About S-Variables and Descriptor Systems

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Abstract

The S-Variable approach has been central in my collaborative work with Japanese colleagues in the past 20 years. The terminology took some time to be settled. We have used terms such as "Dilated-LMIs" (Yoshio Ebihara), "Slack Variable" (Masayuki Sato) and some others, to finally stop iterating with the publication of the book "S-Variable Approach to LMI-Based Robust Control". An other existing terminology in the litterature (mainly by Emilia Fridman) for that same technique is the "Descriptor system approach" and indeed there are interesting extensions to descriptor systems using this technique. In this talk we will recall some previous work on that topic and present some open questions for further collaborative work.

S-procedure

- [Aizerman and Gantmakher, 1964], Lossless [Yakubovitch, 1971]

$$\eta^* Q \eta < 0 \quad \forall \eta^* Q_1 \eta \leq 0 \quad \Leftrightarrow \quad \exists s > 0 \ : \ Q \prec s Q_1$$

- ▲ Extensions: bounded real lemma, passivity, dissipation, KYP

- Lossy

$$\eta^* Q \eta < 0 \quad \forall \left\{ \begin{array}{l} \eta^* Q_1 \eta \leq 0 \\ \eta^* Q_2 \eta \leq 0 \end{array} \right. \quad \Leftarrow \quad \exists s_1 > 0, s_2 > 0 \ : \ Q \prec s_1 Q_1 + s_2 Q_1$$

- ▼ Extensions such as μ-analysis, IQCs, quadratic separation, are (in most cases) conservative

S-Variable (Finsler lemma, Elimination lemma...)

- S-procedure

$$\eta^* Q \eta < 0 \quad \forall \eta^* Q_1 \eta \leq 0 \quad \Leftrightarrow \quad \exists s > 0 \ : \ Q \prec s Q_1$$

- S-Variable approach

$$\eta^* Q \eta < 0 \quad \forall M\eta = 0 \quad \Leftrightarrow \quad M^{\perp T} Q M^{\perp} \prec 0 \quad \Leftrightarrow \quad \exists S \ : \ Q \prec SM + M^* S^* = \{SM\}^{\mathcal{H}}$$

- S-procedure based proof [Ebihara et al., 2015]: $Q_1 = M^* M$ and $S = s\frac{1}{2} M^*$
- ▲ Many new degrees of freedom
- ▼ Increased size of LMIs & many more decision variables

S-Variable, two lossy useful results

- Lossless

$$\eta^* Q \eta < 0 \quad \forall M\eta = 0 \quad \Leftrightarrow \quad \exists S \ : \ Q \prec \{SM\}^{\mathcal{H}}$$

- Lossy but Robust [Peaucelle et al., 2000]

$$\{ \ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0 \ \} \quad \forall \zeta \in [0\ 1] \quad \Leftarrow \quad \exists S \ : \ \left\{ \begin{array}{l} Q^{[0]} \prec \{SM^{[0]}\}^{\mathcal{H}} \\ Q^{[1]} \prec \{SM^{[1]}\}^{\mathcal{H}} \end{array} \right.$$

$$Q(\zeta) = (1-\zeta)Q^{[0]} + \zeta Q^{[1]} \quad , \quad M(\zeta) = (1-\zeta)M^{[0]} + \zeta M^{[1]}$$

- Lossy but good for design (see WUDS 2023 Kyoto [Peaucelle, 2023])

$$\left\{ \begin{array}{ll} \eta^* Q \eta < 0 & \forall M_1 \eta = 0 \\ \eta^* Q \eta < 0 & \forall M_2 \eta = 0 \end{array} \right. \quad \Leftarrow \quad \exists S = M_2^* V \ : \ Q \prec \{SM_1\}^{\mathcal{H}}$$

▲ Heuristics for Control (and Robust Control) Design ▼ Convex only for chosen M_2

S-Variable - Today more about analysis

- Lossless

$$\eta^* Q \eta < 0 \quad \forall M\eta = 0 \quad \Leftrightarrow \quad \exists S \ : \ Q \prec \{SM\}^{\mathcal{H}}$$

