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Macroscopic limit from a structured population model to the Kirkpatrick-Barton model

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Abstract

We consider an ecology model in which the population is structured by a spatial variable and a phenotypic trait. The model combines a parabolic operator on the spatial variable with a kinetic operator on the trait variable. We prove the existence of solutions to that model, and show that these solutions are unique. The kinetic operator present in the model, that represents the effect of sexual reproductions, satisfies a Tanaka-type inequality: it implies a contraction of the Wasserstein distance in the space of phenotypic traits. We combine this contraction argument with parabolic estimates controlling the spatial regularity of solutions to prove the convergence of the population size and the mean phenotypic trait to solutions of the Kirkpatrick-Barton model, which is a well-established model in evolutionary ecology. Specifically, at high reproductive rates, we provide explicit convergence estimates for the moments of solutions of the kinetic model.

Keywords: structured population, infinitesimal model, selection-mutation, asymptotic analysis, macroscopic limit, Wasserstein estimates, parabolic estimates, mathematical ecology.

MSC 2000 subject classification: 35B40, 35K57, 92D15, 92D25, 92D40.

1 Introduction

We are interested in a structured population model that describes the dynamics of a biological population (typically a species of trees subject to climate change). At each time $t \geq 0$ the population is structured by a phenotypic trait $y \in \mathbb{R}$ and a spatial variable $x \in \mathbb{T}^d$ (the $d \in \mathbb{N}^*$ dimensional torus). The population is then represented by a density $n = n(t, x, y)$, and the dynamics of this population is given by the *Spatially structured Infinitesimal Model* (see [41]):

$$\begin{aligned} \partial_t n(t, x, y) = & \Delta_x n(t, x, y) + \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n(t, x, z) dz \right) n(t, x, y) \\ \text{(SIM)} \quad & + \gamma \left(\iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} dy_* dy'_* - n(t, x, y) \right), \end{aligned}$$

where $A > 0$ is the *phenotypic variance at linkage equilibrium* of the population (see [31, 15]), $y_{opt} : \mathbb{T}^d \rightarrow \mathbb{R}$ is a description of the environment (typically $y_{opt}(t, x)$ is the temperature at time t and location x), and $\Gamma_{A/2} : \mathbb{R} \rightarrow \mathbb{R}_+$ denotes the Gaussian distribution with variance $A/2$:

$$\Gamma_{A/2}(y) := \frac{1}{\sqrt{2\pi A}} e^{-\frac{|y|^2}{A}}.$$

Note that generalisations of the (SIM), for instance more general selection operators, could be introduced, but we focus on that specific model in this manuscript, because of its connection to the Kirkpatrick-Barton model that we describe below. (SIM) is composed of parabolic terms, common in ecology models (see the Fisher-KPP equation [36], and [3]), and a kinetic term, with a factor $\gamma > 0$, representing the effect of sexual reproduction. Beyond the importance of this model for applications, (SIM) is an opportunity to develop the analysis methods introduced for other kinetic models (in particular the Boltzmann equation), using an unusual diffusion term in the space variable. This diffusive term enables us to propose a new method to derive a macroscopic limit: using from Wasserstein estimates on the collision operator, we are able to show that when $\gamma > 0$ is large, the dynamics of n can be described by a closed equation on its two first moments.

Indeed, we show that if $\gamma > 0$ is large, the solutions of (SIM) satisfy

$$n(t, x, y) \sim N(t, x) \Gamma_A(y - Z(t, x)),$$

where the macroscopic quantities N and Z asymptotically satisfy the *Kirkpatrick-Barton Model*:

$$\text{(KBM)} \quad \begin{cases} \partial_t N(t, x) - \Delta_x N(t, x) = [1 - \frac{1}{2}(Z(t, x) - y_{opt}(t, x))^2 - N(t, x)] N(t, x), \\ \partial_t Z(t, x) - \Delta_x Z(t, x) = 2 \frac{\nabla_x N \cdot \nabla_x Z}{N}(t, x) - A(Z(t, x) - y_{opt}(t, x)). \end{cases}$$

(KBM), introduced in [46], is used in the evolutionary ecology literature. So far, (SIM) and (KBM) have received limited attention from the mathematical community. In [45] the existence of solutions for models related to (SIM) is discussed. In [41] the propagation fronts for a simplified model are constructed (this article also contains non-rigorous asymptotics related to the present study), while in [40], the long time dynamics of a different simplified model is discussed. In [48], travelling waves and steady distributions have been constructed for (KBM) when a parameter is small. This remarkable result is obtained by a perturbative argument around the case $\varepsilon = 0$ which corresponds to the Fisher-KPP equation [32, 36]: the small parameter $\varepsilon > 0$ could correspond to a weak selection and a weakly heterogeneous environment. Finally, we mention the study of acceleration fronts for sexual reproduction models in [26, 23]. We refer to Section 2.3 for a discussion of the biological aspects of (SIM), (KBM), and the biological implications of this manuscript's results.

In the case of asexual populations, the last term of (SIM) simplifies considerably: it is then replaced by a local term plus a diffusive part (that represents mutations). These asexual population models have recently received considerable attention, and the propagation phenomena that they present are now well understood. The main idea in the asexual case is to consider the model as a semi-linear parabolic equation, to control the non-local competition term through a Harnack inequality, and to use topological fixed-point arguments to construct propagation fronts [3, 7, 11]. Additional difficulties arise when the phenotypic trait y has an impact on the spatial diffusion of individuals in space (see [11, 55, 8]), and these models can lead to acceleration fronts [8, 12]. Finally, when the mutation rate is low, these asexual models can be related to constrained Hamilton-Jacobi equations [13, 55, 10]. Note that in the asexual case, the propagation speed of the population (which plays an important role in biology) is given by a linearisation of the model, and is then explicit in terms of a certain principal eigenvalue problem. This simple characterisation of the propagation velocity is no longer valid in the case of sexual populations, and the macroscopic limit described here can

be used to describe the propagation phenomena for (SIM) (we refer to [18, 50] for a related idea in mathematical physics).

The macroscopic limit we present here is based on the Wasserstein contraction induced by the reproduction operator (see Theorem 5.1). This contraction property exists for a series of operators appearing in physics or econometry [6, 9, 58], and was originally obtained by Tanaka [54]. To our knowledge, few rigorous macroscopic/hydrodynamic results have been derived from it (see [47] for a result without spatial structure). Note that the strategy adopted here is to combine Wasserstein estimates (for the reproduction term) with estimates of a different nature (parabolic estimates for the spatial dimension). This strategy is related to the work of Carlen and Gangbo [24] (see also [1]), who are interested in a kinetic Fokker-Planck equation that combines a hyperbolic transport term in space with a kinetic operator in velocity space. This kinetic operator involves a contraction of the Wasserstein distance. The authors show the long time convergence of solutions to the set of local Maxwellians, but this large-time convergence is not quantitative, due to the lack of regularity estimates in the spatial variable. In the present study, the presence of a diffusive term in the space variable allows us to push the analysis further. Finally, we are also able to handle the selection/competition term to justify the macroscopic limit of (SIM) described above.

Recently, structured population models with sexual reproduction but without a spatial variable have attracted attention. The existence and uniqueness of solutions for such problems has been considered in [51, 34]. A difficult problem, studied in [19, 22, 21, 52] is to show that solutions converge to a unique steady-state when the selection term has a single local maximum. A second approach (in fact closely related to the asymptotic $\gamma \gg 1$ that we consider in this manuscript) is to assume that the reproduction kernel $\Gamma_{A/2}$ has a small variance, see [51, 27, 53, 20, 26]. This small variance approach was also also considered in [34], where an analysis based on moments of the distribution was introduced to describe the dynamics of solutions.

2 Main results and organisation of the paper

In this manuscript, we assume that the optimal phenotypic trait $(t, x) \mapsto y_{opt}(t, x)$ and the initial population $(x, y) \mapsto n^0(x, y)$ satisfy

Assumption 1. *For some $C_1 > 0$,*

(i) $y_{opt} \in C^1(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})$ such that $\|y_{opt}\|_{W^{1,\infty}(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})} < C_1$.

(ii) $n^0 \in C^0(\mathbb{T}^d \times \mathbb{R}, \mathbb{R}_+)$, such that $n^0 > 0$ and

$$\forall (x, y) \in \mathbb{T}^d \times \mathbb{R}, \quad 0 < n^0(x, y) \leq \frac{C_0}{1 + y^{10}},$$

as well as, for $k \in \{0, 1, 4\}$,

$$\left\| x \mapsto \int_{\mathbb{R}} y^k n^0(x, y) dy \right\|_{W^{4,\infty}(\mathbb{T}^d)} \leq C_1.$$

The assumption $n^0(x, y) \leq \frac{C_0}{1+y^{10}}$ seems a strong assumption, but it is actually coherent with biological applications where populations typically have Gaussian tails in the phenotypic

variable. The last regularity assumption is technical: it is a $W^{4,\infty}(\mathbb{T}^d)$ bound on moments of the initial population. For $k = 0$, it corresponds to a regularity assumption on the initial population size N^0 , and on the initial mean phenotypic trait Z^0 for $k = 1$. the case $k = 4$ corresponds to a higher moment that will be useful for asymptotic analysis we will perform in this manuscript. Thanks to these assumptions, we will be able to show that N and Z , that is

$$N(t, x) = \int_{\mathbb{R}} n(t, x, y) dy, \quad Z(t, x) = \int_{\mathbb{R}} y \frac{n(t, x, y)}{\int_{\mathbb{R}} n(t, x, z) dz} dy, \quad (1)$$

are regular in both variables (see Remark 2.7). We will then be able to apply the comparison principle to several equations, which will be crucial in our analysis. Note that $N(t, x)$ and $Z(t, x)$ represent respectively the population size and the mean phenotypic trait at time $t \geq 0$ and position $x \in \mathbb{T}^d$. One could probably consider weaker assumptions on n^0 , but we leave this problem for future research.

2.1 Existence and uniqueness of solutions of (SIM)

In this section, we consider (SIM) with given coefficients, and in particular, $\gamma > 0$ is fixed. We indicate the dependency of the constants in $\gamma > 0$ for readability (see e.g. (4)), but not in the proofs. The next section will be devoted to the asymptotic limit $\gamma \rightarrow \infty$ and the impact of $\gamma > 0$ will then be carefully monitored. We define solutions of (SIM) as follows:

Definition 2.1. *Let $T \in [0, \infty]$, $A, \gamma > 0$, $y_{opt} \in C^1([0, T] \times \mathbb{T}^d)$ and $n^0 \in C^0(\mathbb{T}^d \times \mathbb{R}, \mathbb{R}_+)$. $n \geq 0$ is a solution of (SIM) with initial data n^0 if $n \in L^\infty([0, T] \times \mathbb{T}^d, L^1(\mathbb{R}))$, $n \in L^2([0, T] \times \mathbb{R}, H^1(\mathbb{T}^d))$, $\partial_t n \in L^2([0, T] \times \mathbb{R}, H^{-1}(\mathbb{T}^d))$, and satisfies, for $v \in L^2(\mathbb{R} \times H^1(\mathbb{T}^d))$ and almost every $t \in [0, T]$,*

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t n(t, x, y) v(x, y) dx dy + \int_{\mathbb{R}} \int_{\mathbb{T}^d} \nabla_x n(t, x, y) \cdot \nabla_x v(x, y) dx dy \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n(t, x, z) dz \right) n(t, x, y) v(x, y) dy dx \\ & \quad + \gamma \int_{\mathbb{T}^d} \int_{\mathbb{R}} v(x, y) \left[\iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} dy_* dy'_* - n(t, x, y) \right] dy dx, \end{aligned} \quad (2)$$

it should also satisfy $n(0, \cdot, \cdot) = n^0$ and for any $\tilde{T} \in [0, T]$, $\tilde{T} < \infty$, there should be a constant $\bar{C} > 0$ such that

$$\sup_{y \in \mathbb{R}} \left(\sup_{t \in [0, \tilde{T}]} \|n(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n(\cdot, \cdot, y)\|_{L^2_{loc}([0, \tilde{T}], H^2(\mathbb{T}^d))} + \|\partial_t n(\cdot, \cdot, y)\|_{L^2_{loc}([0, \tilde{T}] \times \mathbb{T}^d)} \right) < \bar{C} < \infty. \quad (3)$$

This notion of solution is closely connected to the notion of strong solutions in e.g. [37]. We prove the existence of such solutions of (SIM) in the following proposition:

Proposition 2.2. *Let $y_{opt} \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{T}^d)$, n^0 satisfying Assumption 1, $A > 0$, and $\gamma > 2 + A + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}$. There is a unique global solution n of (SIM) with the initial*

data n^0 , in the sense of definition 2.1. More precisely, for a constant C_γ that may depend on γ and $(t, x, y) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}$,

$$n(t, x, y) \leq \frac{C_\gamma}{1 + y^{10}}, \quad (4)$$

while for any $T > 0$, there is $C_{\gamma, T} > 0$ such that

$$\sup_{t \in [0, T]} \|n(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n(\cdot, \cdot, y)\|_{L^2([0, T], H^2(\mathbb{T}^d))} + \|\partial_t n(\cdot, \cdot, y)\|_{L^2([0, T] \times \mathbb{T}^d)} \leq \frac{C_\gamma(1 + \gamma)}{1 + y^8}. \quad (5)$$

Moreover, for $(t, x, y) \in \mathbb{R}_+ \times \mathbb{T}^d \times [-1, 1]$,

$$n(t, x, y) \geq \left(\min_{\mathbb{R}_+ \times \mathbb{T}^d \times [-1, 1]} n^0 \right) e^{-C_\gamma t}. \quad (6)$$

Remark 2.3. In Section 5.4, we show that the macroscopic quantities N (resp. Z, V) defined by (1) (resp. (1), (18)) from the solution n of (SIM) given by Proposition 2.2 are C^1 in t and C^2 in x , and they satisfy (1), (18) seen as equalities between continuous functions. In particular the comparison principle (Corollary 7.4 p. 159 in [37]) applies to (17) and (19) when the terms $\frac{\nabla_x N(t, x)}{N(t, x)}$ and $\tilde{n}(t, x, y)$ are considered as given coefficients (we refer to (1) and (14) for the definition of N and \tilde{n}). The comparison principle will play a crucial role in Section 4.

The existence of solutions to (KBM) that are C^1 in t and C^2 in x will be shown in Section 5.7 (using Proposition 4.5), and the uniqueness of such solutions is proven in Proposition 5.7.

We prove Proposition 2.2 in Section 3. A preliminary step is to prove the existence of global solutions n_R to the following truncated problem for $R > 0$:

$$\begin{aligned} \partial_t n_R(t, x, y) - \Delta_x n_R(t, x, y) &= \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n_R(t, x, z) dz \right) n_R(t, x, y) \\ &+ \gamma \left(1_{|y| \leq R} \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* - n_R(t, x, y) \right), \quad (7) \end{aligned}$$

together with $n_R(0, x, y) = n^0(x, y) 1_{|y| \leq R}$. We do so in Section 5.3: in Lemma 5.3, we use a Cauchy-Lipschitz-type construction to construct solutions of (7) on a short time interval. We extend these to prove the existence of global solutions of (7) in Proposition 5.4, and we obtain uniform estimates on the tails of solutions in the phenotypic variable y in Proposition 5.5. In Section 3, we show that n_R converges weakly to a solution n of (SIM) when $R \rightarrow \infty$, which proves Proposition 2.2. Note that to show the convergence of the non-linear birth term, we use the Dunford-Pettis Theorem: since the functions n_R are uniformly bounded, their weak convergence implies their strong convergence. This idea is reminiscent of weak compactness arguments in kinetic theory [60], where the L^∞ bound on solutions of the truncated problem is replaced by estimates on the entropy of solutions and the De La Vallée-Poussin equi-integrability Criterion.

In the next proposition, we show that solutions of (SIM) are unique:

Proposition 2.4. *Let y_{opt} , n^0 satisfying Assumption 1, $A > 0$, and $\gamma > 0$. Let n, \tilde{n} solutions of (SIM), in the sense of definition 2.1, defined on $[0, T] \times \mathbb{T}^d \times \mathbb{R}$ with the same initial data $n^0 \in L^1(\mathbb{T}^d \times \mathbb{R}, \mathbb{R}_+)$, such that*

$$\max_{x \in \mathbb{T}^d} \int (1 + y^4) n^0(x, y) dy < \infty.$$

If there is a constant $C > 0$ such that

$$\forall (t, x) \in [0, T] \times \mathbb{T}^d, \quad \int_{\mathbb{R}} n(t, x, y) (1 + y^4) dy \leq C, \quad \int_{\mathbb{R}} \tilde{n}(t, x, y) (1 + y^4) dy \leq C, \quad (8)$$

then $n = \tilde{n}$.

Remark 2.5. *The assumption (8) on moments of n and \tilde{n} could be replaced by the assumption $n^0(x, y) \leq \frac{C}{1+y^6}$. Indeed, we can use an argument introduced in the proof of Proposition 2.2 (see Step 1) to show that this bound is propagated for $t \in [0, T]$, and (8) is then always satisfied by n and \tilde{n} .*

The proof of Proposition 2.4 is detailed in Section 5.7. It follows an argument developed by J. Guerand, M. Hillairet and S. Mirrahimi in [34] (see section A.3) for a model without spatial structure. We adapt their proof to include the spatial variable x in (SIM) with minimal adjustments.

2.2 Macroscopic limits of solutions of (SIM)

Our main asymptotic result, stated below, shows that when $\gamma > 0$ is large, n satisfies:

$$n(t, x, y) \underset{\gamma \gg 1}{\sim} N(t, x) \Gamma_A(y - Z(t, x)),$$

and the couple (N, Z) , given by (1), is close to the solution of (KBM) with initial data

$$(N(0, \cdot), Z(0, \cdot)) = (N^0, Z^0).$$

Theorem 2.6. *Let y_{opt} , n^0 satisfying Assumption 1 and $A > 0$. For $\gamma > 0$, let $n_\gamma \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ the solution of (SIM) with initial data n^0 .*

There exist $\bar{\gamma} > 0$, $C > 0$ and $\theta \in (0, 1)$ such that for any $\gamma > \bar{\gamma}$, there exist $\varphi_{N, \gamma}, \varphi_{Z, \gamma} : \mathbb{R}_+ \times \mathbb{T}^d \rightarrow \mathbb{R}$ satisfying

$$\|\varphi_{N, \gamma}(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|\varphi_{Z, \gamma}(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^\theta} + C \mathbf{1}_{[0, C \ln \gamma / \gamma]}(t) \quad (9)$$

such that the functions N_γ and Z_γ defined from n_γ by (1) satisfy the following equations as equalities between continuous functions:

$$\begin{cases} \partial_t N_\gamma(t, x) - \Delta_x N_\gamma(t, x) = \left[1 - \frac{1}{2}(Z_\gamma(t, x) - y_{opt}(t, x))^2 - N_\gamma(t, x) + \varphi_{N, \gamma}(t, x)\right] N_\gamma(t, x), \\ \partial_t Z_\gamma(t, x) - \Delta_x Z_\gamma(t, x) = 2 \frac{\nabla_x N_\gamma \cdot \nabla_x Z_\gamma}{N_\gamma}(t, x) - A(Z_\gamma(t, x) - y_{opt}(t, x)) + \varphi_{Z, \gamma}(t, x), \end{cases} \quad (10)$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$. Moreover,

$$\max_{(t, x) \in [\theta \ln \gamma / \gamma, \infty) \times \mathbb{T}^d} W_2 \left(\frac{n_\gamma(t, x, \cdot)}{N_\gamma(t, x)}, \Gamma_A(\cdot - Z_\gamma(t, x)) \right) \leq \frac{C}{\gamma^\theta}, \quad (11)$$

where W_2 stands for the Wasserstein distance, as defined in Section 5.1.

