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A new proof of superadditivity and of the density conjecture for Activated Random Walks on the line

Nicolas Forien*

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Abstract

In two recent works [HJJ24a, HJJ24b], Hoffman, Johnson and Junge proved the density conjecture, the hockey stick conjecture and the ball conjecture for Activated Random Walks in dimension 1, showing an equality between several different definitions of the critical density of the model. This establishes a kind of self-organized criticality, that was originally predicted for the Abelian Sandpile Model.

The proof of Hoffman, Johnson and Junge uses a comparison with a percolation process, which exhibits a superadditivity property.

In the present note, we revisit their argument by providing a new proof of superadditivity directly for Activated Random Walks, without relying on a percolation process. The proof relies on a simple comparison between the stabilization of two neighbouring segments and that of their union. We then explain how this superadditivity property implies the three mentioned conjectures.

Yet, so far it does not seem that this approach yields as much information as does the percolation technology developed by Hoffman, Johnson and Junge, which yields an exponential concentration bound on the stationary density, whereas the superadditivity property alone only ensures an exponential bound on the lower tail.

1 Introduction

1.1 Presentation of the model

Activated Random Walks is a model of interacting particles which is defined as follows. A configuration of the model on a graph consists of a certain number of particles on each vertex, each particle being either active or sleeping. Active particles perform independent continuous-time random walks with jump rate 1, according to a certain jump kernel on the graph. When an active particle is alone on a site, it can fall asleep at a certain rate $\lambda > 0$. A sleeping particle stops moving and is instantaneously reactivated as soon as it shares its site with at least one other particle. If reactivated, the particle resumes its continuous-time random walk.

We say that the system is active if every site of the graph is visited infinitely many times, and otherwise we say that the system fixates. The model on \mathbb{Z}^d , for every $d \geq 1$, undergoes the following phase transition:

Theorem 1. *In any dimension $d \geq 1$, for every sleep rate $\lambda > 0$ and every translation-invariant jump kernel on \mathbb{Z}^d which generates all \mathbb{Z}^d , there exists $\rho_c \in (0, 1)$ such that, for every translation-ergodic initial distribution with no sleeping particles and an average density of active particles ρ , the Activated Random Walk model on \mathbb{Z}^d with sleep rate λ almost surely fixates if $\rho < \rho_c$, whereas it almost surely stays active if $\rho > \rho_c$.*

This result is due to Rolla, Sidoravicius and Zindy [RS12, RSZ19] who showed the existence of the threshold density and its dependence on the mean density of particles only, and to a series of works [RS12, ST17, ST18, Tag19, BGH18, HRR23, FG24, Hu22, AFG24] which established the non-triviality of the threshold, that is to say, that it is strictly between 0 and 1. Note that, unlike some other models, the behaviour of Activated Random Walks is not trivial even in one dimension.

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1.2 Content of this paper

This paper presents a new proof, in the one-dimensional case, of three results known as the density conjecture, the hockey stick conjecture and the ball conjecture. These results are presented in Section 2.

These conjectures were already proved by Hoffman, Johnson and Junge in [HJJ24a, HJJ24b], but using a different technique based on a percolation process. They use a superadditivity property of the percolation process to obtain concentration bounds for the density of particles under the stationary distribution of the driven-dissipative chain (see Section 2 for precise definitions).

The main added value of the present paper consists in a new simple proof of this superadditivity property, without relying on a percolation process. This superadditivity property is presented in Section 3.1 and proved in Section 4. An elementary corollary of this superadditivity is then established in Section 5.

We also explain how to deduce the density conjecture, the hockey stick conjecture and the ball conjecture from this superadditivity property, yielding a new self-contained proof of these results. Yet, the results that we obtain are a bit less precise than those established by Hoffman, Johnson and Junge. In particular, we show an exponential bound only on the lower tail of the stationary density, while they also establish an exponential bound for the upper tail, which is more involved. As a consequence, we obtain less detailed convergence results, and we do not get a control on the transition from polynomial to exponential stabilization time of the model on a cycle, for which our one-sided concentration bound does not seem to be enough: see Proposition 8.12 in [HJJ24a].

Sections 3.2 to 3.4 introduce several earlier results on which we rely: a technique of [LL24] allowing to easily sample from the stationary distribution, results of [RT18] and [For24] relating the phase transition of the model to the number of particles jumping out of a segment, and the Abelian property of the model. Several open challenges are then outlined in Section 3.5. Lastly, Sections 6 to 8 are devoted to the deduction of main conjectures from the superadditivity property.

In the remainder of the paper, we restrict the presentation to the one-dimensional case and we fix once and for all a sleep rate $\lambda > 0$ and a translation-invariant nearest-neighbour jump kernel on \mathbb{Z} , which simply boils down to the choice of a probability to jump to the left, denoted by $p \in (0, 1)$. All the statements hold for every $\lambda > 0$ and for every $p \in (0, 1)$.

2 The conjectures that we reprove

We now present the main conjectures for which we give new proofs. We refer to [LS24] for a more general presentation of these conjectures (where they correspond to Conjectures 1, 11, 12 and 17), along with other nice predictions on the model.

2.1 Density conjecture

The density conjecture, which is the content of Theorem 2 below, connects the phase transition described above with another version of the model called the driven-dissipative chain. Fix an integer $n \geq 1$, and consider the segment $V_n = \{1, \dots, n\} \subset \mathbb{Z}$. A configuration of the model on V_n can be represented by a vector $\eta : V_n \rightarrow \mathbb{N} \cup \{\mathfrak{s}\}$, where $\eta(x) = k \in \mathbb{N}$ means that there are k active particles at x and $\eta(x) = \mathfrak{s}$ means that there is one sleeping particle at x . A configuration is called stable if it does not contain any active particle.

The driven-dissipative system consists of a Markov chain on the finite set $\{0, \mathfrak{s}\}^{V_n}$ of all stable configurations on V_n . At each time step of the Markov chain, an active particle is added to a site of V_n chosen uniformly at random, and we leave the system evolve, with particles being killed when they jump out of V_n (by the left or right exit), until a new stable configuration is reached. This new stable configuration gives the state of the Markov chain at the following time step. This defines an irreducible and aperiodic Markov chain, which is called driven-dissipative because the system is driven by addition of active particles and there is dissipation at the borders of V_n . We denote by S_n the number of sleeping particles in a configuration sampled from the stationary distribution of this Markov chain. The density conjecture states that, when the length of the segment tends to infinity, the density of particles S_n/n in this stationary distribution concentrates around the critical density of Theorem 1. More precisely, we show the following result:

Theorem 2. *We have $S_n/n \rightarrow \rho_c$ in probability as $n \rightarrow \infty$. Moreover, for every $\rho < \rho_c$ there exists $c > 0$ such that for every $n \geq 1$, $\mathbb{P}(S_n \leq \rho n) \leq e^{-cn}$.*

In [HJJ24a] a stronger result is obtained, with an exponential bound also on the upper tail of S_n/n (see their Proposition 8.6), and an interesting consequence is established in [BHS24], namely that the model on \mathbb{Z} with supercritical density but only one particle active remains active with positive probability.

