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Validity of the Fröhlich model for a mobile impurity in a Bose-Einstein condensate

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We analyze the many-body Hamiltonian describing a mobile impurity immersed in a Bose-Einstein condensate (BEC). Using exact unitary transformations and rigorous error estimates, we show the validity of the Bogoliubov-Fröhlich Hamiltonian for the Bose polaron in the regime of moderately strong, repulsive interactions with a dilute BEC. Moreover, we calculate analytically the universal logarithmic correction to the ground state energy that arises from an impurity mediated phonon-phonon interaction.

INTRODUCTION

The concept of a polaron, a quasi-particle formed through the interaction of an impurity with a medium, is ubiquitous in condensed matter physics [1–3]. Among the many examples, the Bose polaron stands out by the degree of control over the interactions allowed by cold atomic gases. For this reason, it is seen as a possible platform for the simulation of polaron physics in analogous systems, and in particular allows to access the large-coupling regime [4–6]. The system also serves a similar role for theory, as a many-body quantum system whose constituents are simple enough that one can understand its complex behavior and effective degrees of freedom from first principles and has thus attracted considerable attention [7–20]. For the same reason, the rigorous justification of the effective description of fluctuations in dilute Bose gases by Bogoliubov phonons is an active area of mathematical research [22–36], with recent results also considering impurities [37–39].

Polarons are often modeled by the Fröhlich Hamiltonian, in which the impurity interacts with the field of low-energy excitations of the medium by linear coupling. However, the validity of this model for the Bose polaron has been questioned [11]. Given the importance of the Bose polaron as a model system, it is crucial to understand precisely the regime of validity of the Fröhlich Hamiltonian and control the modelling errors. In this letter we give conditions under which the Fröhlich Hamiltonian provides a good approximation to the full many-body spectrum of a dilute Bose gas with an impurity, and provide rigorous error estimates. This allows the model to be used with a great deal of confidence, and certifies the analogies to systems with a similar Hamiltonian.

We find that the total energy of the system has a logarithmic correction to the mean-field terms that is still universal, i.e., depends on the interaction only via the impurity-boson scattering length. Compared to similar terms in the pure Bose gas [40–43], it might be more accessible experimentally, by tuning the impurity-boson interaction.

We consider an impurity interacting with a dilute Bose gas ($\varrho_B a_{BB}^3 \ll 1$; ϱ_B is the boson density, a_{BB} the boson-

boson scattering length) at zero temperature. Both the boson-boson and boson-impurity interactions are assumed to be repulsive. In this situation, the relevant low-energy excitations are the Bogoliubov phonons. In order for the Fröhlich Hamiltonian to provide a good approximation, the impurity-boson interaction should not be so strong as to deplete the Bose-Einstein condensate. On the other hand, it should also be strong enough that the impurity-phonon interaction cannot simply be neglected. This corresponds to the regime in which the dimensionless polaronic coupling, which we express in terms of the scattering lengths and the healing length $\xi = 1/\sqrt{8\pi\varrho_{BB}a_{BB}}$ of the Bose gas as

$$\alpha = a_{IB}^2 \xi \varrho_B = \frac{a_{IB}^2}{8\pi a_{BB} \xi}$$

is of order one (definitions in [2, 7] differ by a factor of 8π). That is, the impurity-boson interaction has considerably longer range than the boson-boson interaction. In experiments probing the strong-coupling regime of the Bose polaron [4–6], this corresponds to moderate interactions.

THE MODEL

We now give a concrete model realizing the situation described above and for which we will derive an asymptotic expansion of the eigenvalues in $\varrho_B a_{BB}^3 \ll 1$, featuring the Bogoliubov-Fröhlich Hamiltonian.

We consider a single impurity interacting with N bosons in a box of size $L = \xi$ with periodic boundary conditions at zero temperature. In units where $\hbar = \xi = 1$, and also the impurity mass is $M = 1$, the Hamiltonian is

$$\hat{H} = \frac{1}{2} \hat{P}^2 + \sum_{j=1}^N \frac{\hat{p}_j^2}{2m} + \sum_{1 \leq i < j \leq N} V_{BB}(\mathbf{x}_i - \mathbf{x}_j) + \sum_{j=1}^N V_{IB}(\mathbf{X} - \mathbf{x}_j),$$

where uppercase letters denote quantities pertaining to the impurity, lower case letters those of the bosons. As $8\pi a_{BB}/\xi = N^{-1}$, we write

$$V_{BB}(\mathbf{x}) = N^2 \tilde{V}_{BB}(N\mathbf{x}), \quad (1)$$

where now $\tilde{a}_{\text{BB}} = Na_{\text{BB}} = 1/8\pi$ (cf. the definition [44]). Similarly, $a_{\text{IB}}/\xi = \sqrt{\alpha}N^{-1/2}$ and we write

$$V_{\text{IB}}(\mathbf{X}) = N\tilde{V}_{\text{IB}}(\sqrt{N}\mathbf{X}) \quad (2)$$

with $\tilde{a}_{\text{IB}} = \sqrt{N}a_{\text{IB}} = \sqrt{\alpha}$. We consider $\tilde{V}_{\text{BB}}, \tilde{V}_{\text{IB}}$ to be fixed while we make asymptotic expansions in N .

