Willems' Lemma Reformulations: Which Operators preserve LTI System Behavior?

Alexandre Faye-Bedrin, Stanislav Aranovskiy, Paul Chauchat, Romain Bourdais

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Alexandre Faye-Bédrin[1], Stanislav Aranovskiy[1], Paul Chauchat[2], and Romain Bourdais[1]

***Abstract*— In the behavioral approach, dynamical systems are abstracted as sets of trajectories. This approach gave birth to Willems' Fundamental Lemma, which has sparked significant interest in recent years. Indeed, the Lemma has uses in data-driven control: it provides a simple data-driven representation of any LTI system based on a single input-output record. Reformulations of the Lemma have been proposed in the literature, for instance, using frequency-domain data, each time with a new and specific proof. In this note, we show that all reformulations are necessarily based on linear shift-invariant transformations, which have the fundamental property that they preserve the trajectory space of all LTI systems.**

I. Introduction

The behavioral approach to system theory [1] defines a dynamical system as a set of trajectories called behavior. This abstract perspective allows to separate the representation of systems (e.g., input-output transfer function, state-space representation, convolution) from the study of their fundamental properties (e.g., linearity, complexity, dissipativity). Extensive literature revolves around linear time-invariant (LTI) systems, which are defined as shift-invariant subspaces, as detailed in Section II-A.

A result from the behavioral approach has recently gained interest: the Fundamental Lemma [2] (which we recall in Theorem 1 in Section II-B) allows for the behavior (set of possible trajectories) of any controllable LTI system to be described solely with raw records of input-output trajectories organized in a Hankel matrix. This simple and powerful data-driven representation is at the core of a model-free predictive control method [3], referred to as DeePC (Data-EnablEd Predictive Control). Alternative formulations of the Lemma have been developed in the literature, using transformations of input-output sequences instead of raw data. Authors in [4] propose two formulations: in the first one, the data matrix is based on cross-correlation of input-output signals, and the second one uses cross-spectrum (i.e., the Fourier transform of cross-correlation). In [5], [6], the authors propose a formulation where the data matrix contains frequency-domain data. In both cases, authors prove the correctness of their formulations by applying a similar procedure as for the multiple dataset case [7]. The similarity in the proofs is a crucial observation, as it suggests a fundamental property of the data transformations: they preserve the data structure induced by system dynamics.

[1] IETR, CentraleSupélec, Rennes, France surname.name@centralesupelec.fr
[2] Aix-Marseille Univ, CNRS, LIS, Marseille, France paul.chauchat@lis-lab.fr

This note aims to generalize the aforementioned reformulations: what common kind of data transformation are they based on? To this objective, we address the following question: what is the set of transformations that preserve the trajectory space of any LTI system? Characterizing these transformations will clarify the relationship between the Lemma and its reformulations. To this end, we state in Theorem 2 of this note that the set of data transformations that preserve the behavior of all LTI systems is the set of linear shift-invariant (LSI) operators. We also show the connection with the Fourier transform.

The paper has the following structure. First, Section II presents some preliminaries. Next, we show our main results in Section III and study its connection to a family of operators related to the Fourier transform in Section IV. A perspective on continuous-time LTI systems is then presented in Section V, and some applications of our results are given in Section VI, which include the three reformulations of the Fundamental Lemma (with frequency domain data, cross-correlation, and cross-spectrum). Finally, the conclusion is in Section VII.

Notation

- The operator $\mathrm{col}\,(\cdot)$ stacks up its vector arguments.
- A set of numbers excluding 0 is denoted with $(\cdot)^*$, e.g. $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
- The space of sequences (discrete-time signals) on time axis $\mathbb{Z}$ with values in $\mathbb{R}^q$ is denoted $(\mathbb{R}^q)^{\mathbb{Z}} = \{\mathbb{Z} \to \mathbb{R}^q\}$. For $q = 1$, we write $\mathbb{R}^{\mathbb{Z}}$ or $\mathbb{C}^{\mathbb{Z}}$.
- The set of discrete-time LTI systems of order n with m inputs and $q - m$ outputs is $\mathbb{B}^{q,n}_m$. When the input-output partition is not specified, we write $\mathbb{B}^{q,n}$, and $\mathbb{B}^{q,n}_m \subseteq \mathbb{B}^{q,n}$. When not relevant to the discussion, n is not specified, e.g., $\mathbb{B}^q$ or $\mathbb{B}^q_m$.
- For a sequence s with $s(t) \in \mathbb{R}^q$ or $s(t) \in \mathbb{C}$ and $a \leq b$ we use the following notation for a stacked window:
$$s_{[a,b]} = \mathrm{col}\,(s(a), s(a+1), \dots s(b))\,.$$
- For a finite sequence $\{s(t)\}_{t=1}^T$ with $s(t) \in \mathbb{R}^q$, we denote the Hankel matrix $H_L(s) \in \mathbb{R}^{qL \times (T-L+1)}$ as
$$H_L(s) = \begin{bmatrix} s_{[1,L]} & s_{[2,L+1]} & \dots & s_{[T-L+1,T]} \end{bmatrix}.$$
- The range space of a matrix or linear operator A is denoted $\mathrm{im}(A)$, and its null space is $\ker(A)$.
- Application of operator A to s is denoted $A[s]$, or As when there is no ambiguity.
- The shift operator σ is defined, with any sequence s:
$$\forall t, \sigma[s](t) = s(t+1)$$

