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FINITE TIME BLOW-UP FOR A NONLINEAR PARABOLIC EQUATION WITH SMOOTH COEFFICIENTS

OSCAR JARRÍN AND GASTÓN VERGARA-HERMOSILLA

ABSTRACT. In this article, we consider an n -dimensional parabolic partial differential equation with a smooth coefficient term in the nonlinear gradient term. This equation was first introduced and analyzed in [E. Issoglio, On a non-linear transport-diffusion equation with distributional coefficients, *Journal of Differential Equations*, Volume 267, Issue 10 (2019)], where one of the main *open questions* is the possible *finite-time blow-up* of solutions. Here, leveraging a *virial-type estimate*, we provide a *positive answer* to this question within the framework of smooth solutions.

1. INTRODUCTION AND MAIN RESULT

This article investigates the finite-time blow-up of smooth solutions to the following nonlinear parabolic equation:

$$(1) \quad \begin{cases} \partial_t u = \Delta u + |\nabla u|^2 \mathfrak{b}, \\ u(0, x) = u_0(x), \end{cases}$$

where, given a time $T > 0$ and $n \geq 1$, $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the unknown solution, $\mathfrak{b} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a prescribed coefficient term, and $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ stands for the initial datum.

The primary motivation for studying this class of models stems from different physical and probabilistic contexts. In the latter, specific choices of singular coefficients $\mathfrak{b}(t, x)$ are particularly relevant when modeling the realization of random noise. We refer to the interested reader to [10, 11, 12] for further details.

In the deterministic case, the equation (1) was first introduced and studied in [13]. In order to describe some of the main results in such paper, it is convenient to recall that, for any parameter $\gamma \in \mathbb{R}$, the Hölder-Zygmund spaces $\mathcal{C}^\gamma(\mathbb{R}^n)$ can be characterized through the general framework of Besov spaces yielding the equivalence $\mathcal{C}^\gamma(\mathbb{R}^n) \approx B_{\infty, \infty}^\gamma(\mathbb{R}^n)$ (see for instance [19, Chapter 2]). Within this framework, the main contribution of [13] is to establish the existence and uniqueness of *local-in-time* solutions $u \in \mathcal{C}_t \mathcal{C}_x^{\alpha+1}$, which arise from initial data in Hölder-Zygmund spaces of positive order $\mathcal{C}^{\alpha+1}(\mathbb{R}^n)$, for appropriate values of $\alpha > 0$, and coefficient terms in Hölder-Zygmund spaces of negative order $\mathcal{C}^\beta(\mathbb{R}^n)$, for suitably chosen $\beta < 0$. In particular, as mentioned above, this singular coefficient terms are motivated by a probabilistic interpretation.

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More precisely, such solutions are constructed by using the following (formally equivalent) mild formulation:

$$u(t, \cdot) = h_t * u_0 + \int_0^t h_{t-\tau} * (|\nabla u(\tau, \cdot)|^2 \mathfrak{b}(\tau, \cdot)) d\tau,$$

where h_t denotes the well-known heat kernel. In this formulation, we highlight that the nonlinear expression $|\nabla u(t, x)|^2$, together with the coefficient term $\mathfrak{b}(t, x)$, make the nonlinear term delicate to handle. To overcome this issue, some suitably constraints on the parameters $\beta < 0 < \alpha$ are required. Specifically, the technical relationship $-1/2 < \beta < 0 < \alpha < 1$, together with the assumptions that $u_0 \in \mathcal{C}_x^{\alpha+1}$ and $\mathfrak{b} \in \mathcal{C}_t \mathcal{C}_x^\beta$, allowed the author of [13] to prove a suitably control on the nonlinear term above. Consequently, using a contraction principle, the solution $u \in \mathcal{C}_t \mathcal{C}_x^{\alpha+1}$ to the problem above is rigorously obtained for a time $T_0 > 0$, which is sufficiently small respect with the quantity $\|u_0\|_{\mathcal{C}^{\alpha+1}}$ and the parameters α, β . For more details, we refer to [13, Theorem 3.7].

Due to the restrictions on the existence time T_0 , which are essentially imposed by the nonlinear term $|\nabla u|^2$, one of the main *open questions* in [13] concerns the global-in-time or blow-up in finite time of solutions to the equation (1). In this context, in [13, Proposition 3.13], it is proven a first blow-up criterion, essentially stating that if blow-up occurs then it must holds in a finite time $t_* > 0$.

Thus, with the aim of studying the possible blow-up of solutions, in [3] the authors analyzed the following toy model for equation (1):

$$(2) \quad \begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + |(-\Delta)^{\frac{1}{2}} u|^2 * \mathfrak{b}, & 0 < \alpha \leq 2, \\ u(0, x) = u_0(x). \end{cases}$$

Here, the classical Laplacian operator is substituted by its fractional power, which is defined at the Fourier level by the symbol $|\xi|^\alpha$. However, the main featured of this model is given by the *modified nonlinear term*, where the expression $|\nabla u|^2$ becomes the nonlocal expression with the same derivative order $|(-\Delta)^{\frac{1}{2}} u|^2$; and the product with the coefficient term $\mathfrak{b}(t, x)$ becomes a product of convolution.