- Lossy but Robust [Peaucelle et al., 2000]

$$\{ \ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0 \ \} \quad \forall \zeta \in [0\ 1] \quad \Leftarrow \quad \exists S \ : \ \left\{ \begin{array}{l} Q^{[0]} \prec \{SM^{[0]}\}^{\mathcal{H}} \\ Q^{[1]} \prec \{SM^{[1]}\}^{\mathcal{H}} \end{array} \right.$$

$$\left[\begin{array}{c} Q(\zeta) \\ M(\zeta) \end{array} \right] = (1-\zeta) \left[\begin{array}{c} Q^{[0]} \\ M^{[0]} \end{array} \right] + \zeta \left[\begin{array}{c} Q^{[1]} \\ M^{[1]} \end{array} \right] \in \mathrm{CO} \left\{ \left[\begin{array}{c} Q^{[0]} \\ M^{[0]} \end{array} \right], \left[\begin{array}{c} Q^{[1]} \\ M^{[1]} \end{array} \right] \right\}$$

▲ Efficient for analysis of parametric uncertain systems, including polynomial and rational

Example for Robustness analysis

$$V(x,\zeta) = x^* P(\zeta) x > 0 \qquad \dot{V}(x,\zeta) < 0 \qquad \forall \ddot{y} + \alpha_1(\zeta)\dot{y} + \frac{1}{\alpha_2(\zeta)} y = 0 \qquad \alpha_i(\zeta) \text{ affine of } \zeta \in [0\ 1]$$

- Affine descriptor version (alternative to building LFTs ▲ smaller size)

$$\dot{V}(x,\zeta) < 0 \qquad \forall \left\{ \begin{array}{l} \ddot{y} + \alpha_1(\zeta)\dot{y} + \pi = 0 \\ \alpha_2(\zeta)\pi = y \end{array} \right.$$

- State-space descriptor version with $x = \begin{pmatrix} y & \dot{y} \end{pmatrix}^T$

$$\eta^* \underbrace{\begin{bmatrix} 0 & 0 & P(\zeta) \\ 0 & 0 & 0 \\ P(\zeta) & 0 & 0 \end{bmatrix}}_{Q(P(\zeta))} \eta < 0 \qquad \forall \underbrace{\left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & \alpha_1(\zeta) \\ 0 & 0 & \alpha_2(\zeta) & -1 & 0 \end{array}\right]}_{M(\zeta)} \underbrace{\begin{pmatrix} \dot{x} \\ \pi \\ x \end{pmatrix}}_{\eta} = 0$$

▲ (Lossy) S-variable result: Robust stability holds true if $\exists P^{[0]} \succ 0$, $P^{[1]} \succ 0$ and S :

$$Q(P^{[0]}) \prec \{SM^{[0]}\}^{\mathcal{H}} \quad , \quad Q(P^{[0]}) \prec \{SM^{[1]}\}^{\mathcal{H}}$$

Conservatism reduction by lifting

- Lifting by taking higher order derivatives:

$$\begin{bmatrix} 1 \\ \frac{d}{dt} \end{bmatrix} (y^{(2)} + \alpha_1(\zeta) y^{(1)} + \frac{1}{\alpha_2(\zeta)} y) = \begin{bmatrix} y^{(2)} + \alpha_1(\zeta) y^{(1)} + \frac{1}{\alpha_2(\zeta)} y \\ y^{(3)} + \alpha_1(\zeta) y^{(2)} + \frac{1}{\alpha_2(\zeta)} y^{(1)} \end{bmatrix} = 0$$

- State-space descriptor version with $\hat{x}_1 = \begin{pmatrix} y & y^{(1)} & y^{(2)} \end{pmatrix}^T$

$$\left[\begin{array}{ccc|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & \alpha_1(\zeta) & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & \alpha_1(\zeta) \\ 0 & 0 & 0 & \alpha_2(\zeta) & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2(\zeta) & 0 & -1 & 0 \end{array}\right] \begin{pmatrix} \dot{\hat{x}}_1 \\ \hat{\pi}_1 \\ \hat{x}_1 \end{pmatrix} = 0$$

- ▲ (Lossy) S-variable result applies and gives a less conservative, larger size, LMI
- ▲ Proof involves a higher order parameter-dependent Lyapunov function ▼ Initial conditions ?