We will compare the Hamiltonian \hat{H} with the effective Fröhlich Hamiltonian, given by

$$\begin{aligned} \hat{H}_{\text{BF}} = & \frac{1}{2}\hat{\mathbf{P}}^2 + \sum_{\mathbf{k} \neq 0} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \\ & + \frac{2\pi\sqrt{\alpha}}{\mu} \sum_{\mathbf{k} \neq 0} e^{-i\mathbf{X} \cdot \mathbf{k}} \sqrt{\frac{k^2}{2m\omega_{\mathbf{k}}}} (b_{\mathbf{k}}^\dagger + b_{-\mathbf{k}}) \end{aligned} \quad (3)$$

where $\mu^{-1} = 1 + m^{-1}$ is the reduced mass, $b_{\mathbf{k}}^\dagger, b_{\mathbf{k}}$ create and annihilate Bogoliubov phonons, whose dispersion relation is

$$\omega_{\mathbf{k}} = \sqrt{k^2/(2m)(k^2/2m + 8\pi Na_{\text{BB}}/m)} = \frac{1}{2m} \sqrt{k^4 + 2k^2}.$$

Note that the expression (3) requires renormalization [45], so what we really mean by \hat{H}_{BF} is the renormalized version of (3) (constructed in [39, 45] and explained below).

Asymptotic energy

Theorem 1. *Let $\tilde{V}_{\text{BB}}, \tilde{V}_{\text{IB}}$ be positive, have finite support and be square-integrable, and assume Bose-Einstein condensation occurs. Then, as $N = \xi/(8\pi a_{\text{BB}}) \rightarrow \infty$ the ground state energy in units of $1/\xi^2$ is*

$$\begin{aligned} E_0 = & \frac{1}{4m}N + \frac{2\pi}{\mu}\sqrt{\alpha}N^{1/2} \\ & - \frac{16\pi\alpha^2}{\mu} \left(\mu^{-1} \arcsin \mu - \sqrt{1 - \mu^2} \right) \log N + \mathcal{O}(1). \end{aligned}$$

Moreover, the excited eigenvalues satisfy

$$E_n - E_0 = E_n^{\text{BF}} - E_0^{\text{BF}} + \mathcal{O}(N^{-\delta})$$

for some $\delta > 0$.

In this expansion, the leading term corresponds to the usual Gross-Pitaevskii energy $4\pi a_{\text{BB}}\varrho_B^2 L^3$. The following term is the analogue for the boson-impurity interaction $8\pi a_{\text{IB}}\varrho_B\varrho_I L^3$. The term of order $\log N$ arises from an effective phonon-phonon interaction mediated by the impurity, and thus vanishes in the limit of a heavy impurity, $M \rightarrow \infty$. Such a term has been observed in the context of the Fröhlich-Hamiltonian [10, 45], and our study shows that it is the leading correction to the mean-field approximation. Similar terms can be found in the expansion of the energy in a_{BB}/ξ (without impurity) [40–43], and in a_{IB}/ξ [11], but in our parameter regime these are of lower order ($N^{-1} \log N$, resp. $N^{-1/2} \log N$). This energy shift

may thus be much easier to detect experimentally than the corresponding terms in the pure Bose gas, which so far is only theoretical. Indeed, the shift is even larger than the Lee-Huang-Yang [46] term, which is of order one in our units and has been observed [47–50].

The 2+1-body problem

The contribution to E_0 of order $\log N$ can be understood as a three-body effect due to the interaction of the impurity and two bosons, like the similar term for the pure Bose gas [40–42]. We now give a heuristic derivation of its value by considering only the 2+1-body problem. An account in the N -body problem, and its relation to the renormalization of the Bogoliubov-Fröhlich Hamiltonian will be given later.

In relative coordinates $\mathbf{r}_i = \mathbf{X} - \mathbf{x}_i$, $i = 1, 2$, the Hamiltonian with $V_{\text{BB}} = 0$ is

$$\hat{H}_{2+1} = -\frac{1}{2\mu}(\Delta_{\mathbf{r}_1} + \Delta_{\mathbf{r}_2}) + \nabla_{\mathbf{r}_1} \cdot \nabla_{\mathbf{r}_2} + V_{\text{IB}}(\mathbf{r}_1) + V_{\text{IB}}(\mathbf{r}_2).$$

Given the short range $a_{\text{IB}} = N^{-1/2}\sqrt{\alpha}$ of the interaction, two-body effects will dominate and we make the ansatz $\Psi(\mathbf{r}_1, \mathbf{r}_2) = f(\mathbf{r}_1)f(\mathbf{r}_2)$ for a low-energy state, with f normalized, periodic. Calculating the energy yields

$$\langle \Psi, \hat{H}_{2+1} \Psi \rangle = -\frac{1}{\mu} \int \overline{f(\mathbf{r})} \Delta f(\mathbf{r}) + 2 \int V_{\text{IB}}(\mathbf{r}) f(\mathbf{r})^2 d^3r, \quad (4)$$

as the term with $\nabla_{\mathbf{r}_1} \cdot \nabla_{\mathbf{r}_2}$ integrates to zero. The minimizer f of the right hand side solves a periodized version of the zero-energy scattering equation ($f \propto 1 + \varphi_{\text{IB}}$ from (9)). Inserting this into (4) gives the leading contribution to the energy $2E_{2\text{-bd}} \sim 4\pi N^{-1/2}\sqrt{\alpha}/\mu$. In order to obtain a more precise result, we refine the ansatz by taking into account 3-body effects, $\Psi(\mathbf{r}_1, \mathbf{r}_2) = f(\mathbf{r}_1)f(\mathbf{r}_2)\Phi(\mathbf{r}_1, \mathbf{r}_2)$, where Φ is normalized and differs significantly from one only where $r_1, r_2 \lesssim a_{\text{IB}}$. Then, the mixed derivatives no longer integrate to zero and play the role of an effective potential

$$V_{\text{eff}}(\mathbf{r}_1, \mathbf{r}_2) = \nabla f(\mathbf{r}_1) \cdot \nabla f(\mathbf{r}_2). \quad (5)$$

