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**CANONICAL COMMUTATION RELATIONS: A QUICK PROOF
OF THE STONE-VON NEUMANN THEOREM AND AN
EXTENSION TO GENERAL RINGS**

BACHIR BEKKA

ABSTRACT. Let R be a (not necessary commutative) ring with unit, $d \geq 1$ an integer, and λ a unitary character of the additive group $(R, +)$. A pair (U, V) of unitary representations U and V of R^d on a Hilbert space \mathcal{H} is said to satisfy the canonical commutation relations (relative to λ) if

$$U(a)V(b) = \lambda(a \cdot b)V(b)U(a)$$

for all $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in R^d$, where $a \cdot b = \sum_{k=1}^d a_k b_k$. We give a new and quick proof of the classical Stone von Neumann Theorem about the essential uniqueness of such a pair in the case where R is a local field (e.g. $R = \mathbf{R}$). Our methods allow us to give the following extension of this result to a general locally compact ring R . For a unitary representation U of R^d on a Hilbert space \mathcal{H} , define the inflation $U^{(\infty)}$ of U as the (countably) infinite multiple of U on $\mathcal{H}^{(\infty)} = \bigoplus_{i \in \mathbf{N}} \mathcal{H}$.

Let $(U_1, V_1), (U_2, V_2)$ be two pairs of unitary representations of R^d on corresponding Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ satisfying the canonical commutation relations (relative to λ). Provided that λ satisfies a mild faithful condition, we show that the inflations $(U_1^{(\infty)}, V_1^{(\infty)}), (U_2^{(\infty)}, V_2^{(\infty)})$ are approximately equivalent, that is, there exists a sequence $(\Phi_n)_n$ of unitary isomorphisms $\Phi_n : \mathcal{H}_1^{(\infty)} \rightarrow \mathcal{H}_2^{(\infty)}$ such that

$$\lim_n \|U_2^{(\infty)}(a) - \Phi_n U_1^{(\infty)}(a) \Phi_n^*\| = 0 \quad \text{and} \quad \lim_n \|V_2^{(\infty)}(b) - \Phi_n V_1^{(\infty)}(b) \Phi_n^*\| = 0,$$

uniformly on compact subsets of R^d .

1. INTRODUCTION

Heisenberg's original canonical commutation relations (see [Hei27]) for a system with d degrees of freedom involve position operators q_1, \dots, q_d and momentum operators p_1, \dots, p_d , which are unbounded operators on a complex Hilbert space satisfying the relations

$$p_k q_l - q_l p_k = \frac{h}{2\pi i} \delta_{kl} I,$$

where h is Planck's constant. Considering the unitary operators

$$U(t) = \exp\left(\sum_{k=1}^d \frac{2\pi i}{h} t_k p_k\right) \quad \text{and} \quad V(s) = \exp\left(\sum_{l=1}^d \frac{2\pi i}{h} s_l q_l\right)$$

on a Hilbert space \mathcal{H} for $t = (t_1, \dots, t_d)$ and $s = (s_1, \dots, s_d)$ in \mathbf{R}^d , H. Weyl (see [Wey27, p. 26]) gave the canonical commutation relations the following form:

$$U(t)V(s) = e^{\frac{2\pi i}{h} t \cdot s} V(s)U(t) \tag{CCR}$$

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where $t \cdot s = \sum_{k=1}^d t_k s_k$. A solution of (CCR) is given by the so-called Schrödinger representation and the classical Stone-von Neumann Theorem asserts that this solution is essentially unique (see below for precise statements). We are going to give a quick proof of this theorem and to extend it to more general rings.