II. Preliminaries

A. LTI systems in the behavioral framework

In the behavioral framework (see, e.g., [1]), a discrete-time LTI system (or *behavior*) $\mathcal{B} \in \mathbb{B}^q$ is a shift-invariant subspace of the space of signals $(\mathbb{R}^q)^{\mathbb{Z}}$. In other words, it is a space of trajectories with values in $\mathbb{R}^q$, with the property that $w \in \mathcal{B} \iff \sigma w \in \mathcal{B}$, where $w \in \mathcal{B}$ means that w is a trajectory of $\mathcal{B}$. Such a system $\mathcal{B}$ can be described with the so-called kernel representation

$$\mathcal{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma)w = 0 \right\} = \ker(R(\sigma))$$

where $R(\xi)$ is a matrix polynomial with indeterminate ξ. An LTI system with input-output partitioning $\mathcal{B} \in \mathbb{B}^q_m$ can also be described as

$$\mathcal{B} = \left\{ u \in (\mathbb{R}^m)^{\mathbb{Z}}, y \in (\mathbb{R}^{q-m})^{\mathbb{Z}} \mid R_y(\sigma)y = R_u(\sigma)u \right\}.$$

The restricted behavior $\mathcal{B}|_L$ of a system $\mathcal{B} \in \mathbb{B}^q$ is defined as the set of trajectories with length L of $\mathcal{B}$

$$\mathcal{B}|_L = \left\{ w_{[1,L]} \mid w \in \mathcal{B} \right\}$$

or with an input-output partition

$$\mathcal{B}|_L = \left\{ \begin{bmatrix} u_{[1,L]} \\ y_{[1,L]} \end{bmatrix} \mid (u,y) \in \mathcal{B} \right\}.$$

B. Willems' Fundamental Lemma

We recall Willems' Fundamental Lemma extended for multiple datasets [7], which mobilizes the idea of collective persistence of excitation.

Definition 1: A collection of sequences $s = \{s_i = \{s_k(t)\}_{t=1}^{T_k}\}_{k=1}^{N}$ with $s_k(t) \in \mathbb{R}^q$ is said to be collectively persistently exciting of order M if

$$\operatorname{rank}\left(\begin{bmatrix} H_M(s_1) & H_M(s_2) & \dots & H_M(s_N) \end{bmatrix} \right) = qM.$$

We then write $s \in \mathrm{CPE}(M)$.

Let $\mathcal{B} \in \mathbb{B}^{q,n}_m$ a controllable[1] LTI system of order n. Let $\left(u^D = \left\{ \{u_k^D(t)\}_{t=1}^{T_k} \right\}_{k=1}^{N}, \; y^D = \left\{ \{y_k^D(t)\}_{t=1}^{T_k} \right\}_{k=1}^{N} \right)$ be a collection of finite-length input-output sequences of $\mathcal{B}$.

Theorem 1 ([7]): If $u^D \in \mathrm{CPE}(L+n)$ then

$$\mathcal{B}|_L = \operatorname{im} \begin{bmatrix} H_L(u_1^D) & \dots & H_L(u_N^D) \\ H_L(y_1^D) & \dots & H_L(y_N^D) \end{bmatrix}. \tag{1}$$

Remark 1: Theorem 1 has been extended to the case of uncontrollable systems and under looser conditions [8]. It would be straightforward to connect this extended theorem to our results, however the more complicated formulation would hinder clarity while not adding significant value. The same can be said for other kinds of data matrices or different definitions of PE, such as in [9].

III. Operators and LTI systems

In this section, we state our main result: the characterization of operators that preserve the behavior of all LTI systems.

[1]Controllability has a specific definition in the behavioral framework [1], that it is equivalent to the usual definition for state-space systems.