Minimizing in Φ , the three-body contribution is a generalized scattering energy of V_{eff} [51]. It evaluates approximately to

$$\begin{aligned} E_{3\text{-bd}} \approx & \sum_{\mathbf{k}_1, \mathbf{k}_2} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 |\hat{f}(\mathbf{k}_1)|^2 |\hat{f}(\mathbf{k}_2)|^2}{\frac{1}{2\mu}(k_1^2 + k_2^2) + \mathbf{k}_1 \cdot \mathbf{k}_2} \\ \approx & 32\pi\alpha^2 \mu^{-1} \left(\mu^{-1} \arcsin \mu - \sqrt{1 - \mu^2} \right) N^{-2} \log N. \end{aligned}$$

The term in the expansion of the N -body energy is obtained by multiplying $E_{3\text{-bd}}$ by $N^2/2$, or the number of boson pairs.

DERIVATION OF THE FRÖHLICH HAMILTONIAN

To compare the N -body and the Bogoliubov-Fröhlich Hamiltonian we will conjugate \hat{H} by a series of unitary transformations.

We start by rewriting the Hamiltonian in second quantization and separate the condensate particles with $\mathbf{k} = 0$ from the excitations with $\mathbf{k} \neq 0$, which yields

$$\hat{H} = \frac{1}{2}N_0(N_0 - 1)\hat{V}_{\text{BB}}(0) + N_0\hat{V}_{\text{IB}}(0) \quad (6a)$$

$$+ \frac{1}{2}\hat{P}^2 + \frac{1}{2m}\sum_{\mathbf{k}\neq 0}k^2 \quad (6b)$$

$$+ N_0\sum_{\mathbf{k}\neq 0}(\hat{V}_{\text{BB}}(\mathbf{k}) + \hat{V}_{\text{BB}}(0))a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \quad (6c)$$

$$+ \frac{1}{2}\sqrt{N_0(N_0 - 1)}\sum_{\mathbf{k}\neq 0}\hat{V}_{\text{BB}}(\mathbf{k})a_{\mathbf{k}}a_{-\mathbf{k}} + \text{h.c.} \quad (6d)$$

$$+ \sqrt{N_0}\sum_{\mathbf{k}\neq 0}\hat{V}_{\text{IB}}(\mathbf{k})e^{-i\mathbf{X}\cdot\mathbf{k}}a_{\mathbf{k}} + \text{h.c.} \quad (6e)$$

$$+ \sum_{\mathbf{k},\ell\neq 0}\hat{V}_{\text{IB}}(\mathbf{k})e^{-i\mathbf{X}\cdot\mathbf{k}}a_{\mathbf{k}+\ell}^\dagger a_{\ell} \quad (6f)$$

$$+ \frac{1}{2}\sqrt{N_0}\sum_{\mathbf{k},\ell\neq 0}\hat{V}_{\text{BB}}(\mathbf{k})a_{\mathbf{k}+\ell}^\dagger a_{\mathbf{k}}a_{\ell} + \text{h.c.} \quad (6g)$$

$$+ \frac{1}{2}\sum_{\mathbf{k},\ell,\neq 0}\hat{V}_{\text{BB}}(\mathbf{k})a_{\mathbf{k}+\ell}^\dagger a_{\ell}^\dagger a_{\mathbf{m}}a_{\mathbf{m}-\mathbf{k}} \quad (6h)$$

where $N_0 = N - \sum_{\mathbf{k}\neq 0}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ is the number of condensate particles.

Our condensation hypothesis on \hat{V}_{BB} means that on low-energy states $N_0 \geq N - c\sqrt{N}$. In view of the scaling relations (1)–(2), we are thus tempted to neglect the terms (6f)–(6h). We would then arrive at a Fröhlich-type Hamiltonian and an energy expansion similar to Theorem 1. However, these would feature $\hat{V}_{\text{BB}}(0)$, $\hat{V}_{\text{IB}}(0)$ instead of the scattering lengths a_{BB} , a_{IB} and thus be incorrect even at the leading order. Because interactions in \hat{H} are strong on short length scales, particles correlate to balance the kinetic energy with the interaction. As a result, although most modes have a momentum of order one (that is, ξ^{-1} in our units), a vanishing number of them have large momenta and are responsible for a shift in energy to leading order. We first need to factor out the short scale correlation structure, making the scattering length appear in the couplings instead of the bare potentials and changing the kinetic energy from order N to order one on low energy states.