2.2 Hockey stick conjecture

While Theorem 2 shows a convergence for the stationary measure of the driven-dissipative Markov chain, the hockey stick conjecture complements this information with a prediction about the behaviour of this Markov chain at all times. Fix an integer $n \geq 1$. For every $t \in \mathbb{N}$, we denote by Y_t the number of particles remaining in V_n after t steps of the driven-dissipative Markov chain on V_n , started with the empty configuration at time $t = 0$. The hockey stick conjecture consists in the following result:

Theorem 3. *For every $\rho > 0$ we have $Y_{\lceil \rho n \rceil} / n \rightarrow \min(\rho, \rho_c)$ in probability when $n \rightarrow \infty$.*

The denomination hockey stick refers to the shape of the curve of the function $\rho \mapsto \min(\rho, \rho_c)$. This result, also conjectured more generally in [LS24], was first proved in [HJJ24b], using the technology developed by [HJJ24a].

2.3 Ball conjecture

Another interesting setting consists of k active particles started at the origin of the line \mathbb{Z} , with no particles on the other sites of \mathbb{Z} . We then denote by A_k the random set of sites which are visited at least once when the system evolves from this initial configuration. This set is called the aggregate, and we show the following:

Theorem 4. *We have $|A_k|/k \rightarrow 1/\rho_c$ in probability as $k \rightarrow \infty$.*

Propositions 8.7 and 8.8 in [HJJ24a] make this result more precise by showing that with probability at least $1 - e^{-ck}$ the aggregate contains a ball and is contained in a slightly larger ball, both centered on the origin. Note that in our setting the limit of the aggregate is not necessarily centered on the origin because we also include the case of biased walks (for which the technology of [HJJ24a] is expected to extend).

This property is known as the ball conjecture, because in higher dimensions it is conjectured that the limiting shape of the aggregate is a Euclidean ball centered on the origin, with a density ρ_c of sleeping particles inside.

2.4 Self-organized criticality and sandpiles

The main motivation for the study of Activated Random Walks is the quest for a simple model which exhibits self-organized criticality, a concept coined in by the physicists Bak, Tang and Wiesenfeld [BTW87] to describe the behaviour of certain systems which are spontaneously attracted to a critical-like state. Unlike ordinary phase transitions, where the critical regime is only observed for a very special choice of the parameters, self-organized criticality means that a critical regime is reached without fine tuning of the parameters.

In Section 1.1, we saw that the conservative dynamics of Activated Random Walks on the infinite lattice (without particle addition or dissipation) exhibits a phase transition in the usual sense, with two phases separated by a threshold density ρ_c (which, if considered as a function of the sleep rate λ , can also be seen as a critical curve).

Self-organized criticality comes into play when one considers the driven-dissipative chain described in Section 2.1. Thanks to these two mechanisms of addition of particles and dissipation of particles at the boundaries when the segment becomes too crowded to accommodate more particles, the system is able to self-tune to the critical density. Moreover, the hockey stick conjecture shows that the critical density is not only reached as the limit after a very large number of steps of the chain, but is achieved as soon as at least a critical density of particles has been added to the system.

A stronger version of this conjecture, recently settled in [HJJM25], shows that the driven-dissipative Markov chain exhibits cutoff exactly at the critical density. This shows that not only the density is rapidly self-tuned to the critical density, but the distribution of the sleeping particles quickly resembles the stationary distribution.

The idea to achieve self-organized criticality through the introduction of driving and dissipation in a conservative model which presents a usual phase transition is more general. It was already present in the Abelian Sandpile Model, which was suggested by Bak, Tang and Wiesenfeld to exemplify their concept. In this model, there is only one type of particles, and a vertex of the graph is declared unstable when the number of particles on it exceeds the degree of the vertex. Unstable sites may topple, sending one particle to each neighbouring site, which may in turn make some of these neighbours become unstable. As for Activated Random Walks, one may construct a driven-dissipative Markov chain for the Abelian Sandpile and study its stationary distribution.

Due to its more deterministic nature, the Abelian Sandpile Model is more amenable to exact computations. This enables the study of its stationary distribution, which indeed presents critical-like features such as power-law correlations and fractal structures [Dha06, Red06, Jár18]. However, the density conjecture and the hockey stick conjecture, which were originally predicted for the Abelian Sandpile, turn out to fail for this model: this has been proved mathematically on some particular graphs and suggested with numerical simulations on the two-dimensional lattice [FLW10a, FLW10b, JJ10].

In view of these defects, Activated Random Walks emerged as a variant of the Abelian Sandpile Model involving more randomness, along with another variant called the Stochastic Sandpile Model, which was less studied but is expected to behave similarly. Conjectures about the self-critical behaviour of these two models were formulated progressively in [DMVZ00, DRS10, Rol20, LS24]. And recent results about the mixing time of Activated Random Walks suggest an explanation for the fact that this model behaves better than the Abelian sandpile, because it mixes faster [LL24, BS24]. For a broader comparative overview of sandpile models and Activated Random Walks, we refer the reader to [HJJ24b] and to the references therein.

Yet, an important question which remains open for now is that of the correlations in the stationary distribution of the driven-dissipative system. It is expected that the stationary distributions converge to a limiting distribution on the infinite lattice, and that the correlations in this limiting distribution decay as power-laws in the distance, indicating the absence of a characteristic scale. This would qualify the state reached by the system as truly critical.

3 Main ingredients and some perspectives

We now present the external ingredients on which we rely, so that the rest of the paper is self-contained.

3.1 The crucial point: superadditivity

Recall the notation S_n introduced in Section 2.1 for the number of particles in the segment $V_n = \{1, \dots, n\}$ under the stationary distribution of the driven-dissipative chain. The main result of this paper is the following proposition, which shows an almost superadditivity property for S_n . Such a superadditivity argument was already at the heart of the proof of the density conjecture in [HJJ24a], but we present a different approach to establish it.

Proposition 1. *For every $n, m \geq 1$ the variable S_{n+m+1} stochastically dominates the sum of S_n and of a copy of S_m which is independent of S_n .*

Note that this is only a property of the distribution of S_n , but we state it with random variables for concreteness. The proof of this proposition, which is the main added value of this paper, is presented in Section 4. We welcome any ideas to obtain a similar superadditivity property without the +1 term, that is to say, to show that S_{n+m} dominates an independent sum of S_n and S_m . Yet, this small defect is harmless for what we are interested in, because if for every $n \geq 1$ we consider $X_n = S_{2n-1}$, then Proposition 1 entails that this sequence $(X_n)_{n \geq 1}$ is stochastically superadditive, in the following sense:

Definition 1. We say that a sequence of real variables $(X_n)_{n \geq 1}$ is stochastically superadditive if for every $n, m \geq 1$ the variable X_{n+m} stochastically dominates the sum of X_n and of a copy of X_m which is independent of X_n .

This superadditivity property has the following consequence, whose elementary proof (which may be already present in the literature) is presented in Section 5.