Given a topological group G and a complex Hilbert space \mathcal{H} , recall that a unitary representation π of G on \mathcal{H} is a continuous homomorphism $\pi : G \rightarrow U(\mathcal{H})$, where $U(\mathcal{H})$ is the group of unitary operators of \mathcal{H} equipped with the weak (or strong) operator topology. For a finite or infinite countable cardinal $n \in \{1, 2, \dots, \infty\}$, we denote by

$$\mathcal{H}^{(n)} = \bigoplus_{i \in \{1, 2, \dots, n\}} \mathcal{H}$$

the Hilbert space direct sum of n many copies of \mathcal{H} and by $\pi^{(n)}$ the corresponding *inflation* of π ; so, $\pi^{(n)} : G \rightarrow U(\mathcal{H}^{(n)})$ is the unitary representation of G on $\mathcal{H}^{(n)}$ defined by

$$\pi^{(n)}(g) = \bigoplus_{i \in \{1, 2, \dots, n\}} \pi(g) \quad \text{for all } g \in G.$$

Let R be a locally compact ring with unit; notice that we do not assume that R is commutative. Fix a unitary character λ of the additive group $(R, +)$, that is, a one dimensional unitary representation of $(R, +)$. We say that a pair (U, V) of unitary representations U and V of R^d on a Hilbert space \mathcal{H} satisfies the canonical commutation relations (relative to λ) if

$$U(a)V(b) = \lambda(a \cdot b)V(b)U(a) \quad (\text{CCR})_\lambda$$

for all $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$ in R^d , where $a \cdot b = \sum_{k=1}^d a_k b_k$.

It is quickly checked that a solution of $(\text{CCR})_\lambda$, called *Schrödinger pair* (or *Schrödinger representation*), is given by the pair $(U_{\text{Schr}}, V_{\text{Schr}})$ defined on $L^2(R^d, \mu)$ by

$$(U_{\text{Schr}}(a)f)(x) = f(x+a) \quad \text{and} \quad (V_{\text{Schr}}(b)\xi)(x) = \lambda(x \cdot b)f(x) \quad \text{for } f \in L^2(R^d, \mu),$$

where μ is a Haar measure on $(R^d, +)$.

There is an obvious equivalence relation on pairs of unitary representations of R^d : two such pairs (U_1, V_1) and (U_2, V_2) acting on respective Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 are *equivalent* if there exists a unitary isomorphism $\Phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$\Phi U_1(a) \Phi^* = U_2(a) \quad \text{and} \quad \Phi V_1(b) \Phi^* = V_2(b)$$

for all $a, b \in R^d$.

Let \widehat{R} be the dual group of R , that is, the group of all unitary characters of $(R, +)$. Given λ in \widehat{R} , properties of the continuous homomorphism

$$\nabla_\lambda : R \mapsto \widehat{R}, \quad a \mapsto (x \mapsto \lambda(x \cdot a))$$

will play an important role in the sequel.

Theorem A. (Stone-von Neumann Theorem) *Let R be a unital second-countable locally compact ring and let $\lambda \in \widehat{R}$. We assume that*

- $\lambda(ab) = \lambda(ba)$ for all $a, b \in R$ (Sym)
- $\nabla_\lambda : R \mapsto \widehat{R}$ is an isomorphism. (Isom)

Let (U, V) be a pair of unitary representations of R^d on a separable Hilbert space satisfying $(\text{CCR})_\lambda$. Then the inflation $(U^{(\infty)}, V^{(\infty)})$ of (U, V) is equivalent to the inflation $(U_{\text{Schr}}^{(\infty)}, V_{\text{Schr}}^{(\infty)})$ of the Schrödinger pair.

The following more familiar form of the Stone-von Neumann Theorem is an immediate consequence of Theorem A.

Corollary B. (*Stone-von Neumann Theorem-bis*) Let R and λ be as in Theorem A. Let (U, V) be a pair of unitary representations of R^d on a separable Hilbert space \mathcal{H} satisfying $(\text{CCR})_\lambda$. Then there exist a subset J of \mathbf{N} , an orthogonal decomposition $\mathcal{H} = \bigoplus_{j \in J} \mathcal{H}_j$ of \mathcal{H} into closed $U(R^d)$ - and $V(R^d)$ -invariant subspaces \mathcal{H}_j as well as unitary isomorphisms $\Phi_j : L^2(R^d) \rightarrow \mathcal{H}_j$ such that

$$U(t)|_{\mathcal{H}_j} = \Phi_j U_{\text{Schr}}(a) \Phi_j^* \quad \text{and} \quad V(s)|_{\mathcal{H}_j} = \Phi_j V_{\text{Schr}}(b) \Phi_j^*,$$

for all $a, b \in R^d$ and all $j \in J$.