A. Characterization of behavior-preserving operators

Let $\mathbb{A} \subseteq \{\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}\}$ be the set of linear shift-invariant (LSI) operators on real-valued sequences. For $A \in \mathbb{A}$, the linearity implies that for any sequences $x, y \in \mathbb{R}^{\mathbb{Z}}$ and any scalars $a, b \in \mathbb{R}$ we have

$$A(ax + by) = aA(x) + bA(y),$$

and the shift-invariance implies that A commutes with the shift operator, i.e. for any $x \in \operatorname{dom} A$

$$A[\sigma x] = \sigma[Ax].$$

Our main result is the following theorem, where we highlight a fundamental property of LTI systems.

Theorem 2: $\mathbb{A}$ is the set of operators that preserve the behavior of all LTI systems:

$$\mathbb{A} = \{A | \forall q \in \mathbb{N}^*, \forall \mathcal{B} \in \mathbb{B}^q,\, w \in \mathcal{B} \Rightarrow Aw \in \mathcal{B}\}$$

where Aw is the component-wise application of A to w.

Proof: First, we show that all operators in $\mathbb{A}$ preserve all LTI behaviors. Take $w \in \mathcal{B}$:

$$R(\sigma)[Aw] = A[R(\sigma)w] = 0 \Rightarrow Aw \in \mathcal{B}$$

due to linearity and shift-invariance of A.

Now, we prove that all LTI behavior-preserving operators belong to $\mathbb{A}$. To this end, we show that any operator A such that

$$\forall q \in \mathbb{N}^*,\, \forall \mathcal{B} \in \mathbb{B}^q,\, (w \in \mathcal{B} \Rightarrow Aw \in \mathcal{B}) \tag{2}$$

is both (a) linear and (b) shift-invariant, and thus $A \in \mathbb{A}$.

(a) *Linearity:* If A is not linear, then there exist constants $(a,b) \in \mathbb{R}$ and sequences $(\bar{u}_1, \bar{u}_2) \in \mathbb{R}^{\mathbb{Z}}$ such that

$$A[a\bar{u}_1 + b\bar{u}_2] \neq aA\bar{u}_1 + bA\bar{u}_2.$$

With such a and b, let $\mathcal{B} \in \mathbb{B}^3_2$ defined by $y = au_1 + bu_2$. Then, with $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$,

$$(Au, Ay) \neq (Au, aAu_1 + bAu_2) \text{ i.e. } (Au, Ay) \notin \mathcal{B}$$

contradicting (2), so A must be linear.

(b) *Shift-invariance:* If A is not shift-invariant, then there exists a sequence $\bar{y} \in \mathbb{R}^{\mathbb{Z}}$ such that

$$A[\sigma\bar{y}] \neq \sigma[A\bar{y}].$$

Let $\mathcal{B} \in \mathbb{B}^2_1$, defined by $\sigma y = u$. Then, with $u = \sigma\bar{y}$, we get $y = \bar{y}$ and

$$(Au, Ay) \neq (\sigma Ay, Ay) \text{ i.e. } (Au, Ay) \notin \mathcal{B}$$

contradicting (2), so A must be shift-invariant.

■

Theorem 2 gives the whole set of operators that preserve the trajectory space of all LTI systems. It is tempting to conclude that the three reformulations of Willems' lemma cited in Section I (using cross-correlation, cross-spectrum, and frequency-domain data) must be based on LSI operators. However, these reformulations are about *restricted* behaviors: we still need to clarify the link between restricted and unrestricted behaviors.

B. Operators that preserve restricted behaviors

We introduce a new set of operators that, when applied to trajectories of any given LTI system, yield restricted trajectories of the same system. We then look at the relationship between these new operators and behavior-preserving operators. Let $L \in \mathbb{N}^*$, we denote $\mathbb{A}|_L \subseteq \left\{\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^L\right\}$ the set of operators defined such that, if $\bar{A} \in \mathbb{A}|_L$, for all $q \in \mathbb{N}^*$ and any LTI system $\mathcal{B} \in \mathbb{B}^q$

$$w \in \mathcal{B} \Rightarrow \bar{A}w = \begin{bmatrix} \bar{A}[w](1) \\ \vdots \\ \bar{A}[w](L) \end{bmatrix} \in \mathcal{B}|_L.$$

Now we formulate the question: what is the link between $\mathbb{A}|_L$ and $\mathbb{A}$? The answer is given in the following theorem.