This somehow strong modification on the nonlinear term, allow the authors of [3] to apply a Fourier-based method to prove the blow-up in finite time of solutions. First, for any initial datum $u_0 \in H^1(\mathbb{R}^n)$, in [3, Theorem 1] the authors construct local-in-time mild solutions $u \in L^\infty([0, T_0], H^1(\mathbb{R}^n))$ to the equation (2). These solutions are obtained from an interesting relationship between the fractional power α and the regularity in spatial variable assumed on the coefficient $\mathfrak{b}(t, x)$. More precisely, the weak smoothing effects of the fractional Laplacian operator in the case $0 < \alpha < 1$ are compensated by stronger regular coefficients $\mathfrak{b} \in L_t^\infty H_x^\gamma$ with $0 \leq 1 - \alpha \leq \gamma < 1$. In contrast, stronger smoothing effects in the case when $1 < \alpha \leq 2$ allow to consider singular coefficients $\mathfrak{b} \in L_t^\infty H_x^{-\gamma}$, with $0 \leq \gamma < \alpha - 1$. Having constructed mild solutions $u \in L_t^\infty H_x^1$, the main result of [3] states that, in the Fourier variable, one can define well-prepared initial data $\widehat{u_0}(\xi)$ and coefficient terms $\widehat{\mathfrak{b}}(t, \xi)$ to obtain the blow-up of the associated solution $u(t, \cdot)$ at a finite time $t_* > 0$, which explicitly depends on the fractional power α by the expression $t_* = \frac{\ln(2)}{2^\alpha}$.

This approach is mainly inspired by the method of Montgomery-Smith [17], which was firstly considered to study the blow-up of the well-known cheap Navier-Stokes equations. See also [16, Section 11.2] and [4, 8] for more details and related results. In this method, since the operator

$(-\Delta)^{\frac{1}{2}}$ has the positive symbol $|\xi|$ in the Fourier level, along with the well-known fact that the convolution product with $\mathfrak{b}(t, x)$ becomes a pointwise product with $\widehat{\mathfrak{b}}(t, \xi)$, the authors of [3] are able to derive suitably lower bounds for the mild formulation of the solution in the Fourier variable:

$$\widehat{u}(t, \xi) = e^{-|\xi|^\alpha t} \widehat{u_0}(\xi) + \int_0^t e^{-|\xi|^\alpha(t-\tau)} \left((|\xi| \widehat{u} * |\xi| \widehat{u}) \widehat{\mathfrak{b}} \right) (\tau, \xi) d\tau.$$

These lower bounds, involving well-chosen weight functions defined in the separate variables $\xi \in \mathbb{R}^n$ and $t > 0$, yield the finite blow-up of the expression $\|u(t, \cdot)\|_{H^1}$, when $t \rightarrow t_*$.

However, the method of Montgomery-Smith seems to break down for the original equation (1). Essentially, in contrast to the fractional Laplacian operator $(-\Delta)^{\frac{1}{2}}$, now each component of the gradient operator ∇ in the nonlinear term has the imaginary symbol $i\xi_j$. Consequently, when deriving lower bound on the expression $\widehat{u}(t, \xi)$, one loses the sign information in the nonlinear term, which is one of the key elements in this method. Consequently, the previous results [3, 13] left open the possible blow-up of solution to the equation (1).

In this context, this article gives a positive answer to this problematic of equation (1). Our main contribution is to show that, in this equation, one can define suitable chosen initial data $u_0(x)$ and coefficient terms $\mathfrak{b}(t, x)$ which, together with the nonlinear effects of the term $|\nabla u|^2$, produces a blow-up in finite time in the solution.

To achieve such result, we apply a different approach mainly inspired by [5], in the setting of a nonlocal transport equation, and [6], in the setting of the fractional Burgers equation. As we will elaborate in more detail in the following section, this method fundamentally relies on the differential formulation of equation (1), where, for a well-chosen weight function, we are able to derive a *virial-type estimate* on the solution yielding its blow-up in finite time.

The main result. To implement the method mentioned above, we will work with smooth solutions of equation (1). Specifically, we will require certain local boundedness and continuity properties of their derivatives with respect to the spatial variable. For the completeness of this article, we will start by constructing these solutions, which arise from initial data and coefficient terms in Sobolev spaces.

Proposition 1.1. *For $s > n/2 + 3$, let $u_0 \in H^s(\mathbb{R}^n)$ be an initial datum. Additionally, assume that the coefficient $\mathfrak{b}(t, x)$ satisfies*

$$(3) \quad \mathfrak{b} \in \mathcal{C} \cap L^\infty([0, +\infty[, H^{s-1}(\mathbb{R}^n)).$$

Then, there exists a time $T_0 > 0$, which depends on u_0 and \mathfrak{b} , and there exists a function

$$u \in \mathcal{C}([0, T_0], H^s(\mathbb{R}^n)),$$

which is the unique solution to the equation (1).

By the assumption $s > n/2 + 3$, the Sobolev embedding $H^s(\mathbb{R}^n) \subset \mathcal{C}^3(\mathbb{R}^n)$ holds. Then, it follows that $u_0 \in \mathcal{C}^3(\mathbb{R}^n)$, $\mathfrak{b} \in \mathcal{C}^1([0, +\infty[, \mathcal{C}^2(\mathbb{R}^n))$ and $u \in \mathcal{C}([0, T_0], \mathcal{C}^3(\mathbb{R}^n))$. Particularly, the solution $u(t, x)$ verifies the the equation (1) in the classical sense. It is worth mentioning that these solutions could also be obtained using the more general framework of Besov spaces. Additionally, we do not study the optimality of the relationship $s > n/2 + 3$ for obtaining smooth solutions. As explained, our main objective is to focus directly on the blow-up issue.

Now, with this information at hand, we briefly describe the strategy to prove the finite blow-up of smooth solutions $u(t, x)$ to equation (1). In this equation, the quadratic term $|\nabla u|^2$ suggests to introduce the vector field $\mathbf{v}(t, x) := \nabla u(t, x)$, which is a solution of the following nonlinear system:

$$(4) \quad \begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} + \nabla(|\mathbf{v}|^2 \mathbf{b}), \\ \mathbf{v}(0, \cdot) = \nabla u_0. \end{cases}$$

Since $u \in \mathcal{C}_t H_x^s$ with $s > n/2 + 3$, it follows that $\mathbf{v} \in \mathcal{C}_t H_x^{s-1} \subset \mathcal{C}_t \mathcal{C}_x^2$. Consequently, the vector field $\mathbf{v}(t, x)$ also solves this system in the classical sense.