Here are examples of rings R for which there exists $\lambda \in \widehat{R}$ such that Conditions (Sym) and (Isom) above hold (see Section 3 below; for some of these rings, even every non trivial λ satisfies these two conditions):

- (1) $R = \mathbf{k}$ is a local field \mathbf{k} , that is, a locally compact non discrete field; examples include $\mathbf{k} = \mathbf{R}$ and $\mathbf{k} = \mathbf{Q}_p$ (the fields of p -adic numbers).
- (2) $R = \mathbf{A}$ is the ring of adèles of \mathbf{Q} .
- (3) $R = M_n(\mathbf{k})$ is the ring of $n \times n$ -matrices over a local field \mathbf{k} .
- (4) $R = \mathbf{Z}/n\mathbf{Z}$ is the ring of integers modulo $n \geq 1$.

A few words about the proofs of the Stone-von Neumann Theorem are in order. The original proof in the case of $R = \mathbf{R}$ is due to von Neumann ([vN31]); his methods were later extended to cover other local fields by Segal [Seg63] and Weil [Wei64]. In all these approaches, the pair (U, V) acting on \mathcal{H} is “integrated” to a representation of a certain algebra of functions on $R^d \times R^d$ by an algebra of operators on \mathcal{H} and one shows that this latter algebra contains sufficiently many rank-one operators. Mackey stated and proved a far-reaching generalization of this theorem ([Mac49]); his proof uses a different approach which amounts to the classification of so-called transitive systems of imprimitivity based on R^d (in fact, on an arbitrary second countable locally compact group). In comparison, our proof is more elementary; the basic idea is an imitation of a trick, called Fell’s absorption principle (see Proposition 2 below), and the only sophisticated tool we use is Plancherel’s theorem about the Fourier transform of the abelian group R^d . Our approach confirms somehow the observation made by Howe [How80, p.827] that the Stone-von Neumann and the Plancherel Theorems are equivalent.

Rings for which there exists no $\lambda \in \widehat{R}$ such that Condition (Isom) holds include the ring of integers \mathbf{Z} and, more generally, every countable infinite ring with the discrete topology. Theorem A fails for such rings. Nevertheless, our approach allows us to prove for them the following weaker version of the Stone-von Neumann Theorem.

We need to define a second equivalence relation on pairs of unitary representations of R^d ; this definition is tailored after a similar notion of approximately equivalent representations of C^* -algebras (see [Con00, Definition 41.1]).

Definition 1. Two pairs (U_1, V_1) and (U_2, V_2) of unitary representations of R^d acting on respective Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 are *approximately equivalent* if there exists a sequence $(\Phi_n)_n$ of unitary isomorphisms $\Phi_n : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that, for every compact subset K of R^d , we have

$$\lim_{n \rightarrow \infty} \sup_{a \in K} \|U_2(a) - \Phi_n U_1(a) \Phi_n^*\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{b \in K} \|V_2(b) - \Phi_n V_1(b) \Phi_n^*\| = 0.$$

Theorem C. Let R be a unital second-countable locally compact ring. Let $\lambda \in \widehat{R}$ satisfying Condition (Sym) from Theorem A and such that

$$\text{there is no non-zero two-sided ideal of } R \text{ contained in } \ker \lambda. \quad (\text{Faith})$$

For an integer $d \geq 1$, let (U, V) be a pair of unitary representations of R^d on a separable Hilbert space \mathcal{H} satisfying the commutation relation $(\text{CCR})_\lambda$. Then $(U^{(\infty)}, V^{(\infty)})$ is approximately equivalent to $(U_{\text{Schr}}^{(\infty)}, V_{\text{Schr}}^{(\infty)})$.