Theorem 3: The set of operators $\mathbb{A}|_L$ that preserve restricted behaviors is equivalent to the set of LSI operators $\mathbb{A}$ in the sense that

$$\mathbb{A}|_L = \left\{(s \mapsto A[s]_{[1,L]}) \,|A \in \mathbb{A}\right\}. \tag{3}$$

Proof: First, it is obvious that if we take $A \in \mathbb{A}$

$$w \in \mathcal{B} \Rightarrow A[w]_{[1,L]} \in \mathcal{B}|_L$$

which implies the right inclusion $\supseteq$ in (3). We will show that each $\bar{A} \in \mathbb{A}|_L$ has a corresponding extension $A \in \mathbb{A}$. Take $\bar{A} \in \mathbb{A}|_L$. Due to the properties of $\bar{A}$ and $\mathcal{B}$, we have

$$w \in \mathcal{B} \;\Rightarrow\; \forall t \in \mathbb{Z}, \sigma^t w \in \mathcal{B} \;\Rightarrow\; \forall t \in \mathbb{Z}, \bar{A}[\sigma^t w] \in \mathcal{B}|_L.$$

Now define the operator $A : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ as

$$\forall t \in \mathbb{Z}, \forall s \in \mathbb{R}^{\mathbb{Z}}, A[s](t) = \bar{A}[\sigma^{t-1}s](1).$$

By this definition, A is shift-invariant, and it is easy to check that

$$\forall s \in \mathbb{R}^{\mathbb{Z}}, \bar{A}[s] = A[s]_{[1,L]}.$$

Finally, we prove that A must be linear: to this end, we show that $\bar{A}$ is linear. Let $\mathcal{B} \in \mathbb{B}_2^3$, defined by $y = au_1 + bu_2$. If $\bar{A}$ is not linear, then there exists constants $(a, b) \in \mathbb{R}$ and sequences $(\bar{u}_1, \bar{u}_2) \in \mathbb{R}^{\mathbb{Z}}$ such that

$$\bar{A}[a\bar{u}_1 + b\bar{u}_2] \neq a\bar{A}\bar{u}_1 + b\bar{A}\bar{u}_2.$$

With such a and b, let $\mathcal{B} \in \mathbb{B}_2^3$ defined by $y = au_1 + bu_2$. Then, if we take $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$

$$\begin{bmatrix} \bar{A}u \\ \bar{A}y \end{bmatrix} \neq \begin{bmatrix} \bar{A}u \\ a\bar{A}u_1 + b\bar{A}u_2 \end{bmatrix} \text{ i.e. } \begin{bmatrix} \bar{A}u \\ \bar{A}y \end{bmatrix} \notin \mathcal{B}|_L$$

contradicting $A \in \mathbb{A}|_L$ with $\mathbb{A}|_L$ defined in (3), so $\bar{A}$ must be linear. It follows that A is also linear, so we have $A \in \mathbb{A}$. Hence, we have the left inclusion $\subseteq$ in (3).

■

Theorem 3 makes clear that for each reformulation of the Fundamental Lemma cited in Section I, there are corresponding LSI operators. This means that there is no need for a separate, specific theoretical study of each reformulation because they are natural declinations of Theorem 1 that arise from the fundamental properties of LSI operators.

IV. Geometric sequences and the Fundamental Lemma

In this section, we introduce a set of LSI operators that are closely related to the Fourier transform, and we show how they interact with the Fundamental Lemma.

Let us introduce a family of linear forms $\mathbb{G}^z \subseteq \{\mathbb{C}^{\mathbb{Z}} \to \mathbb{C}\}$ that maps trajectories to complex numbers, with the special property that any $G \in \mathbb{G}^z$ satisfies

$$\forall s \in \mathbb{C}^{\mathbb{Z}}, G[\sigma s] = zGs.$$

where $z \in \mathbb{C}^*$. Note that this is a partial characterization of the Z-transform, which is defined as

$$\forall s \in \mathbb{C}^{\mathbb{Z}}, \mathcal{Z}[s](z) = \sum_{k \in \mathbb{Z}} z^{-k} s(k).$$

By definition, we have $(s \mapsto \mathcal{Z}[s](z)) \in \mathbb{G}^z$. Then, an example of such a linear form is the discrete-time Fourier transform discussed in Section VI-B.

Let $G \in \mathbb{G}^z$ and $\bar{z} \in \mathbb{C}^{\mathbb{Z}}$ a geometric sequence of common ratio z (i.e. $\sigma\bar{z} = z\bar{z}$). Consider the operators

$$\mathcal{R} : s \mapsto \Re(\bar{z}Gs) \quad \text{and} \quad \mathcal{I} : s \mapsto \Im(\bar{z}Gs)$$

where $\Re$ and $\Im$ denote, respectively, the real and imaginary parts of their argument. Note that G applies to w element-wise, hence $Gw \in \mathbb{C}^q$, and we read $\bar{z}Gw$ as such: at any time t, the k-th component of $(\bar{z}Gw)(t)$ equals the k-component of $\bar{z}(t)Gw$.