Conservatism reduction by lifting

- Lifting by causal / non causal, rationnally parameter-dependent, filters :

$$\begin{bmatrix} 1 \\ H(s,\zeta) \end{bmatrix} (y^{(2)} + \alpha_1(\zeta) y^{(1)} + \frac{1}{\alpha_2(\zeta)} y) = 0$$

- ▲ (Lossy) S-variable result applies and gives a less conservative, larger size, LMI
- ▲ Stable causal filters can be chosen with zero initial conditions. Link with recent work by Scherer?
- ▲ Connexion to dynamic IQC multipliers?
- ▼ Choice of filters?
- ▼ Parameter-dependent lifting and SOS type hierarchies?
- ▼ Hierarchies of LMI? Convergence to exactness?

Worst-case extraction

■ The S-variable Robustness conditions for $\{ \ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0 \ \} \quad \forall \zeta \in [0\ 1]$ involve LMI constraints on vertices

$$Q(P^{[0]}) \prec \{SM^{[0]}\}^{\mathcal{H}} \quad , \quad Q(P^{[1]}) \prec \{SM^{[1]}\}^{\mathcal{H}}$$

■ For each constraint exist a dual variable $H^{[0]} \succeq 0$ and $H^{[1]} \succeq 0$ respectively.

▲ [Ebihara et al., 2015] If the primal LMIs are unfeasible (or marginally feasible) and dual variables satisfy

$$\exists \zeta \in [0\ 1] \quad : \quad (1-\zeta)H^{[0]} + \zeta H^{[1]} = 0$$

then ζ is a worst-case such that $\exists M(\zeta)\eta \quad : \quad \eta^* Q(\zeta)\eta \geq 0$.

▲ In practice, solve convex 2nd order cone problem $\zeta^\star = \min_\zeta \|(1-\zeta)H^{[0]} + \zeta H^{[1]}\|_2$. Gives rather good close to "worst case" values.

Robustness analysis w.r.t polytopic uncertainties

- One polytopic uncertainty

$$\eta^* Q(P(\zeta))\eta < 0 \qquad \forall M(\zeta)\eta = 0 \qquad M(\zeta) = \sum_{v=1}^{\bar{v}} \zeta_v M^{[v]} \quad \zeta_v \geq 0 \quad \sum_{v=1}^{\bar{v}} \zeta_v = 1$$

- Holds true if $\exists P^{[v=1\ldots\bar{v}]} \succ 0$ and S :

$$Q(P^{[v]}) \prec \{SM^{[v]}\}^{\mathcal{H}} \quad \forall v = 1 \ldots \bar{v}.$$

- Proof: By convexity, the LMIs imply

$$Q(P(\zeta)) \prec \{SM(\zeta)\}^{\mathcal{H}} \quad \forall \zeta_v \geq 0 \quad \sum_{v=1}^{\bar{v}} \zeta_v = 1$$

where $P(\zeta) = \sum_{v=1}^{\bar{v}} \zeta_v P^{[v]}$, and concludes the proof using the S-variable result.

Robustness analysis w.r.t polytopic uncertainties

- Two (or more) polytopic uncertainties [Is this new?]

$$\eta^* Q(P(\zeta_1,\zeta_2))\eta < 0 \qquad \forall \begin{bmatrix} M_1(\zeta_1) \\ M_2(\zeta_2) \end{bmatrix} \eta = 0 \qquad M_i(\zeta_i) = \sum_{1}^{\bar{v}_i} \zeta_{i,v_i} M_i^{[v_i]} \quad \zeta_{i,v_i} \geq 0 \quad \sum_{v_1=1}^{\bar{v}_i} \zeta_{i,v_i} = 1$$

- Holds true if $\exists P^{[v_1=1\dots\bar{v}_1, v_2=1\dots\bar{v}_2]} \succ 0$, $S_1^{v_2=1\dots\bar{v}_2}$ and $S_2^{v_1=1\dots\bar{v}_1}$:

$$Q(P^{[v_1,v_2]}) \prec \left\{ S_1^{[v_2]} M_1^{[v_1]} + S_2^{[v_1]} M_2^{[v_2]} \right\}^{\mathcal{H}} \qquad \forall v_i = 1 \dots \bar{v}_i$$