Condition (Faith) is a weak faithfulness condition on λ ; provided λ is non trivial, it holds for instance when R is a simple ring.

As is nowadays well-known, an appropriate way to formulate the commutation relations is by means of unitary representations of a Heisenberg group associated to R and d (see [Fol89, Chap.1, §5] in the case $R = \mathbf{R}$). We will recall the definition of this group and rephrase our results in terms of its representations in Section 4.

2. PROOF OF THEOREM A AND THEOREM C

Let λ be an arbitrary unitary character of R . We will use in a crucial way the following pair $(U_{\text{reg}}, V_{\text{reg}})$ of unitary representations of R^d satisfying $(\text{CCR})_\lambda$, which we call the *regular pair* for reasons we will explain below (Remark 4). Set

$$S := R^d \times R^d$$

and consider the Hilbert space $L^2(S) = L^2(S, \mu \otimes \mu)$, where μ is a fixed Haar measure on R^d . Let U_{reg} and V_{reg} be defined on $L^2(S)$ by

$$(U_{\text{reg}}(a)\xi)(x, y) = \xi(x + a, y) \quad \text{and} \quad (V_{\text{reg}}(b)\xi)(x) = \lambda(x \cdot b)\xi(x, y + b)$$

for all $\xi \in L^2(S)$ and $a, b, x, y \in R^d$. One checks immediately that $(U_{\text{reg}}, V_{\text{reg}})$ satisfies the relation $(\text{CCR})_\lambda$.

Fix a pair (U, V) of unitary representations of R^d on a separable Hilbert space \mathcal{H} satisfying the commutation relation $(\text{CCR})_\lambda$. Let $m, n \in \{1, 2, \dots, \infty\}$ be the dimensions of the Hilbert spaces $L^2(S)$ and \mathcal{H} respectively. The Hilbert space $L^2(S, \mathcal{H}) = L^2(S, \mathcal{H}, \mu \otimes \mu)$ of (equivalence classes of) square-integrable measurable maps $S \rightarrow \mathcal{H}$ is isomorphic to $\mathcal{H}^{(m)}$ as well as to $L^2(S)^{(n)}$; the inflations $(U^{(m)}, V^{(m)})$ of (U, V) and $(U_{\text{reg}}^{(n)}, V_{\text{reg}}^{(n)})$ of $(U_{\text{reg}}, V_{\text{reg}})$ are realized on $L^2(S, \mathcal{H})$ by the formulae

$$(U^{(m)}(a)F)(x, y) = U(a)(F(x, y)) \quad \text{and} \quad (V^{(m)}(b)F)(x) = V(b)F(x, y)$$

and

$$(U_{\text{reg}}^{(n)}(a)F)(x, y) = F(x + a, y) \quad \text{and} \quad (V_{\text{reg}}^{(n)}(b)F)(x) = \lambda(x \cdot b)F(x, y + b)$$

for all $F \in L^2(S, \mathcal{H})$ and $a, b, x, y \in R^d$.

We associate to (U, V) another pair (\tilde{U}, \tilde{V}) of unitary representations of R^d on $L^2(S, \mathcal{H})$ defined by

$$(\tilde{U}(a)F)(x, y) = U(a)(F(x + a, y)) \quad \text{and} \quad (\tilde{V}(b)F)(x, y) = V(b)(F(x, y + b))$$

for all $F \in L^2(S, \mathcal{H})$ and $a, b, x, y \in R^d$. Again, one checks immediately that (\tilde{U}, \tilde{V}) satisfies the relation $(\text{CCR})_\lambda$.