We show in Section 5.7 that this theorem implies the convergence of (N_γ, Z_γ) to the solution (\bar{N}, \bar{Z}) of (KBM):

$$\begin{aligned} N &\xrightarrow[\gamma \rightarrow \infty]{} \bar{N} && \text{in } L_{loc}^\infty((0, +\infty), L^\infty(\mathbb{T}^d)), \\ Z &\xrightarrow[\gamma \rightarrow \infty]{} \bar{Z} && \text{in } L_{loc}^\infty((0, +\infty), L^\infty(\mathbb{T}^d)). \end{aligned} \tag{12}$$

Remark 2.7. *The estimates given by Theorem 2.6 are global in time, even though $N(t, \cdot)$ may converge to 0 when $t \rightarrow \infty$. This is possible because the last term of (SIM) (ie the "kinetic" operator) scales linearly with n . It therefore also holds for populations that are going extinct.*

Estimate (10) is a stronger result (more quantitative) than what is typically obtained in macroscopic limits, through e.g. a truncated Hilbert expansion [17] or compactness arguments [14]. This is possible thanks to the diffusion operator in x present in (SIM) that brings some regularity to solutions.

In Section 3, we prove the existence of global solutions for (SIM). In Section 4.1, we show that an $L^\infty([0, \tau] \times \mathbb{T}^d)$ bound on Z (with $\tau \geq 0$) implies an estimate on the fourth moment of $y \mapsto n(t, x, \cdot)$ for $t \in [0, \tau + \bar{\tau}]$, with $\bar{\tau} > 0$. In Section 4.2 we show that Z is Hölder continuous, provided we have a bound on $\|Z\|_{L^\infty}$. This regularity is used in Section 4.3 together with a Tanaka-type inequality (see Theorem 5.1 in the Appendix) to show that $\frac{n(t, x, \cdot)}{N(t, x)}$ is close to $\Gamma_A(\cdot - Z(t, x))$ for the Wasserstein distance W_2 when $\gamma > 0$ is large enough. Finally in Section 4.4 we use the estimates mentioned above to obtain a uniform bound on $\|Z\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}$, through a contradiction argument showing that the maximal time τ where this estimate hold is actually $\tau = \infty$. This estimate implies the macroscopic limit described in Theorem 2.6.

2.3 Biological interpretation of the model and impact for ecology

The first term on the right-hand side of (SIM), $\Delta_x n(t, x, y)$, represents the dispersion of individuals in space. The term $(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2) n(t, x, y)$ represents the effect of natural selection: the individuals whose phenotypic trait y is far from the optimal trait $y_{opt}(t, x)$ have a high mortality rate. The function y_{opt} should therefore be considered as a description of the environment and is a given function. For instance the trait y is could be the temperature to which an individual is best adapted to, and y_{opt} is then the predicted map of temperatures. The term $-(\int n(t, x, z) dz) n(t, x, y)$ in (SIM) represents competition: all individuals present at time $t \geq 0$ and location $x \in \mathbb{T}^d$ are competing for resources. The last term describes the effect of sexual reproductions: when parents give birth to an offspring, the phenotypic trait of the offspring is drawn from a normal distribution with a fixed variance $A/2$ centred on the mean of the parents' traits. This model for the effect of sexual reproduction on a continuous phenotypic trait is known as the *Infinitesimal Model*. It was introduced by Fisher in 1919 [30], and is used in population genetics either for theoretical purpose [15, 56, 5] or for practical purposes [39, 57]. The $\gamma \gg 1$ limit corresponds to a short generation time and it can be seen as the implicit assumption behind the classical *Linkage Equilibrium* assumption used in population genetics (see for instance [15]): in the framework of the *Infinitesimal Model* the Linkage Equilibrium assumption implies that the population distribution $\tilde{n}(t, x, \cdot)$ is Gaussian with fixed variance. Numerical simulations (see [41]) suggest that the macroscopic limit model (KBM) provides a good description of the dynamics of solutions of (SIM) even when γ is not very large: for $\gamma = 2$, the information provided by (KBM) is already relevant.

We expect (SIM) to be related to a well-chosen Individual Based Model via a large number of individuals argument, but as far as we know, no such asymptotics exists at present. This type of derivation exists for asexual models [25], but here an additional difficulty arises: describing (SIM) as a large population limit of an Individual Based Model will require a precise understanding of the link between explicit genetic models and the *Infinitesimal Model* (which is at the root of the reproduction operator appearing in (SIM)). We refer to [5] for more information on this limit.

(KBM) was introduced by Kirkpatrick and Barton in 1997 [46], and is used to model the range dynamics of populations, particularly when these populations are subject to climate change, see for example [16, 2]. The success of (KBM) comes from the complex dynamics it presents [46, 41]: even for a very simple environment described by $y_{opt}(t, x) = Bx$ (and $x \in \mathbb{R}$), the population can either become extinct, survive without spreading, or spread (see [46, 48]). From a mathematical point of view, these dynamics raise a number of difficult questions. Several simplified models exist (see [44, 41]), and we refer to [40, 41] for analysis of some of these simplified models.

A good understanding of the connections between (SIM) and (KBM) (and other connections with stochastic models) has practical implications: the different scales (such as the mesoscopic scale of (SIM) and the macroscopic scale of the (KBM)) are not clearly distinct in most biological systems, and easy navigation between the different scales of description is an essential feature of the theory. This can be seen in [2], where the macroscopic limit from (SIM) to (KBM) plays an important role. We believe that these models will play an important role in understanding the effect of climate change on species, and are a valuable complement to Species Distribution Models (see e.g. [35]) that currently prevail.

2.4 Preliminary: equations satisfied by solutions of the (SIM)

Let $n : (t, x, y) \mapsto n(t, x, y)$ a solution of (SIM). In this section we derive heuristically equations satisfied by n normalized by $\int n(t, x, y) dy$ and other moments of n . The existence and regularity results stated in Section 2.1 (see Proposition 2.2 and Remark 2.7) will provide a rigorous framework for these partial differential equations.

If we integrate (SIM) along the variable y , we get that the population size N (see (1) for its definition) satisfies, for $t \geq 0$ and $x \in \mathbb{T}^d$,

$$\partial_t N(t, x) - \Delta_x N(t, x) = \left[1 + \frac{A}{2} - N(t, x) \right] N(t, x) - \frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 n(t, x, y) dy. \quad (13)$$

We define the normalized profile of the population,

$$\tilde{n}(t, x, y) = \frac{n(t, x, y)}{N(t, x)}, \quad (14)$$

which satisfies

$$\begin{aligned} & \partial_t \tilde{n}(t, x, y) - \Delta_x \tilde{n}(t, x, y) \\ &= 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \tilde{n}(t, x, y) + \gamma (T(\tilde{n}(t, x, \cdot)) - \tilde{n}(t, x, y)) \\ & \quad + \frac{1}{2} \tilde{n}(t, x, y) \left(\int_{\mathbb{R}} (z - y_{opt}(t, x))^2 \tilde{n}(t, x, z) dz - (y - y_{opt}(t, x))^2 \right), \end{aligned} \quad (15)$$

where T , the Infinitesimal operator, is defined by

$$T(\tilde{n})(y) := \int_{\mathbb{R}} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \tilde{n}(t, y_*) \tilde{n}(t, y'_*) dy_* dy'_*. \quad (16)$$

In the Appendix (Section 5.2), we detail the important properties of this operator. From this expression, we can deduce the following equation on the mean phenotypic trait of the population Z (see (1) for its definition):

$$\begin{aligned} & \partial_t Z(t, x) - \Delta_x Z(t, x) \\ &= 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x Z(t, x) - \frac{1}{2} \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy. \end{aligned} \quad (17)$$

We define

$$V(t, x) := \int_{\mathbb{R}} |y|^4 \tilde{n}(t, x, y) dy, \quad (18)$$

and thanks to (15), we show that V satisfies

$$\begin{aligned} & \partial_t V(t, x) - \Delta_x V(t, x) \\ &= 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x V(t, x) + \frac{1}{2} \int_{\mathbb{R}} (V(t, x) - |y|^4) (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy \\ & \quad + \gamma \left(\int_{\mathbb{R}} |y|^4 T(\tilde{n}(t, x, \cdot))(y) dy - V(t, x) \right). \end{aligned} \quad (19)$$

3 Existence of solutions for (SIM)

In this section, we prove Proposition 2.2. For $R > 0$, we denote by n_R the solution of (7) provided by Proposition 5.4.

Step 1: We show that the sequence $(n_R)_{R>0}$ converges to a limit n

An integration of (7) shows that $N_R(t, x) := \int n_R(t, x, y) dy$ satisfies

$$\begin{aligned} \partial_t N_R(t, x) - \Delta_x N_R(t, x) &= \left[1 + \frac{A}{2} - N_R(t, x) \right] N_R(t, x) - \frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 n_R(t, x, y) dy \\ & \quad + \gamma \left(\iint_{\mathbb{R}^2} \left(\int_{-R}^R \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) dy \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* - N_R(t, x) \right). \end{aligned} \quad (20)$$

Thanks to (86) (that also implies a bound on N_R), the right hand side of this equation is uniformly bounded (independently from $R > 0$):

$$\begin{aligned} |\partial_t N_R(t, x) - \Delta_x N_R(t, x)| &\leq \left[1 + \frac{A}{2} + N_R(t, x) \right] N_R(t, x) + \frac{1}{2} \int_{\mathbb{R}} (1 + y^2) n_R(t, x, y) dy \\ & \quad + \gamma \left(2R \Gamma_{A/2}(0) \left(\int_{\mathbb{R}} \frac{n_R(t, x, y_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* \right) \left(\int_{\mathbb{R}} n_R(t, x, y'_*) dy'_* \right) + N_R(t, x) \right) < C. \end{aligned}$$

Theorem 5 in Section 7.1.3 of [29] implies that N_R is a *strong* solution (in the sense of [29]) of this heat equation with the right hand side as a 0-th order term: for any $T > 0$, there exists a constant $C_\gamma > 0$, independent from $R > 0$ (but that depends on $\gamma > 0$) such that

$$\sup_{t \in [0, T]} \|N_R(t, \cdot)\|_{H^1(\mathbb{T}^d)} + \|N_R\|_{L^2([0, T], H^2(\mathbb{T}^d))} + \|\partial_t N_R\|_{L^2([0, T] \times \mathbb{T}^d)} \leq C_\gamma. \quad (21)$$

We recall that n_R also satisfies a regularity estimate independent from $R > 0$, see (80).

Thanks to (81), the family of measures n_R is tight on $[0, T] \times \mathbb{T}^d \times \mathbb{R}$, for any bounded time interval $[0, T]$. Thanks to Prokhorov's theorem, $(n_R)_{R>1}$ converges when $R \rightarrow +\infty$, up to an extraction (R_k) , for the weak topology of measures, to a limit n . Thanks to (81), the sequence $(n_{R_k})_k$ is uniformly bounded in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R})$, and thus equi-integrable. We can therefore apply the Dunford-Pettis Theorem to show that $(n_{R_k})_k$ converges strongly to n in $L^1([0, T] \times \mathbb{T}^d \times \mathbb{R})$, when $k \rightarrow +\infty$, up to an extraction. Note that we can use a diagonal argument to show that this convergence holds for any $T > 0$. This implies in particular the convergence of N_{R_k} to $N(t, x) := \int_{\mathbb{R}} n(t, x, y) dy$ in $L^1([0, T] \times \mathbb{T}^d)$:

$$\int_0^T \int_{\mathbb{T}^d} |N_{R_k}(t, x) - N(t, x)| dx dt \leq \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} |n_{R_k}(t, x, y) - n(t, x, y)| dy dx dt.$$

The uniformity of (80) implies that the limit n of n_{R_k} satisfies (3), and similarly the estimate (21) on N_{R_k} implies that N satisfies the same estimate. Next, we prove a lower bound estimate on n_{R_k} . Thanks to (7), for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$ and $|y| \leq 1$,

$$\begin{aligned} & \partial_t n_{R_k}(t, x, y) - \Delta_x n_{R_k}(t, x, y) \\ & \geq - \left(\frac{1}{2} \left(R + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \right)^2 + \|N_{R_k}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + \gamma \right) n_{R_k}(t, x, y), \end{aligned} \quad (22)$$

and we notice that for any fixed $y \in [-1, 1]$,

$$(t, x) \mapsto \left(\min_{\mathbb{R}_+ \times \mathbb{T}^d \times [-1, 1]} n^0 \right) e^{-\left(\frac{1}{2} \left(R + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \right)^2 + \|N_{R_k}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + \gamma \right) t}$$

is a sub-solution of (22), and we can use the comparison principle to show that for any $(t, x, y) \in \mathbb{R}_+ \times \mathbb{T}^d \times [-1, 1]$,

$$n_{R_k}(t, x, y) \geq \left(\min_{\mathbb{R}_+ \times \mathbb{T}^d \times [-1, 1]} n^0 \right) e^{-\left(\frac{1}{2} \left(R + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \right)^2 + \|N_{R_k}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + \gamma \right) t}. \quad (23)$$

Since this estimate is uniform in $k \in \mathbb{N}$, the limit n of n_{R_k} satisfies the same estimate, which proves (6).

Step 2: We show that n is a solution of (SIM)

For $\phi \in C_c^0([0, \infty) \times \mathbb{T}^d \times \mathbb{R})$ such that $\partial_t \phi, \nabla_x \phi, \Delta_x \phi \in C_c^0(\mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R})$. Since ϕ is compactly supported, there is $B > 0$ such that $\text{supp } \phi \subset [0, B] \times \mathbb{T}^d \times [-B, B]$ for some $B > 0$. Since $\partial_t \phi, \Delta_x \phi \in L^1([0, \infty) \times \mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_t \phi(t, x, y) + \Delta_x \phi(t, x, y)) n_{R_k}(t, x, y) dy dx dt \\ & - \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_t \phi(t, x, y) + \Delta_x \phi(t, x, y)) n(t, x, y) dy dx dt \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned} \quad (24)$$

We can also estimate the following quantity:

$$\begin{aligned}
I_0 &:= \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \left(\left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n_{R_k}(t, x, z) dz \right) n_{R_k}(t, x, y) \right. \right. \\
&\quad \left. \left. - \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n(t, x, z) dz \right) n(t, x, y) \right) n(t, x, y) dy dx dt \right| \\
&\leq C \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - N(t, x) \right) (n_{R_k}(t, x, y) - n(t, x, y)) dy dx dt \right| \\
&\quad + \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \phi(t, x, y) n_{R_k}(t, x, y) dy \right) (N(t, x) - N_{R_k}(t, x)) dx dt \right| = |I_1| + |I_2|. \quad (25)
\end{aligned}$$

To estimate $|I_1|$, we notice that $(t, x, y) \mapsto \phi(t, x, y) \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - N(t, x) \right)$ is a compactly supported bounded function that does not depend on $k \in \mathbb{N}$. The convergence of n_{R_k} to n in $L^1([0, T] \times \mathbb{T}^d \times \mathbb{R})$ then implies $|I_1| \rightarrow 0$ as $k \rightarrow \infty$. To estimate $|I_2|$, we take advantage of $\text{supp } \phi \subset [0, B] \times \mathbb{T}^d \times [-B, B]$:

$$|I_2| \leq \|\phi n_{R_k}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))} \|N - N_{R_k}\|_{L^1([0, B] \times \mathbb{T}^d)} \xrightarrow[k \rightarrow \infty]{} 0,$$

thanks to the fact that $\phi \in C_c^0([0, \infty) \times \mathbb{T}^d \times \mathbb{R})$, the uniform bound (86) on n_{R_k} , and the convergence of N_{R_k} to N in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{T}^d)$. We can therefore conclude the estimate (25):

$$I_0 \xrightarrow[k \rightarrow \infty]{} 0. \quad (26)$$

To estimate the difference between the birth terms for n_{R_k} and n , we notice that

$$\begin{aligned}
I_3 &:= \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \left[1_{|y| \leq R_k} \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_{R_k}(t, x, y_*) n_{R_k}(t, x, y'_*)}{\int_{\mathbb{R}} n_{R_k}(t, x, z) dz} dy_* dy'_* \right. \right. \\
&\quad \left. \left. - \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} dy_* dy'_* \right] dy dx dt \right| \\
&= \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \left[(1_{|y| \leq R_k} - 1) \frac{n_{R_k}(t, x, y_*) n_{R_k}(t, x, y'_*)}{\int_{\mathbb{R}} n_{R_k}(t, x, z) dz} \right. \right. \\
&\quad + \frac{(n_{R_k}(t, x, y_*) - n(t, x, y_*)) n_{R_k}(t, x, y'_*)}{\int_{\mathbb{R}} n_{R_k}(t, x, z) dz} \\
&\quad + \frac{n(t, x, y_*) n_{R_k}(t, x, y'_*)}{\left(\int_{\mathbb{R}} n(t, x, z) dz \right) \left(\int_{\mathbb{R}} n_{R_k}(t, x, z) dz \right)} (N(t, x) - N_{R_k}(t, x)) dy_* dy'_* \\
&\quad \left. \left. + \frac{n(t, x, y_*) (n_{R_k}(t, x, y'_*) - n(t, x, y'_*))}{\int_{\mathbb{R}} n(t, x, z) dz} \right] dy_* dy'_* dy dx dt \right| \leq |I_4| + |I_5| + |I_6| + |I_7|. \quad (27)
\end{aligned}$$

The test function ϕ is compactly supported and then $\phi(t, x, y) (1_{|y| \leq R_k} - 1) \equiv 0$ if $k \in \mathbb{N}$ is

large enough, which implies $I_4 = 0$. To estimate I_6 , we notice that

$$\begin{aligned}
|I_6| &= \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \phi(t, x, y) \frac{\int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) n(t, x, y_*) n_{R_k}(t, x, y'_*) dy_* dy'_*}{N(t, x) N_{R_k}(t, x)} dy \right) \right. \\
&\quad \left. (N(t, x) - N_{R_k}(t, x)) dx dt \right| \leq \|N - N_{R_k}\|_{L^1([0, B] \times \mathbb{T}^d)} \\
&\quad \left\| (t, x, y) \mapsto \phi(t, x, y) \frac{\int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) n(t, x, y_*) n_{R_k}(t, x, y'_*) dy_* dy'_*}{N(t, x) N_{R_k}(t, x)} dy \right\|_{L^1([0, B] \times \mathbb{T}^d, L^1([-B, B]))} \\
&\leq \sqrt{2B} \|\phi\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R})} \Gamma_{A/2}(0) \|N - N_{R_k}\|_{L^2([0, B] \times \mathbb{T}^d)} \xrightarrow{k \rightarrow \infty} 0, \tag{28}
\end{aligned}$$

where we have used the fact that $\frac{\int_{\mathbb{R}} n(t, x, y_*) dy_*}{N(t, x)} = \frac{\int_{\mathbb{R}} n_{R_k}(t, x, y'_*) dy'_*}{N_{R_k}(t, x)} = 1$ and the convergence of N_{R_k} to N in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{T}^d)$. To estimate I_5 , we notice that

$$\begin{aligned}
|I_5| &= \left| \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{-B}^B \int_{\mathbb{R}} \phi(t, x, y) \frac{n_{R_k}(t, x, y'_*)}{N_{R_k}(t, x)} \right. \\
&\quad \left. \left(\int_{\mathbb{R}} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) (n_{R_k}(t, x, y_*) - n(t, x, y_*)) dy_* \right) dy'_* dy dx dt \right| \\
&\leq \left\| (t, x, y'_*) \mapsto \left(\int_{-B}^B \phi(t, x, y) dy \right) \frac{n_{R_k}(t, x, y'_*)}{N_{R_k}(t, x)} \right\|_{L^\infty([0, B] \times \mathbb{T}^d \times \mathbb{R})} \\
&\quad \Gamma_{A/2}(0) \|n_{R_k} - n\|_{L^1([0, B] \times \mathbb{T}^d \times \mathbb{R})} \xrightarrow{k \rightarrow \infty} 0, \tag{29}
\end{aligned}$$

where the first factor on the right hand side of that inequality is bounded thanks to (81) and (23), while the last one converges to 0. Finally, the argument above (to estimate I_5) can be reproduced to show that $I_7 \rightarrow 0$ when $k \rightarrow \infty$.