- Proof: By convexity, the LMIs imply

$$Q(P(\zeta_1,\zeta_2)) \prec \{S_1(\zeta_2)M_1(\zeta_1) + S_2(\zeta_1)M_2(\zeta_2)\}^{\mathcal{H}} = \left\{ \begin{bmatrix} S_1(\zeta_2) & S_2(\zeta_1) \end{bmatrix} \begin{bmatrix} M_1(\zeta_1) \\ M_2(\zeta_2) \end{bmatrix} \right\}^{\mathcal{H}}$$

where $P(\zeta_1,\zeta_2) = \sum_{v_1=1}^{\bar{v}_1} \sum_{v_2=1}^{\bar{v}_2} \zeta_{1,v_1}\zeta_{2,v_2} P^{[v_1,v_2]}$, $S_i(\zeta_k) = \sum_{v_k=1}^{\bar{v}_k} \zeta_{k,v_k} S_i^{[v_k]}$
and concludes the proof using the S-variable result.

Example for Robustness analysis

$$\ddot{y} + \alpha_1(\zeta_1)\dot{y} + \frac{1}{\alpha_2(\zeta_2)} y = 0 \qquad \alpha_i(\zeta_i) \text{ affine of } \zeta_i \in [0\ 1]$$

$$\underbrace{\eta^* \begin{bmatrix} 0 & 0 & P(\zeta) \\ 0 & 0 & 0 \\ P(\zeta) & 0 & 0 \end{bmatrix} \eta}_{Q(P(\zeta))} < 0 \qquad \forall \underbrace{\left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & \alpha_1(\zeta_1) \\ 0 & 0 & \alpha_2(\zeta_2) & -1 & 0 \end{array}\right]}_{M(\zeta)} \underbrace{\begin{pmatrix} \dot{x} \\ \pi \\ x \end{pmatrix}}_{\eta} = 0$$

- Multi-polytopic result applies : $Q(P^{[\nu_1,\nu_2]}) \prec \left\{ \begin{bmatrix} S_1^{[\nu_2]} & S_2^{[\nu_1]} \end{bmatrix} \begin{bmatrix} M_1^{[\nu_1]} \\ M_2^{[\nu_2]} \end{bmatrix} \right\}^{\mathcal{H}}$ at the 4 vertices.
- Less conservative than having $\begin{bmatrix} S_1^{[\nu_2]} & S_2^{[\nu_1]} \end{bmatrix} = S$ over all vertices.
- At the expense of increasing the size of π, and reordering of rows,
  the $M(\zeta) = \begin{bmatrix} M_1(\zeta_1) \\ M_2(\zeta_2) \end{bmatrix}$ is always achievable.

Example for negativity of matrix polynomials

$$T(\zeta) = N^*(\zeta) R N(\zeta) \prec 0 \qquad N(\zeta) \text{ is polynomial (or rational) of } \{\zeta_1, \dots \zeta_{\bar{k}}\}$$

- Equivalently written as

$$w^* R w < 0 \qquad \forall w = N(\zeta) z$$

- Descriptor version

$$\begin{pmatrix} w \\ z \end{pmatrix}^* \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 0 \qquad \forall \begin{bmatrix} -I & N(\zeta) \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = 0$$

- At the expense of including additional algebraically constrained vectors π

$$\eta^* \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \eta = \eta^* Q \eta < 0 \qquad \forall \begin{bmatrix} M_1(\zeta_1) \\ \vdots \\ M_{\bar{k}}(\zeta_{\bar{k}}) \end{bmatrix} \eta = 0 \qquad M_k(\zeta_k) \text{ affine of } \zeta_k$$

Example for building affine descriptor forms

$$N(\zeta) = \begin{bmatrix} (1+2\zeta_1)(4+\zeta_2) & (1+2\zeta_1)\zeta_2 \\ (4+\zeta_2) & \zeta_1+\zeta_2 \end{bmatrix}$$