Lemma 1: The operators $\mathcal{R}$ and $\mathcal{I}$ are linear and shift-invariant.

Proof: Take $s \in \mathbb{C}^{\mathbb{Z}}$, then for the real part

$$\sigma\left[\Re\left(\bar{z}Gs\right)\right] = \Re\left(\sigma[\bar{z}Gs]\right) = \Re\left(z\bar{z}Gs\right) = \Re\left(\bar{z}G[\sigma s]\right)$$

and the same holds for the imaginary part $\Im(\bar{z}Gs)$. Hence the operators $\mathcal{R}$ and $\mathcal{I}$ belong to $\mathbb{A}$. ■

Let N scalars $\{z_k \in \mathbb{C}^*\}_{k=1}^N$, their corresponding linear forms $\{G_k \in \mathbb{G}^{z_k}\}_{k=1}^N$ and geometric sequences $\{\bar{z}_k \in \mathbb{C}^{\mathbb{Z}} : \bar{z}_k(t) = z_k^{t-1}\}_{k=1}^N$. Let $\left(u = \{u_k\}_{k=1}^N, y = \{y_k\}_{k=1}^N\right)$ be a collection of N input-output sequences of a controllable system $\mathcal{B} \in \mathbb{B}_m^{q,n}$, i.e., for any k, $u_k \in (\mathbb{R}^m)^{\mathbb{Z}}$. We denote for a sequence $s_k \in (\mathbb{R}^r)^{\mathbb{Z}}$ (standing for u_k or y_k)

$$\xi_k^s = \bar{z}_k G_k s_k \in (\mathbb{C}^r)^{\mathbb{Z}}$$

and

$$\mathscr{H}_L(s) = \begin{bmatrix} \xi_{1,[1,L]}^s & \cdots & \xi_{N,[1,L]}^s \end{bmatrix} \in \mathbb{C}^{rL \times N}.$$

The following theorem is a consequence of Theorem 1.

Theorem 4: Suppose that

$$\operatorname{rank}\begin{bmatrix} \Re(\mathscr{H}_{L+n}(u)) & \Im(\mathscr{H}_{L+n}(u)) \end{bmatrix} = (L+n)m \tag{4}$$

then

$$\mathcal{B}|_L = \operatorname{im}\begin{bmatrix} \Re(\mathscr{H}_L(u)) & \Im(\mathscr{H}_L(u)) \\ \Re(\mathscr{H}_L(y)) & \Im(\mathscr{H}_L(y)) \end{bmatrix}. \tag{5}$$

Proof: Condition (4) implies that $\{\Re(\xi_{k,[1,L+n]}^u)\}_{k=1}^N \cup \{\Im(\xi_{k,[1,L+n]}^u)\}_{k=1}^N \in \mathrm{CPE}(L+n)$, see Definition 1. Then according to Theorem 1

$$\mathcal{B}|_L = \operatorname{im}\begin{bmatrix} \mathcal{A}_1 u & \cdots & \mathcal{A}_N u \\ \mathcal{A}_1 y & \cdots & \mathcal{A}_N y \end{bmatrix} \tag{6}$$

where $\mathcal{A}_k$ is defined, with s standing for u or y, as

$$\mathcal{A}_k s = \begin{bmatrix} \Re\left(H_L\left(\xi^s_{k,[1,L+n]}\right)\right) & \Im\left(H_L\left(\xi^s_{k,[1,L+n]}\right)\right) \end{bmatrix}.$$

Then, note that for all k and any sequence $s \in (\mathbb{R}^r)^{\mathbb{Z}}$:

$$H_L\left(\xi^s_{k,[1,T]}\right) = \begin{bmatrix} \xi^s_k(1) & z_k\xi^s_k(1) & \cdots & z_k^{T-L}\xi^s_k(1) \\ \xi^s_k(2) & z_k\xi^s_k(2) & \cdots & z_k^{T-L}\xi^s_k(2) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^s_k(L) & z_k\xi^s_k(L) & \cdots & z_k^{T-L}\xi^s_k(L) \end{bmatrix}$$
$$= \begin{bmatrix} \xi^s_k(1) \\ \xi^s_k(2) \\ \vdots \\ \xi^s_k(L) \end{bmatrix} \begin{bmatrix} 1 & z_k & \cdots & z_k^{T-L} \end{bmatrix}$$

and so

$$\operatorname{im}\begin{bmatrix} \Re\left(H_L\left(\xi^s_{k,[1,T]}\right)\right) & \Im\left(H_L\left(\xi^s_{k,[1,T]}\right)\right) \end{bmatrix} = \operatorname{im}\begin{bmatrix} \Re\left(\xi^s_{k,[1,L]}\right) & \Im\left(\xi^s_{k,[1,L]}\right) \end{bmatrix}.$$