Lemma 1. Let $(X_n)_{n \geq 1}$ be a sequence of non-negative random variables which is stochastically superadditive. Then, defining $\rho_* = \sup_{n \geq 1} \mathbb{E}X_n/n \in [0, \infty]$, we have the convergence in probability $X_n/n \rightarrow \rho_*$ as $n \rightarrow \infty$. Moreover, we have the following exponential bound on the lower tail: for every $\rho < \rho_*$ there exists $c > 0$ such that, for every $n \geq 1$,

$$\mathbb{P}\left(\frac{X_n}{n} \leq \rho\right) \leq e^{-cn}. \quad (1)$$

In the case of S_n , this Lemma enables to deduce that S_n/n converges in probability as $n \rightarrow \infty$, with an exponential bound on the lower tail. Note that no exponential bound on the upper tail follows from stochastic superadditivity in general, as shows the counter-example of

$$X_n = \sum_{k=1}^{\lfloor \ln n / \ln 2 \rfloor} 2^k Y_{n,k},$$

where the variables $(Y_{n,k})_{n,k}$ are i.i.d. with $Y_{n,k}$ having a binomial distribution with parameters $\lfloor n/2^k \rfloor$ and $1/2^k$. This sequence $(X_n)_{n \geq 1}$ is stochastically superadditive and $X_n/n \rightarrow 1$ in probability, but for $\varepsilon > 0$ the rate of decay of $\mathbb{P}(X_n/n \geq 1 + \varepsilon)$ is not exponential in n .

However, an exponential bound on the upper tail is not required to obtain the results presented in Section 2, for which convergence in probability of S_n/n is enough.

3.2 Exact sampling

We now state a nice key result of [LL24] which gives a convenient way to sample S_n :

Lemma 2. Fix an integer $n \geq 1$. Consider the initial configuration with one active particle on each site of V_n and let the system evolve, with particles being killed when they jump out of V_n , until no active particle remains in V_n . Then the distribution of the resulting stable configuration is exactly the stationary distribution of the driven-dissipative Markov chain on V_n . In particular, the number of sleeping particles remaining in V_n is distributed as S_n .

This property is not specific to the one-dimensional case and holds more generally. Its quite elementary proof goes as follows: let η be the random configuration obtained after stabilizing the initial configuration with one active particle per site, and let η' be the configuration obtained after adding one active particle to η at a site $X \in V_n$ chosen uniformly at random, and stabilizing. Then the Abelian property (see Section 3.4) shows that η' can also be obtained by directly stabilizing the configuration $\mathbf{1}_{V_n} + \delta_X$, and this can be done by first letting the extra particle at X walk until it jumps out of V_n , and then stabilizing the remaining n particles. This shows that η' has the same distribution as η , implying that the distribution of η is the invariant distribution of the driven-dissipative chain.

3.3 Fraction jumping out of a segment

Once the convergence in law of S_n/n is established, to identify the limit as ρ_c (Theorem 2) and to obtain the hockey stick conjecture (Theorem 3), we rely on the two following results. They give information on M_n , which is the number of particles jumping out of V_n during stabilization of V_n with particles being killed when they exit V_n (but not necessarily starting with one active particle per site). The first one is a consequence of a result of [RT18]:

Lemma 3. For each $\rho < \rho_c$, if the initial configuration is i.i.d. with density of particles ρ , then $\mathbb{E}M_n = o(n)$.

The second one is a result of [For24] which is a kind of a reciprocal:

Lemma 4. *For every $\rho > \rho_c$ there exists $\varepsilon > 0$ and $c > 0$ such that for every $n \geq 1$, for every deterministic initial configuration $\eta : V_n \rightarrow \mathbb{N}$ with at least pn particles, all active, we have $\mathbb{P}(M_n > \varepsilon n) \geq c$.*

Our proofs of Theorems 2 and 3 using these two results are presented in Sections 6 and 7. Note that Theorem 2 (along with a monotonicity property given by Lemma 7 below) implies a stronger estimate, namely that for every $\varepsilon < \rho - \rho_c$ we have $\mathbb{P}(M_n > \varepsilon n) \rightarrow 1$.

Then, in Section 8 we present the proof of Theorem 4, about the point source case. For the outer bound, we rely on another result of [For24], which shows that if only few particles jump out of a segment, then with high probability no particle leaves a slightly larger segment:

Lemma 5. *For every $n \geq 1$, for every deterministic initial configuration $\eta : V_n \rightarrow \mathbb{N}$, for any $i, j \in \mathbb{N}$ we have*

$$\mathbb{P}(A(\eta) \subset \{1 - 2j, \dots, n + 2j\}) \geq \mathbb{P}(M_n \leq i) \times \mathbb{P}(G_1 + \dots + G_i \leq j),$$

where $A(\eta)$ denotes the set of sites which are visited during the stabilization of η in \mathbb{Z} and G_1, \dots, G_i are i.i.d. geometric variables with parameter $\lambda/(1 + \lambda)$.

3.4 Abelian property and monotonicity

The Abelian property is a central tool to study Activated Random Walks. It comes with a graphical representation of this kind of interacting particles system, often called the Diaconis and Fulton representation [DF91, RS12].

A configuration of the model on \mathbb{Z} is encoded into a vector $\eta : \mathbb{Z} \rightarrow \mathbb{N} \cup \{\mathfrak{s}\}$, where s represents a sleeping particles and numbers in $\mathbb{N} \setminus \{0\}$ represent active particles. If $V \subset \mathbb{Z}$ we denote by $\|\eta\|_V$ the total number of particles (active or sleeping) in V in the configuration η .

For every site $x \in \mathbb{Z}$ consider an infinite sequence $(\tau_{x,j})_{j \geq 1}$ of “instructions”, where each instruction $\tau_{x,j}$ can either be a sleep instruction or a jump instruction to some site $y \in \mathbb{Z}$.

Let $\tau = (\tau_{x,j})_{x \in \mathbb{Z}, j \geq 1}$ be such an array of instructions, let $\eta : \mathbb{Z} \rightarrow \mathbb{N} \cup \{\mathfrak{s}\}$ be a particle configuration, and let $h : \mathbb{Z} \rightarrow \mathbb{N}$ be an array called odometer, which counts how many instructions of τ have already been used at each site. A site $x \in \mathbb{Z}$ is called unstable in η if $\eta(x) \geq 1$, that is to say if there is at least one active particle at x . If $x \in \mathbb{Z}$ is unstable, we say that it is legal to topple the site x . Toppling the site x means applying the next instruction from the array τ at x , namely $\tau_{x, h(x)+1}$, to the configuration η . If this instruction is a jump instruction to some site $y \in \mathbb{Z}$, then we make one particle jump from x to y , waking up the sleeping particle at y if any. If this instruction is a sleep instruction, then the particle at x falls asleep if $\eta(x) = 1$, and nothing happens if $\eta(x) \geq 2$. This gives a new configuration η' and a new odometer $h' = h + \delta_x$.

If $\alpha = (x_1, \dots, x_k)$ is a finite sequence of sites of \mathbb{Z} , we say that α is a legal toppling sequence if it is legal to topple these sites in this order. The odometer of a toppling sequence α , which counts how many times each site appears in α , is defined as $m_\alpha = \delta_{x_1} + \dots + \delta_{x_k}$.

For a fixed configuration η and a fixed array of instructions τ , for every $V \subset \mathbb{Z}$ we can define the stabilization odometer of V , which is given by

$$m_{V, \eta} = \sup_{\alpha \subset V, \alpha \text{ legal}} m_\alpha,$$

where the notation $\alpha \subset V$ simply means that all the sites of α belong to V .

Another useful notion is that of acceptable topplings. We say that it is acceptable to topple a site x if η contains at least one particle at x , which may be sleeping. When we perform an acceptable toppling at a site which contains a sleeping particle, we first wake it up. We say that a sequence of topplings $\alpha = (x_1, \dots, x_k)$ is acceptable if it is acceptable to topple these sites in this order.