Using the information $\mathbf{v} \in \mathcal{C}_t \mathcal{C}_x^2$, the key idea in our method is to first show that the solution $\mathbf{v}(t, x)$ blows-up in a finite time $t_* > 0$, which in turn yields the blow-up of the quantity $\|u(t, \cdot)\|_{H^s}$ as $t \rightarrow t_*$.

The blow-up phenomenon of the solution $\mathbf{v}(t, x)$ arises from the nonlinear effects of the term $\nabla(|\mathbf{v}|^2 \mathbf{b})$, along with certain suitable conditions on the coefficient $\mathbf{b}(t, x)$. Specifically, for simplicity, we will assume that this coefficient depends only on the spatial variable. Additionally, for any $x = (x_1, \dots, x_i, \dots, x_n) \in \mathbb{R}^n$, we assume that

$$(5) \quad \mathbf{b}(x) = \prod_{i=1}^n \mathbf{b}_i(x_i),$$

where each term $\mathbf{b}_i(x_i)$ is a *non-constant* function satisfying:

$$(6) \quad \mathbf{b}_i \in \mathcal{S}(\mathbb{R}), \quad \mathbf{b}_i(x_i) \geq 0 \quad \text{for any } x_i \in \mathbb{R}, \quad \mathbf{b}_i(0) = 0.$$

Note that from the expression (5) it holds that $\mathbf{b} \in \mathcal{C} \cap L^\infty([0, +\infty[, \mathcal{S}(\mathbb{R}^n))$. Consequently, this coefficient verifies the assumption (3).

In expressions (11)–(12) below, we define a suitable vector field $\mathbf{w}(x)$ localized over the compact domain $[-1, 1]^n$, which is essentially chosen to illustrate our main computations. Then, given the solution $\mathbf{v}(t, x)$ of the system (4), the virial-type estimates that we shall derive allow us to study the time evolution of the functional

$$I(t) := \int_{[-1, 1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx.$$

The assumption (5) on the coefficient $\mathbf{b}(x)$ yields that this functional satisfies an ordinary differential inequality with a quadratic nonlinearity. From this, the expression $I(t)$ blows up in finite time as long as the initial data $u_0(x)$ satisfies the conditions:

$$(7) \quad \int_{[-1, 1]^n} \nabla u_0(x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx > 0,$$

and

$$(8) \quad \left(\int_{[-1, 1]^n} \nabla u_0(x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx \right)^2 \geq \mathfrak{C} \times \begin{cases} \|u_0\|_{H^s}, & \text{when } \|u_0\|_{H^s} \leq 1, \\ \|u_0\|_{H^s}^2, & \text{when } \|u_0\|_{H^s} > 1, \end{cases}$$

for some constant $\mathfrak{C} > 0$ essentially depending on n , the (technical) parameter κ introduced in expression (12) below, and the quantities $\|\mathbf{b}\|_{L^\infty}$ and $\|\nabla \mathbf{b}\|_{L^\infty}$. In this context, our main result reads as follows:

Theorem 1.1. *Under the same assumptions as in Proposition 1.1, suppose that the coefficient $\mathbf{b}(t, x)$ satisfies conditions (5)-(6), and the initial datum $u_0(x)$ adheres to conditions (7)-(8).*

Then, for any given number $m_ \gg 1$ there exists a finite time $t_* = t_*(m_*) > 0$, depending on m_* and explicitly provided in the expression (28) below, such that the quantity $\|u(t, \cdot)\|_{H^s}$ blows-up at t_* .*

The following comments are in order. First, as previously mentioned, we show that smooth solutions of equation (1) can develop a blow-up phenomenon in the case of a family of well-chosen coefficient terms satisfying assumption (5) and a family of well-chosen initial data satisfying (7) and (8). In particular, this type of assumption on initial data appears in previous blow-up results [1, 6, 15] for different one-dimensional models. In this setting, our result can also be seen as a generalization of this method to the n -dimensional case. On the other hand, it is interesting to point out that a functional similar to $I(t)$ was previously considered in the study of the Dyadic model of the Navier-Stokes equations in [4].

In the particular case of equation (2) when $\mathbf{b} \equiv 1$, it is interesting to make a comparison with the model:

$$\partial_t u = \Delta u + |\nabla u|^2.$$

In [9], the authors prove the existence of a unique *global-in-time* explicit solution arising from any bounded and continuous initial datum. This fact shows that *non-constant* coefficient terms \mathbf{b} plays a substantial role in the time dynamics of equation (2). See also [7, 18] for related results.

Organization of the rest of the article. In Section 2, we recall some well-known tools that we will use in our proofs. Section 3 is devoted to providing a proof of Proposition 1.1. Finally, in Section 4, we prove our main result stated in Theorem 1.1.

2. PRELIMINARIES

In order to keep this paper reasonably self-contained, in this section we collect some results for the proof of our main results. Thus, we begin by presenting well-known facts about Sobolev spaces $H^s(\mathbb{R}^n)$.

Proposition 2.1 (H^s estimate). *Let $s_1, s_2 \geq 0$. Then, there is a constant $C > 0$, which depends on the dimension $n \in \mathbb{N}^*$ and the parameter s_2 , such that*

$$\|h_t * \varphi\|_{H^{s_1+s_2}} \leq C (1 + t^{-s_2}) \|\varphi\|_{H^{s_1}}.$$

A proof of this result can be consulted in [14].

Proposition 2.2 (Banach algebra). *Let $f, g \in H^s(\mathbb{R}^n)$, for $s > n/2$. Then, $H^s(\mathbb{R}^n)$ is a Banach algebra. In particular, there exists $C > 0$, which depends on the dimension n and the parameter s , such that*

$$\|fg\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}.$$

A proof of this result can be consulted in the appendix of book [2].