These estimates on I_3 (see (27), as well as (25), (26) and (24) show that for any $\phi \in C_c^0(\mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned}
0 &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_t \phi(t, x, y) + \Delta_x \phi(t, x, y)) n_{R_k}(t, x, y) dy dx dt \\
&\quad + \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \left[\left(1 + \frac{A}{2} - \frac{1}{2} (y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n_{R_k}(t, x, z) dz \right) n_{R_k}(t, x, y) \right. \\
&\quad \left. + \gamma \left(\mathbf{1}_{|y| \leq R} \int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_{R_k}(t, x, y_*) n_{R_k}(t, x, y'_*)}{\int_{\mathbb{R}} n_{R_k}(t, x, z) dz} dy_* dy'_* - n_{R_k}(t, x, y) \right) \right] dy dx dt \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_t \phi(t, x, y) + \Delta_x \phi(t, x, y)) n(t, x, y) dy dx dt \\
&\quad + \int_{\mathbb{R}_+} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \phi(t, x, y) \left[\left(1 + \frac{A}{2} - \frac{1}{2} (y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n(t, x, z) dz \right) n(t, x, y) \right. \\
&\quad \left. + \gamma \left(\mathbf{1}_{|y| \leq R} \int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} dy_* dy'_* - n(t, x, y) \right) \right] dy dx dt.
\end{aligned}$$

This equality and the regularity estimate (3) imply that n is a solution of (SIM) in the sense of Definition 2.1. Estimate (4) is a consequence of (86) and the convergence of n_{R_k} to n in $L^1([0, T] \times \mathbb{T}^d \times \mathbb{R})$, for any $T > 0$.

4 Macroscopic limits of solutions of (SIM)

4.1 Tail estimates uniform in $\gamma > 0$ for solutions of (SIM)

In this section, we show that a bound on $\|Z\|_{L^\infty([0,\tau]\times\mathbb{T}^d)}$ implies a bound on $\|V\|_{L^\infty([0,\tau]\times\mathbb{T}^d)}$. This is the beginning of a bootstrap argument that will unfold in Proposition 4.5: we assume $\|Z\|_{L^\infty([0,\tau]\times\mathbb{T}^d)} \leq \kappa$ and will show that it implies a stronger estimate on $\|Z\|_{L^\infty([0,\tau]\times\mathbb{T}^d)}$. This bootstrap will imply a uniform estimate on $\|Z\|_{L^\infty([0,\tau]\times\mathbb{T}^d)}$, validating all the estimates obtained in Proposition 4.1, 4.3, and 4.4.

Proposition 4.1. *Let $\alpha > 0$, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$, $C_\kappa > 0$ and $\bar{\tau}_\kappa > 0$ such that if y_{opt} , n^0 satisfies Assumption 1 and $\gamma > \bar{\gamma}$, then the following statement holds.*

If the solution $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ of (SIM) with initial condition n^0 and if it satisfies $\|Z\|_{L^\infty([0,\tau]\times\mathbb{T}^d)} \leq \kappa$ for some $\tau \in [0, +\infty]$, then

$$\forall (t, x) \in [0, \tau + \bar{\tau}_\kappa) \times \mathbb{T}^d, \quad \int_{\mathbb{R}} |y|^4 \frac{n(t, x, y)}{\int_{\mathbb{R}} n(t, x, z) dz} dy \leq C_\kappa.$$

Remark 4.2. *Note that the estimate proven in Proposition 4.1 is uniform in $\gamma > 0$ and is therefore not implied by (4). To obtain it, it is necessary to assume that this quantity is finite at $t = 0$, but this is implied by the assumption $n^0(x, y) \leq \frac{C_0}{1+y^{10}}$ made in Assumption 1.*

Under the assumptions of the proposition above, (30) and Proposition 4.1 imply the following estimate, that will be useful on several occasions in the manuscript:

$$\int_{\mathbb{R}} |y|^4 T(\tilde{n}(t, x, \cdot))(y) dy \leq C_\kappa$$

Proof of Proposition 4.1. The dynamics of V is given by (19), and to estimate the last term of that equation, we take advantage of the fact that $T(\Gamma_A(\cdot - Z)) = \Gamma_A(\cdot - Z)$ (see (67)), and Corollary 5.2: for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$,

$$\begin{aligned} \int_{\mathbb{R}} |y|^4 T(\tilde{n}(t, x, \cdot))(y) dy &= W_4(T(\tilde{n}(t, x, \cdot)), \delta_0)^4 \\ &\leq [W_4(T(\tilde{n}(t, x, \cdot)), T(\Gamma_A(Z(t, x) - \cdot))) + W_4(\Gamma_A(Z(t, x) - \cdot), \delta_0)]^4 \\ &\leq \left[\frac{1}{2^{1/4}} W_4(\tilde{n}(t, x, \cdot), \Gamma_A(Z(t, x) - \cdot)) + W_4(\Gamma_A(Z(t, x) - \cdot), \delta_0) \right]^4 \\ &\leq \left(\frac{1}{2^{1/4}} W_4(\tilde{n}(t, x, \cdot), \delta_0) + 2W_4(\delta_0, \Gamma_A(Z(t, x) - \cdot)) \right)^4 \leq \left(\frac{1}{2^{1/4}} W_4(\tilde{n}(t, x, \cdot), \delta_0) + 2Z(t, x) + C \right)^4 \\ &\leq \frac{2}{3} W_4^4(\tilde{n}(t, x, \cdot), \delta_0) + C (Z(t, x)^4 + 1), \end{aligned} \tag{30}$$

for some constant $C > 0$, thanks to a Young inequality. The last term of (19) then satisfies

$$\gamma \left(\int_{\mathbb{R}} |y|^4 T(\tilde{n}(t, x, \cdot))(y) dy - V(t, x) \right) \leq \gamma \left(C (|Z(t, x)|^4 + 1) - \frac{1}{3} V(t, x) \right).$$

To estimate the second term on the right hand side of (19), we use a Cauchy-Schwarz inequality as follows

$$\begin{aligned} \int_{\mathbb{R}} (V(t, x) - |y|^4) (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy &\leq V(t, x) \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy \\ &\leq CV(t, x) \int (|y|^2 + 1) \tilde{n}(t, x, y) dy \leq C \left(1 + \sqrt{V(t, x)} \right) V(t, x). \end{aligned}$$

Thanks to these estimates, (19) becomes, for $(t, x) \in [0, \tau] \times \mathbb{T}^d$,

$$\begin{aligned} \partial_t V(t, x) - \Delta_x V(t, x) &\leq 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x V(t, x) + C \left(1 + \sqrt{V(t, x)}\right) V(t, x) \\ &\quad + \gamma \left(C (|Z(t, x)|^4 + 1) - \frac{1}{3} V(t, x) \right). \end{aligned} \quad (31)$$

Let

$$\bar{V} := \max \left(\|V(0, \cdot)\|_{L^\infty(\mathbb{T}^d)}, 7C ((\kappa + 1)^4 + 1) \right).$$

As soon as $\gamma \geq C (1 + \sqrt{\bar{V}})$, we have

$$C (1 + \sqrt{\bar{V}}) \bar{V} + \gamma \left(C ((\kappa + 1)^4 + 1) - \frac{1}{3} \bar{V} \right) \leq 0,$$

and $\phi(t, x) \equiv \bar{V}$ satisfies $V(0, x) \leq \phi(0, x)$ for $x \in \mathbb{T}^d$, as well as

$$\begin{aligned} \partial_t \phi(t, x) - \Delta_x \phi(t, x) &\geq 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \phi(t, x) + C \left(1 + \sqrt{\phi(t, x)}\right) \phi(t, x) \\ &\quad + \gamma \left(C ((\kappa + 1)^4 + 1) - \frac{1}{3} \phi(t, x) \right), \end{aligned}$$

and ϕ is a super-solution of (30) for $x \in \mathbb{T}^d$ and $t \in [0, \tau']$, where τ' is such that $\|Z\|_{L^\infty([0, \tau'] \times \mathbb{T}^d)} \leq \kappa + 1$. Note that the assumption $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$ implies $\kappa' \geq \kappa$. We may apply the parabolic comparison principle (see Remark 2.3), and then, for $(t, x) \in [0, \tau'] \times \mathbb{T}^d$, $V(t, x) \leq \bar{V}$.

Thanks to (17), we have

$$\partial_t Z(t, x) - \Delta_x Z(t, x) - 2 \frac{\nabla_x N(t, x) \cdot \nabla_x Z(t, x)}{N(t, x)} \leq C (1 + V(t, x)),$$

and we define $\psi(t, x) = \kappa + C (1 + \bar{V}) (t - \tau)$ which satisfies $Z(\tau, x) \leq \psi(\tau, x)$ for $x \in \mathbb{T}^d$, as well as

$$\partial_t \psi(t, x) - \Delta_x \psi(t, x) - 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \psi(t, x) = C (1 + \bar{V}) \geq C (1 + V(t, x)),$$

for $t \in [\tau, \tau']$. We may thus apply the parabolic comparison principle (see Remark 2.3) to show $Z(t, x) \leq \kappa + C (1 + \bar{V}) (t - \tau)$ for $(t, x) \in [\tau, \tau'] \times \mathbb{T}^d$. We may then choose

$$\tau' = \tau + \frac{1}{C(1 + \bar{V})} = \tau + \frac{1}{C \left(1 + \max \left(\|V(0, \cdot)\|_{L^\infty(\mathbb{T}^d)}, 7C ((\kappa + 1)^4 + 1) \right)\right)}.$$

□

4.2 Regularity estimates uniform in $\gamma > 0$ for N and Z

In the proposition below, we prove some regularity estimates on N and Z that are uniform in $\gamma > 0$. It is this uniformity that sets them apart from the regularity results obtained in Section 5.4.

Proposition 4.3. *Let $\alpha > 0$, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$, $\theta \in (0, 1)$ and $C_\kappa > 0$ such that if y_{opt} , n^0 satisfies Assumption 1 and $\gamma > \bar{\gamma}$, then the following statement holds.*

Let $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ the solution of (SIM) with initial condition n^0 , and N and Z are defined by (1). If it satisfies $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$ for some $\tau \in (0, +\infty]$, then for any $s, t \in [0, \tau)$ and $x, y \in \mathbb{T}^d$,

$$\frac{|Z(t, x) - Z(s, y)|}{(|t - s| + |x - y|)^\theta} + \frac{|N(t, x) - N(s, y)|}{(|t - s| + |x - y|)^\theta} \leq C_\kappa.$$

Moreover,

$$\left\| \frac{\nabla_x N}{N} \right\|_{L^{d+3}([0, \tau] \times \mathbb{T}^d)} \leq C_\kappa. \quad (32)$$

Note that C_κ in this statement is independent from τ . This is a consequence of Step 3 of the proof. It relies on the application of a Harnack inequality (that was used to obtain (35)), and of (39), that is a corollary of the Harnack inequality. This uniformity in τ is essential for to conclude the bootstrap argument described at the beginning of Section 4.1

Proof of Proposition 4.3. Let $\bar{\gamma} > 0$ as in Proposition 4.1.

Step 1: Lower bound on $N(t, x)$

Since $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$, Proposition 4.1 implies that $\int |y|^4 \tilde{n}(t, x, y) dy$ is uniformly bounded on $[0, \tau) \times \mathbb{T}^d$, and there exists a constant $C_\kappa > 0$ such that for $(t, x) \in [0, \tau) \times \mathbb{T}^d$,

$$\left| \left[1 + \frac{A}{2} - N(t, x) \right] N(t, x) - \frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 n(t, x, y) dy \right| \leq C_\kappa N(t, x), \quad (33)$$

where we have also used the uniform bound on N provided by Proposition 2.2. Thanks to (33) and the comparison principle used in (13), for $t \in [0, 1] \cap [0, \tau)$,

$$N(t, x) \geq e^{-C_\kappa t} \inf_{\mathbb{T}^d} N(0, \cdot) \geq C_\kappa, \quad (34)$$

thanks to Assumption 1. Estimate (33) also provides a uniform bound on the coefficients of (13), we can apply the Harnack inequality for $t \in [0, \tau) \setminus [0, 1]$ (see [38], or Theorem 3 in [4]): there exists $C_\kappa > 0$ such that for any $t \in [0, \tau) \setminus [0, 1]$,

$$\max_{(s, x) \in [t-3/4, t-1/2] \times \mathbb{T}^d} N(s, x) \leq C_\kappa \min_{(s, x) \in [t-1/3, t] \times \mathbb{T}^d} N(s, x).$$

Since $\partial_t N - \Delta_x N \leq (1 + A/2)N$, we may consider the super-solution

$$(s, x) \mapsto \left(\max_{x \in \mathbb{T}^d} N(t - 1/2, x) \right) e^{(1+A/2)(s-(t-1/2))},$$

and the comparison principle implies, for $t \in [0, \tau) \setminus [0, 1]$,

$$\max_{(s, x) \in [t-3/4, t] \times \mathbb{T}^d} N(s, x) \leq C_\kappa \min_{(s, x) \in [t-1/3, t] \times \mathbb{T}^d} N(s, x). \quad (35)$$

Step 2: L^{d+3} estimate on $\frac{\nabla_x N(t, x)}{N(t, x)}$ for $t \in [0, 1]$

We notice that for $(t, x) \in (-\infty, \tau) \times \mathbb{R}$, $N(t, x) = (N(0, x) + \mathcal{N}(t, x)) 1_{t \geq 0}$, where \mathcal{N} is a solution of

$$\partial_t \mathcal{N}(t, x) - \Delta_x \mathcal{N}(t, x) = \mu_N(t, \pi(x)) 1_{t \geq 0}, \quad (t, x) \in (-\infty, \tau) \times \mathbb{R}^d, \quad (36)$$

where $\pi(x)$ is the standard projection of $x \in \mathbb{R}^d$ on \mathbb{T}^d , and

$$\mu_N(t, x) = \Delta_x N^0(x) + \left(1 + \frac{A}{2} - \frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy - N(t, x)\right) N(t, x).$$

Note that $\mathcal{N}(t, \cdot) \equiv 0$ for $t \leq 0$. Thanks to (33) and Assumption 1, we have $\|\mu_N\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} < C_\kappa$, and we can apply Theorem 7.22 of [37] to obtain

$$\|\partial_x \mathcal{N}\|_{L^{d+3}([t-1/4, t] \times \mathbb{T}^d)} \leq C_\kappa \left(\|\mathcal{N}\|_{L^{d+3}([t-1/3, t] \times \mathbb{T}^d)} + 1 \right), \quad (37)$$

for any $t \in \mathbb{R}$. For $t \in [0, 1]$, since N is uniformly bounded and thanks to the lower estimate (34), we obtain

$$\left\| \frac{\nabla_x N}{N} \right\|_{L^{d+3}([0, 1] \times \mathbb{T}^d)} \leq C_\kappa. \quad (38)$$

Step 3: L^{d+3} estimate on $\frac{\nabla_x N(t, x)}{N(t, x)}$ for $t \in [0, \tau] \setminus [0, 1]$

The argument here is similar to the one developed for step 2, but on equation (13) instead of (36). Theorem 7.22 of [37] applied to (13) implies that for $t \geq 1$,

$$\|\nabla_x N\|_{L^{d+3}([t-1/4, t] \times \mathbb{T}^d)} \leq C_\kappa \|N\|_{L^{d+3}([t-1/3, t] \times \mathbb{T}^d)}, \quad (39)$$

which we combine to (35) to obtain, for $t \geq 1$,

$$\begin{aligned} \left\| \frac{\nabla_x N}{N} \right\|_{L^{d+3}([t-1/4, t+1/4] \cap [0, \tau] \times \mathbb{T}^d)} &\leq \frac{\|\nabla_x N\|_{L^{d+3}([t-1/4, t+1/4] \cap [0, \tau] \times \mathbb{T}^d)}}{\min_{(s, x) \in [t-1/3, t] \times \mathbb{T}^d} N(s, x)} \\ &\leq C_\kappa \frac{\|N\|_{L^{d+3}([t-1/3, t] \times \mathbb{T}^d)}}{\|N\|_{L^\infty([t-3/4, t] \times \mathbb{T}^d)}} \leq C_\kappa, \end{aligned} \quad (40)$$

Since $\|N\|_{L^{d+3}([t-1/3, t] \times \mathbb{T}^d)} \leq \|N\|_{L^\infty([t-3/4, t] \times \mathbb{T}^d)}$.

Step 4: Regularity of N and Z

Just as we have done for $\mathcal{N}(t, x) = N(t, x) - N^0(x)$ (see (36)), we can define $\mathcal{Z} = (Z(t, x) - Z(0, x)) 1_{t \geq 0}$, solution of

$$\partial_t \mathcal{Z}(t, x) - \Delta_x \mathcal{Z}(t, x) = 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \mathcal{Z}(t, x) + \mu_Z(t, \pi(x)) 1_{t \geq 0}, \quad (t, x) \in (-\infty, \tau) \times \mathbb{R}^d,$$

where $\|\mu_Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} < C_\kappa$ thanks to Proposition 4.1 and Assumption 1, and $\frac{\nabla_x N}{N}$ satisfies (38), (40). This equation then has the structure of equation (5) in [4] and we can apply satisfy Theorem 4 from that reference to obtain a Hölder estimate on \mathcal{Z} to prove the Hölder continuity of \mathcal{Z} . This theorem can also be applied to \mathcal{N} (since all its coefficients are bounded), which concludes the proof of the proposition. \square

4.3 Distance of solutions of (SIM) to local Maxwellians

Proposition 4.4. *Let $\alpha > 0$, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$, $\theta \in (0, 1)$ and $C_\kappa > 0$ such that if y_{opt} , n^0 satisfies Assumption 1 and $\gamma > \bar{\gamma}$, then the following statement holds.*

Let $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ the solution of (SIM) with initial condition n^0 , and Z defined by (1). If it satisfies $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$ for some $\tau \in (0, +\infty]$, then

$$\forall t \in \left[C_\kappa \frac{\ln \gamma}{\gamma}, \tau \right), \quad \max_{x \in \mathbb{T}^d} W_2^2 \left(\tilde{n}(t, x, \cdot), \Gamma_A(\cdot - Z(t, x)) \right) \leq \frac{C_\kappa}{\gamma^\theta}, \quad (41)$$

where \tilde{n} is given by (14) and Γ_A is defined by (68).

Note that in this statement, the constant $C_\kappa > 0$ is independent from $\gamma \geq \bar{\gamma}$, so that after a boundary layer $(0, C_\kappa \ln \gamma / \gamma)$, the functions $y \mapsto \tilde{n}(t, x, y)$ becomes close to Gaussian distributions in the trait space, for any $x \in \mathbb{T}^d$.