After this first step, we are left with quadratic and linear terms acting on low momenta, which up to a scalar

contributions is essentially an ultraviolet cutoff version of

$$\begin{aligned} \hat{H}_{\text{Bog}} &= \frac{1}{2}\hat{P}^2 + \frac{1}{2m}\sum_{\mathbf{k}\neq 0}(k^2 + 8\pi Na_{\text{BB}})a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \\ &+ \frac{1}{2m}\sum_{\mathbf{k}\neq 0}4\pi Na_{\text{BB}}\left(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}}a_{-\mathbf{k}}\right) \\ &+ \frac{1}{2\mu}\sum_{\mathbf{k}\neq 0}8\pi\sqrt{N}a_{\text{IB}}e^{-i\mathbf{X}\cdot\mathbf{k}}\left(a_{\mathbf{k}}^\dagger + a_{-\mathbf{k}}\right) \end{aligned} \quad (7)$$

and which is equivalent to the Bogoliubov-Fröhlich Hamiltonian (3) by passing to the phonon creation/annihilation operators $b_{\mathbf{k}}^\dagger, b_{\mathbf{k}}$, cf. (13).

Finally, we can renormalize this operator and obtain the correct energy expansion including the $\log N$ -contribution.

Approximate diagonalization

Large momenta are those with $k \gtrsim N^\tau$, for some small $0 < \tau < 1/2$. The unitary transformations are obtained by exponentiating linear, quadratic and cubic operators

$$U_q = \exp\left(\frac{1}{2}\sum_{k>N^\tau}N\varphi_{\text{BB}}(\mathbf{k})a_{\mathbf{k}}a_{-\mathbf{k}} - \text{h.c.}\right),$$

$$U_c = \exp\left(\sum_{\substack{k>N^\tau \\ \ell\leq N^\tau}}\chi(N_+)\sqrt{N}\varphi_{\text{BB}}(\mathbf{k})a_{\mathbf{k}+\ell}^\dagger a_{-\mathbf{k}}a_{\ell} - \text{h.c.}\right),$$

$$U_W = \exp\left(\sum_{k>N^\tau}\sqrt{N}\varphi_{\text{IB}}(\mathbf{k})e^{-i\mathbf{X}\cdot\mathbf{k}}a_{\mathbf{k}}^\dagger - \text{h.c.}\right),$$

where $N\varphi_{\text{BB}}$ and $N\varphi_{\text{IB}}$ are approximate zero-energy scattering solutions with periodic boundary conditions

$$\frac{1}{m}k^2\varphi_{\text{BB}}(\mathbf{k}) + \sum_{\ell\neq 0}\hat{V}_{\text{BB}}(\mathbf{k}-\ell)\varphi_{\text{BB}}(\ell) = -\hat{V}_{\text{BB}}(\mathbf{k}), \quad (8)$$

$$\frac{1}{2\mu}k^2\varphi_{\text{IB}}(\mathbf{k}) + \sum_{\ell\neq 0}\hat{V}_{\text{IB}}(\mathbf{k}-\ell)\varphi_{\text{IB}}(\ell) = -\hat{V}_{\text{IB}}(\mathbf{k}) \quad (9)$$

for $\mathbf{k} \neq 0$, and $\chi(N_+)$ truncates the excitation number.

The transformations U_q and U_c take care of creation and annihilation of the so-called hard and soft pairs of bosons with large momenta (the sum of which is either zero or small) and the map U_W deals with the large momentum scattering between the bosons and the impurity.

From the equation (8), we see that $\varphi_{\text{BB}}(\mathbf{k})$ behaves similarly to $-\hat{V}_{\text{BB}}(\mathbf{k})/(mk^2)$, and thus $\sum_{k>N^\tau}|N\varphi_{\text{BB}}(\mathbf{k})|^2 \sim N^{-\tau}$. The unitaries are therefore close to the identity. However, since, e.g., $\sum_{k>N^\tau}k^2|N\varphi_{\text{BB}}(\mathbf{k})|^2 \sim N$, they affect the energy to leading order.

In order to evaluate the action of the unitaries on the Hamiltonian, we use the exact adjoint expansion

$$e^{-B}Ae^B = A + [A, B] + \int_0^1 du \int_0^u dt e^{-tB}[[A, B], B]e^{tB}.$$

The renormalization of the energy and the couplings becomes apparent when evaluating these commutators and simplifying the result using the scattering equations (8), (9). The error terms are small relative to the kinetic operators in (6c) and the positive interaction terms (6f), (6h).

For this reason we need to first implement U_q to renormalize the order N of the energy, followed by U_W which renormalizes the order $N^{1/2}$. At this stage the kinetic energy will be of order N^τ on low energy states and we can implement U_c . Let us explain this in more detail.

First Bogoliubov transformation. This transformation implements the correlations due to hard boson pairs. It acts non-trivially on the boson Hamiltonian, namely on the terms (6b),(6d) and (6h). Using the scattering equation (8) leads to the replacements

$$\frac{1}{2}N^2\hat{V}_{\text{BB}}(0) \mapsto 2\pi N^2 a_{\text{BB}}/m = N/(4m)$$

$$(6d) \mapsto \frac{1}{4m} \sum_{0 < k < N^\tau} (a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}}) - \frac{1}{2m} N_+,$$

and extracts the energy contribution $\sum_{0 < k < N^\tau} \frac{(4\pi N a_{\text{BB}})^2}{m k^2}$ corresponding to the correction to the Born approximation which will renormalize the Bogoliubov Hamiltonian [30, 46]. After this step, the impurity-boson interaction becomes dominant and the kinetic energy is of order $N^{1/2}$ on low energy states.