Finally, we recall a classical abstract result which will be considered to construct an unique (local in time) solution of (1).

Theorem 2.1 (Banach-Picard principle). *Consider a Banach space $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ and a bounded bilinear application $B : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$:*

$$\|B(e, e)\|_{\mathcal{E}} \leq C_B \|e\|_{\mathcal{E}}^2.$$

Given $e_0 \in \mathcal{E}$ such that $\|e_0\|_{\mathcal{E}} \leq \gamma$ with $0 < \gamma < \frac{1}{4C_B}$, then the equation

$$e = e_0 - B(e, e),$$

admits a unique solution $e \in \mathcal{E}$ which satisfies $\|e\|_{\mathcal{E}} \leq 2\gamma$.

3. LOCAL WELLPOSEDNESS RESULT: PROOF OF PROPOSITION 1.1

In order to prove Proposition 1.1, in this section we will use the Banach-Picard principle for a time $0 < T < +\infty$ small enough. To this end we consider the Banach space

$$\mathcal{E}_T = \mathcal{C}([0, T], H^s(\mathbb{R}^n)),$$

which is endowed with its natural norm. Thus, with this framework at hand, we begin by considering the following classical estimate for the initial data

$$(9) \quad \|h_t * u_0\|_{\mathcal{E}_T} \leq C_1 \|u_0\|_{H^s},$$

where $C_1 > 0$ is a constant. Then, we study the nonlinear term:

Lemma 3.1. *Let $\mathfrak{b} \in \mathcal{C} \cap L^\infty([0, +\infty], H^{s-1}(\mathbb{R}^n))$, with $s > n/2 + 3$. Then, there exists a constant $C_{\mathcal{B}}(T) > 0$, which depends on the time T and the norm $\|\mathfrak{b}\|_{L_t^\infty H_x^{s-1}}$, such that*

$$(10) \quad \left\| \int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds \right\|_{\mathcal{E}_T} \leq C_{\mathcal{B}}(T) \|u\|_{\mathcal{E}_T}^2.$$

Proof. To start, we consider the H^s -norm to the term $\int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds$ to get

$$\left\| \int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds \right\|_{H^s} \leq \int_0^t \|h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot)\|_{H^s} ds.$$

Then, by using Proposition 2.1 with $s_1 = s - 1$ and $s_2 = 1$, we can write

$$\int_0^t \|h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot)\|_{H^s} ds \leq \int_0^t \left(C + \frac{C}{(t-s)^{\frac{1}{2}}} \right) \| |\nabla u|^2 \mathfrak{b}(s, \cdot) \|_{H^{s-1}} ds.$$

Now, since $H^\sigma(\mathbb{R}^n)$ is a Banach algebra for $\sigma > n/2$ (see Proposition 2.2 with $\sigma = s - 1$), we obtain

$$\left\| \int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds \right\|_{H^s} \leq \int_0^t \left(C + \frac{C}{(t-s)^{\frac{1}{2}}} \right) \|\nabla u(s, \cdot)\|_{H^{s-1}}^2 \|\mathfrak{b}(s, \cdot)\|_{H^{s-1}} ds.$$

Thus, by taking the L^∞ -norm on the time interval $[0, T]$, we get

$$\left\| \int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds \right\|_{\mathcal{E}_T} \leq \left(\int_0^T C + \frac{C}{(T-s)^{\frac{1}{2}}} ds \right) \|\mathfrak{b}\|_{L_t^\infty H_x^{s-1}} \|u\|_{\mathcal{E}_T}^2,$$

and then, by considering our assumption on \mathfrak{b} , we conclude the estimate

$$\left\| \int_0^t h_{t-s} * |\nabla u|^2 \mathfrak{b}(s, \cdot) ds \right\|_{\mathcal{E}_T} \leq C(T + \sqrt{T}) \|\mathfrak{b}\|_{L_t^\infty H_x^{s-1}} \|u\|_{\mathcal{E}_T}^2.$$

Therefore, by considering $C_{\mathcal{B}}(T) = C(T + \sqrt{T}) \|\mathbf{b}\|_{L_t^\infty H_x^{s-1}}$, we deduce (10) and we conclude the proof of this lemma. \square

End of the proof. Gathering together the hypothesis assumed in the statement of Proposition 1.1 and the estimates obtained in estimates (9) and (10), we conclude the existence of a time $0 < T_0 < +\infty$ such that $4C_1 C_{\mathcal{B}}(T_0) \|u_0\|_{H^s} < 1$. Thus, by applying the Banach-Picard principle stated in Theorem 2.1 we obtain a unique (local in time) solution of (1). This concludes the proof of Proposition 1.1.

4. BLOW-UP RESULT: PROOF OF THEOREM 1.1

For $s > n/2 + 3$ and $T_0 > 0$, let $u \in \mathcal{C}([0, T_0], H^s(\mathbb{R}^n))$ be the smooth solution to the equation (1), which is obtained in Proposition 1.1, from any initial datum $u_0 \in H^s(\mathbb{R}^n)$. In order to prove Theorem 1.1, we argue by contradiction and we assume the following key hypothesis

(H) : the solution $u(t, x)$ can be extended to the entire interval of time $[0, +\infty[$.

In this framework, we define the vector field $\mathbf{v}(t, x) := \nabla u(t, x)$, which verifies the system (4), and from hypothesis (H), it exists globally-in-time.

Now, for any $x = (x_1, \dots, x_i, \dots, x_n) \in \mathbb{R}^n$, we define the following vector field of weights

$$(11) \quad \mathbf{w}(x) = (w_1(x_1), \dots, w_i(x_i), \dots, w_n(x_n)),$$

with

$$(12) \quad w_i(x_i) := \begin{cases} \text{sign}(x_i)(|x_i|^{-\kappa} - 1), & -1 < x_i < 1, \quad x_i \neq 0, \quad 0 < \kappa < 1, \\ 0, & \text{otherwise.} \end{cases}$$

From (12) we deduce that each component is an odd function supported in the interval $[-1, 1]$, and thus we can see that $\text{supp}(\mathbf{w}) \subseteq [-1, 1]^n$.