Proof of Proposition 4.4. In this proof, we use the linear problems and estimates presented in Section 5.6 of the Appendix. In particular, we define $(t, x) \mapsto \phi_{s, z, y}(t, x)$ as the solution of

$$\begin{cases} \partial_t \phi_{s, z, y}(t, x) - \Delta_x \phi_{s, z, y}(t, x) \\ \quad = 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \phi_{s, z, y}(t, x) - \frac{1}{2} (y - y_{opt}(t, x))^2 \phi_{s, z, y}(t, x), & (t, x) \in [s, \tau] \times \mathbb{T}^d, \\ \phi_{s, z, y}(s, x) = \delta_z(x), & x \in \mathbb{T}^d. \end{cases} \quad (42)$$

This solution exists since $\frac{\nabla_x N(t, x)}{N(t, x)}$ is a continuous function (see Section 5.4) and y is a parameter here, therefore all coefficients of this linear parabolic equation are bounded and continuous. Alternatively, it is possible to build explicit solutions from a heat equation, we refer to (99) for this argument. For $t \geq 0$, we can use a Duhamel formula to write \tilde{n} (we recall that \tilde{n} satisfies (15)) as follows

$$\begin{aligned} \tilde{n}(t, x, y) &= e^{-\gamma t} \int_{\mathbb{R}} \phi_{0, z, y}(t, x) \tilde{n}(0, z, y) dz \\ &+ \frac{1}{2} \int_0^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \phi_{s, z, y}(t, x) \tilde{n}(s, z, y) \left(\int_{\mathbb{R}} (w - y_{opt}(s, z))^2 \tilde{n}(s, z, w) dw \right) dz ds \\ &+ \gamma \int_0^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \phi_{s, z, y}(t, x) T(\tilde{n}(s, z, \cdot))(y) dz ds. \end{aligned}$$

Since $\tilde{n}(t, x, \cdot)$ is a probability measure, the y -integral of the right hand side of the equation above sums up to one and the right hand side can be seen as a convex combinations of three probability distributions. The convexity properties of the squared Wasserstein distance W_2^2 ,

that we detail in Section 5.1 in the Appendix (see (66)), then implies:

$$\begin{aligned}
& W_2^2(\tilde{n}(t, x, \cdot), \Gamma_A(\cdot - Z(t, x))) \leq e^{-\gamma t} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{0,z,y}(t, x) \tilde{n}(0, z, y) dy \right) \\
& W_2^2 \left(\frac{\phi_{0,z,\cdot}(t, x) \tilde{n}(0, z, \cdot)}{\int_{\mathbb{R}} \phi_{0,z,y}(t, x) \tilde{n}(0, z, y) dy}, \Gamma_A(\cdot - Z(t, x)) \right) dz \\
& + \frac{1}{2} \int_0^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{s,z,y}(t, x) \tilde{n}(s, z, y) dy \right) \left(\int_{\mathbb{R}} (w - y_{opt}(s, z))^2 \tilde{n}(s, z, w) dw \right) \\
& W_2^2 \left(\frac{\phi_{s,z,\cdot}(t, x) \tilde{n}(s, z, \cdot)}{\int_{\mathbb{R}} \phi_{s,z,y}(t, x) \tilde{n}(s, z, y) dy}, \Gamma_A(\cdot - Z(t, x)) \right) dz ds \\
& + \gamma \int_0^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{s,z,y}(t, x) T(\tilde{n}(s, z, \cdot))(y) dy \right) \\
& W_2^2 \left(\frac{\phi_{s,z,\cdot}(t, x) T(\tilde{n}(s, z, \cdot))}{\int_{\mathbb{R}} \phi_{s,z,y}(t, x) T(\tilde{n}(s, z, \cdot))(y) dy}, T(\Gamma_A(\cdot - Z(t, x))) \right) dz ds. \tag{43}
\end{aligned}$$

Note that we have used that $\Gamma_A(\cdot - Z(t, x))$ is a fixed point for T (see (67)). To estimate the first two terms on the right hand side of (43), a rough estimate is sufficient: for any $(s, z) \in [0, \infty) \times \mathbb{T}^d$ and $(t, x) \in [s, \infty) \times \mathbb{T}^d$,

$$\begin{aligned}
& W_2^2 \left(\frac{\phi_{s,z,\cdot}(t, x) \tilde{n}(s, z, \cdot)}{\int_{\mathbb{R}} \phi_{s,z,y}(t, x) \tilde{n}(s, z, y) dy}, \Gamma_A(\cdot - Z(t, x)) \right) \\
& \leq \left(W_2 \left(\frac{\phi_{s,z,\cdot}(t, x) \tilde{n}(s, z, \cdot)}{\int_{\mathbb{R}} \phi_{s,z,y}(t, x) \tilde{n}(s, z, y) dy}, \delta_0 \right) + W_2(\delta_0, \Gamma_A(\cdot - Z(t, x))) \right)^2 \\
& \leq 2 \int_{\mathbb{R}} |y|^2 \frac{\phi_{s,z,y}(t, x) \tilde{n}(s, z, y)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t, x) \tilde{n}(s, z, y') dy'} dy + 2 \int_{\mathbb{R}} |y|^2 \Gamma_A(y - Z(t, x)) dy \leq C_\kappa, \tag{44}
\end{aligned}$$

where the final estimate follows from Section 5.6 in the Appendix: if we define R by (100) and R' as in (102) (note that $|R'| \leq C_\kappa$), then (101), (103) and Proposition 4.1 imply

$$\begin{aligned}
& \int_{\mathbb{R}} |y|^2 \frac{\phi_{s,z,y}(t, x) \tilde{n}(s, z, y)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t, x) \tilde{n}(s, z, y') dy'} dy \\
& \leq \int_{[-R', R']^c} |y|^2 \frac{(\min_{|\tilde{y}| \leq R} \phi_{s,z,\tilde{y}}(t, x)) \tilde{n}(s, z, y)}{\int_{-R}^R \phi_{s,z,y'}(t, x) \tilde{n}(s, z, y') dy'} dy + (R')^2 \int_{-R'}^{R'} \frac{\phi_{s,z,y}(t, x) \tilde{n}(s, z, y)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t, x) \tilde{n}(s, z, y') dy'} dy \\
& \leq \int_{\mathbb{R}} |y|^2 \frac{\tilde{n}(s, z, y)}{1/2} dy + (R')^2 \leq C_\kappa. \tag{45}
\end{aligned}$$

We repeat the estimate (44) (using additionally the estimate of Remark 4.2) to control the last term of (43) for $s \leq t - \varepsilon$, for some $\varepsilon > 0$ that we will define later on. We obtain then, for $s \leq t - \varepsilon$,

$$W_2^2 \left(\frac{\phi_{s,z,\cdot}(t, x) T(\tilde{n}(s, z, \cdot))}{\int_{\mathbb{R}} \phi_{s,z,y}(t, x) T(\tilde{n}(s, z, \cdot))(y) dy}, T(\Gamma_A(\cdot - Z(t, x))) \right) \leq C_\kappa. \tag{46}$$

For $s \in [t - \varepsilon, t]$, we need a more precise estimate, which we will obtain with the following

coupling π . We define $\bar{\phi}_{s,z}(t,x)$ by (99), and

$$\begin{aligned}\pi(y_1, y_2) &= \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)} T(\tilde{n}(s,z,\cdot))(y_1) \delta_{y_1=y_2} \\ &\quad + \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1) \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'}.\end{aligned}$$

π is then a probability measure on $\mathbb{R} \times \mathbb{R}$ (note that $\phi_{s,z,y_1}(t,x) \leq \bar{\phi}_{s,z}(t,x)$, thanks to (104)), with marginals

$$\pi|_1(y_1) = T(\tilde{n}(s,z,\cdot))(y_1) \quad \text{and} \quad \pi|_2(y_2) = \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'}.$$

Then,

$$\begin{aligned}W_2^2 \left(\frac{\phi_{s,z,y}(t,x) T(\tilde{n}(s,z,\cdot))(y)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'}, T(\tilde{n}(s,z,\cdot)) \right) &\leq \iint_{\mathbb{R}^2} |y_1 - y_2|^2 d\pi(y_1, y_2) \\ &\leq \iint_{\mathbb{R}^2} |y_1 - y_2|^2 \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1) \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'} dy_1 dy_2 \\ &\leq 2 \iint_{\mathbb{R}^2} (y_1^2 + y_2^2) \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1) \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'} dy_1 dy_2 \\ &\leq 2 \int_{\mathbb{R}} y_1^2 \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1) dy_1 \\ &\quad + 2 \left(1 - \frac{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'}{\bar{\phi}_{s,z}(t,x)}\right) \int_{\mathbb{R}} y_2^2 \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'} dy_2.\end{aligned}\tag{47}$$

We estimate below the first integral term of (47), where $s < t$. We estimate the integral by separating it into two integral terms. The first integral one can then be controlled thanks to a Chebyshev's inequality (we recall Remark 4.2), while we use the estimate (104), derived in the Appendix, to estimate the second integral term:

$$\begin{aligned}\int_{\mathbb{R}} y_1^2 \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1) dy_1 &\leq \int_{|y_1| \geq (t-s)^{-1/3}} y_1^2 T(\tilde{n}(s,z,\cdot))(y_1) dy_1 \\ &\quad + \int_{|y_1| \leq (t-s)^{-1/3}} y_1^2 \left(1 - e^{-(t-s)^{1/2}(y_1 + \mathcal{O}(1))^2}\right) T(\tilde{n}(s,z,\cdot))(y_1) dy_1 \\ &\leq (t-s)^{2/3} \int_{|y_1| \geq (t-s)^{-1/3}} y_1^4 T(\tilde{n}(s,z,\cdot))(y_1) dy_1 \\ &\quad + \left(1 - e^{-(t-s)^{2/3}}\right) \int_{|y_1| \leq (t-s)^{-1/3}} y_1^2 T(\tilde{n}(s,z,\cdot))(y_1) dy_1 \\ &\leq C_{\kappa} (t-s)^{2/3} + C_{\kappa} \left(1 - e^{-(t-s)^{2/3}}\right) \leq C_{\kappa} (t-s)^{2/3},\end{aligned}\tag{48}$$

provided $t - s > 0$ is small enough. We estimate the last term of (47) as follows, provided $|t - s|$ is small enough:

$$\begin{aligned}2 \left(1 - \frac{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'}{\bar{\phi}_{s,z}(t,x)}\right) \int_{\mathbb{R}} y_2^2 \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y') dy'} dy_2 \\ \leq \left(\int_{\mathbb{R}} \left(1 - \frac{\phi_{s,z,y'}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y') dy'\right) C_{\kappa} \leq C_{\kappa} (t-s)^{2/3},\end{aligned}\tag{49}$$

where the first inequality is justified by (45), and the second inequality can be obtained through the argument performed in (48) (with 1 instead of y_1^2). Thanks to (48) and (49), the estimate (47) becomes

$$W_2^2 \left(\frac{\phi_{s,z,y}(t,x)T(\tilde{n}(s,z,\cdot))(y)}{\int_{\mathbb{R}} \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y') dy'}, T(\tilde{n}(s,z,\cdot)) \right) \leq C_\kappa (t-s)^{2/3}.$$

This estimate combined to the regularity estimates on N and Z obtained in Proposition 4.3 lead to

$$\begin{aligned} & W_2^2 \left(\frac{\phi_{s,z,\cdot}(t,x)T(\tilde{n}(s,z,\cdot))}{\int_{\mathbb{R}} \phi_{s,z,y}(t,x)\tilde{n}(s,z,y) dy}, T(\Gamma_A(\cdot - Z(t,x))) \right) \\ & \leq W_2^2 \left(\frac{\phi_{s,z,\cdot}(t,x)T(\tilde{n}(s,z,\cdot))}{\int_{\mathbb{R}} \phi_{s,z,y}(t,x)\tilde{n}(s,z,y) dy}, T(\tilde{n}(s,z,\cdot)) \right) \\ & \quad + W_2^2(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))) + |Z(t,x) - Z(s,z)| \\ & \leq W_2^2(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))) + C_\kappa |t-s|^\theta + C_\kappa |x-z|^\theta, \end{aligned} \quad (50)$$

for some $\theta \in (0, 1)$, provided $\gamma > 0$ is large enough. We are now ready to consider the original estimate (43): thanks to (44), (46) and (50), the estimate (43) implies

$$\begin{aligned} W_2^2(\tilde{n}(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))) & \leq e^{-\gamma t} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{0,z,y}(t,x)\tilde{n}(0,z,y) dy \right) C_\kappa dz \\ & \quad + \frac{C_\kappa}{2} \int_0^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{s,z,y}(t,x)\tilde{n}(s,z,y) \left(\int_{\mathbb{R}} (w - y_{opt}(s,z))^2 \tilde{n}(s,z,w) dw \right) dy \right) dz ds \\ & \quad + \gamma \int_0^{t-\varepsilon} e^{-\gamma(t-s)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{s,z,y}(t,x)T(\tilde{n}(s,z,\cdot))(y) dy \right) C_\kappa dz ds \\ & \quad + \gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_{s,z,y}(t,x)T(\tilde{n}(s,z,\cdot))(y) dy \right) \\ & \quad \left(W_2^2(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))) + C_\kappa |t-s|^\theta + C_\kappa |x-z|^\theta \right) dz ds. \end{aligned}$$

We can now use the estimate (104) (and Proposition 4.1) to obtain

$$\begin{aligned} W_2^2(\tilde{n}(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))) & \leq e^{-\gamma t} \left(\int_{\mathbb{R}} \bar{\phi}_{0,z}(t,x) dz \right) C_\kappa \\ & \quad + \frac{1}{2} \int_0^t e^{-\gamma(t-s)} \left(\int_{\mathbb{R}} \bar{\phi}_{s,z}(t,x) dz \right) C_\kappa ds + \gamma \int_0^{t-\varepsilon} e^{-\gamma(t-s)} \left(\int_{\mathbb{R}} \bar{\phi}_{s,z}(t,x) dz \right) C_\kappa dz ds \\ & \quad + \gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} \left(\int_{\mathbb{R}} \bar{\phi}_{s,z}(t,x) dz \right) \max_{z \in \mathbb{T}^d} W_2^2(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))) ds \\ & \quad + \gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} \int_{\mathbb{R}} \bar{\phi}_{s,z}(t,x) \left(C_\kappa |t-s|^\theta + C_\kappa |x-z|^\theta \right) dz ds. \end{aligned}$$

Thanks to (107), we have $\int \bar{\phi}_{0,z}(t,x) dz = 1$, while (105) shows that $\int \bar{\phi}_{s,z}(t,x) |x-z|^\theta \leq C_\kappa |t-s|^{\frac{\theta}{2}}$. Then,

$$\begin{aligned} W_2^2(\tilde{n}(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))) & \leq C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \\ & \quad + \gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} \max_{z \in \mathbb{T}^d} W_2^2(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))) ds, \end{aligned} \quad (51)$$

where we have used the change of variable $\tilde{s} = \gamma(t - s)$ to show

$$\gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} (t-s)^{\theta/2} ds = \int_0^{\gamma\varepsilon} e^{-s} \left(\frac{s}{\gamma}\right)^{\theta/2} ds \leq C\gamma^{\theta/2}.$$

Since the right hand side of (51) is independent of $x \in \mathbb{T}^d$, we can consider the maximum over that variable. If moreover we apply the Tanaka inequality (see Theorem 5.1), we obtain

$$I(t) \leq C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} + \frac{\gamma}{2} \int_{t-\varepsilon}^t e^{-\gamma(t-s)} I(s) ds,$$

where $I(s) := \max_{x \in \mathbb{T}^d} W_2^2\left(\tilde{n}(s, x, \cdot), \Gamma_A(\cdot - Z(s, x))\right)$. Thanks to a Grönwall inequality (see e.g. [28]),

$$\begin{aligned} I(t) &\leq C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \\ &\quad + \frac{\gamma}{2} e^{-\gamma t} \int_{t-\varepsilon}^t \left(C_\kappa e^{-\gamma s} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \right) e^{\gamma s} e^{\frac{\gamma}{2}(t-s)} ds \\ &\leq C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} + \left(C_\kappa e^{-\gamma(t-\varepsilon/2)} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \right). \end{aligned}$$

We can chose $\varepsilon := \frac{\theta \ln \gamma}{2\gamma}$ to obtain

$$I(t) \leq C_\kappa e^{-\gamma t} \gamma^{\theta/4} + \frac{C_\kappa}{\gamma^{\theta/2}},$$

so that finally, for any $\gamma > 0$ large enough,

$$\max_{t \in [\theta \ln \gamma / \gamma, \tau)} I(t) \leq \frac{C_\kappa}{\gamma^{\theta/2}}.$$

The result follows (note that we need to define a slightly different parameter θ : $\tilde{\theta} := \theta/2 > 0$). \square

4.4 Macroscopic limit from (SIM) to (KBM)

Let $\alpha > 0$, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$, $C_\kappa > 0$, and $\theta \in (0, 1)$ such that if y_{opt} , n^0 satisfies Assumption 1 and $\gamma > \bar{\gamma}$, then the following statement holds.

If $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ is the solution of (SIM) with initial condition n^0 and if it satisfies $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$, then

Proposition 4.5. *Let $\alpha > 0$, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$ such that if y_{opt} , n^0 satisfies Assumption 1 and $\gamma > \bar{\gamma}$, then the solution $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ of (SIM) with initial condition n^0 satisfies*

$$\|Z\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \leq \|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + 1, \quad (52)$$

where Z is defined by (1).

Proof of Proposition 4.5. Let

$$\kappa := \|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + 1, \quad (53)$$

and

$$\tau = \max\{t \geq 0; \|Z\|_{L^\infty([0,t] \times \mathbb{T}^d)} \leq \kappa\}. \quad (54)$$

Note that if $\tau = +\infty$, the estimate (52) holds and the proof is completed. We can therefore consider the other case, where $\tau < \infty$. More precisely, we will use a contradiction argument: we assume that $\tau < \infty$ and prove that it is not possible.

We have $\|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \kappa$. Thanks to Proposition 4.1, for some $\bar{\tau}_\kappa > 0$ independent from $\gamma > \bar{\gamma}$,

$$\int_{\mathbb{R}} |y|^4 \frac{n(t, x, y)}{\int_{\mathbb{R}} n(t, x, z) dz} dy \leq C_\kappa, \quad (55)$$

for $(t, x) \in [0, \tau + \bar{\tau}_\kappa] \times \mathbb{T}^d$, as soon as $\gamma > \bar{\gamma}$. Our goal is to show that $\|Z(t)\|_{L^\infty(\mathbb{T}^d)} < \kappa$ for $t \in [0, \tau + \bar{\tau}]$, for some $\bar{\tau}$.

If $\tau < \bar{\tau}_\kappa$, then we can then apply Proposition 4.3, which ensures the Hölder regularity of Z (uniformly for $\gamma > 0$ large enough), and then, in particular,

$$\|Z(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} + Ct^\theta,$$

and then, up to a reduction of $\bar{\tau}_\kappa > 0$ into $\bar{\tau} > 0$,

$$\|Z\|_{L^\infty([0, \tau + \bar{\tau}] \times \mathbb{T}^d)} < \|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + 1 = \kappa. \quad (56)$$

In particular, $\|Z(t)\|_{L^\infty(\mathbb{T}^d)} < \kappa$ for $t \in [0, \bar{\tau}]$, which completes this initialisation step of this proof.