Weyl transformation. The next transformation plays a similar role to U_q , but for V_{IB} instead of V_{BB} . For fixed \mathbf{X} , it is a Weyl transformation, mapping $U_W^\dagger a_{\mathbf{k}} U_W = a_{\mathbf{k}} + \sqrt{N} \varphi_{\text{IB}}(\mathbf{k}) e^{-i\mathbf{X} \cdot \mathbf{k}}$. The action of this unitary on the boson-boson interaction is essentially negligible, and the action on the kinetic and impurity-boson terms (6e) and (6f) has the effect of replacing

$$N\hat{V}_{\text{IB}}(0) \mapsto 2\pi N a_{\text{IB}}/\mu = 2\pi \sqrt{\alpha N}/\mu \quad (10)$$

$$(6e) \mapsto \frac{2\pi \sqrt{\alpha}}{\mu} \sum_{0 < k < N^\tau} e^{-i\mathbf{X} \cdot \mathbf{k}} a_{\mathbf{k}}^\dagger + \text{h.c.} \quad (11)$$

and adding a correction term $\sum_{0 < k < N^\tau} \frac{8\pi^2 \alpha}{\mu k^2}$. The action on \hat{P}^2 also gives rise to new interaction terms, including

$$\frac{N}{2} \sum_{k, \ell > N^\tau} \varphi_{\text{IB}}(\mathbf{k})(\mathbf{k} \cdot \boldsymbol{\ell}) \varphi_{\text{IB}}(\boldsymbol{\ell}) e^{-i\mathbf{X} \cdot (\mathbf{k} + \boldsymbol{\ell})} a_{\mathbf{k}}^\dagger a_{\boldsymbol{\ell}}^\dagger + \text{h.c.} \quad (12)$$

which corresponds to the effective three-body potential (5) and will contribute to the $\log N$ -term.

Cubic transformation Now that the kinetic energy is of order N^τ , we can perform the cubic transformation, whose role is to take into account correlations of soft boson pairs. Acting mainly on the terms (6b),(6g) and (6h), it factors out contributions from the cubic term and renormalizes the diagonal quadratic terms of the boson-boson interaction

$$(6d) \mapsto \frac{1}{2m} 16\pi a_{\text{BB}} N N_+ = \frac{1}{m} N_+.$$

Renormalization of the Fröhlich Hamiltonian

At this stage, ignoring cubic and quartic contributions, the resulting Hamiltonian is unitarily equivalent to \hat{H}_{Bog} cut off at momentum N^τ , via a Weyl transformation similar to U_W^\dagger .

The renormalization at the order N^τ its ultraviolet divergence is performed again via a Bogoliubov and a Weyl transformation, which are however not perturbations of the identity.

Second Bogoliubov transformation. To diagonalize the boson Hamiltonian, we define the usual phonon operators

$$b_{\mathbf{k}} = u_k a_{\mathbf{k}} + v_k a_{\mathbf{k}}^\dagger, \quad b_{\mathbf{k}}^\dagger = u_k a_{\mathbf{k}}^\dagger + v_k a_{\mathbf{k}}, \quad (13)$$

with $u_k = \cosh(\theta_k)$, $v_k = \sinh(\theta_k)$ for $\theta_k = \frac{1}{2} \log(k^2/(2m\omega_k))$ if $k < N^\tau$; otherwise $\theta_k = 0$. The Bogoliubov Hamiltonian becomes $\sum_{\mathbf{k}} \omega_k b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ plus its ground state energy of order N^τ , which combines with the scalar term created by U_q to yield the usual Lee-Huang-Yang [46] correction term, which is of order one in our regime. The linear coupling between impurity and the bosons changes to

$$(11) \mapsto \frac{2\pi \sqrt{\alpha}}{\mu} \sum_{0 < k < N^\tau} \sqrt{\frac{k^2}{2m\omega_k}} e^{-i\mathbf{X} \cdot \mathbf{k}} b_{\mathbf{k}}^\dagger + \text{h.c.} .$$

The transformation also creates terms of the form $\sum_{\mathbf{k}, \boldsymbol{\ell}} \hat{V}_{\text{IB}}(\mathbf{k}) u_{\mathbf{k}+\boldsymbol{\ell}} v_{\boldsymbol{\ell}} b_{\mathbf{k}+\boldsymbol{\ell}}^\dagger b_{\boldsymbol{\ell}}^\dagger$ from the quadratic interaction term (6d), which are responsible for the logarithmic correction to the energy found in [11]. In our case this is of the lower order $N^{-1/2} \log N$, due to the decay of $v_k \sim k^{-2}$ for large k (compare [39, Lem.4.25]).