- Decomposed in canonic monomial basis [Peaucelle and Sato, 2009]

$$N(\zeta)z = \begin{bmatrix} 4+8\zeta_1 & 0 \\ 4 & \zeta_1 \end{bmatrix} z + \begin{bmatrix} 1+2\zeta_1 & 1+2\zeta_1 \\ 1 & 1 \end{bmatrix} \zeta_2 z$$

gives a systematic way of building affine descriptor form for $w = N(\zeta)z$

$$\left[\begin{array}{cc|cc|cc} -1 & 0 & 1+2\zeta_1 & 1+2\zeta_1 & 4+8\zeta_1 & 0 \\ 0 & -1 & 1 & 1 & 4 & \zeta_1 \\ 0 & 0 & -1 & 0 & \zeta_2 & 0 \\ 0 & 0 & 0 & -1 & 0 & \zeta_2 \end{array}\right] \begin{pmatrix} w \\ \pi \\ z \end{pmatrix} = 0$$

Example for building affine descriptor forms

$$N(\zeta) = \begin{bmatrix} (1+2\zeta_1)(4+\zeta_2) & (1+2\zeta_1)\zeta_2 \\ (4+\zeta_2) & \zeta_1+\zeta_2 \end{bmatrix}$$

- Use Linear-Fractional Transformation (eg. in Robust Control Toolbox or LFR toolbox)

$$N(\zeta) = N_{11} + N_{12}\Delta(I - N_{22}\Delta)^{-1}N_{21} \qquad \Delta \text{ diagonal matrix of } \zeta_1, \zeta_2$$

gives an easy (but usually large) descriptor form for $w = N(\zeta)z$

$$\begin{bmatrix} -I & N_{11} & N_{12}\Delta \\ 0 & N_{12} & I - N_{22}\Delta \end{bmatrix} \eta = 0$$

at the expense or reordering the rows the uncertainties appear as needed.

Example for building affine descriptor forms

$$N(\zeta) = \begin{bmatrix} (1+2\zeta_1)(4+\zeta_2) & (1+2\zeta_1)\zeta_2 \\ (4+\zeta_2) & \zeta_1+\zeta_2 \end{bmatrix}$$

▲ No need to decompose in canonic monomial basis, by exploiting the structure:

$$\left[\begin{array}{cc|c|cc} -1 & 0 & 1+2\zeta_1 & 0 & 0 \\ 0 & -1 & 1 & 0 & \zeta_1 \\ 0 & 0 & -1 & 4+\zeta_2 & \zeta_2 \end{array}\right] \begin{pmatrix} w \\ \hat{\pi} \\ z \end{pmatrix} = 0$$

S-Variable - reducing the computational burden

$$\{\ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0\ \}\ \ \forall \zeta \in [0\ 1] \quad \Leftarrow \quad \exists S\ :\ Q(P^{[v]}) \prec \{S M^{[v]}\}^{\mathcal{H}}$$

- S is of the size of M^*. May become very large.
- Need for efficient software to build limited size $M(\zeta)$, project to do so in Julia with Ch. Louembet
- Need to find systematic reduction (exact / approximate) tools for affine descriptor models
- Non conservative size reduction of LMIs if $M(\zeta) = T \begin{bmatrix} \hat{M}(\zeta) \\ \breve{M} \end{bmatrix}$, T invertible

$$\exists \hat{S}\ :\ \begin{cases} \breve{M}^{\perp T} Q_0 \breve{M}^{\perp} \prec \{\hat{S}\hat{M}_0\breve{M}^{\perp}\}^{\mathcal{H}} \\ \breve{M}^{\perp T} Q_1 \breve{M}^{\perp} \prec \{\hat{S}\hat{M}_1\breve{M}^{\perp}\} \end{cases}$$

- Exists also a (conservative but efficient) size reduction of LMIs

if $M(\zeta) = \begin{bmatrix} \hat{M}(\zeta) & \breve{M} \end{bmatrix} T$ and $\begin{bmatrix} 0 \\ I \end{bmatrix}^* T^{-*} Q(\zeta) T^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = \hat{Q}_{22} \succeq 0.$