From (17) we get, for $(t, x) \in [0, \tau + \bar{\tau}] \times \mathbb{T}^d$,

$$\begin{aligned} & \partial_t Z(t, x) - \Delta_x Z(t, x) - 2 \frac{\nabla_x N(t, x) \cdot \nabla_x Z(t, x)}{N(t, x)} \\ &= -\frac{1}{2} \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 \Gamma_A(y - Z(t, x)) dy \\ & \quad + \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy. \end{aligned} \quad (57)$$

The first term on the right hand side of this equation can be simplified as follows

$$\begin{aligned} & -\frac{1}{2} \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 \Gamma_A(y - Z(t, x)) dy \\ &= -(Z(t, x) - y_{opt}(t, x)) \int_{\mathbb{R}} |y|^2 \Gamma_A(y) dy = -A(Z(t, x) - y_{opt}(t, x)), \end{aligned} \quad (58)$$

and to estimate the last term of (57), we introduce for some $R > 0$ and a Lipschitz function

$\phi_R : \mathbb{R} \mapsto [0, 1]$ such that $\phi_R|_{[-R, R]} = 1$, $\phi_R|_{[-R-1, R+1]} = 0$ and $\|\phi'_R\|_{L^\infty(\mathbb{R})} < 2$. Then,

$$\begin{aligned} & \left| \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right| \\ & \leq \left| \int_{\mathbb{R}} \phi_R(y) (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right| \\ & \quad + \left| \int_{\mathbb{R}} (1 - \phi_R(y)) (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right| \\ & \leq \max_{y \in \mathbb{R}} \left| \frac{d}{dy} [\phi_R(y) (y - Z(t, x)) (y - y_{opt}(t, x))^2] \right| W_1(\tilde{n}(t, x, \cdot), \Gamma_A(\cdot - Z(t, x))) \\ & \quad + C_\kappa \int_{|y| \geq R} |y + \kappa|^3 \tilde{n}(t, x, y) dy + C_\kappa \int_{|y| \geq R} |y + \kappa|^3 \Gamma_A(y - Z(t, x)) dy, \end{aligned}$$

where $\kappa > 0$ is defined by (53) and where we have used the Kantorovich-Rubinstein estimate (see Section 5.1 in the Appendix) to obtain the first term on the right hand side of the estimate above. We use next the fact that ϕ_R as well as its derivative ϕ'_R , is supported in $[-R-1, R+1]$ and the Chebyshev's inequality to obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right| \\ & \leq C(R + \kappa)^3 W_2(\tilde{n}(t, x, \cdot), \Gamma_A(\cdot - Z(t, x))) + \frac{C}{R} \int |y|^4 \tilde{n}(t, x, y) dy \\ & \quad + \frac{C}{R} \int_{\mathbb{R}} |y|^4 \Gamma_A(y - Z(t, x)) dy, \end{aligned}$$

To estimate the three terms that appear in the estimate above, we use Proposition 4.4 and (55) to obtain

$$\left| \int_{\mathbb{R}} (y - Z(t, x)) (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right| \leq \frac{C_\kappa R^3}{\gamma^\theta} + \frac{C_\kappa}{R} \leq \frac{C_\kappa}{\gamma^{\theta/4}}, \quad (59)$$

for $(t, x) \in [\theta \ln \gamma / \gamma, \tau + \bar{\tau}] \times \mathbb{T}^d$, provided we chose $R = \gamma^{\theta/4}$. Note that for $\gamma > 0$ large enough, $C_\kappa \frac{\ln \gamma}{\gamma} < k\sigma$, so that $[\bar{\tau}, \tau + \bar{\tau}] \subset [\theta \ln \gamma / \gamma, \tau + \bar{\tau}]$. Thanks to (58) and (59), we obtain that for $t \in [\theta \ln \gamma / \gamma, \tau + \bar{\tau}]$ and $\gamma \geq \bar{\gamma}$ (this may require to increase the value of $\bar{\gamma} > 0$, but this new value of $\bar{\gamma}$ remains independent of $\tau \geq \bar{\tau}$),

$$\partial_t Z(t, x) - \Delta_x Z(t, x) = 2 \frac{\nabla_x N(t, x) \cdot \nabla_x Z(t, x)}{N(t, x)} - A(Z(t, x) - y_{opt}(t, x)) + \mathcal{O}(1),$$

where $|\mathcal{O}(1)| \leq A$. This estimate combined to (56) and a comparison of $Z(t, x)$ with $\varphi(t, x) \equiv \pm \kappa$ thanks to the comparison principle (see Remark 2.3) proves that $\|Z\|_{L^\infty([0, (k+1)\sigma] \times \mathbb{T}^d)} < \kappa$. This is in contradiction with (54) if $\tau \neq +\infty$, which concludes the proof. \square

We are now ready to prove Theorem 2.6:

Proof of Theorem 2.6. Thanks to Proposition 4.5, there exists a solution $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1((1 + |y|^4) dy))$ of the SIM with initial condition n^0 such that

$$\|Z\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \leq \kappa := \|Z(0, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} + 1. \quad (60)$$

Thanks to (13) and (17), we get the following expressions for the functions φ_N and φ_Z appearing in (10):

$$\varphi_N(t, x) = \left(-\frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy + \frac{A}{2} + \frac{1}{2} (Z(t, x) - y_{opt}(t, x))^2 \right) N(t, x). \quad (61)$$

$$\varphi_Z(t, x) = -\frac{1}{2} \int_{\mathbb{R}} (y - Z(t, x))(y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy + A(Z(t, x) - y_{opt}(t, x)). \quad (62)$$

Thanks to (60), we can apply Proposition 4.1 with $[0, \tau) = [0, \infty)$, and there exists a constant $C > 0$ such that $\int |y|^4 \tilde{n}(t, x, y) dy \leq C$, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$ and $\gamma > \bar{\gamma}$ large enough. This combined to the boundedness of Z provided by (60) implies the existence of a constant $C > 0$ such that

$$\forall t \geq 0, \quad \|\varphi_N(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} + \|\varphi_Z(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

To show (9), we need to show that after an initial layer, this estimate can be improved. For φ_Z , we can use an estimate derived in the proof of Proposition 4.5: (59) and (58) imply

$$\forall t \geq C \frac{\ln \gamma}{\gamma}, \quad \|\varphi_Z(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^{\theta/4}}.$$

To estimate $\|\varphi_N(t, \cdot)\|_{L^\infty(\mathbb{T}^d)}$, we note that

$$\frac{1}{2} \int_{\mathbb{R}} (y - y_{opt}(t, x))^2 \Gamma_A(y - Z(t, x)) dy = \frac{A}{2} + \frac{1}{2} (Z(t, x) - y_{opt}(t, x))^2,$$

and then

$$\varphi_N(t, x) = \frac{1}{2} \left(\int_{\mathbb{R}} (y - y_{opt}(t, x))^2 (\Gamma_A(y - Z(t, x)) - \tilde{n}(t, x, y)) dy \right) N(t, x).$$

We can repeat the argument developed in (58)-(59) to estimate the integral term, and then,

$$\forall t \geq C \frac{\ln \gamma}{\gamma}, \quad \|\varphi_N(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^{\theta/4}}.$$

To conclude the proof, we notice that (11) is a consequence of Proposition 4.4. To obtain estimate (9), we define a slightly different parameter $\theta: \tilde{\theta} := \frac{\theta}{4} > 0$. Finally, $(N, Z) = (N(t, x), Z(t, x))$, for some $\gamma > 0$ large, are C^1 in t and C^2 in x thanks to Section 4.2, and satisfy (10), according to (13) and (17) and the definitions (61), (62) of φ_N, φ_Z . \square

5 Appendix

5.1 Wasserstein distances

In this section, we review the definition of the Wasserstein distance and several useful formula. We refer to [59] for more on this topic. Let $p \geq 1$, and $\mathcal{P}_p(\mathbb{R})$ the set of probability measures with finite p -moment, that is the set of probability measures μ over \mathbb{R} such that

$$\int_{\mathbb{R}} |y|^d d\mu(y) < \infty. \quad (63)$$

If π is a probability measure over \mathbb{R}^2 , we call marginals the probability measures $\pi|_1$ and $\pi|_2$ such that for any Borelian $A \subset \mathbb{R}$,

$$\pi(A \times \mathbb{R}) = \pi|_1(A), \quad \pi(\mathbb{R} \times A) = \pi|_2(A).$$

For $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$, we call transference plans the probability measures π over \mathbb{R}^2 such that $\pi|_1 = \tilde{n}$ and $\pi|_2 = \tilde{m}$, and $\Pi(\tilde{n}, \tilde{m})$ the set of such plans:

$$\Pi(\tilde{n}, \tilde{m}) := \{ \pi \in \mathcal{P}(\mathbb{R}^2); \pi|_1 = \tilde{n}, \pi|_2 = \tilde{m} \}. \quad (64)$$

We can now define the p - Wasserstein distance between two measures $\tilde{n}, \tilde{m} \in \mathcal{P}_p(\mathbb{R})$ as follows

$$W_p(\tilde{n}, \tilde{m}) = \left(\inf_{\pi \in \Pi(\tilde{n}, \tilde{m})} \iint_{\mathbb{R}^2} |y_1 - y_2|^p d\pi(y_1, y_2) \right)^{\frac{1}{p}}.$$

Note that $W_p(\tilde{n}, \delta_{\bar{y}}) = \int |y - \bar{y}|^p d\tilde{n}(y)$, for any $\bar{y} \in \mathbb{R}$ and $\tilde{n} \in \mathcal{P}_p(\mathbb{R})$.

For $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$ and $f \in W^{1,\infty}(\mathbb{R})$, the Kantorovich-Rubinstein is the following useful estimate:

$$\left| \int_{\mathbb{R}} f(y) d\tilde{n}(y) - \int_{\mathbb{R}} f(y) d\tilde{m}(y) \right| \leq \|f'\|_{L^\infty(\mathbb{R})} W_1(\tilde{n}, \tilde{m}).$$

For $\tilde{n}, \tilde{m} \in \mathcal{P}_p(\mathbb{R})$ (with $p \geq 1$), the Kantorovich duality provides the following equality

$$W_p(\tilde{n}, \tilde{m}) = \left(\sup_{(\varphi, \psi) \in F} \int_{\mathbb{R}} \varphi(y) d\tilde{n}(y) + \int_{\mathbb{R}} \psi(Y) d\tilde{m}(Y) \right)^{\frac{1}{p}}, \quad (65)$$

where $F = \{(\varphi, \psi) \in (C_b^0(\mathbb{R}, \mathbb{R}))^2; \forall y, Y \in \mathbb{R}, \varphi(y) + \psi(Y) \leq |y - Y|^p\}$.

Finally, we will also use the convexity of the squared Wasserstein distance W_2 . Let $\tilde{n}_1, \tilde{m} \in \mathcal{P}_2(\mathbb{R}) \cap L^1(\mathbb{R})$ and, $\tilde{n}_2 \in L^\infty([0, t] \times \mathbb{T}^d, \mathcal{P}_2(\mathbb{R}) \cap L^1(\mathbb{R}))$, for some $t > 0$. For any $\alpha \in [0, 1]$ and $\beta \in L^1([0, t] \times \mathbb{T}^d)$ such that $\int_{[0, t] \times \mathbb{T}^d} \beta = 1 - \alpha$, we have

$$\begin{aligned} & W_2^2 \left(\alpha \tilde{n}_1 + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \tilde{n}_2(t, x, \cdot) dx d\sigma, \tilde{m} \right) \\ & \leq \alpha W_2^2(\tilde{n}_1, \tilde{m}) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) W_2^2(\tilde{n}_2(\sigma, x, \cdot), \tilde{m}) dx d\sigma. \end{aligned} \quad (66)$$

To obtain this estimate, let $(\varphi, \psi) \in F$ with $p = 2$. Then,

$$\begin{aligned} & \int_{\mathbb{R}} \varphi(y) \left(\alpha \tilde{n}_1(y) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \tilde{n}_2(\sigma, x, y) dx d\sigma \right) dy + \int_{\mathbb{R}} \psi(Y) \tilde{m}(Y) dY \\ & \leq \alpha \left(\int_{\mathbb{R}} \varphi(y) \tilde{n}_1(y) dy + \int_{\mathbb{R}} \psi(Y) \tilde{m}(Y) dY \right) \\ & \quad + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \left(\int_{\mathbb{R}} \varphi(y) \tilde{n}_2(\sigma, x, y) dy + \int_{\mathbb{R}} \psi(Y) \tilde{m}(Y) dY \right) dx d\sigma \\ & \leq \alpha W_2^2(\tilde{n}_1, \tilde{m}) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) W_2^2(\tilde{n}_2(\sigma, x, \cdot), \tilde{m}) dx d\sigma, \end{aligned}$$

and (66) follows thanks to (65), if we consider the supremum over $(\varphi, \psi) \in F$.

5.2 Properties of the Infinitesimal operator

In (16), we have defined the Infinitesimal operator T . More precisely, we define this operator on the space $\mathcal{P}_2(\mathbb{R})$ (see Section 5.1) by (16). Then, for any $\tilde{n} \in \mathcal{P}_2(\mathbb{R})$,

$$\int_{\mathbb{R}} T(\tilde{n})(y) dy = \int_{\mathbb{R}} \tilde{n}(y) dy = 1, \quad \int_{\mathbb{R}} y T(\tilde{n})(y) dy = \int_{\mathbb{R}} y \tilde{n}(y) dy,$$

and for any $Z \in \mathbb{R}$,

$$\forall y \in \mathbb{R}, \quad T(\Gamma_A(\cdot - Z))(y) = \Gamma_A(y - Z). \quad (67)$$

where

$$\Gamma_A(y) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{|y|^2}{2A}}. \quad (68)$$

T induces a contraction for the Wasserstein distance W_2 , which can be seen as a version of the Tanaka inequality [54] (see also [6, 9]):

Theorem 5.1 (A Tanaka inequality). *Let $A > 0$, $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$ such that $\int y\tilde{n}(y) dy = \int y\tilde{m}(y) dy$, and T defined by (16). Then*

$$W_2(T(\tilde{n}), T(\tilde{m})) \leq \frac{1}{\sqrt{2}} W_2(\tilde{n}, \tilde{m}).$$

Proof of the Theorem 5.1. We consider φ, ψ such that for any $y, Y \in \mathbb{R}$, $\varphi(y) + \psi(Y) \leq |y - Y|^2$, and $\pi \in \Pi(\tilde{n}, \tilde{m})$. Then,

$$\begin{aligned} & \int_{\mathbb{R}} \varphi(y) T(\tilde{n})(y) dy + \int_{\mathbb{R}} \psi(Y) T(\tilde{m})(Y) dY \\ &= \iiint_{\mathbb{R}^3} \varphi(y) \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \tilde{n}(y_*) \tilde{n}(y'_*) dy_* dy'_* dy \\ & \quad + \iiint_{\mathbb{R}^3} \psi(Y) \Gamma_{A/2} \left(Y - \frac{Y_* + Y'_*}{2} \right) \tilde{n}(Y_*) \tilde{n}(Y'_*) dY_* dY'_* dY \\ &= \iiint_{\mathbb{R}^3} \varphi \left(y + \frac{y_* + y'_*}{2} \right) \Gamma_{A/2}(y) \tilde{n}(y_*) \tilde{n}(y'_*) dy_* dy'_* dy \\ & \quad + \iiint_{\mathbb{R}^3} \psi \left(y + \frac{Y_* + Y'_*}{2} \right) \Gamma_{A/2}(Y) \tilde{n}(Y_*) \tilde{n}(Y'_*) dY_* dY'_* dY \\ &= \int_{\mathbb{R}} \Gamma_{A/2}(y) \iiint_{\mathbb{R}^4} \varphi \left(y + \frac{y_* + y'_*}{2} \right) + \psi \left(y + \frac{Y_* + Y'_*}{2} \right) d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) dy \\ &\leq \int_{\mathbb{R}} \Gamma_{A/2}(y) \iiint_{\mathbb{R}^4} \left| \left(y + \frac{y_* + y'_*}{2} \right) - \left(y + \frac{Y_* + Y'_*}{2} \right) \right|^2 d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) dy \\ &\leq \frac{1}{4} \iiint_{\mathbb{R}^4} |(y_* - Y_*) + (y'_* - Y'_*)|^2 d\pi(y_*, Y_*) d\pi(y'_*, Y'_*). \end{aligned} \quad (69)$$

We notice that

$$\iiint_{\mathbb{R}^4} (y_* - Y_*)(y'_* - Y'_*) d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) = \left(\int_{\mathbb{R}} y\tilde{n}(y) dy - \int_{\mathbb{R}} y\tilde{m}(Y) dY \right)^2 = 0,$$

and then

$$\begin{aligned}
& \int_{\mathbb{R}} \varphi(y)T(\tilde{n})(y) dy + \int_{\mathbb{R}} \psi(Y)T(\tilde{m})(Y) dY \\
& \leq \frac{1}{4} \iiint \iiint_{\mathbb{R}^4} [(y_* - Y_*)^2 + 2(y_* - Y_*)(y'_* - Y'_*) + (y'_* - Y'_*)^2] d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) \\
& = \frac{1}{2} \iint_{\mathbb{R}^2} (y - Y)^2 d\pi(y, Y).
\end{aligned}$$

Since this inequality holds for any $\pi \in \Pi(\tilde{n}, \tilde{m})$, we can consider the infimum of over these, to obtain, thanks to the definition of the Wasserstein distance:

$$\int_{\mathbb{R}} \varphi(y)T(\tilde{n})(y) dy + \int_{\mathbb{R}} \psi(Y)T(\tilde{m})(Y) dY \leq \frac{1}{2}W_2^2(\tilde{n}, \tilde{m}).$$

We can now take the supremum of this inequality over the functions φ, ψ satisfying $\varphi(y) + \psi(Y) \leq |y - Y|^2$ and conclude, thanks to the Kantorovich duality formula (65). \square

Corollary 5.2 (A Tanaka inequality for W_4). *Let $A > 0$, $\tilde{n}, \tilde{m} \in \mathcal{P}_4(\mathbb{R})$ such that $\int y\tilde{n}(y) dy = \int y\tilde{m}(y) dy$, and T defined by (16). Then*

$$W_4(T(\tilde{n}), T(\tilde{m})) \leq \frac{1}{2^{1/4}}W_4(\tilde{n}, \tilde{m}).$$

Proof of the Corollary 5.2. We can reproduce the proof of Theorem 5.1 until (69), and obtain that for any φ, ψ satisfying $\varphi(y) + \psi(Y) \leq |y - Y|^4$ and $\pi \in \Pi(\tilde{n}, \tilde{m})$,

$$\begin{aligned}
& \int_{\mathbb{R}} \varphi(y)T(\tilde{n})(y) dy + \int_{\mathbb{R}} \psi(Y)T(\tilde{m})(Y) dY \\
& \leq \frac{1}{16} \iiint \iiint_{\mathbb{R}^4} |(y_* - Y_*) + (y'_* - Y'_*)|^4 d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) \\
& = \frac{1}{16} \iiint \iiint_{\mathbb{R}^4} \left[(y_* - Y_*)^4 + 4(y_* - Y_*)^3(y'_* - Y'_*) + 6(y_* - Y_*)^2(y'_* - Y'_*)^2 \right. \\
& \quad \left. + 4(y_* - Y_*)(y'_* - Y'_*)^3 + (y'_* - Y'_*)^4 \right] d\pi(y_*, Y_*) d\pi(y'_*, Y'_*) \\
& = \frac{1}{8} \left(\iint_{\mathbb{R}^2} (y - Y)^4 d\pi(y, Y) \right) + \frac{3}{8} \left(\iint_{\mathbb{R}^2} (y - Y)^2 d\pi(y, Y) \right)^2 \\
& \leq \frac{1}{2} \left(\iint_{\mathbb{R}^2} (y - Y)^4 d\pi(y, Y) \right).
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 5.1. \square

5.3 Existence theory for a truncated version of (SIM)

In this section, we prove the existence of global solutions to (7), the truncated version of (SIM). To do so, we first construct local in time (i.e. for $t \in [0, \bar{t}]$, $\bar{t} > 0$) solutions of the truncated equation (7):

Lemma 5.3. *Let $y_{opt} \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})$, $A > 0$, $\gamma \geq 1$ and n^0 satisfying Assumption 1. There is $C > 0$ such that if $R > 1$ and if $n^0 \in C^0(\mathbb{T}^d \times \mathbb{R}, \mathbb{R}_+)$ satisfies $n^0(x, y) > 0$ for*

$(x, y) \in \mathbb{T}^d \times [-R, R]$, then there is a unique solution $n_R = n_R(t, x, y)$ of (7) for $t \in [0, \bar{t}]$ together with the initial data $n_R(0, x, y) = n^0(x, y)1_{|y| \leq R}$, where

$$\bar{t} = \frac{1}{C(\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} + \gamma)R + 1}.$$