Now, considering the coefficient $\mathbf{b}(x)$ defined in (5) and (6), we take the \mathbb{R}^n -inner product in each term of the system (4) with $\mathbf{b}(x) \mathbf{w}(x)$. Then, integrating over \mathbb{R}^n and noting that $\text{supp}(\mathbf{w}) \subseteq [-1, 1]^n$, we obtain

$$(13) \quad \begin{aligned} \frac{d}{dt} \left(\int_{[-1, 1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) \, dx \right) &= \int_{[-1, 1]^n} \Delta \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) \, dx \\ &+ \int_{[-1, 1]^n} \nabla(|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \cdot (\mathbf{b}(x) \mathbf{w}(x)) \, dx. \end{aligned}$$

Thus, in the following technical propositions, we derive a lower bound for each term on the right-hand side of (13).

To state the first proposition below, recall that the system (4) is directly related to the equation (1) through the relationship $\mathbf{v}(t, x) := \nabla u(t, x)$. Thus, we will shift from one to the other as needed, without distinction.

Proposition 4.1. *For $s > n/2 + 3$, let $u_0 \in H^s(\mathbb{R}^n)$ be the initial datum in equation (1). Assume the hypothesis (H), which yields that for any given time $T > 0$, the solution $\mathbf{v}(t, x)$ of the system equation (4) is defined over the interval $[0, T]$. Additionally, let $0 < \kappa < 1$ be the parameter introduced in expression (12).*

Then, there exists a constant $\mathbf{C}_1(T) = \mathbf{C}_1(n, \kappa, \|\mathbf{b}\|_{L^\infty}, T) > 0$, such that for any time $0 < t \leq T$ the following lower bound holds:

$$(14) \quad \int_{[-1,1]^n} \Delta \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \geq -\mathbf{C}_1(T) \|u_0\|_{H^s}.$$

Proof. Considering (11) and (12), since $0 < \kappa < 1$, we can directly deduce that $\mathbf{w} \in L^1(\mathbb{R}^n)$. Additionally, we can find a constant $C_{n,\kappa} > 0$ such that $\|\mathbf{w}\|_{L^1} \leq C_{n,\kappa} < +\infty$.

Then, applying Hölder inequalities and recalling that $\mathbf{v}(t, x) := \nabla u(t, x)$, we write

$$(15) \quad \left| \int_{[-1,1]^n} \Delta \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \right| \leq \|\Delta \mathbf{v}(t, \cdot)\|_{L^\infty} \|\mathbf{b}\|_{L^\infty} \|\mathbf{w}\|_{L^1} \\ \leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} \|\Delta \mathbf{v}(t, \cdot)\|_{L^\infty}$$

Thus, using Sobolev embeddings (with a parameter $n/2 < \sigma < s - 3$), we obtain

$$(16) \quad \left| \int_{[-1,1]^n} \Delta \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \right| \leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} C \|\Delta \mathbf{v}(t, \cdot)\|_{H^\sigma} \\ \leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} C \|u(t, \cdot)\|_{H^{\sigma+3}} \\ \leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} C \|u(t, \cdot)\|_{H^s}.$$

Here, we still need to estimate the term $\|u(t, \cdot)\|_{H^s}$. Note that, from Hypothesis (H), the solution $u(t, x)$ to equation (1) is extended to the interval $[0, +\infty[$ by the following standard iterative argument. For $k \in \mathbb{N}$, and suitably chosen times $0 < T_k < T_{k+1}$, particularly $T_{k+1} - T_k$ must be sufficiently small, we consider the initial datum $u(T_k, \cdot)$, and applying again Picard's fixed point scheme in the space $\mathcal{C}_t H_x^s$ we extend the solution to the interval of time $[T_k, T_{k+1}]$. Furthermore, we know that there exists a constant $c_k > 0$ such that $\sup_{T_k \leq t \leq T_{k+1}} \|u(t, \cdot)\|_{H^s} \leq c_k \|u(T_k, \cdot)\|_{H^s}$.

Iterating these estimates, we can find a constant $C_k > 0$ big enough, where $C_k > \prod_{j=0}^k c_j$, such that the following estimate holds

$$\sup_{T_k \leq t \leq T_{k+1}} \|u(t, \cdot)\|_{H^s} \leq c_k \|u(T_k, \cdot)\|_{H^s} \leq C_k \|u_0\|_{H^s}.$$

Thus, for any given time $T > 0$, there exists $k = k(T) \in \mathbb{N}$ such that

$$(17) \quad \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^s} \leq \sum_{j=0}^{k(T)} \sup_{T_j \leq t \leq T_{j+1}} \|u(t, \cdot)\|_{H^s} \leq \left(\sum_{j=0}^{k(T)} C_j \right) \|u_0\|_{H^s} =: C(T) \|u_0\|_{H^s}.$$

With this control at hand, we return to (16) to obtain

$$\left| \int_{[-1,1]^n} \Delta \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \right| \leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} C(T) \|u_0\|_{H^s} =: \mathbf{C}_1(T) \|u_0\|_{H^s},$$

which yields the desired estimate (14). \square

Proposition 4.2. *Within the framework of Proposition 4.1, for any $0 \leq t \leq T$, there exists a constant $\mathbf{C}_2(T) = \mathbf{C}_2(n, \kappa, \|\mathbf{b}\|_{L^\infty}, \|\nabla \mathbf{b}\|_{L^\infty}, T) > 0$ such that the following lower estimate holds*

$$(18) \quad \int_{[-1,1]^n} \nabla(|v(t,x)|^2 \mathbf{b}(x)) \cdot (\mathbf{b}(x)w(x)) \, dx \geq \frac{\kappa}{2^{n+1}} \left(\int_{[-1,1]^n} v(t,x) \cdot (\mathbf{b}(x)w(x)) \, dx \right)^2 - \mathbf{C}_2(T) \|u_0\|_{H^s}^2.$$