S-Variable - case of unbounded uncertainties

- Bounded uncertainties, where $Q(\zeta) = (1-\zeta)Q^{[0]} + \zeta Q^{[1]}, \quad M(\zeta) = (1-\zeta)M^{[0]} + \zeta M^{[1]}$

$$\{ \ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0 \ \} \ \boxed{\forall \zeta \in [0\ 1]} \quad \Leftarrow \quad \exists S \ : \ \begin{cases} Q^{[0]} \prec \{SM^{[0]}\}^{\mathcal{H}} \\ Q^{[1]} \prec \{SM^{[1]}\}^{\mathcal{H}} \end{cases}$$

- Unbounded uncertainties [Peaucelle and Sato, 2009] where $M(\zeta) = M^{[0]} + \zeta M^{[\infty]}$

$$\{ \ \eta^* Q\eta < 0 \quad \forall M(\zeta)\eta = 0 \ \} \ \boxed{\forall \zeta} \quad \Leftarrow \quad \exists G = -G^* \ : \ Q \prec \{M^{[\infty]*} G M^{[0]}\}^{\mathcal{H}}$$

- Surprisingly simple proof

$$Q \prec \{M^{[\infty]*} G M^{[0]}\}^{\mathcal{H}} = \{M^{[\infty]*} G M(\zeta)\}^{\mathcal{H}} - \zeta \{M^{[\infty]*} G M^{[\infty]}\}^{\mathcal{H}}$$

last term is zero $M^{[\infty]*} G M^{[\infty]}\}^{\mathcal{H}} = M^{[\infty]*}(G + G^*)M^{[\infty]} = 0$. Hence

$$Q \prec \{SM(\zeta)\}^{\mathcal{H}} \quad , \quad \text{with } S = M^{[\infty]*} G$$

S-Variable - case of unbounded uncertainties

- Bounded uncertainties, where $Q(\zeta) = (1-\zeta)Q^{[0]} + \zeta Q^{[1]}, \quad M(\zeta) = (1-\zeta)M^{[0]} + \zeta M^{[1]}$

$$\{\ \eta^* Q(\zeta)\eta < 0 \quad \forall M(\zeta)\eta = 0\ \} \ \boxed{\forall \zeta \in [0\ 1]} \quad \Leftarrow \quad \exists S \ : \ \begin{cases} Q^{[0]} \prec \{S M^{[0]}\}^{\mathcal{H}} \\ Q^{[1]} \prec \{S M^{[1]}\}^{\mathcal{H}} \end{cases}$$

- Unbounded uncertainties [Peaucelle and Sato, 2009] where $M(\zeta) = M^{[0]} + \zeta M^{[\infty]}$

$$\{\ \eta^* Q\eta < 0 \quad \forall M(\zeta)\eta = 0\ \} \ \boxed{\forall \zeta} \quad \Leftarrow \quad \exists G = -G^* \ : \ Q \prec \{M^{[\infty]*} G M^{[0]}\}^{\mathcal{H}}$$

- Positive real uncertainties, where $M(\zeta) = M^{[0]} + \zeta M^{[\infty]}$

$$\{\ \eta^* Q\eta < 0 \quad \forall M(\zeta)\eta = 0\ \} \ \boxed{\forall \zeta \geq 0} \quad \Leftarrow \quad \exists G = -G^* \quad D \succeq 0 \ : \ Q \prec \{M^{[\infty]*}(D+G)M^{[0]}\}^{\mathcal{H}}$$

- Positive complex uncertainties, where $M(\zeta) = M_0 + \zeta M_\infty$

$$\{\ \eta^* Q\eta < 0 \quad \forall M(\zeta)\eta = 0\ \} \ \boxed{\forall \mathrm{Re}(\zeta) \geq 0} \quad \Leftarrow \quad \exists D \succeq 0 \ : \ Q \prec \{M^{[\infty]*} D M^{[0]}\}^{\mathcal{H}}$$