Second Weyl transform. We can now finish regularizing the linear terms (11) with another Weyl transform, taking into account the correct dispersion relation ω for low-energy phonon-impurity scattering. Its action is given by

$$b_{\mathbf{k}} \mapsto b_{\mathbf{k}} + e^{-i\mathbf{X} \cdot \mathbf{k}} \frac{2\pi \sqrt{\alpha}}{\mu(\frac{1}{2}k^2 + \omega_k)} \sqrt{\frac{k^2}{2m\omega_k}}. \quad (14)$$

This extracts a scalar of order N^τ which combines with the one created by U_W , yielding again a contribution of order one. It also transforms the linear terms (11) to more regular quadratic terms, completing the ones in (12). Calling U the product of all the unitary transformations and taking $w(\mathbf{k})$ equal to $2\pi \sqrt{\alpha k}/(\mu(\frac{1}{2}k^2 + \omega_k)\sqrt{2m\omega_k})$ for $k \leq N^\tau$, and $\sqrt{N} \varphi_{\text{IB}}$ for $k > N^\tau$, we obtain up to small errors

$$U^\dagger \hat{H} U \approx \frac{1}{4m} N + \frac{2\pi}{\mu} \sqrt{\alpha} N^{1/2} + E + \hat{K} \quad (15)$$

$$+ \frac{1}{2} \sum_{k, \ell \neq 0} w(\mathbf{k})(\mathbf{k} \cdot \boldsymbol{\ell}) w(\boldsymbol{\ell}) e^{-i\mathbf{X} \cdot (\mathbf{k} + \boldsymbol{\ell})} b_{\mathbf{k}}^\dagger b_{\boldsymbol{\ell}}^\dagger + \text{h.c.} + \hat{R},$$

where E is a scalar of order one,

$$\hat{K} = \frac{1}{2}\hat{P}^2 + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}, \ell \neq 0} \hat{V}_{\text{IB}}(\mathbf{k}) e^{-i\mathbf{X} \cdot \mathbf{k}} b_{\mathbf{k}+\ell}^\dagger b_{\ell},$$

and \hat{R} is the additional interaction term

$$\begin{aligned} \hat{R} = & \sum_{\mathbf{k}, \ell \neq 0} w(\mathbf{k})(\mathbf{k} \cdot \ell) w(\ell) e^{-i\mathbf{X} \cdot (\mathbf{k}+\ell)} b_{\mathbf{k}}^\dagger b_{-\ell} \\ & + \sum_{\mathbf{k} \neq 0} w(\mathbf{k}) e^{-i\mathbf{X} \cdot \mathbf{k}} b_{\mathbf{k}}^\dagger \hat{P} + \text{h.c.} . \end{aligned}$$

Renormalization of $U^\dagger \hat{H} U$ The Hamiltonian (15) can be renormalized non-perturbatively [39, 45]. The terms responsible for the divergence are those creating or annihilating pairs of phonons. To deal with these, we set

$$G = -\frac{1}{2\hat{K}} \sum_{\mathbf{k}, \ell \neq 0} w(\mathbf{k})(\mathbf{k} \cdot \ell) w(\ell) e^{-i\mathbf{X} \cdot (\mathbf{k}+\ell)} b_{\mathbf{k}}^\dagger b_{\ell}^\dagger,$$

and rewrite the Hamiltonian as

$$(1 - G^\dagger) \hat{K} (1 - G) - G^\dagger \hat{K} G + \hat{R},$$

which eliminates the problematic interaction terms. The new term $G^\dagger \hat{K} G$ contains the divergent self energy. Putting this term into normal order, we find the scalar

$$\langle \emptyset | \hat{G}^\dagger \hat{K}^{-1} \hat{G} | \emptyset \rangle \approx \frac{1}{2} \sum_{\mathbf{k}, \ell \neq 0} \frac{w(\mathbf{k})^2 w(\ell)^2 (\mathbf{k} \cdot \ell)^2}{\frac{1}{2} |\mathbf{k} + \ell|^2 + \omega_{\mathbf{k}} + \omega_{\ell}}$$

plus corrections of order one due to $\hat{V}_{\text{IB}}(\mathbf{k})$. Evaluating the sum for large N yields the $\log N$ -term from Theorem 1. One easily confirms that $\langle \emptyset | \hat{R}^\dagger \hat{K}^{-1} \hat{R} | \emptyset \rangle$ is of order one, i.e., \hat{R} is not singular. The normal ordered term $:G^\dagger \hat{K} G:$ no longer suffers from singularities and gives a finite operator also for $N = \infty$ (since it contains an additional factor of \hat{K}^{-1}). This is also true for \hat{R} . We thus have a limiting operator for $N \rightarrow \infty$ that is unitarily equivalent to the renormalized Bogoliubov-Fröhlich Hamiltonian via the inverse of the Weyl transformation (14). Note that the term with $\hat{V}_{\text{IB}}(\mathbf{k})$ disappears from \hat{K} in the limit. This term gives a global energy shift order one, but does not change the differences of eigenvalues.

The proof of Theorem 1 relies on min-max formulas which require bounds on various error terms accumulated in the calculations. As we have explained, these are small relative to the kinetic energies, and the positive interaction terms (6h), (6f). They may thus be neglected for a lower bound on the energy. An upper bound is obtained by constructing trial states using the eigenvectors of the Bogoliubov-Fröhlich Hamiltonian and the unitary transformations.

CONCLUSION

We have obtained an expansion of the energy for a dilute Bose gas coupled to an impurity with effective coupling $\alpha \sim 1$. By analyzing precisely the renormalization

of the model parameters due to the emergence of a point-potential, we have shown the presence of a correction due to an effective three-body interaction. We have also established that the excitation spectrum is accurately described by the Bogoliubov-Fröhlich Hamiltonian.