For every fixed η and τ , we have the following relation between acceptable and legal topplings:

Lemma 6 (Lemma 2.1 in [Rol20]). *For every $V \subset \mathbb{Z}$, if α is an acceptable sequence of topplings that stabilizes η in V , and $\beta \subset V$ is a legal sequence of topplings for η , then $m_\alpha \geq m_\beta$. Thus, if α is an acceptable sequence of topplings that stabilizes η in V , then $m_\alpha \geq m_{V, \eta}$.*

This entails in particular that if α and β are two legal sequences of topplings in V which stabilize η in V , then $m_\alpha = m_\beta = m_{V, \eta}$ (hence the name stabilization odometer), and it is not hard to convince oneself that the resulting final configurations are also equal. This is known as the abelian property, which allows us to choose whatever order to perform the topplings, as soon as they are legal, or acceptable if we only look for upper bounds on the stabilization odometer.

For a given set $V \subset \mathbb{Z}$, a configuration $\eta : V \rightarrow \mathbb{N} \cup \{\mathfrak{s}\}$ and an array $\tau = (\tau_{x,j})_{x \in V, j \geq 1}$ we define $\text{Stab}_V(\eta, \tau)$ as the number of (sleeping) particles which remain in V after applying any legal sequence of topplings in V which stabilizes η in V (so that this corresponds to ignoring particles once they jump out of V).

To relate this construction to the dynamics of Activated Random Walks, one needs to make the array of instructions random. Recalling that we fixed throughout this article a sleep rate $\lambda > 0$ and $p \in (0, 1)$ a probability to jump to the left, we consider i.i.d. stacks of instructions, where each instruction is a sleep instruction with probability $\lambda/(1 + \lambda)$, a jump instruction to the left neighbouring site with probability $p/(1 + \lambda)$, and a jump instruction to the right with probability $(1 - p)/(1 + \lambda)$. Throughout the article probabilities will all be denoted by \mathbb{P} , which will refer most of the time to this distribution on the arrays, but that we will also abusively use when there are other random elements than just the array τ (for example the initial configuration).

Lastly, we will make use of the following consequence of Lemma 6:

Lemma 7. *Let $n \geq 1$ and let $\eta, \xi : V_n \rightarrow \mathbb{N}$ be two deterministic configurations of particles on V_n which contain only active particles and which are such that $\eta \geq \xi$. Let τ be a random array of instructions with distribution \mathbb{P} . Then $\text{Stab}_V(\eta, \tau)$ stochastically dominates $\text{Stab}_V(\xi, \tau)$.*

Proof. It is enough to consider the case $\eta = \xi + \delta_x$ for a certain $x \in V$. Then, this configuration $\xi + \delta_x$ in V can be stabilized by first forcing one particle to walk from x until it exits from V , using acceptable topplings, and then stabilizing the remaining configuration ξ in V . This yields an acceptable sequence of topplings which stabilizes $\xi + \delta_x$ in V and which leaves a number of sleeping particles in V which is equal to $\text{Stab}_V(\xi, \tau)$ in distribution. By virtue of Lemma 6, this shows the desired stochastic domination. \square

3.5 Some open questions

Let us recall that our results are limited to the one-dimensional case, with nearest-neighbour jumps. Not only is our proof of superadditivity very specific to dimension 1, but also our proof of the upper bound $\rho_\star \leq \rho_c$ in Section 6.2, which uses Lemma 4 which is only known in dimension 1, and also our proof of the outer bound in the ball conjecture in Section 8.2, which uses Lemma 5 which is also only proved in dimension 1.

Thus, a natural open question is the following: which of these results also hold in higher dimension, on more general graphs or with more general jump distributions? In particular, in view of the crucial role played by the superadditivity property, it would be particularly interesting to know whether a similar superadditivity property still holds. And, if this is the case, which results could one deduce from such a superadditivity property?

Besides, here is another natural question: is there a simple way to establish the exponential bound on the upper tail of S_n/n ? Lastly, a smaller open problem is: can one show the superadditivity property without the +1 defect?

Lastly, let us recall that many of the open problems presented in [LS24] remain open.

4 Superadditivity: proof of Proposition 1

4.1 The setting: two segments on both sides of the origin

Let $n, m \geq 1$, let $V = \{-n, \dots, +m\}$ and let $\tau = (\tau_{x,j})_{x \in V, j \geq 1}$ be a random array of instructions with the distribution described in Section 3.4. Let us write $L = \{-n, \dots, -1\}$ and $R = \{1, \dots, m\}$. Recalling the

notation Stab introduced in Section 3.4, which counts the number of particles left after stabilizing a given configuration, let us define

$$S_V = \text{Stab}_V(\mathbf{1}_V, \tau) \stackrel{d}{=} S_{n+m+1}, \quad S_L = \text{Stab}_L(\mathbf{1}_L, \tau) \stackrel{d}{=} S_n \quad \text{and} \quad S_R = \text{Stab}_R(\mathbf{1}_R, \tau) \stackrel{d}{=} S_m,$$

where the equalities in distribution follow from the exact sampling result given by Lemma 2. Notice that S_L and S_R are independent because they depend on instructions of τ on two disjoint sets of sites. Thus, our goal is to show that S_V stochastically dominates $S_L + S_R$. Note that there is no hope to show more than a stochastic domination there, because it is not true that $S_V \geq S_L + S_R$ for all the possible realizations of the array of instructions.

At this point, the heuristics is the following: $S_L + S_R$ corresponds to the number of particles which remain sleeping in the segment V if in some sense the site 0 becomes a sink or, formulated differently, if all instructions at this site 0 are changed into jump instructions which make particles jump directly out of V . Then, the intuitive idea is that changing these instructions to make them point out of V decreases, at least in distribution, the number of particles which remain in V . Although intuitive, this assertion is less obvious than it seems, and its proof below uses the trick to stabilize the segment from one side to the other side, which is very specific to the one-dimensional case. Ideas about what happens in higher dimension are most welcome!

4.2 Turning the origin into an ejector seat

For every $k \in \mathbb{N}$, let τ_k denote the array obtained from τ by replacing all the instructions with number $j \geq k$ at the site 0 with a jump instruction pointing out of V , i.e., for example a jump instruction from 0 to $m+1$. We then consider the number of particles left sleeping in V when stabilizing $\mathbf{1}_V$ using these modified arrays of instructions, defining $N_k = \text{Stab}_V(\mathbf{1}_V, \tau_k)$ for every $k \in \mathbb{N}$.

On the one hand, since in τ_0 all the instructions at the site 0 are jumps directly to the exterior of V , when using instructions in τ_0 any particle which visits the site 0 will jump out of V without visiting $L \cup R$. Therefore, we have $N_0 = S_L + S_R$.

On the other hand, if k is strictly larger than the number of instructions used at 0 to stabilize $\mathbf{1}_V$ in V with instructions read from τ , then $N_k = S_V$ because the modified instructions are not used. Since this number of instructions used at 0 is almost surely finite, we deduce that N_k almost surely converges to S_V when $k \rightarrow \infty$.

Thus, the proof of the proposition will be complete if we show that for every $k \in \mathbb{N}$ the variable N_{k+1} stochastically dominates N_k . That is to say we are going to change the instructions one by one.