Proof. To begin, we consider expression (11), to write

$$(19) \quad \begin{aligned} & \int_{[-1,1]^n} \nabla(|v(t,x)|^2 \mathbf{b}(x)) \cdot (\mathbf{b}(x)w(x)) \, dx \\ &= \sum_{i=1}^n \int_{[-1,1]^n} \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}(x)w_i(x_i)) \, dx \\ &= \sum_{i=1}^n \int_{-1}^1 \cdots \int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}(x)w_i(x_i)) \, dx_1 \cdots dx_i \cdots dx_n \\ &= \sum_{i=1}^n \int_{-1}^1 \cdots \left(\int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}(x)w_i(x_i)) \, dx_i \right) dx_1, \cdots, dx_{i-1} dx_{i+1} \cdots dx_n. \end{aligned}$$

Then, for a fixed $1 \leq i \leq n$, we will study the term $\int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}(x)w_i(x_i)) \, dx_i$. First, from the explicit definition of the coefficient term $\mathbf{b}(x)$ given in (5), we write

$$(20) \quad \begin{aligned} & \int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}(x)w_i(x_i)) \, dx_i \\ &= \int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) \left(\prod_{j=1}^n \mathbf{b}_j(x_j) w_i(x_i) \right) \, dx_i \\ &= \left(\int_{-1}^1 \partial_i(|v(t,x)|^2 \mathbf{b}(x)) (\mathbf{b}_i(x_i) w_i(x_i)) \, dx_i \right) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j). \end{aligned}$$

Thereafter, to compute the integral above, we split it into the intervals $[-1, 0]$ and $[0, 1]$. Note that, from expression (12), the the weight $w_i(x_i)$ verifies $w_i(-1) = w_i(1) = 0$. Additionally, from the assumption (6), the coefficient \mathbf{b}_i verifies $\mathbf{b}_i(0) = 0$. Therefore, performing integration by parts and

rearranging terms, we obtain

$$\begin{aligned}
& \int_{-1}^1 \partial_i (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) (\mathbf{b}_i(x_i) w_i(x_i)) dx_i \\
&= \int_{-1}^0 \partial_i (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) (\mathbf{b}_i(x_i) w_i(x_i)) dx_i + \int_0^1 \partial_i (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) (\mathbf{b}_i(x_i) w_i(x_i)) dx_i \\
(21) \quad &= - \int_{-1}^0 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i'(x_i) dx_i - \int_0^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i'(x_i) dx_i \\
&\quad - \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i'(x_i) w_i(x_i) dx_i \\
&= I_1 - \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i'(x_i) w_i(x_i) dx_i,
\end{aligned}$$

where

$$I_1 := - \int_{-1}^0 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i'(x_i) dx_i - \int_0^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i'(x_i) dx_i.$$

In the following technical lemma, we derive a lower bound for I_1 .

Lemma 4.1. *The following estimate holds:*

$$(22) \quad I_1 \geq \frac{\kappa}{2} \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i^2(x_i) dx_i.$$

Proof. From (12) it follows that

$$w_i'(x_i) = \begin{cases} -\kappa(-x_i)^{-\kappa-1}, & -1 < x_i < 0, \\ -\kappa x_i^{-\kappa-1}, & 0 < x_i < 1, \end{cases}$$

hence, we can write

$$(23) \quad I_1 = \kappa \int_{-1}^0 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) (-x_i)^{-\kappa-1} dx_i + \kappa \int_0^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) x_i^{-\kappa-1} dx_i.$$

On the other hand, always by (12), the following pointwise estimates holds

$$\begin{cases} (-x_i)^{-\kappa-1} \geq \frac{1}{2} w_i^2(x_i), & -1 < x_i < 0, \\ x_i^{-\kappa-1} \geq \frac{1}{2} w_i^2(x_i), & 0 < x_i < 1. \end{cases}$$

In fact, concerning the first estimate above, for any $-1 < x_i < 0$, we write $w_i^2(x_i) = ((-x_i)^{-\kappa} - 1)^2 \leq (-x_i)^{-2\kappa} + 1$. Since $0 < \kappa < 1$ and $0 < -x_i < 1$, it follows that $(-x_i)^{-2\kappa} \leq (-x_i)^{-\kappa-1}$ and $1 \leq (-x_i)^{-\kappa-1}$. Thus, we obtain $w_i^2(x_i) \leq 2(-x_i)^{-\kappa-1}$. The second estimate follows from analogous computations.

Now, taking into account these pointwise estimates and the assumption (6) that ensures $\mathbf{b}_i(x_i) \geq 0$, we revisit the identity (23) to write

$$I_1 \geq \frac{\kappa}{2} \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i^2(x_i) dx_i,$$

this last inequality yields the desired estimate (22). This finishes the proof. \square

Thus, considering the estimate (22) and (21), we obtain

$$\begin{aligned} \int_{-1}^1 \partial_i (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) (\mathbf{b}_i(x_i) w_i(x_i)) dx_i &\geq \frac{\kappa}{2} \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i^2(x_i) dx_i \\ &\quad - \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}'_i(x_i) w_i(x_i) dx_i. \end{aligned}$$

Then, we come back to the identity (20). Recalling that from the assumption (6) it follows that

$$\prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \geq 0, \text{ and from the assumption (5) we have } \mathbf{b}(x) = \prod_{j=1}^n \mathbf{b}_j(x_j), \text{ we can write}$$

$$\begin{aligned} &\int_{-1}^1 \partial_i (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) (\mathbf{b}(x) w_i(x_i)) dx_i \\ &\geq \left(\frac{\kappa}{2} \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}_i(x_i) w_i^2(x_i) dx_i \right) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \\ &\quad - \left(\int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}'_i(x_i) w_i(x_i) dx_i \right) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \\ &= \frac{\kappa}{2} \int_{-1}^1 |\mathbf{v}(t, x)|^2 \mathbf{b}^2(x) w_i^2(x_i) dx_i - \int_{-1}^1 (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \mathbf{b}'_i(x_i) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) w_i(x_i) dx_i. \end{aligned}$$