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- [1] A. S. Alexandrov and J. T. Devreese, *Advances in polaron physics*, Springer Series in Solid-State Science, Vol. 159 (Springer, 2010).
- [2] F. Grusdt and E. Demler, New theoretical approaches to Bose polarons, in *Proceedings of the international school of physics "Enrico Fermi"*, edited by M. Inguscio, W. Ketterle, S. Stringari, and G. Roati (2016) pp. 325–411.
- [3] F. Grusdt, N. Mostaan, E. Demler, and L. A. P. Ardila, Impurities and polarons in bosonic quantum gases: a review on recent progress, arXiv preprint arXiv:2410.09413 (2024).
- [4] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinsen, R. S. Christensen, G. M. Bruun, and J. J. Arlt, Observation of attractive and repulsive polarons in a Bose-Einstein condensate, *Phys. Rev. Lett.* **117**, 055302 (2016).
- [5] M.-G. Hu, M. J. Van de Graaff, D. Kedar, J. P. Corson, E. A. Cornell, and D. S. Jin, Bose polarons in the strongly interacting regime, *Phys. Rev. Lett.* **117**, 055301 (2016).
- [6] Z. Z. Yan, Y. Ni, C. Robens, and M. W. Zwierlein, Bose polarons near quantum criticality, *Science* **368**, 190 (2020).
- [7] J. Tempere, W. Casteels, M. Oberthaler, S. Knoop, E. Timmermans, and J. Devreese, Feynman path-integral treatment of the BEC-impurity polaron, *Phys. Rev. B* **80**, 184504 (2009).
- [8] S. P. Rath and R. Schmidt, Field-theoretical study of the Bose polaron, *Phys. Rev. A* **88**, 053632 (2013).
- [9] W. Li and S. Das Sarma, Variational study of polarons in Bose-Einstein condensates, *Phys. Rev. A* **90**, 013618 (2014).
- [10] F. Grusdt, Y. Shchadilova, A. Rubtsov, and E. Demler, Renormalization group approach to the Fröhlich polaron model: application to impurity-BEC problem, *Sci. Rep.* **5**, 12124 (2015).
- [11] R. Christensen, J. Levinsen, and G. Bruun, Quasiparticle properties of a mobile impurity in a Bose-Einstein condensate, *Phys. Rev. Lett.* **115**, 160401 (2015).