Hence, (6) is equivalent to (5). ∎

V. Continuous time LTI systems

A. In the behavioral framework

Up to this point, we have focused exclusively on discrete-time (DT) LTI systems, but it is possible to transpose some results to the continuous-time (CT) setup. We write $\bar{\mathbb{B}}^q \subseteq (\mathbb{R}^q)^{\mathbb{R}}$ the set of q-variate differential CT LTI systems. In the behavioral framework, such a system $\mathcal{B} \in \bar{\mathbb{B}}^q$ can be described with a kernel representation (see [10])

$$\mathcal{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{R}} | R\left(\frac{d}{dt}\right) w = 0 \right\}$$

where R is, as in the DT case, a matrix polynomial. It is worth mentioning that other, larger classes of CT LTI systems exist, such as systems with delays [11] or leveraging the idea of distributional behavior [12]. For the sake of conciseness, we only consider differential CT systems, but we infer that our approach would yield analogous results.

Compared to the DT setup, the signal space changes from $(\mathbb{R}^q)^{\mathbb{Z}}$ to $(\mathbb{R}^q)^{\mathbb{R}}$ (the time axis becomes the set of real numbers), and the shift operator σ is substituted with the time derivative $\frac{d}{dt}$. We elude on the precise definition of CT signals: we only assume they are compatible with the operations we use, so our propositions are not impacted. Depending on the context, signals may be considered as (infinitely) differentiable functions with respect to time, but they can also be defined as distributions (to allow for discontinuities), or (locally) square integrable functions. Some discussion can be found in [12] and references therein.

There exist versions of the Fundamental Lemma for CT systems [13], [14], but a detailed discussion on the differences between the DT and the several CT cases is out of scope for this paper.

B. Behavior-preserving operators, complex exponentials

We can draw similar results to those of Theorem 2 and Lemma 1 in the CT setup.

For the first result, we introduce the CT equivalent of DT LSI operators. Let $\bar{\mathbb{A}} \subseteq \{\mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}\}$ be the set of linear operators such that any $A \in \bar{\mathbb{A}}$ satisfies

$$\forall s \in \mathbb{R}^{\mathbb{R}}, \forall \tau \in \mathbb{R}, A[\sigma^\tau s] = \sigma^\tau [As]$$

where σ denotes, with an abuse of notation, the delay operator in the CT setup:

$$\forall s \in \mathbb{R}^{\mathbb{R}}, \forall (\tau, t) \in \mathbb{R}, \sigma^\tau[s](t) = s(t+\tau).$$

For any signal $s \in \mathbb{R}^{\mathbb{R}}$ that is differentiable with respect to time, the definition of A implies that it commutes with $\frac{d}{dt}$:

$$A\left[\frac{d}{dt}s\right] = A\left[\lim_{t\to 0}\frac{\sigma^t - 1}{t}s\right] = \frac{d}{dt}[As]. \tag{7}$$

Conversely, if A commutes with $\frac{d}{dt}$, then it is time-invariant.

Theorem 5: The set $\bar{\mathbb{A}}$ is the set of operators that preserve the behavior of all CT LTI systems:

$$\bar{\mathbb{A}} = \left\{ A | \forall q \in \mathbb{N}^*, \forall \mathcal{B} \in \bar{\mathbb{B}}^q,\ w \in \mathcal{B} \Rightarrow Aw \in \mathcal{B} \right\}.$$

For the second result, we introduce the CT equivalent of $\mathbb{G}^z$. Let $z \in \mathbb{C}^*$ and let $\bar{\mathbb{G}}^z \subseteq \{\mathbb{C}^{\mathbb{R}} \to \mathbb{C}\}$ the set of linear forms such that any $G \in \bar{\mathbb{G}}z$ satisfies

$$\forall s \in \mathbb{C}^{\mathbb{R}}, G\left[\frac{d}{dt}s\right] = zGs.$$

Lemma 2: Let $\mathcal{B}$ a CT LTI system and $A \in \bar{\mathbb{G}}^z$. Let $\bar{z} \in \mathbb{C}^{\mathbb{R}}$ satisfying the differential equation $\frac{d}{dt}\bar{z} = z\bar{z}$. Then

$$w \in \mathcal{B} \Rightarrow \begin{cases} \Re(\bar{z}Aw) \in \mathcal{B} \\ \Im(\bar{z}Aw) \in \mathcal{B} \end{cases}.$$