S-Variable - KYP lemma

- Prove $N^*(j\omega)QN(j\omega) \prec 0$ where $N(j\omega) = \begin{bmatrix} (j\omega - A)^{-1}B \\ I \end{bmatrix}$ for all $\omega \in \mathbb{R}$
- Equivalent to $\left\{ \eta^* Q \eta < 0 \quad \forall \begin{bmatrix} j\omega - A & -B \end{bmatrix} \eta = 0 \right\} \quad \forall \omega$
- S-variable result: $\exists G = -G^* \quad : \quad Q \prec \left\{ \begin{bmatrix} -jI \\ 0 \end{bmatrix} G \begin{bmatrix} -A & -B \end{bmatrix} \right\}^{\mathcal{H}}$
- Exactly identical to KYP lemma: $\exists P = P^* (= -jG) \quad : \quad Q + \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \prec 0$

- Encourages to re-examine the central control results with the S-variable / descriptor viewpoint

S-Variable - Static-output feedback design

- At the expense of increasing the size of π, uncertainties can be separated from design variables

$$\begin{bmatrix} E_x & E_\pi & A & B \\ 0 & KF_\pi & KC & KD-I \end{bmatrix} \begin{pmatrix} \dot{x} \\ \pi \\ x \\ u \end{pmatrix} = \begin{bmatrix} (1-\zeta)M^{[0]} + \zeta M^{[1]} \\ KM_2 + M_3 \end{bmatrix} \eta = 0 \qquad \zeta \in [0\ 1]$$

- S-Variable robust stability analysis condition: $\exists P^{[v]} \succ 0, S, S^{[v]}$ such that

$$Q(P^{[v]}) \prec \left\{ SM^{[v]} + S^{[v]}(KM_2 + M_3) \right\} \qquad v = 0, 1$$

S-Variable - Static-output feedback design

[Sadabadi and Peaucelle, 2016]

$$\begin{bmatrix} E_x & E_\pi & A & B \\ 0 & KF_\pi & KC & KD-I \end{bmatrix}\eta = \begin{bmatrix} (1-\zeta)M^{[0]}+\zeta M^{[1]} \\ KM_2+M_3 \end{bmatrix}\eta = 0 \qquad \zeta \in [0\ 1]$$

- Heuristic robust design condition: choose $K_\pi^{[v]}$ and $K_x^{[v]}$ such that the following 2 systems are stable

$$\begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & A^{[v]} & B^{[v]} \\ 0 & K_\pi^{[v]} & K_x^{[v]} & -I \end{bmatrix}\eta = \begin{bmatrix} M^{[v]} \\ M_4^{[v]} \end{bmatrix}\eta = 0 \qquad v = 0,1$$

any solution $K = V^{-1}W$ of the following LMIs robustly stabilizes the uncertain plant

$$Q(P^{[v]}) \prec \left\{ SM^{[v]} + M_4^{[v]*}(WM_2+VM_3) \right\}^{\mathcal{H}} \qquad v = 0,1$$

- Good/Best choice of the full information feedback gains $K_\pi^{[v]}$ and $K_x^{[v]}$?

S-Variable - Static full-information feedback design

■ Choose full-information feedback $K_\pi^{[v]}$, $K_x^{[v]}$ such that the following 2 systems are stable ?

$$\left[\begin{array}{ccc} E_x^{[v]} & E_\pi^{[v]} + B^{[v]} K_\pi^{[v]} & A^{[v]} + B^{[v]} K_x^{[v]} \end{array} \right] \hat{\eta} = 0 \qquad v = 0, 1$$

▼ and probably some more constraints (to be determined)

▲ Possible elegant solution in [Peaucelle et al., 2024], solve

$$\hat{Q}(R^{[v]}) \prec \left\{ \hat{M}^* \left[\begin{array}{ccc} E_x^{[v]} U_x^{[v]} & E_\pi^{[v]} U_\pi^{[v]} + B^{[v]} W_\pi^{[v]} & A^{[v]} U_x^{[v]} + B^{[v]} W_x^{[v]} \end{array} \right] \right\}^{\mathcal{H}} \qquad v = 0, 1$$

where $\hat{M}\hat{\eta} = 0$ describes a stable target system
and get $K_\pi^{[v]} = W_\pi^{[v]} (U_\pi^{[v]})^{-1}$, $K_x^{[v]} = W_x^{[v]} (U_x^{[v]})^{-1}$.

Conclusions

- Thank you for the good discussions and ideas!
- Looking forward for more collaborative results!

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