More precisely, n_R satisfies the following estimate for some constant $C_R > 0$ may depend on $R > 0$ and $\gamma > 0$:

$$\sup_{y \in \mathbb{R}} \left(\sup_{t \in [0, \bar{t}]} \|n_R(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n_R(\cdot, \cdot, y)\|_{L^2_{loc}([0, \bar{t}], H^2(\mathbb{T}^d))} + \|\partial_t n_R(\cdot, \cdot, y)\|_{L^2_{loc}([0, \bar{t}] \times \mathbb{T}^d)} \right) \leq C_{R, \gamma}.$$

Proof of Lemma 5.3. Step 1: Definition of the set \mathcal{F}_R and the application $F_{R, \bar{t}}$

For $0 < \bar{t} < \frac{1}{3\gamma(1+A/2)}$, let

$$\mathcal{F}_{R, \bar{t}} := \left\{ m \in L^\infty([0, \bar{t}] \times \mathbb{T}^d \times \mathbb{R}); m \geq 0, m(t, x, y) = 0 \text{ if } |y| \geq R, \right. \\ \left. \|m\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d \times \mathbb{R})} \leq 3\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} \right\}. \quad (70)$$

We introduce the operator $F_{R, \bar{t}}$, that is defined by $F_{R, \bar{t}}(m) = n$ for $m \in \mathcal{F}_{R, \bar{t}}$ and where n is the solution of

$$\partial_t n(t, x, y) = \Delta_x n(t, x, y) + \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} m(t, x, z) dz \right) n(t, x, y) \\ + \gamma \left(1_{|y| \leq R} \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{m(t, x, y_*) m(t, x, y'_*)}{\int_{\mathbb{R}} m(t, x, z) dz} dy_* dy'_* - n(t, x, y) \right), \quad (71)$$

together with $n(0, x, y) = n^0(x, y) 1_{|y| \leq R}$ for $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, so that $n(t, x, y) = 0$ if $|y| > R$. Note that a different application could probably be used. Notice that in (71), y can be seen as a simple coefficient, and the equation can be solved independently for each $y \in \mathbb{R}$. We also notice that for any fixed $y \in [-R, R]$, the coefficients of the parabolic equation (71) are bounded when $m \in \mathcal{F}_{R, \bar{t}}$ (we recall that $\gamma > 0$ is here a fixed constant), so that a non-negative weak solution $((t, x) \mapsto n(t, x, y)) \in L^2([0, T]_+, H^1(\mathbb{T}^d))$ (such that $((t, x) \mapsto \partial_t n(t, x, y)) \in L^2([0, T], H^{-1}(\mathbb{T}^d))$) exists thanks to standard arguments (see Theorem 3 in Section 7.1.2 of [29]). The bounded coefficients actually imply that this is a *strong* solution (in the sense of [29]), thanks to Theorem 5 in Section 7.1.3 of [29] applied for each $y \in \mathbb{R}$. More precisely,

$$\sup_{t \in [0, \bar{t}]} \|n(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n(\cdot, \cdot, y)\|_{L^2([0, \bar{t}], H^2(\mathbb{T}^d))} + \|\partial_t n(\cdot, \cdot, y)\|_{L^2([0, \bar{t}] \times \mathbb{T}^d)} \leq C_R, \quad (72)$$

where the constant C_R is related to a bound on the coefficient $1_{|y| \leq R} \frac{1}{2}(y - y_{opt}(t, x))^2$, and it therefore depends on $R > 0$ in this estimate.

Step 2: The set $\mathcal{F}_{R, \bar{t}}$ is stable under $F_{R, \bar{t}}$ provided $\bar{t} \leq 1/C$

We notice that n satisfies

$$\partial_t n(t, x, y) - \Delta_x n(t, x, y) \leq \left(1 + \frac{A}{2} \right) n(t, x, y) + \gamma \Gamma_{A/2}(0) \|m\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d, L^1([-R, R]))}. \quad (73)$$

Let

$$\begin{aligned} \phi(t, x, y) := & 1_{|y| \leq R} \left[\left(\sup_{\mathbb{T}^d \times [-R, R]} n^0 \right) e^{(1+\frac{A}{2})t} \right. \\ & \left. + \left(\|m\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d, L^1([-R, R]))} \right) \gamma \Gamma_{A/2}(0) \frac{e^{(1+\frac{A}{2})t} - 1}{1 + A/2} \right], \end{aligned}$$

that is a super-solution of (73). Since additionally $\phi(0, x, y) \geq n^0(x, y)1_{|y| \leq R}$ for $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, we can use the maximum principle (see Corollary 7.4 p. 159 in [37]) to compare $(t, x) \mapsto n(t, x, y)$ and $(t, x) \mapsto \phi(t, x, y)$, for any $y \in [-R, R]$. We then show that for $(t, x, y) \in [0, \bar{t}] \times \mathbb{T}^d \times \mathbb{R}$,

$$n(t, x, y) \leq \phi(t, x, y) \leq \left(1_{|y| \leq R} e^{(1+\frac{A}{2})t} + 3\gamma \Gamma_{A/2}(0) \left(e^{(1+\frac{A}{2})t} - 1 \right) \right) \|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])}, \quad (74)$$

where we have used the estimate on $\|m\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d, L^1(\mathbb{R}))}$ provided by the definition of $\mathcal{F}_{R, \bar{t}}$. If

$$\bar{t} \leq \frac{\ln \left(1 + \min(1/(3\gamma \Gamma_{A/2}(0)), 1) \right)}{1 + A/2}, \quad (75)$$

then $\left(1_{|y| \leq R} e^{(1+\frac{A}{2})t} + 3\gamma \Gamma_{A/2}(0) \left(e^{(1+\frac{A}{2})t} - 1 \right) \right) \leq 3$, which implies

$$\|n\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d \times [-R, R])} \leq 3 \|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])}. \quad (76)$$

We have proven that for any $m \in \mathcal{F}_{R, \bar{t}}$, we have $F_{R, \bar{t}}(m) = n \in \mathcal{F}_{R, \bar{t}}$ provided (75) is satisfied.

Step 3: $F_{R, \bar{t}}$ is a contraction for the norm $\|\cdot\|_{L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R})}$ on $\mathcal{F}_{R, \bar{t}}$

Let $m, \tilde{m} \in \mathcal{F}_{R, \bar{t}}$ (see (70)), and $n := F_{R, \bar{t}}(m)$, $\tilde{n} := F_{R, \bar{t}}(\tilde{m})$. Then $(n - \tilde{n})(0, x, y) = 0$ for $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, $(n - \tilde{n})(t, x, y) = 0$ for $(t, x) \in [0, \bar{t}] \times \mathbb{T}^d$ and $y \notin [-R, R]$. We therefore consider $(t, x) \in [0, \bar{t}] \times \mathbb{T}^d$ and $y \in [-R, R]$ from now on. We can estimate the difference between birth terms as follows:

$$\begin{aligned} & \left| \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \left[\frac{m(t, x, y_*) m(t, x, y'_*)}{\int_{\mathbb{R}} m(t, x, z) dz} - \frac{\tilde{m}(t, x, y_*) \tilde{m}(t, x, y'_*)}{\int_{\mathbb{R}} \tilde{m}(t, x, z) dz} \right] dy_* dy'_* \right| \\ &= \left| \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \left[\frac{(m(t, x, y_*) - \tilde{m}(t, x, y_*)) m(t, x, y'_*)}{\int_{\mathbb{R}} m(t, x, z) dz} \right. \right. \\ & \quad + \frac{\tilde{m}(t, x, y_*) m(t, x, y'_*)}{\left(\int_{\mathbb{R}} m(t, x, z) dz \right) \left(\int_{\mathbb{R}} \tilde{m}(t, x, z) dz \right)} \left(\int_{\mathbb{R}} \tilde{m}(t, x, z) - m(t, x, z) dz \right) \\ & \quad \left. + \frac{\tilde{m}(t, x, y_*) (m(t, x, y'_*) - \tilde{m}(t, x, y'_*))}{\int_{\mathbb{R}} \tilde{m}(t, x, z) dz} \right] dy_* dy'_* \right| \\ & \leq 3\Gamma_{A/2}(0) \int_{\mathbb{R}} |\tilde{m}(t, x, z) - m(t, x, z)| dz = 3\Gamma_{A/2}(0) \int_{-R}^R |\tilde{m}(t, x, z) - m(t, x, z)| dz. \end{aligned}$$

$(n - \tilde{n})$ satisfies:

$$\begin{aligned} \partial_t(n - \tilde{n})(t, x, y) - \Delta_x(n - \tilde{n})(t, x, y) & \leq \left(1 + \frac{A}{2} \right) (n - \tilde{n})(t, x, y) \\ & \quad + \left(\int_{\mathbb{R}} (m(t, x, z) - \tilde{m}(t, x, z)) dz \right) \tilde{n}(t, x, y) + 3\gamma \Gamma_{A/2}(0) \int_{-R}^R |m(t, x, z) - \tilde{m}(t, x, z)| dz \\ & \leq \left(1 + \frac{A}{2} \right) (n - \tilde{n})(t, x, y) + 6R \left(\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} + \gamma \Gamma_{A/2}(0) \right) \|m - \tilde{m}\|_{L^\infty([0, \tau] \times \mathbb{T}^d \times [-R, R])}, \end{aligned} \quad (77)$$

where we have used (76) to estimate $0 \leq \tilde{n}(t, x, y)$ from above. We notice that

$$(t, x) \mapsto 6R \left(\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} + \gamma \Gamma_{A/2}(0) \right) \|m - \tilde{m}\|_{L^\infty([0, \tau] \times \mathbb{T}^d \times [-R, R])} e^{(1+A/2)t}$$

is a super-solution of the parabolic equation (77) for any fixed $y \in [-R, R]$, and then the comparison principle implies

$$\begin{aligned} & \max_{[0, \bar{t}] \times \mathbb{T}^d \times [-R, R]} (n - \tilde{n}) \\ & \leq 6R \left(\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} + \gamma \Gamma_{A/2}(0) \right) \|m - \tilde{m}\|_{L^\infty([0, \tau] \times \mathbb{T}^d \times [-R, R])} \frac{e^{(1+A/2)\bar{t}} - 1}{1 + A/2} \\ & \leq \frac{1}{2} \|m - \tilde{m}\|_{L^\infty([0, \tau] \times \mathbb{T}^d \times [-R, R])}, \end{aligned} \quad (78)$$

provided we set \bar{t} as follows

$$\bar{t} = \frac{1}{C(\|n^0\|_{L^\infty(\mathbb{T}^d \times [-R, R])} + \gamma)R + 1}, \quad (79)$$

where $C > 0$ is chosen large enough for (75) and (78) to hold. If $(n - \tilde{n})(t, x, y) \leq 0$, a similar argument can be made on $(\tilde{n} - n)$, since n and \tilde{n} have symmetric properties. We then obtain, for \bar{t} defined by (79),

$$\|n - \tilde{n}\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d \times [-R, R])} \leq \frac{1}{2} \|m - \tilde{m}\|_{L^\infty([0, \bar{t}] \times \mathbb{T}^d \times [-R, R])}.$$

The Banach fixed-point theorem then shows that there is a unique fixed point n_R of $F_{R, \bar{t}}$ in $\mathcal{F}_{R, \bar{t}}$. That fixed point $n_R \in \mathcal{F}_{R, \bar{t}}$ is a solution of (7) that satisfies (72). \square

We can now construct global (i.e. for $t \in \mathbb{R}_+$) solutions of the truncated model (7):

Proposition 5.4. *Let $y_{opt} \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})$, $A > 0$, $\gamma > 2 + A + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}^2$, and n^0 satisfying Assumption 1. For $R > 1$, there is a unique global solution n_R of (7) together with the initial data $n_R(0, x, y) = n^0(x, y)1_{|y| \leq R}$. More precisely, for $T > 0$, there is a constant $C_\gamma > 0$ independent from $R > 0$ (but that depends on γ) such that n_R satisfies*

$$\sup_{y \in \mathbb{R}} \left(\sup_{t \in [0, T]} \|n_R(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n_R(\cdot, \cdot, y)\|_{L^2([0, T], H^2(\mathbb{T}^d))} + \|\partial_t n_R(\cdot, \cdot, y)\|_{L^2([0, T] \times \mathbb{T}^d)} \right) \leq C_\gamma, \quad (80)$$

and for $(t, x, y) \in [0, \infty) \times \mathbb{T}^d \times \mathbb{R}$, there is $\bar{C} > 0$ independent from both R and γ , such that

$$n_R(t, x, y) \leq \frac{\bar{C}\gamma}{1 + y^2}. \quad (81)$$

Proof of Proposition 5.4. Let $\tau \geq 0$. We assume that n_R is a solution of (7) for $t \in [0, \tau]$ with initial data $(x, y) \mapsto n^0(x, y)1_{|y| \leq R}$. Thanks to an integration of (7) along $y \in \mathbb{R}$,

$N_R(t, x) = \int n_R(t, x, y) dy$ is a *strong* solution of the following parabolic equation (in the sense of Theorem 5(i) in Section 7.1.3 of [29]) with n_R as a given coefficient:

$$\begin{aligned} \partial_t N_R(t, x) - \Delta_x N_R(t, x) &\leq \left[1 + \frac{A}{2} - N_R(t, x) \right] N_R(t, x) - \frac{1}{2} \int_{-R}^R (y - y_{opt}(t, x))^2 n_R(t, x, y) dy, \\ &\leq \left[1 + \frac{A}{2} - N_R(t, x) \right] N_R(t, x). \end{aligned} \quad (82)$$

We notice that $\phi : (t, x) \mapsto \max \left(\int_{\mathbb{R}} \frac{C_0}{1+y^2} dy, 1 + \frac{A}{2} \right)$ satisfies $N_R(0, x) \leq \phi(0, x)$ thanks to Assumption 1, and is a super-solution of (82). We can then apply the comparison principle to show that $N_R(t, x) \leq \phi(t, x)$ for $(t, x) \in [0, \tau] \times \mathbb{T}^d$, that provides a uniform bound $\int_{\mathbb{R}} n_R(t, x, y) dy \leq C$ for some constant $C > 0$ that only depends on A and the constant C_0 from Assumption 1. Then,

$$\begin{aligned} \partial_t n_R(t, x, y) - \Delta_x n_R(t, x, y) &\leq \left(1 + \frac{A}{2} - \frac{1}{2} (y_{opt}(t, x)^2 - 2y y_{opt}(t, x) + y^2/2) - \frac{\gamma}{2} \right) n_R(t, x, y) \\ &\quad - \left(\frac{\gamma}{2} + \frac{y^2}{4} \right) n_R(t, x, y) + \gamma \Gamma_{A/2}(0) \int_{\mathbb{R}} n_R(t, x, y) dy \quad (83) \\ &= \left(\left(1 + \frac{A}{2} + \frac{y_{opt}(t, x)^2}{2} - \frac{\gamma}{2} \right) - \frac{1}{2} \left(\sqrt{2} y_{opt}(t, x) - \frac{y}{\sqrt{2}} \right)^2 \right) n_R(t, x, y) \\ &\quad - \left(\frac{\gamma}{2} + \frac{y^2}{4} \right) n_R(t, x, y) + \gamma \Gamma_{A/2}(0) N_R(t, x) \\ &\leq - \left(\frac{\gamma}{2} + \frac{y^2}{4} \right) n_R(t, x, y) + C\gamma, \end{aligned} \quad (84)$$

since we have assumed $\gamma \geq 2 \left(1 + \frac{A}{2} + \frac{1}{2} \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}^2 \right)$. We notice that for any $y \in [-R, R]$, $(t, x) \mapsto \psi(x, y) := \frac{4C}{1+y^2} \gamma$ is a super-solution of (84) that satisfies $\psi(x, y) \geq n^0(x, y)$ for $(x, y) \in \mathbb{T}^d \times [-R, R]$, provided $C > C_0/\gamma$, thanks to Assumption 1. For $y \in [-R, R]$, the comparison principle applied to (84) shows $n_R(t, x, y) \leq \psi(y)$ for $(t, x) \in [0, \tau] \times \mathbb{T}^d$. Since this holds for any $y \in [-R, R]$ and $n_R(t, x, y) = 0$ if $|y| > R$, we have

$$\forall (t, x, y) \in [0, \tau] \times \mathbb{T}^d \times \mathbb{R}, \quad n_R(t, x, y) \leq \frac{4C}{1+y^2} \gamma. \quad (85)$$

Thanks to Lemma 5.3, we can extend the solution $n_R = n_R(t, x, y)$ into a solution of (7) on $[0, \tau + \bar{t}] \times \mathbb{T}^d \times [-R, R]$, with $n_R(\tau + t, x, y) := \tilde{n}_R(t, x, y)$, and \bar{t} as in Lemma 5.3. We notice that \bar{t} is independent of $\tau \geq 0$, so that this extension argument can be iterated to construct a global solution of (7) satisfying (85). Thanks to (81), for any fixed $\bar{y} \in [-R, R]$, we can see $(t, x) \mapsto n_R(t, x, \bar{y})$ (where n_R is the solution of (7)) as a solution of the heat equation $\partial_t n_R(t, x, \bar{y}) - \Delta_x n_R(t, x, \bar{y}) = f_{\bar{y}}(t, x)$ with a bounded 0-order term: $|f_{\bar{y}}(t, x)| \leq C_\gamma$, where $C_\gamma > 0$ is independent from R . $(t, x) \mapsto n_R(t, x, \bar{y})$ is then a *strong* solution (in the sense of [29]) of that heat equation and Theorem 5 in Section 7.1.3 of [29] implies the regularity estimate (80). □

In Proposition 5.4, we have proven the bound (81), that is independent from $R > 0$, which will be very useful to consider the limit $R \rightarrow \infty$ of n_R to construct solutions of (SIM) and

prove Proposition 2.2, see Section 3. We can actually improve this estimates on tails of n_R , as we show in the following proposition:

Proposition 5.5. *Let $y_{opt} \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})$, $A > 0$, $\gamma > 2 + A + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}^2$, and n^0 satisfying Assumption 1. There is $\bar{C} > 0$ (independent from $\gamma > 0$) such that for any $R > 1$, the global solution n_R of (7) with the initial data $n_R(0, x, y) = n^0(x, y)1_{|y| \leq R}$ satisfies*

$$n_R(t, x, y) \leq \frac{\bar{C}\gamma}{1 + y^{10}}. \quad (86)$$

Moreover, for $T > 0$, there is a constant $C_{T,\gamma}$ that may depend on T and γ , but that is uniform in R , such that

$$\sup_{t \in [0, T]} \|n_R(t, \cdot, y)\|_{H^1(\mathbb{T}^d)} + \|n_R(\cdot, \cdot, y)\|_{L^2([0, T], H^2(\mathbb{T}^d))} + \|\partial_t n_R(\cdot, \cdot, y)\|_{L^2([0, T] \times \mathbb{T}^d)} \leq \frac{C_{T,\gamma}}{1 + y^8}. \quad (87)$$

Proof of Proposition 5.5. Note that estimate (81) provides an upper bound on $y^2 n_R(t, x, y)$ that is uniform in $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$ and that does not depend on $R > 0$. To improve this tail estimate further, we decompose the birth term as follows:

$$\begin{aligned} & \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ &= \iint_{[-|y|/4, |y|/4]^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ &+ \iint_{\mathbb{R}^2 \setminus [-|y|/4, |y|/4]^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_*. \end{aligned} \quad (88)$$

