same estimate as in (91), we get

$$|A_3| \leq Cs \int_0^T \int_{\mathcal{R}} |v|^2 \varphi^3 e^{-2s\eta} dx dt. \quad (104)$$

By applying the Carleman estimate given in Theorem 9 (since $[cv_x(t)]_0 = -[\xi \tilde{y}_{tx}(t)]_0 = 0$ due to (H_c)), we obtain

$$\begin{aligned} s \iint_Q |v|^2 \varphi^3 e^{-2s\eta} dx dt &\leq C \left(\iint_Q \varphi^2 e^{-2s\eta} |\xi \tilde{y}_{tx}|^2 dx dt \right. \\ &\quad \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)|^2 dt \right). \end{aligned} \quad (105)$$

As in the previous section, we can get with $T' = T/2$ and (94)

$$\begin{aligned} |A_3| &\leq C \left[\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx \right. \\ &\quad \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)|^2 dt \right]. \end{aligned} \quad (106)$$

We clearly have (96) and (97). Therefore by applying the Carleman estimate given in Theorem 9 to v_t , we obtain

$$\begin{aligned} \iint_Q s^{-1} \varphi^{-1} e^{-2s\eta} |v_t|^2 dx dt &\leq Cs^{-4} \iint_Q (s\varphi)^3 e^{-2s\eta} |v_t|^2 dx dt \\ &\leq C \left[s^{-2} \int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx \right. \\ &\quad \left. + s^{-3} \int_0^T \sum_{j=1}^n \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{ttx}(l_j, t) + c(l_j) v_{xt}(l_j, t)|^2 dt \right]. \end{aligned} \quad (107)$$

With (106) and (107), we obtain an upper bound for A_1 for s sufficiently large:

$$\begin{aligned}
A_1 \leq & C \left[\int_{\mathcal{R}} |\varphi(T')|^2 e^{-2s\eta(T')} |\xi|^2 dx \right. \\
& + s^{-3} \int_0^T \sum_{j=1}^n \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{ttx}(l_j, t) + c(l_j) v_{xt}(l_j, t)|^2 dt \\
& \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)|^2 dt \right]. \quad (108)
\end{aligned}$$

We now search for lower bounds for A_1 . First we have that

$$v(T') = (cy_x - \tilde{c}\tilde{y}_x)_x(T'). \quad (109)$$

Thanks to the study of the stationary case and Proposition 15, we can write

$$\begin{aligned}
A_1 &= \int_{\mathcal{R}} |(cy_x - \tilde{c}\tilde{y}_x)_x(T')|^2 e^{-2s\eta(T')} dx \\
&\geq Cs^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx \\
&\quad - C(s) \left(\|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})}^2 + \sum_{j=1}^n e^{-2s\eta_j(l_j, T')} |y_{j,x}(l_j, T') - \tilde{y}_{j,x}(l_j, T')|^2 \right). \quad (110)
\end{aligned}$$

Thus with (108), (110) and (H_{Ext}) , we get for s sufficiently large

$$\begin{aligned}
& s^2 \int_{\mathcal{R}} e^{-2s\eta(T')} |\xi|^2 dx \\
& \leq C(s) \left[\|y(T') - \tilde{y}(T')\|_{H^1(\mathcal{R})}^2 + \sum_{j=1}^n e^{-2s\eta_j(l_j, T')} |y_{j,x}(l_j, T') - \tilde{y}_{j,x}(l_j, T')|^2 \right. \\
& \quad + s^{-3} \int_0^T \sum_{j=1}^n \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{ttx}(l_j, t) + c(l_j) v_{xt}(l_j, t)|^2 dt \\
& \quad \left. + \int_0^T \sum_{j=1}^n s^{-1} \varphi_j(l_j, t) e^{-2s\eta_j(l_j, t)} |\xi(l_j) \tilde{y}_{tx}(l_j, t) + c(l_j) v_x(l_j, t)|^2 dt \right].
\end{aligned}$$

This completes the proof of Theorem 2, since $\xi(l_j) \tilde{y}_{tx}(l_j) = 0$ for $j \in \llbracket 1, n \rrbracket$ (see (H_{Ext})). \square

4 Conclusion

As it was said in the introduction, these results can be extended to general tree-shaped networks: with boundary measurements at all exterior nodes except one, the root of the tree, or with internal measurements at all exterior edges of the tree except one. The key point is in the choice of the weights in the Carleman estimates. More precisely, we obtain, by integration by parts, instead of one matrix A_β defined in Lemma 3, several matrices of the same kind, corresponding to each interior node. Then, we take second order polynomial functions like (8) for external edges and affine functions like (7) for all remaining edges. The construction is made by induction in a similar way as in [3]. The main difficulty here is just to use clear and understandable notations.

However the case of a general network with a cycle remains an open question.

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