- [12] Y. Shchadilova, R. Schmidt, F. Grusdt, and E. Demler, Quantum dynamics of ultracold Bose polarons, *Phys. Rev. Lett.* **117**, 113002 (2016).
- [13] F. Grusdt, R. Schmidt, Y. E. Shchadilova, and E. Demler, Strong-coupling Bose polarons in a Bose-Einstein condensate, *Phys. Rev. A* **96**, 013607 (2017).
- [14] M. Drescher, M. Salmhofer, and T. Enss, Real-space dynamics of attractive and repulsive polarons in Bose-Einstein condensates, *Phys. Rev. A* **99**, 023601 (2019).
- [15] T. Ichmoukhamedov and J. Tempere, Feynman path-integral treatment of the Bose polaron beyond the Fröhlich model, *Phys. Rev. A* **100**, 043605 (2019).
- [16] J. Levinsen, L. A. P. Ardila, S. M. Yoshida, and M. M. Parish, Quantum behavior of a heavy impurity strongly coupled to a Bose gas, *Phys. Rev. Lett.* **127**, 033401 (2021).
- [17] P. Massignan, N. Yegovtsev, and V. Gurarie, Universal aspects of a strongly interacting impurity in a dilute Bose condensate, *Phys. Rev. Lett.* **126**, 123403 (2021).
- [18] K. Seetharam, Y. Shchadilova, F. Grusdt, M. B. Zvonarev, and E. Demler, Dynamical quantum Cherenkov transition of fast impurities in quantum liquids, *Phys. Rev. Lett.* **127**, 185302 (2021), 2101.00030.
- [19] T. Ichmoukhamedov and J. Tempere, General memory kernels and further corrections to the variational path integral approach for the Bogoliubov-Fröhlich Hamiltonian, *Phys. Rev. B* **105**, 104304 (2022).
- [20] N. Mostaan, N. Goldman, and F. Grusdt, A unified theory of strong coupling Bose polarons: From repulsive polarons to non-Gaussian many-body bound states, arXiv preprint arXiv:2305.00835 (2023).
- [21] K. Seetharam, Y. Shchadilova, F. Grusdt, M. Zvonarev, and E. Demler, Quantum Cherenkov transition of finite-momentum Bose polarons, *Phys. Rev. A* **110**, 063306 (2024).
- [22] I. Rodnianski and B. Schlein, Quantum fluctuations and rate of convergence towards mean field dynamics, *Commun. Math. Phys.* **291**, 31 (2009).
- [23] H.-T. Yau and J. Yin, The second order upper bound for the ground energy of a Bose gas, *J. Stat. Phys.* **136**, 453 (2009).
- [24] M. G. Grillakis, M. Machedon, and D. Margetis, Second-order corrections to mean field evolution of weakly interacting bosons. I, *Commun. Math. Phys.* **294**, 273 (2010).
- [25] M. G. Grillakis, M. Machedon, and D. Margetis, Second-order corrections to mean field evolution of weakly interacting bosons. II, *Adv. Math.* **228**, 1788 (2011).
- [26] R. Seiringer, The excitation spectrum for weakly interacting bosons, *Comm. Math. Phys.* **306**, 565 (2011).
- [27] M. Lewin, P. T. Nam, and B. Schlein, Fluctuations around Hartree states in the mean-field regime, *Amer. J. Math.* **137**, 1613 (2015).
- [28] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, Bogoliubov spectrum of interacting Bose gases, *Comm. Pure Appl. Math.* **68**, 413 (2015), 1211.2778.
- [29] C. Boccato, S. Cenatiempo, and B. Schlein, Quantum many-body fluctuations around nonlinear Schrödinger dynamics, *Ann. Henri Poincaré* **18**, 113 (2017).
- [30] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, Bogoliubov theory in the Gross-Pitaevskii limit, *Acta Math.* **222**, 219 (2019).
- [31] S. Fournais and J. P. Solovej, The energy of dilute Bose gases, *Ann. Math. (2)* **192**, 893 (2020).
- [32] S. Fournais and J. P. Solovej, The energy of dilute Bose gases II: The general case, *Invent. Math.* **232**, 863 (2023).
- [33] P. T. Nam and A. Triay, Bogoliubov excitation spectrum of trapped Bose gases in the Gross-Pitaevskii regime, *J. Math. Pures Appl. (9)* **176**, 18 (2023).
- [34] C. Brennecke, B. Schlein, and S. Schraven, Bogoliubov theory for trapped bosons in the Gross-Pitaevskii regime, *Ann. H. Poincaré* **23**, 1583 (2022).
- [35] F. Haberberger, C. Hainzl, P. T. Nam, R. Seiringer, and A. Triay, The free energy of dilute Bose gases at low temperatures, arXiv preprint arXiv:2304.02405 (2023).
- [36] C. Caraci, J. Oldenburg, and B. Schlein, Quantum fluctuations of many-body dynamics around the Gross-Pitaevskii equation, *Ann. Inst. Henri Poincaré C* (2024).
- [37] K. Myśliwy and R. Seiringer, Microscopic derivation of the Fröhlich Hamiltonian for the Bose polaron in the mean-field limit, *Ann. H. Poincaré* **21**, 4003 (2020).
- [38] J. Lampart and P. Pickl, Dynamics of a tracer particle interacting with excitations of a Bose-Einstein condensate, *Ann. H. Poincaré* **23**, 2855 (2022).
- [39] J. Lampart and A. Triay, The excitation spectrum of a Bose gas with an impurity in the Gross-Pitaevskii regime, arXiv preprint arXiv:2401.14911v2 (2024).
- [40] T. T. Wu, Ground state of a Bose system of hard spheres, *Phys. Rev.* **115**, 1390 (1959).
- [41] K. Sawada, Ground-state energy of Bose-Einstein gas with repulsive interaction, *Phys. Rev.* **116**, 1344 (1959).
- [42] N. Hugenholtz and D. Pines, Ground-state energy and excitation spectrum of a system of interacting bosons, *Phys. Rev.* **116**, 489 (1959).
- [43] C. Caraci, A. Olgiatei, D. S. Aubin, and B. Schlein, Third order corrections to the ground state energy of a Bose gas in the Gross-Pitaevskii regime, arXiv preprint arXiv:2311.07433 (2023).
- [44] We use here the definition of the scattering length of a general potential $4\pi a = \min \int |\nabla\varphi(x)|^2 + 2m_{\text{red}}(1 + \varphi(x))^2 V(x) dx = 2m_{\text{red}} \int V(x)(1 + \varphi_{\min}(x)) dx$.
- [45] J. Lampart, The renormalised Bogoliubov-Fröhlich Hamiltonian, *J. Math. Phys.* **61**, 101902 (2020).
- [46] T. Lee, K. Huang, and C. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties, *Phys. Rev.* **106**, 1135 (1957).
- [47] S. Papp, J. Pino, R. Wild, S. Ronen, C. E. Wieman, D. S. Jin, and E. A. Cornell, Bragg spectroscopy of a strongly interacting Rb85 Bose-Einstein condensate, *Phys. Rev. Lett.* **101**, 135301 (2008).
- [48] N. Navon, S. Pietecki, K. Günter, B. Rem, T. C. Nguyen, F. Chevy, W. Krauth, and C. Salomon, Dynamics and thermodynamics of the low-temperature strongly interacting Bose gas, *Phys. Rev. Lett.* **107**, 135301 (2011).
- [49] T. G. Skov, M. G. Skou, N. B. Jørgensen, and J. J. Arlt, Observation of a Lee-Huang-Yang fluid, *Phys. Rev. Lett.* **126**, 230404 (2021).
- [50] L. Lavoine, A. Hammond, A. Recati, D. Petrov, and T. Bourdel, Beyond-mean-field effects in Rabi-coupled two-component Bose-Einstein condensate, *Phys. Rev. Lett.* **127**, 203402 (2021).
- [51] P. T. Nam, J. Ricaud, and A. Triay, The condensation of a trapped dilute Bose gas with three-body interactions, *Prob. Math. Phys.* **4**, 91 (2023).