4.3 Replacing one instruction with a jump to the exterior

Let $k \in \mathbb{N}$. First, assume that $\tau_{0,k}$ is a sleep instruction. If moreover this instruction is the last instruction used at 0 when stabilizing $\mathbf{1}_V$ using the array τ_{k+1} , then we have $N_{k+1} = N_k + 1$ because the two resulting final configurations only differ by the presence of a sleeping particle at 0. Otherwise, if the stabilization of $\mathbf{1}_V$ using τ_{k+1} uses strictly less or strictly more than k instructions at 0, then $N_{k+1} = N_k$. Therefore, on the event \mathcal{A} that $\tau_{0,k}$ is a sleep instruction we have $N_{k+1} \geq N_k$.

Let now \mathcal{B} be the event that $\tau_{0,k}$ is a jump instruction to the right, so that at this position in τ_{k+1} there is a jump to the right while in τ_k there is a direct jump out of V . Let us consider the following procedure to stabilize the configuration $\mathbf{1}_V$ in V using the instructions in τ_k :

1. During the first step, we always topple the leftmost unstable site in V , and we stop when k instructions have been used at 0, or when there are no more active particles in V (if this happens before using k instructions at 0).
2. During the second step, we stabilize the remaining active particles in V using whatever legal toppling sequence.

Let η_1 and h_1 be the random configuration on V and the random odometer obtained at the end of step 1, and let $\eta : V \rightarrow \mathbb{N} \cup \{\mathfrak{s}\}$ and $h : V \rightarrow \mathbb{N}$ be a fixed configuration and a fixed odometer such that $\mathbb{P}(\mathcal{B}_{\eta,h}) > 0$, where $\mathcal{B}_{\eta,h} = \mathcal{B} \cap \{\eta_1 = \eta, h_1 = h\}$. We wish to show that, conditionally on this event $\mathcal{B}_{\eta,h}$, the variable N_{k+1} stochastically dominates N_k .

First, if $h(0) < k$ then it means that, on the event $\mathcal{B}_{\eta,h}$, step 1 stopped before using k instructions at 0, so the configuration η is stable in V and $N_{k+1} = N_k$.

Assume now that $h(0) = k$ (note that by definition of step 1 it is impossible that $h(0) > k$). This means that (on the event $\mathcal{B}_{\eta,h}$) the last instruction of τ_k used during step 1 was the k -th instruction at 0, which is a jump to the exterior. Because of the priority to the left, this entails that the configuration η is stable in L . Besides, during step 2 all the remaining instructions at 0 are jumps directly to the exterior, so that any particle which visits the site 0 during step 2 then jumps out of V without visiting $L \cup R$. Therefore, on the event $\mathcal{B}_{\eta,h}$ we have

$$N_k = \|\eta\|_L + \text{Stab}_R(\eta, \tau'), \quad (2)$$

where $(\tau')_{x \in R, j \geq 1}$ is the array on the sites of R obtained from τ by removing the instructions below the odometer h and re-indexing the remaining instructions, that is to say, for every $x \in R$ and $j \geq 1$ we have $\tau'_{x,j} = \tau_{x, h(x)+j}$.

Now, recall that N_{k+1} is the number of particles that we obtain if we use instead the array τ_{k+1} . On the event \mathcal{B} this array has a jump to the right at position k at site 0, so that applying the same sequence of topplings that we performed during step 1 but reading the instructions from τ_{k+1} we obtain the configuration $\eta + \delta_1$. Hence, on the event $\mathcal{B}_{\eta,h}$ it holds

$$N_{k+1} = \|\eta\|_L + \text{Stab}_R(\eta + \delta_1, \tau'), \quad (3)$$

with the same array τ' of remaining instructions.

We now wish to apply Lemma 7 to show that, conditionally on $\mathcal{B}_{\eta,h}$, the variable $\text{Stab}_R(\eta + \delta_1, \tau')$ stochastically dominates $\text{Stab}_R(\eta, \tau')$.

First, this Lemma only applies with configurations where all particles are active. Yet, note that the priority rule to the left ensures that during step 1 there can never be a sleeping particle to the right of an active particle. As a consequence, the configuration η does not contain any sleeping particle in R (because the last instruction was used at 0).

Besides, the event $\mathcal{B}_{\eta,h}$ being measurable with respect to the instructions $(\tau_{x,j})_{x \in V, j \leq h(x)}$, it is independent of the array τ' . Therefore, conditionally on $\mathcal{B}_{\eta,h}$, the array τ' has the same distribution as τ .

Therefore, Lemma 7 applies and allows us to deduce that $\text{Stab}_R(\eta + \delta_1, \tau')$ dominates $\text{Stab}_R(\eta, \tau')$. Plugging this into (2) and (3) it follows that, conditionally on $\mathcal{B}_{\eta,h}$, the variable N_{k+1} stochastically dominates N_k .

This is true for every η, h such that $\mathbb{P}(\mathcal{B}_{\eta,h}) > 0$, and the same argument works on the event \mathcal{C} that the instruction $\tau_{0,k}$ is a jump to the left (by considering instead the procedure which always topples the rightmost active site during step 1), so we eventually conclude that N_{k+1} stochastically dominates N_k .

5 Convergence for superadditive sequences: proof of Lemma 1

Let $(X_n)_{n \geq 1}$ be a stochastically superadditive sequence of non-negative variables. First note that Fekete's subadditive lemma [Fek23] entails that $\mathbb{E}X_n/n \rightarrow \rho_*$ when $n \rightarrow \infty$, where $\rho_* = \sup_{n \geq 1} \mathbb{E}X_n/n$.

5.1 Exponential bound on the lower tail

Let us start by directly proving the exponential bound (1) on the lower tail of X_n/n . Let $\rho < \rho_*$ and let $\rho' \in (\rho, \rho_*)$. Since $\mathbb{E}X_n/n \rightarrow \rho_*$ we can take $n \geq 1$ such that $\mathbb{E}X_n/n > \rho'$. Let $(Y_j)_{j \geq 1}$ be a sequence of i.i.d. variables with the same distribution as X_n/n , so that $\mathbb{E}[Y_1] = \mathbb{E}X_n/n > \rho'$. A classical Chernoff bound yields $c > 0$ such that for every $j \geq 1$,

$$\mathbb{P}\left(\frac{Y_1 + \dots + Y_j}{j} \leq \rho'\right) \leq e^{-cj}. \quad (4)$$

Let now $j_0 = \lceil \rho/(\rho' - \rho) \rceil$, let $k \geq j_0 n$ and let $j = \lfloor k/n \rfloor$. Superadditivity and the non-negativity of the variables imply that X_k stochastically dominates X_{jn} . Hence, we can write

$$\mathbb{P}\left(\frac{X_k}{k} \leq \rho\right) \leq \mathbb{P}\left(\frac{X_{jn}}{k} \leq \rho\right) \leq \mathbb{P}\left(\frac{X_{jn}}{(j+1)n} \leq \rho\right).$$

Using now that $j \geq j_0 \geq \rho/(\rho' - \rho)$, which implies that $\rho(j+1) \leq \rho'j$, we get

$$\mathbb{P}\left(\frac{X_k}{k} \leq \rho\right) \leq \mathbb{P}\left(\frac{X_{jn}}{jn} \leq \rho'\right).$$

Then, recalling that by superadditivity $X_{jn}/(jn)$ stochastically dominates $(Y_1 + \dots + Y_j)/j$, we deduce that

$$\mathbb{P}\left(\frac{X_k}{k} \leq \rho\right) \leq \mathbb{P}\left(\frac{Y_1 + \dots + Y_j}{j} \leq \rho'\right) \leq e^{-cj},$$

using (4). Using now that $j \geq k/(2n)$, we obtain

$$\mathbb{P}\left(\frac{X_k}{k} \leq \rho\right) \leq e^{-c'k},$$

with $c' = c/(2n)$, and we proved this for every $k \geq j_0 n$. Decreasing this constant c' if necessary so that the bound holds for every $k \geq 1$, we eventually obtain the exponential bound (1).