So, we have three pairs of unitary representations of R^d acting on $L^2(S, \mathcal{H})$ and satisfying $(\text{CCR})_\lambda$: $(U^{(m)}, V^{(m)})$, (\tilde{U}, \tilde{V}) , and $(U_{\text{reg}}^{(n)}, V_{\text{reg}}^{(n)})$; observe that, for fixed n , the last one is independent of (U, V) .

The following proposition is the crucial tool we will use in the proofs of both Theorem A and Theorem C; Item (i) is reminiscent of a property of the regular representation of a group (see [BdlHV08, Corollary E.2.6]), often called Fell's absorption principle (for the precise relationship with this principle, see Section 4).

Proposition 2.

- (i) The pair (\tilde{U}, \tilde{V}) is equivalent to the pair $(U_{\text{reg}}^{(n)}, V_{\text{reg}}^{(n)})$.

(ii) The pair $(U^{(m)}, V^{(m)})$ is equivalent to the pair $(\overline{U}^m, \overline{V}^m)$ given on $L^2(S, \mathcal{H})$ by $(\overline{U}^m(a)F)(x, y) = \lambda(a \cdot x)U(a)(F(x, y))$ and $(\overline{V}^m(b)F)(x, y) = \lambda(y \cdot b)V(b)(F(x, y))$ for all $F \in L^2(S, \mathcal{H})$ and $a, b, x, y \in R^d$.

Proof. (i) Let $\Phi : L^2(S, \mathcal{H}) \rightarrow L^2(S, \mathcal{H})$ be the unitary isomorphism defined by

$$\Phi(F)(x, y) = U(-x)V(-y)(F(x, y)) \quad \text{for all } F \in L^2(S, \mathcal{H}), (x, y) \in S.$$

For $a, b \in R^d$, we have

$$\begin{aligned} \left((\tilde{U}(a)\Phi)(F) \right) (x, y) &= U(a)(\Phi(F)(x + a, y)) \\ &= U(a)U(-x - a)V(-y)(F(x + a, y)) \\ &= U(-x)V(-y)(F(x + a, y)) \\ &= \Phi(U_{\text{reg}}^{(n)}(a)F)(x, y) \end{aligned}$$

and, similarly,

$$\left((\tilde{V}(b)\Phi)(F) \right) (x, y) = \Phi(V_{\text{reg}}^{(n)}(b)F)(x, y).$$

(ii) Let $\Psi : L^2(S, \mathcal{H}) \rightarrow L^2(S, \mathcal{H})$ be the unitary isomorphism defined by

$$\Psi(F)(x, y) = V(-x)U(y)(F(x, y)) \quad \text{for all } F \in L^2(S, \mathcal{H}), (x, y) \in S.$$

For $a, b \in R^d$, we have

$$\begin{aligned} \Psi(U^{(m)}(a)F)(x, y) &= V(-x)U(y)U(a)(F(x, y)) \\ &= V(-x)U(a)U(y)(F(x, y)) \\ &= \lambda(a \cdot x)U(a)(V(-x)U(y)(F(x, y))) \\ &= (\overline{U}^m(a)(\Psi(F)))(x, y) \end{aligned}$$

and, similarly,

$$\Psi(V^{(m)}(b)F)(x, y) = (\overline{V}^m(b)(\Psi(F)))(x, y).$$

□

2.1. Proof of Theorem A. We assume that λ satisfies Conditions (Sym) and (Isom). Since \widehat{S} is, in the canonical way, topologically isomorphic to \widehat{R}^{2d} , the map

$$\nabla_{\lambda}^{(2d)} : S \mapsto \widehat{S}, (a, b) \mapsto ((x, y) \mapsto \lambda(x \cdot a)\lambda(y \cdot b))$$

is an isomorphism of topological groups. We can therefore identify the (inverse) Fourier transform of S with the map $\mathcal{F} : L^1(S) \rightarrow C(S)$ given by

$$\mathcal{F}(f)(x, y) = \int_S f(t, s)(\nabla_{\lambda}^{(2d)}(x, y))(t, s)d\mu(t)d\mu(s) = \int_S f(t, s)\lambda(t \cdot x)\lambda(s \cdot y)d\mu(t)d\mu(s)$$

for $f \in L^1(S)$. Upon properly normalizing \mathcal{F} , we can assume by Plancherel's theorem (see [Bou23, Chap.II, §1]) that \mathcal{F} extends from $L^1(S) \cap L^2(S)$ to a bijective isometry $\mathcal{F} : L^2(S) \rightarrow L^2(S)$.