Proof: The proofs of Theorem 5 and Lemma 2 follow the exact same structure as the proof of Theorem 2 and Lemma 1, where the shift operator σ is substituted with the time derivative $\frac{d}{dt}$. ∎

VI. Application examples

A. Convolution and cross-correlation

Convolution $f * g$ between scalar-valued signals $(f, g) \in \mathbb{R}^{\mathbb{Z}}$ is defined as

$$f * g = \sum_{k=-\infty}^{\infty} f(-k)\sigma^k g.$$

For any given $f \in \mathbb{R}^{\mathbb{Z}}$, $(s \mapsto f * s)$ is linear and shift-invariant. Take any sequence $f \in \mathbb{R}^{\mathbb{Z}}$ (e.g., a trajectory of an arbitrary non-linear system), then Theorem 2 implies that if w is a trajectory of any LTI system $\mathcal{B}$, $f*w$ is also a trajectory of $\mathcal{B}$. Note that any LSI operator can be represented as a convolution, for instance with $A \in \mathbb{A}$:

$$\forall s \in \mathbb{R}^{\mathbb{Z}}, A[s] = \sum_{k\in\mathbb{Z}} A[\delta](-k)\ \sigma^k s = A[\delta] * s$$

where $\delta(k) = 0$ for all k except $\delta(0) = 1$ and $A[\delta](-k)$ is the value of $A[\delta]$ at the time instance $-k$.

Remark 2: Operators in $\mathbb{A}$ inherit all the properties of convolutions: $\mathbb{A}$ is closed under composition ($\forall A, B \in \mathbb{A}, AB \in \mathbb{A}$) and linear combination ($\forall A, B \in \mathbb{A}, \forall a, b \in \mathbb{R}, aA + bB \in \mathbb{A}$), and it contains the identity and zero elements. Finally, all operators in $\mathbb{A}$ commute: we have for any pair of operators $A, B \in \mathbb{A}$ and any sequence $s \in \mathbb{R}^{\mathbb{Z}}$

$$ABs = A[\delta] * B[\delta] * s = B[\delta] * A[\delta] * s = BAs.$$

Cross-correlation between two signals is similar to convolution: for $(f, g) \in \mathbb{R}^{\mathbb{Z}}$ it is defined as

$$f \star g = \sum_{k=-\infty}^{\infty} f(k)\sigma^k g.$$

Again, given $f \in \mathbb{R}^{\mathbb{Z}}$, $(s \mapsto f \star s)$ is linear and shift-invariant. In the first reformulation given in [4], the input-output signals are correlated with the system inputs. Then, the correlations are used in place of raw data in the mosaic-Hankel matrices of Theorem 1.

B. Discrete-time Fourier transform

The Fourier transform of a sequence $s \in \mathbb{C}^{\mathbb{Z}}$ at frequency ω is defined as

$$\mathcal{F}(\omega)[s] = \sum_{k=-\infty}^{+\infty} s(k)e^{-j\omega k}$$

where j is the imaginary unit, then we note that

$$\mathcal{F}(\omega)[\sigma s] = e^{j\omega}\mathcal{F}(\omega)[s] \Rightarrow \mathcal{F}(\omega) \in \mathbb{G}^{e^{j\omega}}.$$

Using Theorem 4 with $z = e^{j\omega}$, we recover the result of authors in [5], [6]. Note that we express the persistence of excitation condition (4) by separating the real and imaginary part of $\mathscr{H}_{L+n}(u)$, while authors in [5] equivalently use the complex values and their complex conjugates.

Successive application of cross-correlation and Fourier transform yields a formulation with cross-spectrums, similar to the one exposed in [4]. The difference is that the original publication considers complex-valued trajectories, while we only allow for real-valued signals. As noted in Remark 2, the order of transformations does not matter for the theoretical result, so the most practical one should be used in real applications.

C. Dynamic extension of a data matrix

We can draw inspiration from Dynamic Regressor Extension used in parameter estimation schemes (as in the first steps in [15]). Define N operators $\{F_k \in \mathbb{A}|_{T_k}\}_{k=1}^{N}$, for example based on first-order filters as

$$F_k[s] = \left(\frac{1}{\sigma - \lambda_k} s\right)_{[1,T_k]}$$

for any sequence $s \in \mathbb{R}^{\mathbb{Z}}$ and $0 < \lambda_k < 1$. We write the dynamic extension of a Hankel matrix based on these filters

$$H_L^F(s) = \begin{bmatrix} H_L\left(s_{[1,T]}\right) & H_L\left(F_1[s]\right) & \dots & H_L\left(F_N[s]\right) \end{bmatrix}$$

for $s \in \mathbb{R}^{\mathbb{Z}}$. Then, substituting s for input-output data (u, y) of a system $\mathcal{B} \in \mathbb{B}_m^{q,n}$, we can use Theorem 1: if

$$\operatorname{rank} H_{L+n}^F(u) = (L + n)m$$

then

$$\mathcal{B}|_L = \operatorname{im} \begin{bmatrix} H_L^F(u) \\ H_L^F(y) \end{bmatrix}.$$