5.2 Bound on the upper tail

Assume that $\rho_* < \infty$, otherwise there is nothing to prove. Fix $\varepsilon > 0$ and let us show that $\mathbb{P}(X_n/n \geq \rho_* + \varepsilon) \rightarrow 0$ when $n \rightarrow \infty$. Let $\delta > 0$. Let $\varepsilon' = \varepsilon\delta/2$. Let $c > 0$ be such that the bound (1) holds for $\rho = \rho_* - \varepsilon'$. Then, for every $n \geq 1$ we can write

$$\begin{aligned} \rho_* &\geq \mathbb{E}\left[\frac{X_n}{n}\right] \geq (\rho_* + \varepsilon) \mathbb{P}\left(\frac{X_n}{n} \geq \rho_* + \varepsilon\right) + (\rho_* - \varepsilon') \mathbb{P}\left(\rho_* - \varepsilon' \leq \frac{X_n}{n} < \rho_* + \varepsilon\right) \\ &= \rho_* - \varepsilon' + (\varepsilon + \varepsilon') \mathbb{P}\left(\frac{X_n}{n} \geq \rho_* + \varepsilon\right) - (\rho_* - \varepsilon') \mathbb{P}\left(\frac{X_n}{n} < \rho_* - \varepsilon'\right) \\ &\geq \rho_* - \varepsilon' + \varepsilon \mathbb{P}\left(\frac{X_n}{n} \geq \rho_* + \varepsilon\right) - \rho_* e^{-cn}, \end{aligned}$$

which implies that

$$\mathbb{P}\left(\frac{X_n}{n} \geq \rho_* + \varepsilon\right) \leq \frac{\varepsilon' + \rho_* e^{-cn}}{\varepsilon} = \frac{\delta}{2} + \frac{\rho_*}{\varepsilon} e^{-cn} \leq \delta,$$

provided that n is large enough. This concludes the proof of the convergence of X_n/n to ρ_* in probability, thus concluding the proof of Lemma 1.

6 Limit of the driven-dissipative chain: proof of Theorem 2

We now show that $S_n/n \rightarrow \rho_c$ in probability. The superadditivity property given by Proposition 1, along with Lemma 1, already ensures that $S_n/n \rightarrow \rho_*$ in probability, where $\rho_* = \sup_{n \geq 1} \mathbb{E}S_n/n$. Thus, to obtain Theorem 2 there only remains to show that $\rho_* = \rho_c$.

We treat separately the lower bound $\rho_* \geq \rho_c$ and the upper bound $\rho_* \leq \rho_c$. In both cases we rely on Lemma 2 which tells us that S_n is the number of sleeping particles which remain in V_n after stabilization starting with one active particle per site.

6.1 Lower bound

The lower bound is a quite direct consequence of Lemma 3. Let $\rho < \rho_c$ and consider the following procedure to stabilize the segment V_n starting with one active particle per site.

For each particle in the initial configuration, we draw an independent Bernoulli variable with probability ρ . If we obtain 0, we force the particle to walk with acceptable topplings until it jumps out of V_n , and if we obtain 1 we leave the particle at its starting point. After this first step of the procedure, we obtain a configuration which is i.i.d. with mean density ρ , with all particles active. Let N_n be the random number of particles in this configuration (it is a binomial with parameters n and ρ).

In the second step we simply perform legal topplings in whatever order until a stable configuration is reached. Let M_n be the number of particles which jump out of V_n during this second step, so that the final number of sleeping particles remaining in V_n after the two steps is equal to $N_n - M_n$. Since we performed acceptable topplings, by Lemma 6 the odometer of the sequence that we performed is above the legal stabilizing odometer. The number of particles jumping out of V_n being a non-decreasing function of the odometer, we therefore have $N_n - M_n \leq S_n$. Besides, Lemma 3 tells us that $\mathbb{E}M_n = o(n)$ as $n \rightarrow \infty$. Hence, we have

$$\frac{\mathbb{E}S_n}{n} \geq \frac{\mathbb{E}[N_n - M_n]}{n} = \rho - \frac{\mathbb{E}M_n}{n} \xrightarrow{n \rightarrow \infty} \rho,$$

whence $\rho_\star \geq \rho$.

6.2 Upper bound

We now turn to the upper bound $\rho_\star \leq \rho_c$. Let us assume by contradiction that $\rho_\star > \rho_c$. Let $\rho' \in (\rho_c, \rho_\star)$, and let $\varepsilon > 0$ and $c > 0$ be given by Lemma 4 applied with ρ' , so that for every $n \geq 1$, for every deterministic initial configuration $\eta : V_n \rightarrow \mathbb{N}$ with at least $\rho'n$ particles, all active, we have $\mathbb{P}(M_n > \varepsilon n) \geq c$. Decreasing ε if necessary, we assume that $\rho_\star - \varepsilon/3 \geq \rho'$.

Since $S_n/n \rightarrow \rho_\star$ in probability, we can take n large enough so that $\mathbb{P}(S_n/n \leq \rho_\star + \varepsilon/3) \geq 1/2$. Let now L_n and R_n denote the number of particles which respectively jump out of V_n from the left and from the right endpoint during the stabilization of V_n starting with one active particle per site, so that $S_n = n - L_n - R_n$. We claim that

$$\left\{ \frac{S_n}{n} \leq \rho_\star + \frac{\varepsilon}{3} \right\} = \left\{ \frac{L_n + R_n}{n} \geq 1 - \rho_\star - \frac{\varepsilon}{3} \right\} \subset \bigcup_{k=0}^K \mathcal{A}_k,$$

where $K = \lfloor 3/\varepsilon \rfloor$ and for every $k \in \{0, \dots, K\}$ the event \mathcal{A}_k is defined by

$$\mathcal{A}_k = \left\{ \frac{L_n}{n} \geq k \frac{\varepsilon}{3} \right\} \cap \left\{ \frac{R_n}{n} \geq 1 - \rho_\star - (k+2) \frac{\varepsilon}{3} \right\}.$$

Indeed, on the event that $(L_n + R_n)/n \geq 1 - \rho_\star - \varepsilon/3$, if we define $k = \lfloor (L_n/n)/(\varepsilon/3) \rfloor$, then we have

$$\frac{R_n}{n} \geq 1 - \rho_\star - \frac{\varepsilon}{3} - \frac{L_n}{n} \geq 1 - \rho_\star - \frac{\varepsilon}{3} - (k+1) \frac{\varepsilon}{3},$$

which shows that the event \mathcal{A}_k is realized.

Recalling now that $\mathbb{P}(S_n/n \leq \rho_\star + \varepsilon/3) \geq 1/2$, we deduce that there exists $k \in \{0, \dots, K\}$ (which may depend on n) such that $\mathbb{P}(\mathcal{A}_k) \geq 1/(2(K+1))$. We fix such an integer k and we define $\ell = \lceil (k\varepsilon/3)n \rceil$ and $r = \lceil (1 - \rho_\star - (k+2)\varepsilon/3)n \rceil$, so that $\mathcal{A}_k = \{L_n \geq \ell, R_n \geq r\}$.