With this lower bound at hand, returning to the identity (19), we obtain

$$\begin{aligned} &\int_{[-1,1]^n} \nabla (|\mathbf{v}(t, x)|^2 \mathbf{b}(x)) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx \\ &\geq \frac{\kappa}{2} \sum_{i=1}^n \int_{-1}^1 \cdots \left(\int_{-1}^1 |\mathbf{v}(t, x)|^2 \mathbf{b}^2(x) w_i^2(x_i) dx_i \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ (24) \quad &- \sum_{i=1}^n \int_{-1}^1 \cdots \left(\int_{-1}^1 |\mathbf{v}(t, x)|^2 \mathbf{b}(x) \mathbf{b}'_i(x_i) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) w_i(x_i) dx_i \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ &= \frac{\kappa}{2} \int_{[-1,1]^n} |\mathbf{v}(t, x)|^2 |\mathbf{b}(x) \mathbf{w}(x)|^2 dx \\ &\quad - \int_{[-1,1]^n} |\mathbf{v}(t, x)|^2 \mathbf{b}(x) \left(\sum_{i=1}^n \mathbf{b}'_i(x_i) w_i(x_i) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \right) dx. \end{aligned}$$

Here, we define

$$\begin{aligned} I_2 &:= \frac{\kappa}{2} \int_{[-1,1]^n} |\mathbf{v}(t, x)|^2 |\mathbf{b}(x) \mathbf{w}(x)|^2 dx, \\ I_3 &:= - \int_{[-1,1]^n} |\mathbf{v}(t, x)|^2 \mathbf{b}(x) \left(\sum_{i=1}^n \mathbf{b}'_i(x_i) w_i(x_i) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \right) dx. \end{aligned}$$

In the following technical lemmas, we derive a lower bound for these expressions.

Lemma 4.2. *The following lower bound holds:*

$$(25) \quad I_2 \geq \frac{\kappa}{2^{n+1}} \left(\int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \right)^2.$$

Proof. Since the Lebesgue measure of the set $[-1, 1]^n$ is directly computed as $dx([-1, 1]^n) = 2^n$, by the Jensen inequality it follows that

$$\begin{aligned} I_2 &\geq \frac{\kappa}{2} \int_{[-1,1]^n} |\mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x))|^2 \, dx \\ &\geq \frac{\kappa 2^n}{2} \int_{[-1,1]^n} |\mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x))|^2 \frac{dx}{2^n} \\ &\geq \frac{\kappa 2^n}{2} \left(\int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \frac{dx}{2^n} \right)^2 \\ &= \frac{\kappa}{2^{n+1}} \left(\int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x)\mathbf{w}(x)) \, dx \right)^2. \end{aligned}$$

□

Lemma 4.3. *There exists a constant $\mathbf{C}_2(T) = \mathbf{C}_2(n, \kappa, \|\mathbf{b}\|_{L^\infty}, \|\nabla \mathbf{b}\|_{L^\infty}, T) > 0$ such that following lower bound holds*

$$(26) \quad I_3 \geq -\mathbf{C}_2(T) \|u_0\|_{H^s}^2.$$

Proof. Recall that, from (11), a (12) and $0 < \kappa < 1$, it follows that $\|\mathbf{w}\|_{L^1} \leq C_{n,\kappa} < +\infty$. On the other hand, note that from the assumption (5) the following inequality holds $\|\mathbf{b}\|_{L^\infty} \leq \sum_{i=1}^n \|\mathbf{b}_i\|_{L^\infty}$.

Then, using Hölder inequalities, we write

$$\begin{aligned} -I_3 &= \int_{[-1,1]^n} |\mathbf{v}(t, x)|^2 \mathbf{b}(x) \left(\sum_{i=1}^n \mathbf{b}'_i(x_i) w_i(x_i) \prod_{j=1, j \neq i}^n \mathbf{b}_j(x_j) \right) \, dx \\ &\leq \|\mathbf{v}(t, \cdot)\|_{L^\infty}^2 \|\mathbf{b}\|_{L^\infty} \left(\int_{[-1,1]^n} |\nabla \mathbf{b}(x) \cdot \mathbf{w}(x)| \, dx \right) \left(\sum_{i=1}^n \prod_{j=1, j \neq i}^n \|\mathbf{b}_j\|_{L^\infty} \right) \\ &\leq \|\mathbf{v}(t, \cdot)\|_{L^\infty}^2 \left(\sum_{i=1}^n \|\mathbf{b}_i\|_{L^\infty} \right) \|\nabla \mathbf{b}\|_{L^\infty} \|\mathbf{w}\|_{L^1} n \left(\sum_{i=1}^n \|\mathbf{b}_i\|_{L^\infty} \right)^{n-1} \\ &\leq n C_{n,\kappa} \left(\sum_{i=1}^n \|\mathbf{b}_i\|_{L^\infty} \right)^n \|\nabla \mathbf{b}\|_{L^\infty} \|\mathbf{v}(t, \cdot)\|_{L^\infty}^2. \end{aligned}$$

Additionally, from Sobolev embeddings (with the parameter $n/2 < \sigma < s - 1$), along with the identity $\mathbf{v}(t, x) := \nabla u(t, x)$ and the control (17) on $u(t, x)$, it follows that

$$\|\mathbf{v}(t, \cdot)\|_{L^\infty}^2 \leq C \|\mathbf{v}(t, \cdot)\|_{H^\sigma}^2 \leq C \|u(t, \cdot)\|_{H^s}^2 \leq C C^2(T) \|u_0\|_{H^s}^2.$$