We can estimate the first term on the right hand side of (88) as follows:

$$\begin{aligned} & \iint_{[-|y|/4, |y|/4]^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & \leq \left(\max_{y_*, y'_* \in [-|y|/4, |y|/4]} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \right) \iint_{[-|y|/4, |y|/4]^2} \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & \leq \Gamma_{A/2} \left(\frac{3|y|}{4} \right) \int n_R(t, x, z) dz \leq C e^{-\frac{9y^2}{16A}}, \end{aligned} \quad (89)$$

since $y_*, y'_* \in [-|y|/4, |y|/4]$ implies $\Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \leq \Gamma_{A/2} (3|y|/4)$. To estimate the last term of (88), we take advantage of (81) to show:

$$\begin{aligned} & \iint_{\mathbb{R}^2 \setminus [-|y|/4, |y|/4]^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & \leq \int_{\mathbb{R}} \int_{|y_*| \geq |y|/4} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & + \int_{|y'_*| \geq |y|/4} \int_{\mathbb{R}} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & \leq 2 \int_{\mathbb{R}} \left(\int \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) dy_* \right) \left(\max_{z \in [-|y|/4, |y|/4]^c} |n_R(t, x, z)| \right) \frac{n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy'_* \\ & \leq C \max_{z \in [-|y|/4, |y|/4]^c} |n_R(t, x, z)| \leq \frac{C\gamma}{1 + (|y|/4)^2} \leq \frac{C\gamma}{1 + y^2}, \end{aligned} \quad (90)$$

for some constant $C > 0$. We can reproduce here the argument developed in (83)-(84), and use the estimates (88), (89) and (90) on the birth term to show

$$\begin{aligned} & \partial_t n_R(t, x, y) - \Delta_x n_R(t, x, y) \\ & \leq - \left(\frac{\gamma}{2} + \frac{y^2}{4} \right) n_R(t, x, y) + \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \\ & \leq - \left(\frac{\gamma}{2} + \frac{y^2}{4} \right) n_R(t, x, y) + \frac{C\gamma}{1 + y^2}. \end{aligned} \quad (91)$$

We notice that for any $y \in [-R, R]$, $(t, x) \mapsto n_R(t, x, y)$ satisfies (91), while

$$(t, x) \mapsto \psi(y) := \frac{\max(4C\gamma, \bar{C})}{1 + y^4}$$

is a super-solution of (91) that satisfies $\psi(y) \geq n^0(x, y)$ for $x \in \mathbb{T}^d$ thanks to Assumption 1. The comparison principle then implies $n_R(t, x, y) \leq \frac{C}{1+y^4}\gamma$ for $(t, x, y) \in \mathbb{R}_+ \times \mathbb{T}^d \times [-R, R]$ and a constant $C > 0$ independent from $R > 0$. Since this holds for any $y \in \mathbb{R}$, we have

$$\forall (t, x, y) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}, \quad n_R(t, x, y) \leq \frac{C\gamma}{1 + y^4}. \quad (92)$$

This estimate can be used to obtain a better estimate (90). It then becomes

$$I_2(t, x, y) \leq C \max_{z \in [-|y|/4, |y|/4]^c} |n_R(t, x, z)| \leq \frac{C\gamma}{1 + (|y|/4)^4} \leq \frac{C\gamma}{1 + y^4},$$

the argument above can be repeated to show that $n_R(t, x, y) \leq \frac{C\gamma}{1+y^6}$ for some constant $C > 0$ independent of R . This estimate can be used again twice to improve the estimate (90), and iterate the argument once more to show (86).

In the last iteration above, we have derived the following bound on the birth term:

$$\left| \gamma \iint_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \frac{n_R(t, x, y_*) n_R(t, x, y'_*)}{\int_{\mathbb{R}} n_R(t, x, z) dz} dy_* dy'_* \right| \leq \frac{C\gamma^2}{1 + y^8},$$

while the estimate (86) implies

$$\left| \left(1 + \frac{A}{2} - \gamma - \frac{1}{2}(y - y_{opt}(t, x))^2 - \int_{\mathbb{R}} n_R(t, x, z) dz \right) n_R(t, x, y) \right| \leq \frac{C\gamma(1 + \gamma)}{1 + y^8},$$

so that for $y \in [-R, R]$ and $T > 0$, we have on $[0, T] \times \mathbb{T}^d$:

$$|\partial_t n_R(t, x, y) - \Delta_x n_R(t, x, y)| \leq \frac{C\gamma(1 + \gamma)}{1 + y^8}. \quad (93)$$

We can then apply Theorem 5 in Section 7.1.3 of [29] to (93) for each $y \in \mathbb{R}$, which implies (87). □

Remark 5.6. In (93), we have an L^∞ bound on $\partial_t n_R(t, x, y) - \Delta_x n_R(t, x, y)$, which is stronger than the L^2 estimate necessary to apply Theorem 5 in Section 7.1.3 of [29]. We can take advantage of this to obtain a stronger regularity result: Thanks to Theorem 7.22 in [37], we have, for $y \in [-R, R]$, that the function $(t, x) \mapsto n_R(t, x, y)$ belongs to $W_{d+2}^{2,1}([0, T] \times \mathbb{T}^d)$ (we use here the notation $W_{d+2}^{2,1}$ from [37]: the time derivative belongs to L^{d+2} , and its second spatial derivative belongs to L^{d+2}). Moreover, (93) implies that the $W_{d+2}^{2,1}([0, T] \times \mathbb{T}^d)$ norm of $(t, x) \mapsto n_R(t, x, y)$ is dominated by $\frac{C}{1+y^8}$, with a constant $C > 0$ that may depend on T and $\gamma > 0$, but that is independent from $R > 0$. This weighted regularity estimate will be useful in Section 5.4

5.4 Regularity of the macroscopic quantities N , Z and V

Let n a solution of (SIM). Thanks to the uniform regularity of n_R described in Remark 5.6 and an integration against 1 , y and y^4 respectively, $N(t, x)$, $Y(t, x) := \int_{\mathbb{R}} yn(t, x, y) dy$ and $W(t, x) := \int_{\mathbb{R}} y^4 n(t, x, y) dy$ satisfy

$$N, Y, W \in W_{d+2}^{2,1}([0, T] \times \mathbb{T}^d),$$

where we use here the notation $W_{d+2}^{2,1}$ from [37]: the time derivative of the function belongs to L^{d+2} , and its second spatial derivative belongs to L^{d+2} .

Let $\varphi \in L^\infty([0, T] \times \mathbb{T}^d)$, and $R > 0$. If n is a solution of (SIM) and $\varphi \in C^2([0, T] \times \mathbb{T}^d)$, we can use $\varphi(t, x)1_{[-R, R]}$ (resp. $\varphi(t, x)y1_{[-R, R]}$, $\varphi(t, x)y^41_{[-R, R]}$) as a test function, and let $R \rightarrow \infty$ to show that N solves (13), while Y and W solve

$$\partial_t Y(t, x) - \Delta_x Y(t, x) = \left(1 + \frac{A}{2} - N(t, x)\right) Y(t, x) - \frac{1}{2} \int_{\mathbb{R}} y(y - y_{opt})^2 n(t, x, y) dy, \quad (94)$$

$$\partial_t W(t, x) - \Delta_x W(t, x) = \left(1 + \frac{A}{2} - N(t, x)\right) W(t, x) - \frac{1}{2} \int_{\mathbb{R}} y^4 (y - y_{opt})^2 n(t, x, y) dy \quad (95)$$

$$+ \gamma \left(N(t, x) \int_{\mathbb{R}} |y|^4 T(\tilde{n}(t, x, \cdot))(y) dy - W(t, x) \right). \quad (96)$$

We can notice that the right hand side of the equations (13), (94) and (95) after a derivation in t or in x , belong to $L^{d+2}([0, T] \times \mathbb{T}^d)$ uniformly in R , thanks to Remark 5.6. We can then apply Theorem 7.22 in [37] to show that $\partial_t N$, $\partial_t Y$, $\partial_t W$, $\nabla_x N$, $\nabla_x Y$ and $\nabla_x W$ belong to $W_{d+2}^{2,1}([0, T] \times \mathbb{T}^d)$. Morrey's inequality then implies that N, Y, W are C^1 functions in the t variable, and C^2 functions in the x variable. They are then *classical solutions* of (13), (94) and (95): they satisfy these equalities pointwise, as an equality between continuous functions. In particular, ∂_t and Δ_x are the actual differential operators, and we can compute $(\partial_t - \Delta_x) \frac{Z}{N}$ to show (17).

In the above argument, we use interior parabolic estimates, specifically Theorem 7.22 in [37]. These estimates do not hold up to the boundary $\{t = 0\}$ directly, but it is possible to use traditional ideas to get around this difficulty: it is possible to transform these problems into equations that hold for $t \in \mathbb{R}$ thanks to the regularity of the initial condition (provided by Assumption 1), we refer to Step 4 of the proof of Proposition 4.3 where we develop the details of a similar argument.

5.5 Proof of Proposition 2.4: uniqueness of solutions of (SIM)

Since n and \tilde{n} are solutions of (SIM), we can reproduce the lower bound argument made in the proof of Proposition 2.2 (see (22)-(23)), that is based on the comparison principle, to show that there exists $C_T > 0$ such that $\int_{\mathbb{R}} n(t, x, y) dy \geq C_T$ and $\int_{\mathbb{R}} \tilde{n}(t, x, y) dy \geq C_T$.

Thanks to [29] (Theorem 3, p. 287) and using the fact that $n(0, x, y) = \tilde{n}(0, x, y) = n^0(x, y)$ for $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, we obtain that for $t \in [0, T]$,

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(t, x, y) - \tilde{n}(t, x, y))^2 (1 + y^4) dx dy + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\nabla_x n(s, x, y) - \nabla_x \tilde{n}(s, x, y)|^2 (1 + y^4) dx dy ds \\
&= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \left(1 + \frac{A}{2} - \gamma - \frac{1}{2}(y - y_{opt}(s, x))^2 - \int_{\mathbb{R}} n(s, x, z) dz \right) (n(s, x, y) - \tilde{n}(s, x, y))^2 (1 + y^4) dy dx ds \\
&+ \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} n(s, x, z) - \tilde{n}(s, x, z) dz \right) \tilde{n}(s, x, y) (n(s, x, y) - \tilde{n}(s, x, y)) (1 + y^4) dy dx ds \\
&+ \gamma \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(s, x, y) - \tilde{n}(s, x, y)) \left[- (n(s, x, y) - \tilde{n}(s, x, y)) \right. \\
&\quad \left. \int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \left(\frac{n(s, x, y_*) n(s, x, y'_*)}{\int_{\mathbb{R}} n(s, x, z) dz} - \frac{\tilde{n}(s, x, y_*) \tilde{n}(s, x, y'_*)}{\int_{\mathbb{R}} \tilde{n}(s, x, z) dz} \right) dy_* dy'_* \right] (1 + y^4) dx dy ds \\
&\leq \left(1 + \frac{A}{2} - \gamma \right) \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(s, x, y) - \tilde{n}(s, x, y))^2 (1 + y^4) dx dy ds \\
&+ \left(\max_{s \in [0, t], x \in \mathbb{T}^d} \int_{\mathbb{R}} n(s, x, y) \sqrt{1 + y^4} dy \right)^{\frac{1}{2}} \\
&\int_0^t \left(\int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(s, x, y) - \tilde{n}(s, x, y))^2 (1 + y^4) dx dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \int_{\mathbb{T}^d} |n(s, x, y) - \tilde{n}(s, x, y)| dx dy \right) ds \tag{97}
\end{aligned}$$

$$\begin{aligned}
&+ \frac{\gamma}{2} \int_0^t \left\{ \int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(s, x, y) - \tilde{n}(s, x, y))^2 (1 + y^4) dx dy \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} \int_{\mathbb{T}^d} \left[\int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) \right. \right. \\
&\quad \left. \left. \left(\frac{n(s, x, y_*) n(s, x, y'_*)}{\int_{\mathbb{R}} n(s, x, z) dz} - \frac{\tilde{n}(s, x, y_*) \tilde{n}(s, x, y'_*)}{\int_{\mathbb{R}} \tilde{n}(s, x, z) dz} \right) dy_* dy'_* \right]^2 (1 + y^4) dx dy \right\}^{\frac{1}{2}} ds, \tag{98}
\end{aligned}$$

and we can estimate the last term appearing in brackets in (98) as follows:

$$\begin{aligned}
\{\cdot\} &\leq C \int_{\mathbb{T}^d} \left[\max_{y \in \mathbb{R}} \int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) (1 + y^2) \left| \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} - \frac{\tilde{n}(t, x, y_*) \tilde{n}(t, x, y'_*)}{\int_{\mathbb{R}} \tilde{n}(t, x, z) dz} \right| dy_* dy'_* \right] \\
&\quad \left[\int_{\mathbb{R}} \int \int_{\mathbb{R}^2} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) (1 + y^2) \left| \frac{n(t, x, y_*) n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} - \frac{\tilde{n}(t, x, y_*) \tilde{n}(t, x, y'_*)}{\int_{\mathbb{R}} \tilde{n}(t, x, z) dz} \right| dy_* dy'_* dy \right] dx.
\end{aligned}$$

We notice the following estimates satisfied by the reproduction kernel $\Gamma_{A/2}$:

$$\begin{aligned}
\Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) (1 + y^2) &\leq \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) C \left(1 + \left(y - \frac{y_* + y'_*}{2} \right)^2 + y_*^2 + (y'_*)^2 \right) \\
&\leq C (1 + y_*^2 + (y'_*)^2),
\end{aligned}$$

for any $y \in \mathbb{R}$, as well as

$$\begin{aligned} \int_{\mathbb{R}} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) (1 + y^2) dy &\leq \int_{\mathbb{R}} \Gamma_{A/2} \left(y - \frac{y_* + y'_*}{2} \right) C \left(1 + \left(y - \frac{y_* + y'_*}{2} \right)^2 + y_*^2 + (y'_*)^2 \right) dy \\ &\leq C (1 + y_*^2 + (y'_*)^2), \end{aligned}$$

and then,

$$\begin{aligned} &\iint_{\mathbb{R}^2} C (1 + y_*^2 + (y'_*)^2) \left| \frac{n(t, x, y_*)n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} - \frac{\tilde{n}(t, x, y_*)\tilde{n}(t, x, y'_*)}{\int_{\mathbb{R}} \tilde{n}(t, x, z) dz} \right| dy_* dy'_* \\ &\leq \iint_{\mathbb{R}^2} C (1 + y_*^2 + (y'_*)^2) \left| \frac{(n(t, x, y_*) - \tilde{n}(t, x, y_*))n(t, x, y'_*)}{\int_{\mathbb{R}} n(t, x, z) dz} \right. \\ &\quad \left. + \frac{\tilde{n}(t, x, y_*)n(t, x, y'_*)}{\left(\int_{\mathbb{R}} n(t, x, z) dz\right)\left(\int_{\mathbb{R}} \tilde{n}(t, x, z) dz\right)} \left(\int_{\mathbb{R}} \tilde{n}(t, x, z) - n(t, x, z) dz \right) \right. \\ &\quad \left. + \frac{\tilde{n}(t, x, y_*) (n(t, x, y'_*) - \tilde{n}(t, x, y'_*))}{\int_{\mathbb{R}} \tilde{n}(t, x, z) dz} \right| dy_* dy'_* \\ &\leq C \int_{\mathbb{R}} (1 + y^2) |n(t, x, y) - \tilde{n}(t, x, y)| dy. \end{aligned}$$

We can then continue the estimation of the last term appearing in brackets in (98):

$$\begin{aligned} \{\cdot\} &\leq C \int_{\mathbb{T}^d} \left[\int_{\mathbb{R}} (1 + y^2) |n(t, x, y) - \tilde{n}(t, x, y)| dy \right]^2 dx \\ &\leq C \int_{\mathbb{T}^d} \left[\int_{\mathbb{R}} \frac{1}{1 + y^2} dy \right] \left[\int_{\mathbb{R}} (1 + y^2)^2 |n(t, x, y) - \tilde{n}(t, x, y)|^2 dy \right] dx \\ &\leq C \int_{\mathbb{T}^d} \int_{\mathbb{R}} |n(t, x, y) - \tilde{n}(t, x, y)|^2 (1 + y^4) dy dx \end{aligned}$$

Similarly, we can estimate the last factor of (97):

$$\int_{\mathbb{R}} \int_{\mathbb{T}^d} |n(s, x, y) - \tilde{n}(s, x, y)| dx dy \leq \left(\int_{\mathbb{R}} \int_{\mathbb{T}^d} |n(s, x, y) - \tilde{n}(s, x, y)|^2 (1 + y^4) dx dy \right)^{\frac{1}{2}}.$$

Brought together, these estimates imply

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(t, x, y) - \tilde{n}(t, x, y))^2 (1 + y^4) dx dy + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\nabla_x n(s, x, y) - \nabla_x \tilde{n}(s, x, y)|^2 (1 + y^4) dx dy ds \\ &\leq C \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} (n(s, x, y) - \tilde{n}(s, x, y))^2 (1 + y^4) dx dy ds. \end{aligned}$$

In particular $y(t) := \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\tilde{n}(t, x, y) - n(t, x, y))^2 (1 + y^4) dx dy$ satisfies $y(t) \leq C \int_0^t y(s) ds$, and a Gronwall estimate shows that $y(t) = 0$ for $t \in [0, T]$, that is $n = \tilde{n}$, which concludes the proof of Proposition 2.4.

5.6 Technical estimates for some linear problems

In this section, we derive estimates on solutions of linear parabolic problems that are used in Section 4.3 (proof of Proposition 4.4). We consider the assumption made in Proposition 4.4,

and in particular: y_{opt}, n^0 satisfying Assumption 1, $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1(\mathbb{R}))$ a solution of (SIM) with initial condition n^0 , and \tilde{n}, N, Z defined by (14) and (1), and we assume that $\|Z\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq \kappa$, for some $\tau > 0$ and $\kappa > 0$.

Some linear parabolic equations

For $(s, z, y) \in [0, \tau] \times \mathbb{T}^d \times \mathbb{R}$, let $\phi_{s,z,y}(t, x)$ the solution of (42). Let $(t, x) \mapsto \psi_{s,z,y}(t, x) := \phi_{s,z,y}(t, x)N(t, x)$, which satisfies the following linear parabolic equation:

$$\begin{cases} \partial_t \psi_{s,z,y}(t, x) - \Delta_x \psi_{s,z,y}(t, x) \\ \quad = \left(1 + \frac{A}{2} - N(t, x) - \frac{1}{2}(y - y_{opt}(t, x))^2 - \frac{1}{2} \int (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy\right) \psi_{s,z,y}(t, x), \\ \quad (t, x) \in [s, \tau] \times \mathbb{T}^d, \\ \psi_{s,z,y}(s, x) = N(s, z) \delta_z(x), \quad x \in \mathbb{T}^d. \end{cases}$$

Since the factor on the right hand side of the equation satisfied by $\psi_{s,z,y}$ is bounded (see Proposition 4.1), the existence and uniqueness of $\psi_{s,z,y}$ derives from standard methods (see e.g. Theorem 7.3 and Theorem 7.4 in [29]), and this implies the existence and uniqueness of the solution $\phi_{s,z,y}$ of (42).

We also define $\bar{\phi}_{s,z}(t, x)$ as the solution of:

$$\begin{cases} \partial_t \bar{\phi}_{s,z}(t, x) - \Delta_x \bar{\phi}_{s,z}(t, x) = 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x \bar{\phi}_{s,z}(t, x), \quad (t, x) \in [s, \tau] \times \mathbb{T}^d, \\ \bar{\phi}_{s,z}(s, x) = \delta_z(x), \quad x \in \mathbb{T}^d. \end{cases} \quad (99)$$

Thanks to Section 5.4, the quotient $\frac{\nabla_x N(t, x)}{N(t, x)}$ is continuous, the existence and uniqueness of solution then follows from standard arguments. Alternatively, we can notice that $\bar{\psi}_{s,z}(t, x) := \bar{\phi}_{s,z} N(t, x)$ satisfies

$$\begin{cases} \partial_t \bar{\psi}_{s,z}(t, x) - \Delta_x \bar{\psi}_{s,z}(t, x) \\ \quad = \left(1 + \frac{A}{2} - N(t, x) - \frac{1}{2} \int (y - y_{opt}(t, x))^2 \tilde{n}(t, x, y) dy\right) \bar{\psi}_{s,z}(t, x), \quad (t, x) \in [s, \tau] \times \mathbb{T}^d, \\ \bar{\psi}_{s,z}(s, x) = N(s, z) \delta_z(x), \quad x \in \mathbb{T}^d, \end{cases}$$

which is a linear heat equation, and the unique solution is then given by an explicit Duhamel formula. It can then be used to construct $\bar{\phi}_{s,z}$.