Let us now consider the following three-steps procedure to stabilize V_n starting with one active particle per site:

1. During the first step, we always topple the leftmost active particle in V_n , until either ℓ particles jumped out by the left exit, or no active particle remains.
2. If the configuration is already stable after step 1 or if already at least r particles jumped by the right exit during step 1, then we do nothing during step 2. Otherwise, step 2 consists in always toppling the rightmost active particle in V_n until either no active particle remains, or a total of r particles have jumped out by the right exit during steps 1 and 2.

3. If the configuration is still not stable after step 2, during the third and last step we perform legal topplings until the configuration is stable in V_n .

On the event \mathcal{A}_k , step 1 cannot stop before ℓ particles jump out by the left exit. This implies that the last toppling performed during step 1 is a toppling on the leftmost site of V_n , which entails that this site contained an active particle just before the end of step 1. Yet, note that during step 1 there can never be a sleeping particle to the left of an active particle. Hence on the event \mathcal{A}_k step 1 terminates with no sleeping particle in V_n and with ℓ particles having jumped by the left exit.

For the same reasons, on the event \mathcal{A}_k step 2 necessarily terminates with at least r particles having jumped out by the right exit (in total during steps 1 and 2), and with no sleeping particles in V_n .

Let η be the random configuration obtained after step 2. We just showed that on the event \mathcal{A}_k this configuration η contains only active particles, and the number of particles satisfies

$$\|\eta\| \leq n - \ell - r \leq \left(\rho_\star + \frac{2\varepsilon}{3}\right)n.$$

Therefore, we have $\mathcal{A}_k \subset \mathcal{E} \cup \mathcal{F}$, where the events \mathcal{E} and \mathcal{F} are defined by

$$\mathcal{E} = \left\{ \|\eta\| < \left(\rho_\star - \frac{\varepsilon}{3}\right)n \right\} \quad \text{and} \quad \mathcal{F} = \left\{ \left(\rho_\star - \frac{\varepsilon}{3}\right)n \leq \|\eta\| \leq \left(\rho_\star + \frac{2\varepsilon}{3}\right)n, \text{ no sleeping particle in } \eta \right\}.$$

Let M_n be the number of particles which jump out of V_n during step 3, so that $S_n = \|\eta\| - M_n$. Then, it follows from our assumptions that $\mathbb{P}(M_n > \varepsilon n \mid \mathcal{F}) \geq c$. Hence, we can write

$$\mathbb{P}\left(\frac{S_n}{n} \leq \rho_\star - \frac{\varepsilon}{3}\right) \geq \mathbb{P}(\mathcal{E}) + \mathbb{P}(\mathcal{F} \cap \{M_n > \varepsilon n\}) \geq \mathbb{P}(\mathcal{E}) + c\mathbb{P}(\mathcal{F}) \geq c\mathbb{P}(\mathcal{E} \cup \mathcal{F}) \geq c\mathbb{P}(\mathcal{A}_k).$$

Combining this with the bound $\mathbb{P}(\mathcal{A}_k) \geq 1/(2(K+1))$, we deduce that

$$\mathbb{P}\left(\frac{S_n}{n} \leq \rho_\star - \frac{\varepsilon}{3}\right) \geq \frac{c}{2(K+1)}.$$

This being true for every n large enough, we obtain a contradiction with the fact that $S_n/n \rightarrow \rho_\star$ in probability. The proof by contradiction that $\rho_\star \leq \rho_c$ is thereby complete.

7 The hockey stick: proof of Theorem 3

We now prove the hockey stick conjecture, which in fact easily follows from what we just did in Section 6.

Note that, following the Abelian property, for every $t \in \mathbb{N}$ the variable Y_t is the number of particles which remain in V_n after stabilization starting with an initial configuration consisting of t active particles placed independently and uniformly in V_n .

7.1 Upper bound for subcritical densities

The upper bound for subcritical densities is trivial because by definition of Y_t we always have $Y_t \leq t$.

7.2 Upper bound for supercritical densities

Note that for every $t \in \mathbb{N}$, it follows from the monotonicity property given by Lemma 7 that Y_t is stochastically dominated by S_n . Therefore, for every $\rho > \rho_c$ and for every $\varepsilon > 0$ we have

$$\mathbb{P}\left(\frac{Y_{\lfloor \rho n \rfloor}}{n} > \rho_c + \varepsilon\right) \leq \mathbb{P}\left(\frac{S_n}{n} > \rho_c + \varepsilon\right),$$

which tends to 0 as $n \rightarrow \infty$ by Theorem 2.

7.3 Lower bound

The lower bound follows the same line of proof as the lower bound on S_n established in Section 6.1. Let $\rho > 0$ and let $\rho' < \min(\rho, \rho_c)$. Let $n \geq 1$ and let η be a random initial configuration with $\lceil \rho n \rceil$ active particles placed independently and uniformly in V_n . Let ρ'' be such that $\rho' < \rho'' < \min(\rho, \rho_c)$. Then, one may couple this configuration η with an i.i.d. configuration η' with mean ρ'' such that $\eta \geq \eta'$ with probability tending to 1 as $n \rightarrow \infty$. The result then follows using Lemma 7 (monotonicity) and Lemma 3 (starting with an i.i.d. subcritical configuration, the probability that a positive fraction of the particles jump out tends to 0).

8 Growth of a ball: proof of Theorem 4

8.1 Inner bound

We start by proving the following result:

Lemma 8. *For every $n, k \geq 1$, for every $x \in V_n$, we have*

$$\mathbb{P}(x + A_k \subset V_n) \leq \mathbb{P}(S_n \geq k).$$

Proof. Let $n, k \geq 1$ and $x \in V_n$, and consider the initial configuration $\eta = \mathbf{1}_{V_n} + k\delta_x$, with $k + 1$ active particles at x and one active particle at each other site of V_n .

On the one hand, if we first move the k particles from x out of V_n , on top of the other n particles which we do not move, and then we let this remaining carpet of one active particle per site stabilize, then the number of remaining particles is distributed as S_n .

On the other hand, if we start by forcing the n particles of the carpet to move out of V_n , using acceptable topplings, and then let the k particles at x stabilize, then with probability $\mathbb{P}(x + A_k \subset V_n)$, these k particles all remain inside V_n .

Note that in the first scenario we only used legal topplings, while in the second scenario we used acceptable topplings. Thus, the number of particles remaining inside V_n being a non-increasing function of the odometer, the monotonicity property of Lemma 6 allows us to conclude. \square

With this result in hand we now turn to the proof of the inner bound.