We claim that \mathcal{F} induces an equivalence between the pairs $(\overline{U}^m, \overline{V}^m)$ and (\tilde{U}, \tilde{V}) from Proposition 2. Indeed, let $\tilde{\mathcal{F}} : L^2(S, \mathcal{H}) \rightarrow L^2(S, \mathcal{H})$ be given by

$$\tilde{\mathcal{F}}(F)(x, y) = \int_S \lambda(t \cdot x)\lambda(s \cdot y)F(t, s)d\mu(t)d\mu(s)$$

for $F \in L^2(S, \mathcal{H})$; more precisely, $\tilde{\mathcal{F}}$ is defined on the linear span X of $\{f \otimes v : f \in L^2(S), v \in \mathcal{H}\}$, where $(f \otimes v)(x, y) = f(x, y)v$, by

$$\tilde{\mathcal{F}}(f \otimes v) = \mathcal{F}(f)v;$$

since $\tilde{\mathcal{F}}$ is isometric on X and has dense image, it extends to a bijective isometry from $L^2(S, \mathcal{H})$ to $L^2(S, \mathcal{H})$. For $a, b \in R^d$ and $F \in L^2(S, \mathcal{H})$, we have, using Condition (Sym),

$$\begin{aligned} \left(\tilde{U}(a)\tilde{\mathcal{F}}(F) \right) (x, y) &= U(a)(\tilde{\mathcal{F}}(F)(x+a, y)) \\ &= U(a) \left(\int_S \lambda(t \cdot (x+a))\lambda(s \cdot y)F(t, s)d\mu(t)d\mu(s) \right) \\ &= \int_S \lambda(t \cdot a)\lambda(t \cdot x)\lambda(s \cdot y)U(a)(F(t, s))d\mu(t)d\mu(s) \\ &= \int_S \lambda(a \cdot t)\lambda(t \cdot x)\lambda(s \cdot y)U(a)(F(t, s))d\mu(t)d\mu(s) \\ &= \int_S \lambda(t \cdot x)\lambda(s \cdot y)(\overline{U^m}(a)F)(t, s)d\mu(t)d\mu(s) \\ &= \tilde{\mathcal{F}}(\overline{U^m}(a)F)(x, y) \end{aligned}$$

and, similarly,

$$\left(\tilde{V}(b)\tilde{\mathcal{F}}(F) \right) (x, y) = \tilde{\mathcal{F}}(\overline{V^m}(b)F)(x, y).$$

This concludes the proof of Theorem A; indeed, denoting by \sim the equivalence of pairs of unitary representations of R^d on separable Hilbert space \mathcal{H} satisfying $(\text{CCR})_\lambda$, Proposition 2 shows that

$$(U^{(m)}, V^{(m)}) \sim (\overline{U^m}, \overline{V^m}) \quad \text{and} \quad (\tilde{U}, \tilde{V}) \sim (U_{\text{reg}}^{(n)}, V_{\text{reg}}^{(n)}),$$

where $m = \dim S$, $n = \dim(\mathcal{H})$; moreover, we have just proved that

$$(\overline{U^m}, \overline{V^m}) \sim (\tilde{U}, \tilde{V});$$

so $(U^{(m)}, V^{(m)}) \sim (U_{\text{reg}}^{(n)}, V_{\text{reg}}^{(n)})$ and this implies that $(U^{(\infty)}, V^{(\infty)}) \sim (U_{\text{reg}}^{(\infty)}, V_{\text{reg}}^{(\infty)})$. As $(U_{\text{reg}}, V_{\text{reg}})$ does not depend on (U, V) , we have therefore

$$(U^{(\infty)}, V^{(\infty)}) \sim (U_{\text{Schr}}^{(\infty)}, V_{\text{Schr}}^{(\infty)}).$$