Estimate 1

Thanks to Proposition 4.1, there exists $C_\kappa > 0$ such that $\int |y|^4 \tilde{n}(t, x, y) dy \leq C_\kappa$ for any $(t, x) \in [0, \tau] \times \mathbb{T}^d$, and we can define

$$R = (2C_\kappa)^{1/4}. \quad (100)$$

Then, for any $(t, x) \in [0, \tau] \times \mathbb{T}^d$,

$$\int_{-R}^R \tilde{n}(t, x, y) dy = 1 - \int_{[-R, R]^c} \tilde{n}(t, x, y) dy \geq 1 - \frac{1}{R^4} \int_{[-R, R]^c} |y|^4 \tilde{n}(t, x, y) dy \geq \frac{1}{2}. \quad (101)$$

Let also

$$R' = R + \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}. \quad (102)$$

Then, for any $y \in [-R', R']^c$, we have $-\frac{1}{2}(y - y_{opt}(t, x))^2 \leq \min_{\tilde{y} \in [-R, R]} \left(-\frac{1}{2}(\tilde{y} - y_{opt}(t, x))^2\right)$. The maximum principle (see Remark 2.3) applies to (42) (comparing the case $y \in [-R', R']^c$ to the case where $\tilde{y} \in [-R, R]$), and then, for $y \in [-R', R']^c$,

$$\forall (s, z) \in [0, \tau] \times \mathbb{T}^d, \forall (t, x) \in [s, \tau] \times \mathbb{T}^d, \quad \phi_{s,z,y}(t, x) \leq \min_{|\tilde{y}| \leq R} \phi_{s,z,\tilde{y}}(t, x). \quad (103)$$

Estimate 2

For any $y \in \mathbb{R}$,

$$-\frac{1}{2} \left(y + \operatorname{sgn}(y) \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \right)^2 \leq -\frac{1}{2} (y - y_{opt}(t, x))^2 \leq -\frac{1}{2} \left(y - \operatorname{sgn}(y) \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \right)^2$$

Then $\phi_{s,z,y}(t, x) e^{(t-s)\frac{1}{2}(y + \operatorname{sgn}(y) \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)})^2}$ is a super-solution of (99), and thanks to the comparison principle, $\bar{\phi}_{s,z}(t, x) \leq \phi_{s,z,y}(t, x) e^{(t-s)\frac{1}{2}(y + \operatorname{sgn}(y) \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)})^2}$. The reverse estimate can be obtained similarly, and together, those estimates imply for any $(s, z, y) \in [0, \tau) \times \mathbb{T}^d \times \mathbb{R}$ and $(t, x) \in (s, \min(s+1, \tau)) \times \mathbb{T}^d$,

$$\phi_{s,z,y}(t, x) = \bar{\phi}_{s,z}(t, x) e^{-(t-s)\frac{1}{2}(y + \mathcal{O}(1))^2}, \quad (104)$$

where $|\mathcal{O}(1)| \leq \|y_{opt}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)}$.

Estimate 3

$\bar{\psi}_{s,z}$ satisfies $\bar{\psi}_{s,z}(s, \cdot) = N(s, z)\delta_z$ and

$$\partial_t \bar{\psi}_{s,z}(t, x) - \Delta_x \bar{\psi}_{s,z}(t, x) \leq \left(1 + \frac{A}{2}\right) \bar{\psi}_{s,z}(t, x).$$

Thanks to the comparison principle, $\bar{\psi}_{s,z}(t, x) \leq N(s, z) e^{(1+\frac{A}{2})(t-s)} \Gamma_{t-s}(x-z)$, where the notation $(t, x) \mapsto \Gamma_t(x)$ designates the fundamental solution of the heat equation on \mathbb{T}^d , and $x-y$ stands for the subtraction of x by y on that torus. Since $\bar{\psi}_{s,z}(t, x) = \bar{\phi}_{s,z}(t, x)N(t, x)$, we have, for $\theta \in (0, 1)$,

$$\int_{\mathbb{R}} \bar{\phi}_{s,z}(t, x) |z-x|^\theta dz \leq e^{(1+\frac{A}{2})(t-s)} \int_{\mathbb{R}} \Gamma_{t-s}(x-z) \frac{N(s, z)}{N(t, x)} |z-x|^\theta dz.$$

We can use the estimate (35) to show that $\left| \frac{N(s, z)}{N(t, x)} \right| \leq C_\kappa$, as soon as $1 < s \leq t \leq \min(s+1, \tau)$. If $0 \leq s \leq t \leq 2$, we can use the lower bound (34) and the upper bound $\|N\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)} \leq \max\left(1, \|N(0, \cdot)\|_{L^\infty(\mathbb{T}^d)}\right)$ to obtain a similar estimate. Then,

$$\int_{\mathbb{R}} \bar{\phi}_{s,z}(t, x) |z-x|^\theta dz \leq C_\kappa e^{(1+\frac{A}{2})(t-s)} \int_{\mathbb{R}} \Gamma_{t-s}(x-z) |z-x|^\theta dz \leq C_\kappa (t-s)^{\frac{\theta}{2}}, \quad (105)$$

provided $0 < s \leq t \leq \min(s+1, \tau)$.

Estimate 4

For $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{T}^d$, let $u_{\bar{t}, \bar{x}}$ the solution of the following problem (note that the *time variable* is here reversed compared to usual problems)

$$\begin{cases} -\frac{\partial u_{\bar{t}, \bar{x}}}{\partial \bar{t}}(t, x) - \Delta_x u_{\bar{t}, \bar{x}}(t, x) = -2\nabla_x \cdot \left(\frac{\nabla_x N(t, x)}{N(t, x)} u_{\bar{t}, \bar{x}}(t, x) \right), & (t, x) \in (-\infty, \bar{t}] \times \mathbb{T}^d \\ u_{\bar{t}, \bar{x}}(\bar{t}, x) = \delta_{\bar{x}}(x). \end{cases} \quad (106)$$

This problem is indeed the dual problem of (99) in the sense that $\frac{d}{dt} \int_{\mathbb{R}} \bar{\phi}_{s,z}(t, x) u_{\bar{t}, \bar{x}}(t, x) dx = 0$ for $t \in [s, \bar{t}]$. It follows that for any $s < \bar{t}$ and $z \in \mathbb{T}^d$,

$$\int_{\mathbb{R}} \bar{\phi}_{s,z}(s, x) u_{\bar{t}, \bar{x}}(s, x) dx = \int_{\mathbb{R}} \bar{\phi}_{s,z}(t, x) u_{\bar{t}, \bar{x}}(\bar{t}, x) dx,$$

which, given the initial conditions specified in (99) and (106) (note that the reversion of time in this dual problem implies that the initial condition holds for the largest time considered, ie $t = \bar{t}$), is equivalent to

$$u_{\bar{t}, \bar{x}}(s, z) = \bar{\phi}_{s, z}(\bar{t}, \bar{x}).$$

The divergence form of (106) implies that $\int_{\mathbb{R}} u_{\bar{t}, \bar{x}}(s, z) dz = \int_{\mathbb{R}} u_{\bar{t}, \bar{x}}(\bar{t}, z) dz = 1$, and then for any $\bar{t} > s$ and $\bar{x} \in \mathbb{T}^d$,

$$\int_{\mathbb{R}} \bar{\phi}_{s, z}(\bar{t}, \bar{x}) dz = 1. \quad (107)$$

5.7 Existence, uniqueness and stability of solutions of (KBM)

In this section, we first show that Proposition 4.5 implies the existence of solutions to (KBM). Note that there are probably more direct proofs of this result. The stability of solutions of (KBM) is then proven Proposition 5.7, which also implies the uniqueness of solutions. Note that this stability result is important to show (12)

We consider the assumptions made in Proposition 4.5, denote by n_γ the solution of (SIM) and N_γ, Z_γ defined by (1). Thanks to Proposition 4.5, Z_γ is uniformly bounded for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$ and $\gamma > \bar{\gamma}$. We may then apply Proposition 4.3 to show that N_γ, Z_γ are uniformly θ -Lipschitz continuous. Thanks to Ascoli, they converge to a limit (\bar{N}, \bar{Z}) up to an extraction. We consider this subsequence from now on (the uniqueness of solutions proven in Proposition 2.4 will show that the convergence holds without taking a subsequence). This limit is then itself θ -Lipschitz continuous. We can define $Y_\gamma := N_\gamma Z_\gamma, \bar{Y} := \bar{N} \bar{Z}$, and note that $\|N_\gamma\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} + \|Y_\gamma\|_{L^\infty([0, \tau] \times \mathbb{T}^d)} \leq C$. Moreover, (N_γ, Y_γ) satisfies a system of equations where only the 0th order terms are non-linear (note that we already considered Y_γ in Section 5.4, see (94)):

$$\begin{cases} \partial_t N_\gamma(t, x) - \Delta_x N_\gamma(t, x) = \left(1 - \frac{1}{2} (Z_\gamma(t, x) - y_{opt}(t, x))^2 - N_\gamma(t, x) + \varphi_{N, \gamma}(t, x)\right) N_\gamma(t, x), \\ \partial_t Y_\gamma(t, x) - \Delta_x Y_\gamma(t, x) = \left(1 - \frac{1}{2} (Z_\gamma(t, x) - y_{opt}(t, x))^2 - N_\gamma(t, x) + \varphi_{N, \gamma}(t, x)\right) Y_\gamma(t, x) \\ \quad + (-A(Z_\gamma(t, x) - y_{opt}(t, x)) + \varphi_{Z, \gamma}(t, x)) N_\gamma(t, x), \end{cases} \quad (108)$$

Applying Theorem 7.22 in [37] to these equations show that $N_\gamma, Y_\gamma \in W_{d+2}^{2,1}(\mathbb{R}_+ \times \mathbb{T}^d)$ (using the notations of [37]) with a norm that is uniform in $\gamma > \bar{\gamma}$. Since \bar{N} and \bar{Z} are bounded, the factor of N_γ on the right hand side of the first equation in (108) is uniformly bounded, and then

$$N_\gamma(t, x) \geq C e^{-t/C}, \quad \bar{N}(t, x) \geq C e^{-t/C}, \quad (109)$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$. The second inequality being a consequence of the uniform bound on N_γ and the convergence of N_γ to \bar{N} when $\gamma \rightarrow \infty$. This lower bound on \bar{N} can be used in (108) to show that \bar{N} and \bar{Y} satisfy, in a weak sense (integrated against smooth test functions),

$$\begin{aligned} \partial_t \bar{N}(t, x) - \Delta_x \bar{N}(t, x) &= \left(1 - \frac{1}{2} (\bar{Z}(t, x) - y_{opt}(t, x))^2 - \bar{N}(t, x)\right) \bar{N}(t, x), \\ \partial_t \bar{Y}(t, x) - \Delta_x \bar{Y}(t, x) &= \left(1 - \frac{1}{2 \bar{N}(t, x)^2} (\bar{Y}(t, x) - y_{opt}(t, x) \bar{N}(t, x))^2 - \bar{N}(t, x)\right) \bar{Y}(t, x) \\ &\quad - A(\bar{Y}(t, x) - y_{opt}(t, x) \bar{N}(t, x)). \end{aligned} \quad (110)$$

Since $\bar{N}, \bar{Y} \in W_{d+2}^{2,1}(\mathbb{R}_+ \times \mathbb{T}^d)$ the right hand side of these equations belong to $L^{d+2}([0, T] \times \mathbb{T}^d)$ for any $T > 0$. Applying Theorem 7.22 in [37] and Morrey's inequality, we have that \bar{N}, \bar{Y} are C^1 in t and C^2 in x , and (110) is satisfied as an equality between continuous functions. Coming back to (KBM) with $\bar{Z} = \bar{Y}/\bar{N}$ shows that (\bar{N}, \bar{Z}) is a solution of (KBM). This proves the existence of a solution to (KBM) under Assumption 1.

We are now interested in the uniqueness of solutions of (KBM), and use the notation $W_{d+2}^{1,2}(\mathbb{R}_+ \times \mathbb{T}^d)$ of [37] to write the following result:

Proposition 5.7. *Let $y_{opt} \in C^1(\mathbb{R}_+ \times \mathbb{T}^d)$, $N^0, Z^0 \in W^{2,\infty}(\mathbb{T}^d)$, such that $N^0 > 0$.*

Let $\varphi_N, \varphi_Z \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)$ and $N, Z, \bar{N}, \bar{Z} \in W_{d+2}^{1,2}$ that are C^1 in t and C^2 in x , with $N(t, x) > 0, \bar{N}(t, x) > 0$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$. We assume that (N, Z) satisfies

$$\begin{cases} \partial_t N(t, x) - \Delta_x N(t, x) = \left(1 - \frac{1}{2} (Z(t, x) - y_{opt}(t, x))^2 - N(t, x) + \varphi_N(t, x)\right) N(t, x), \\ \partial_t Z(t, x) - \Delta_x Z(t, x) = 2 \frac{\nabla_x N(t, x)}{N(t, x)} \cdot \nabla_x Z(t, x) - A(Z(t, x) - y_{opt}(t, x)) + \varphi_Z(t, x), \end{cases} \quad (111)$$

while (\bar{N}, \bar{Z}) satisfies (KBM), as an equality between continuous functions in both cases, and with the same initial condition $(N(0, \cdot), Z(0, \cdot)) = (\bar{N}(0, \cdot), \bar{Z}(0, \cdot)) = (N^0, Z^0)$. If

$$|\varphi_N(t, x)| + \varphi_Z(t, x) \leq \frac{\bar{C}}{\gamma^\theta} + \bar{C}1_{[0, 1/\gamma^\theta]}(t),$$

for some $\bar{C} > 0$ and $\gamma > 1$. For any $T > 0$, there is $\hat{C} > 0$ independent from $\gamma > 0$ such that

$$\|(N - \bar{N})(t, \cdot)\|_{L^\infty([0, T] \times \mathbb{T}^d)} + \|(Z - \bar{Z})(t, \cdot)\|_{L^\infty([0, T] \times \mathbb{T}^d)} \leq \frac{\hat{C}}{\gamma^\theta}. \quad (112)$$

This proposition shows the uniqueness of solutions of (KBM) under the assumptions above. It also shows the convergence of solutions (N, Z) of (10) to the solution (\bar{N}, \bar{Z}) of (KBM) when $\gamma \rightarrow \infty$, see (12).

Proof of Proposition 5.7. Since $N, Z, \bar{N}, \bar{Z} \in W_{d+2}^{1,2}$, it is possible to use the comparison principle to show explicit upper bounds on $|Z|, |\bar{Z}|$ and N that are uniform in $\gamma > 1$. We can use these to prove a lower bound on N (see (109)) that is also uniform in $\gamma > 1$.

We can define $Y = ZN$ and $\bar{Y} = \bar{Z}\bar{N}$ and write the equations satisfied by Y, \bar{Y} , see (108), (110). Note that in these equation, the non-linear terms do not involve any derivative of the functions. Since N, Y, \bar{N}, \bar{Y} are bounded functions and N, \bar{N} are bounded from below independently from $\gamma > 1$ on $[0, T] \times \mathbb{T}^d$, we can then estimate

$$\begin{aligned} \partial_t(N - \bar{N})(t, x) - \Delta_x(N - \bar{N})(t, x) &= \mathcal{O}(1)(N - \bar{N})(t, x) + \mathcal{O}(1)(Y - \bar{Y})(t, x) + \mathcal{O}(1)\varphi_N(t, x), \\ \partial_t(Y - \bar{Y})(t, x) - \Delta_x(Y - \bar{Y})(t, x) &= \mathcal{O}(1)(N - \bar{N})(t, x) + \mathcal{O}(1)(Y - \bar{Y})(t, x) + \mathcal{O}(1)\varphi_N(t, x) \\ &\quad + \mathcal{O}(1)\varphi_Z(t, x), \end{aligned} \quad (113)$$

where the notation $\mathcal{O}(1)$ denotes functions $[0, T] \times \mathbb{R}$ with an L^∞ norm that is independent from $\gamma > 0$. Let $C > 0$ such that the coefficients on the right hand side of the system above are dominated by $C > 0$, ie $|\mathcal{O}(1)| \leq C$. We consider the following differential equation:

$$y'(t) = 2Cy + C \left(\frac{\bar{C}}{\gamma^\theta} + 2\bar{C}1_{[0, 1/\gamma^\theta]}(t) \right),$$

together with $y(0) = \varepsilon > 0$. The solution to this ODE is

$$\begin{aligned} y(t) &= \varepsilon e^{2Ct} + C e^{2Ct} \int_0^t e^{-2Cs} \left(\frac{\bar{C}}{\gamma^\theta} + 2\bar{C}1_{[0,1/\gamma^\theta]}(t) \right) ds \leq \varepsilon e^{2Ct} + \frac{\bar{C}e^{2Ct}}{2\gamma^\theta} + 2\frac{C\bar{C}}{\gamma^\theta} e^{2Ct} \\ &\leq \varepsilon e^{2Ct} + \frac{\hat{C}}{\gamma^\theta} e^{2Ct}, \end{aligned}$$

for $\gamma > 1$ and $\hat{C} := \bar{C}/2 + 2C\bar{C}$. We can now define $\hat{N}(t, x) := y(t)$ and $\hat{Z}(t, x) := y(t)$, that satisfy

$$\partial_t \hat{N}(t, x) - \Delta_x \hat{N}(t, x) = C\hat{N} + C\hat{Y} + C \left(\frac{\bar{C}}{\gamma^\theta} + 2\bar{C}1_{[0,1/\gamma^\theta]}(t) \right) > C\hat{N}(t, x) + C\hat{Y}(t, x) + C\varphi_N(t, x), \quad (114)$$

as well as $\hat{N}(0, x) > (N - \bar{N})(0, x) = 0$ for $x \in \mathbb{T}^d$. Similarly, $-\hat{N}$ satisfies $-\hat{N}(0, x) < (N - \bar{N})(0, x) = 0$ and

$$\partial_t(-\hat{N})(t, x) - \Delta_x(-\hat{N})(t, x) < C(-\hat{N})(t, x) + C(-\hat{Y})(t, x) - C\varphi_N(t, x). \quad (115)$$

The same estimates can be obtained on \hat{Z} , replacing φ_N by φ_Z . We now use a contradiction argument. At $t = 0$, we have $|(N - \bar{N})(0, x)| = 0 < \hat{N}(0, x)$ and $|(Z - \bar{Z})(0, x)| = 0 < \hat{Z}(0, x)$ for $x \in \mathbb{T}^d$. Since N, Y, \bar{N}, \bar{Y} are C^1 in t and C^2 in x , the first time when $\max(|(N - \bar{N})(t, x)|, |(Z - \bar{Z})(t, x)|) = y(t)$ leads to a contradiction since the inequalities in (114) and (115) (and the similar equations on $\hat{Z}, -\hat{Z}$) are strict while $(N - \bar{N}), (Z - \bar{Z})$ satisfy (113). Thanks to this contradiction argument, $\max(|(N - \bar{N})(t, x)|, |(Z - \bar{Z})(t, x)|) \leq y(t)$ for $t \geq 0$. since this holds for any $\varepsilon > 0$, we have (112) if we redefine \hat{C} as $\hat{C}e^{2CT}$. \square

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