Proof of Theorem 4, inner bound. Let $\rho > \rho_c$, and let's show that $\mathbb{P}(|A_k|/k \leq 1/\rho) \rightarrow 0$ as $k \rightarrow \infty$. Let's fix an intermediate density $\rho' \in (\rho_c, \rho)$. For every $k, n, m \geq 1$, we can write

$$\{|A_k| \leq n\} = \bigcup_{x \in V_n} \{x + A_k \subset V_n\} \subset \bigcup_{y=1}^{\lceil n/m \rceil} \{ym + A_k \subset V_{n+m}\},$$

where the last inclusion is obtained by taking $y = \lceil x/m \rceil$. Combining this with Lemma 8, we deduce that

$$\mathbb{P}(|A_k| \leq n) \leq \sum_{y=1}^{\lceil n/m \rceil} \mathbb{P}(ym + A_k \subset V_{n+m}) \leq \left\lceil \frac{n}{m} \right\rceil \mathbb{P}(S_{n+m} \geq k).$$

Taking now $n = n_k = \lfloor k/\rho \rfloor$ and $m = m_k = \lfloor n_k(\rho - \rho')/\rho' \rfloor$ we can write

$$\mathbb{P}\left(\frac{|A_k|}{k} \leq \frac{1}{\rho}\right) = \mathbb{P}(|A_k| \leq n_k) \leq \left\lceil \frac{n_k}{m_k} \right\rceil \mathbb{P}\left(\frac{S_{n_k+m_k}}{n_k+m_k} \geq \rho'\right),$$

using that $k \geq (n_k + m_k)\rho'$. When $k \rightarrow \infty$ we have $n_k + m_k \rightarrow \infty$, so that the last probability above tends to 0, following the convergence in probability $S_n/n \rightarrow \rho_c$ established in Theorem 2. Besides, we have $n_k/m_k = O(1)$ when $k \rightarrow \infty$, so we eventually deduce that $\mathbb{P}(|A_k|/k \leq 1/\rho) \rightarrow 0$ when $k \rightarrow \infty$, which is the desired inner bound. \square

8.2 Outer bound

Let $\rho < \rho_c$. As for the outer bound in [LS21], we proceed in two steps: first we make the particles spread (using acceptable topplings) to obtain a subcritical particle density, and then we let the particles stabilize, using that at subcritical densities few particles jump out of a segment. Thanks to Lemma 5 this implies that, with high probability, no particle jumps out of a slightly enlarged segment.

Let ρ', ρ'' be such that $\rho < \rho' < \rho'' < \rho_c$. Let $k \geq 1$. Let $\eta : \mathbb{Z} \rightarrow \{0, 1\}$ be an i.i.d. Bernoulli initial configuration with parameter ρ'' . We see the sites x such that $\eta(x) = 1$ as holes, which can be filled by one particle.

We start with k active particles at 0 and no other particles elsewhere. During the first step, we force each of these k particles to walk, with acceptable topplings, until it finds an unoccupied hole. At the end of this first step, we end up with the k particles placed in k consecutive holes. Let I be the set of sites visited during this first step. Note that this set I contains exactly k holes, i.e., we always have $\|\eta\|_I = k$, and the configuration after step 1 is simply $\eta \mathbf{1}_I$.

Then, in the second step we simply leave these k particles stabilize in \mathbb{Z} with legal topplings. Let B_k be the set of sites of \mathbb{Z} which are visited at least once during these two steps. The monotonicity with respect to enforced activation indicated by Lemma 6 entails that $A_k \subset B_k$. Therefore, we aim at showing the outer bound for B_k instead.

Taking $n = \lceil k/\rho' \rceil$, we can write

$$\{|I| \geq n\} = \bigcup_{x \in V_n} \{I \supset V_n - x\} \subset \bigcup_{x \in V_n} \{k = \|\eta\|_I \geq \|\eta\|_{V_n - x}\},$$

so that

$$\mathbb{P}(|I| \geq n) \leq \sum_{x \in V_n} \mathbb{P}(\|\eta\|_{V_n - x} \leq k) = n \mathbb{P}(\|\eta\|_{V_n} \leq k) = n \mathbb{P}\left(\frac{\eta(1) + \dots + \eta(n)}{n} \leq \rho'\right),$$

which tends to 0 when $n \rightarrow \infty$ by the law of large numbers, since $\rho' < \rho'' = \mathbb{E}[\eta(1)]$.

Thus, to prove the desired outer bound, there only remains to prove that $\mathbb{P}(\mathcal{E}_k) \rightarrow 0$ as $k \rightarrow \infty$, where $\mathcal{E}_k = \{|B_k| \geq k/\rho, |I| < n_k\}$ with $n_k = \lceil k/\rho' \rceil$. As we did above for the inner bound, for $n = n_k$ and $m \geq 1$ we may write

$$\mathcal{E}_k \subset \bigcup_{x \in V_n} \left\{ |B_k| \geq \frac{k}{\rho}, I \subset V_n - x \right\} \subset \bigcup_{y=1}^{\lceil n/m \rceil} \mathcal{F}_{k,y}, \quad (5)$$

where the events $\mathcal{F}_{k,y}$ are defined by

$$\mathcal{F}_{k,y} = \left\{ |B_k| \geq \frac{k}{\rho}, I \subset V_{n+m} - ym \right\}.$$

We now choose $\alpha, \beta > 0$ such that

$$\frac{1}{\rho'} + \alpha + 4\beta < \frac{1}{\rho}$$

and we define $m = m_k = \lceil \alpha k \rceil$ and $j = j_k = \lceil \beta k \rceil$ for $k \geq 1$, so that $n + m + 4j < k/\rho$ for k large enough. Let k be large enough so that this holds, and let $y \leq \lceil n/m \rceil$ and $J_y = V_{n+m} - ym$. Defining $K_y = \{1 - ym - 2j, \dots, n + m - ym + 2j\}$ we have $|K_y| = n + m + 4j < k/\rho$, so that

$$\mathcal{F}_{k,y} \subset \{B_k \not\subset K_y, I \subset J_y\}.$$

Now recall the notation $A(\eta)$ for the set of sites visited during the stabilization of a configuration η in \mathbb{Z} . With this notation, we have $B_k = I \cup A(\eta \mathbf{1}_I)$. Since $J_y \subset K_y$, we get

$$\mathcal{F}_{k,y} \subset \{A(\eta \mathbf{1}_I) \not\subset K_y, I \subset J_y\} \subset \{A(\eta \mathbf{1}_{J_y}) \not\subset K_y\}.$$

Thus, we obtain

$$\mathbb{P}(\mathcal{F}_{k,y}) \leq \mathbb{P}(A(\eta \mathbf{1}_{J_y}) \not\subset K_y) = \mathbb{P}(A(\eta \mathbf{1}_{V_{n+m}}) \not\subset K_0).$$

Using now Lemma 5 we deduce that, for every $i \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{F}_{k,y}) \leq 1 - \mathbb{P}(M_{n+m} \leq i) \mathbb{P}(G_1 + \cdots + G_i \leq j),$$

where G_1, \dots, G_i are i.i.d. Geometric variables with parameter $\lambda/(1+\lambda)$. Plugging this into (5), we are left with

$$\mathbb{P}(\mathcal{E}_k) \leq \left\lceil \frac{n}{m} \right\rceil \left(1 - \mathbb{P}(M_{n+m} \leq i) \mathbb{P}(G_1 + \cdots + G_i \leq j) \right).$$

Choosing now $i = i_k = \lceil \gamma k \rceil$ with a certain parameter $\gamma > 0$ such that $\gamma\lambda/(1+\lambda) < \beta$, we then have $\mathbb{P}(G_1 + \cdots + G_i \leq j) \rightarrow 1$ when $k \rightarrow \infty$ by the law of large numbers, and $\mathbb{P}(M_{n+m} \leq i) \rightarrow 1$ by Lemma 3. Since $n_k/m_k = O(1)$ when $k \rightarrow \infty$, we conclude that $\mathbb{P}(\mathcal{E}_k) \rightarrow 0$ as $k \rightarrow \infty$, which completes the proof of the outer bound.

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