2.2. Proof of Corollary B. First, observe that the pair $(U_{\text{Schr}}, V_{\text{Schr}})$ is irreducible (that is, there is no non trivial closed subspace of $L^2(R^d)$ which is simultaneously invariant under $U(R^d)$ and $V(R^d)$). Indeed, let $T : L^2(R^d) \rightarrow L^2(R^d)$ be a bounded operator which commutes with all $U_{\text{Schr}}(a)$'s and all $V_{\text{Schr}}(b)$'s. Since T commutes with all $V_{\text{Schr}}(b)$'s and since $\nabla_\lambda^{(d)}(R^d) = \widehat{R^d}$ is weak*-dense in $L^\infty(R^d)$, the operator T commutes with every operator $m_\varphi : L^2(R^d) \rightarrow L^2(R^d)$ given by multiplication with a function $\varphi \in L^\infty(R^d)$. As is well-known (see e.g. [BdlH20, 1.H.6]), this implies that T is itself of the form $T = m_\varphi$ for some $\varphi \in L^\infty(R^d)$. Since T commutes with all $U_{\text{Schr}}(a)$'s, the function φ is invariant by the translations $x \mapsto x+a$. So, φ is constant and T is a multiple of the identity. Schur's lemma shows that $(U_{\text{Schr}}, V_{\text{Schr}})$ is irreducible.

Next, let (U, V) be a pair of unitary representations of R^d on separable Hilbert space \mathcal{H} , satisfying $(\text{CCR})_\lambda$. Then, by Theorem A, there exists a unitary isomorphism

$$T : \oplus_{i \in \mathbf{N}} \mathcal{H}_i \rightarrow \oplus_{j \in \mathbf{N}} \mathcal{K}_j,$$

which intertwines $(U^{(\infty)}, V^{(\infty)})$ and $(U_{\text{Schr}}^{(\infty)}, V_{\text{Schr}}^{(\infty)})$, where the \mathcal{H}_i 's are copies of \mathcal{H} and the \mathcal{K}_j 's are copies of $L^2(R^d)$. Let J be the set of all $j \in \mathbf{N}$ for which the map

$$T_j := P_j \circ T|_{\mathcal{H}_1} : \mathcal{H}_1 \rightarrow \mathcal{K}_j$$

is non zero. Since T_j intertwines (U, V) and $(U_{\text{Schr}}, V_{\text{Schr}})$ and since $(U_{\text{Schr}}, V_{\text{Schr}})$ is irreducible, the map $\oplus_{j \in J} T_j : \mathcal{H}_1 \rightarrow \oplus_{j \in J} \mathcal{K}_j$ is a unitary isomorphism and sets up an equivalence between (U, V) and $(U_{\text{Schr}}^{(|J|)}, V_{\text{Schr}}^{(|J|)})$.

2.3. A lemma on partitions. For the proof of Theorem C, we will need the following general lemma about partitions.

Lemma 3. *Let G be a second-countable locally compact group and let U be a compact neighbourhood of the identity in G with $U^{-1} = U$. We can find a partition of $G = \coprod_{i \geq 1} \Omega_i$ in countably many Borel subsets $\Omega_i^{(n)}$ with the following properties:*

- (P1) *For every $i \geq 1$ and any elements x, y in Ω_i , we have $y^{-1}x \in U$;*
- (P2) *Ω_i is contained in the closure of the interior $\text{Int}(\Omega_i)$ of Ω_i , for every $i \geq 1$;*