Consequently, we obtain

$$-I_3 \leq n C_{n,\kappa} \left(\sum_{i=1}^n \|\mathbf{b}_i\|_{L^\infty} \right)^n \|\nabla \mathbf{b}\|_{L^\infty} C C^2(T) \|u_0\|_{H^s}^2 := \mathbf{C}_2(T) \|u_0\|_{H^s}^2.$$

□

Finally, the wished lower bound (18) directly follows now from the estimates (25) and (26), along with the inequality (24). This concludes the proof of Proposition 4.2. □

End of the proof. With the information obtained in estimates (14) and (18), we return to the identity (13) where, for any time $0 < T < +\infty$ and for any $0 \leq t \leq T$, we obtain

$$(27) \quad \begin{aligned} & \frac{d}{dt} \int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \cdot \mathbf{w}(x)) dx \\ & \geq -\mathbf{C}_1(T) \|u_0\|_{H^s} + \frac{\kappa}{2^{n+1}} \left(\int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx \right)^2 - \mathbf{C}_2(T) \|u_0\|_{H^s}^2. \end{aligned}$$

With this inequality, following similar ideas of [15], we conclude the blow-up in finite of the solution $u(t, x)$ as follows. First, we define the functional

$$I(t) := \int_{[-1,1]^n} \mathbf{v}(t, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx,$$

the constant

$$c_1 := \frac{\kappa}{2^{n+1}} > 0,$$

and the quantity depending on the time T :

$$\mathbf{C}(T) := \mathbf{C}_1(T) + \mathbf{C}_2(T).$$

With this notation, we recast (27) as the following ordinary differential inequality:

$$I'(t) \geq c_1 I^2(t) - \mathbf{C}(T) \times \begin{cases} \|u_0\|_{H^s}, & \text{when } \|u_0\|_{H^s} \leq 1, \\ \|u_0\|_{H^s}^2, & \text{when } \|u_0\|_{H^s} > 1, \end{cases} \quad 0 < t \leq T.$$

Given a number $m_* \gg 1$, we fix the time $T = m_*$. So, in the quantity $\mathbf{C}(T)$ introduced above we obtain the numerical constant $\mathbf{C}(m_*)$. Additionally, we define the constant

$$c_2 := \mathbf{C}(m_*) \times \begin{cases} \|u_0\|_{H^s}, & \text{when } \|u_0\|_{H^s} \leq 1, \\ \|u_0\|_{H^s}^2, & \text{when } \|u_0\|_{H^s} > 1. \end{cases}.$$

Then, the functional $I(t)$ verifies:

$$I'(t) \geq c_1 I^2(t) - c_2, \quad c_1, c_2 > 0, \quad 0 < t \leq m_*,$$

with initial datum given by

$$I(0) = \int_{[-1,1]^n} \mathbf{v}(0, x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx = \int_{[-1,1]^n} \nabla u_0(x) \cdot (\mathbf{b}(x) \mathbf{w}(x)) dx.$$

To continue, we denote by $J(t)$ the solution of the initial value problem below

$$\begin{cases} J'(t) = c_1 J^2(t) - c_2, \\ J(0) = I(0), \end{cases}$$

which, is explicitly computed as:

$$J(t) = \sqrt{\frac{c_2}{c_1}} \frac{\left(I(0) + \sqrt{\frac{c_2}{c_1}}\right) + \left(I(0) - \sqrt{\frac{c_2}{c_1}}\right) e^{2\sqrt{c_1 c_2} t}}{\left(I(0) + \sqrt{\frac{c_2}{c_1}}\right) - \left(I(0) - \sqrt{\frac{c_2}{c_1}}\right) e^{2\sqrt{c_1 c_2} t}}.$$

Note that this function blows-up at the finite time

$$(28) \quad t_* := \frac{1}{2\sqrt{c_1 c_2}} \ln \left(\frac{I(0)\sqrt{c_1} + \sqrt{c_2}}{I(0)\sqrt{c_1} - \sqrt{c_2}} \right),$$

which is positive as long as the following inequality holds:

$$\frac{I(0)\sqrt{c_1} + \sqrt{c_2}}{I(0)\sqrt{c_1} - \sqrt{c_2}} > 1.$$

From the conditions (7) and (8) assumed on the initial datum u_0 , together with the expressions $I(0)$, c_1 and c_2 defined above, we obtain that $I(0) > 0$ and $I(0)\sqrt{c_1} - \sqrt{c_2} > 0$, respectively. Thus, the wished inequality holds.

Now, by the standard comparison principle of ordinary differential equations, it follows that $I(t) \geq J(t)$, and the functional $I(t)$ also blows-up at the time t_* .

Finally, recalling the bound $\|w\|_{L^1([-1,1]^n)} \leq C_{n,\kappa} < +\infty$, and considering the Hölder inequality, we can write

$$\begin{aligned} I(t) &\leq \|v(t, \cdot)\|_{L^\infty([-1,1]^n)} \|\mathbf{b}\|_{L^\infty([-1,1]^n)} \|w\|_{L^1([-1,1]^n)} \\ &\leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} \|v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Thus, by the Sobolev embeddings (with a parameter $n/2 < \sigma < s - 1$), we obtain

$$\begin{aligned} I(t) &\leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} \|v(t, \cdot)\|_{H^\sigma} \\ &\leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} \|u(t, \cdot)\|_{H^{\sigma+1}} \\ &\leq C_{n,\kappa} \|\mathbf{b}\|_{L^\infty} \|u(t, \cdot)\|_{H^s}. \end{aligned}$$

Consequently, the quantity $\|u(t, \cdot)\|_{H^s}$ blows-up at the finite time t_* , which gives a contradiction. This concludes the proof of Theorem 1.1.

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