D. The case of affine systems

An affine time-invariant system of order n with q variables can be defined, with the state-space formalism, as

$$\mathcal{B} = \left\{ (u, y) | \exists x, \begin{cases} \sigma x = Fx + Bu + e \\ y = Cx + Du + r \end{cases} \right\} \tag{8}$$

where (like in LTI systems) $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ is the state, $u \in (\mathbb{R}^m)^{\mathbb{Z}}$ the input and $y \in (\mathbb{R}^{q-m})^{\mathbb{Z}}$ the output. F, B, C, D are matrices of suitable dimensions, and e, r are constant vectors.

The system (8) is not an LTI, and the previous results do not apply directly. However, it is possible to define an LTI embedding for it, allowing for Willems' lemma adaptation, e.g., as it was done in [16] and [17], [18] with applications in the non-linear system control. In what follows, we show that if an LTI embedding can be defined for a (affine) dynamical system, then it is possible to use LSI data transformations on this embedding. However, because of the constraints that map the embedding back to the original system, such LSI transformations may produce trajectories that do not correspond to the original dynamical system, and thus, extra constraints should be imposed.

An LTI embedding for (8) is defined as an LTI system $\bar{\mathcal{B}}$ with an extended input $\bar{u}$

$$\begin{cases} \sigma x = Fx + \bar{B}\bar{u} \\ y = Cx + \bar{D}\bar{u} \end{cases}$$

where

$$\bar{u} = \begin{bmatrix} u \\ 1 \end{bmatrix}, \bar{B} = \begin{bmatrix} B & e \end{bmatrix}, \bar{D} = \begin{bmatrix} D & r \end{bmatrix}. \tag{9}$$

We can rewrite this LTI system $\bar{\mathcal{B}} \in \mathbb{B}_{m+1}^{q+1}$ as

$$\bar{\mathcal{B}} = \{(\bar{u}, y) | R_u(\sigma)\bar{u} = R_y(\sigma)y\}$$

so the original affine system $\mathcal{B}$ is an affine subspace of $\bar{\mathcal{B}}$:

$$\mathcal{B} = \left\{ (u, y) | (\bar{u}, y) \in \bar{\mathcal{B}}, \bar{u} = \begin{bmatrix} u \\ 1 \end{bmatrix} \right\}.$$

Take an operator $A \in \mathbb{A}$, using Theorem 2 we have that

$$(\bar{u}, y) \in \bar{\mathcal{B}} \Rightarrow (A\bar{u}, Ay) \in \bar{\mathcal{B}}$$

so the only missing condition for A to satisfy

$$(u, y) \in \mathcal{B} \Rightarrow (Au, Ay) \in \mathcal{B}$$

is that the affine constraint is still satisfied:

$$\begin{bmatrix} Au \\ 1 \end{bmatrix} = A\bar{u} = \begin{bmatrix} Au \\ A1 \end{bmatrix}.$$

E. Continuous-time linear sampling functional

Authors in [19] propose a "linear sampling functional": a CT signal w is then "sampled" as $\bar{w}(t) = \int w(\tau + t)g(\tau)d\tau$ where g is a user-defined function or distribution. Ordinary sampling is recovered when g is the Dirac distribution. This sampling method is motivated by the easy computation of time derivatives: $\frac{d\bar{w}}{dt}(t) = -\int w(\tau + t)\frac{dg}{d\tau}(\tau)d\tau$. From our standpoint, this is equivalent to using an LTI operator (convolution with $\tau \mapsto g(-\tau)$) before sampling.

VII. Conclusion

We have shown that the set of transformations that preserve the behavior of all DT LTI systems is the set of LSI operators $\mathbb{A}$. We demonstrated how a family of linear forms, including the Fourier transform, is related to these LSI operators. Finally, we showed that the three reformulations of the Fundamental Lemma (using correlation, cross-spectrum, and frequency domain data) naturally arise from the fundamental properties of LSI operators, eliminating the need for a specific deep theoretical analysis. Besides, we introduced a family of transformations $\bar{\mathbb{A}}$ in CT that also preserves behaviors, opening new directions for research.

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