Proof. Since G is second-countable, G contains a countable dense subset $D = \{d_1, d_2, \dots\}$. Let V be an open neighbourhood of the identity with $V^{-1} = V$ and $V^2 \subset U$. Let $g \in G$; since gV is a neighbourhood of g , we have $gV \cap D \neq \emptyset$, that is, $g \in DV^{-1} = DV$. Therefore, we have $G = \bigcup_{i \geq 1} d_i V$. Set $\Omega_1 := d_1 \bar{V}$ and, for define recursively Ω_i by

$$\Omega_{i+1} = d_{i+1} \bar{V} \setminus \bigcup_{1 \leq j \leq i} \Omega_j \quad \text{for all } i \geq 1.$$

It is obvious that $G = \bigcup_i \Omega_i$ is a partition of G ; moreover, for $x, y \in \Omega_i$, we have

$$y^{-1}x \in \bar{V}^{-1} \bar{V} = \bar{V}^2 \subset \bar{U} = U$$

and so Condition (P1) is satisfied.

We claim that Condition (P2) is also satisfied. Indeed, $\text{Int}(\Omega_1) = d_1 V$ is dense in Ω_1 . Moreover, for every $i \geq 2$,

$$\bigcup_{1 \leq j \leq i-1} \Omega_j = \bigcup_{1 \leq j \leq i-1} d_j \bar{V}$$

is closed in G ; therefore, the set $d_i V \setminus \bigcup_{1 \leq j \leq i-1} \Omega_j$ is open in G and dense in Ω_i . \square

2.4. Proof of Theorem C. When Condition (Isom) from Theorem A is not satisfied, the pairs (\bar{U}^m, \bar{V}^m) and (\tilde{U}, \tilde{V}) from Proposition 2 are no longer equivalent. We will first realize (\tilde{U}, \tilde{V}) on $L^2(\hat{S}, \mathcal{H})$ by means of the Fourier transform. Let ν be the Haar measure on \hat{S} for which Plancherel's theorem holds, that is, for which the Fourier transform $\mathcal{F} : L^2(S, \mu \otimes \mu) \rightarrow L^2(\hat{S}, \nu)$, defined by

$$\mathcal{F}(f)(\chi) = \int_S \overline{\chi(x, y)} f(x, y) d\mu(x) d\mu(y) \quad \text{for all } \chi \in \hat{S},$$

is an isometry. We extend \mathcal{F} to an isometry $L^2(S, \mathcal{H}) \rightarrow L^2(\hat{S}, \mathcal{H})$, which we again denote by \mathcal{F} , by

$$\mathcal{F}(F)(\chi) = \int_S \overline{\chi(x, y)} F(x, y) d\mu(x) d\mu(y) \quad \text{for all } F \in L^2(S, \mathcal{H}), \chi \in \hat{S}.$$

The inverse \mathcal{F}^{-1} of \mathcal{F} is given by

$$\mathcal{F}(H)(x, y) = \int_{\hat{S}} \chi(x, y) H(\chi) d\nu(\chi) \quad \text{for all } H \in L^2(\hat{S}, \mathcal{H}), (x, y) \in S.$$

We transfer the pair (\tilde{U}, \tilde{V}) to the pair $(\mathcal{F}(\tilde{U}), \mathcal{F}(\tilde{V}))$ on $L^2(\hat{S}, \mathcal{H})$ given by

$$\mathcal{F}(\tilde{U})(a) = \mathcal{F}\tilde{U}(a)\mathcal{F}^{-1} \quad \text{and} \quad \mathcal{F}(\tilde{V})(b) = \mathcal{F}\tilde{V}(b)\mathcal{F}^{-1}.$$

Using the translation invariance of μ , one checks that

$$(*) \begin{cases} \mathcal{F}(\tilde{U})(a)(H)(\chi_1, \chi_2) = \chi_1(a)U(a)(H(\chi_1, \chi_2)) \\ \mathcal{F}(\tilde{V})(b)(H)(\chi_1, \chi_2) = \chi_2(b)V(b)(H(\chi_1, \chi_2)) \end{cases}$$

for all $H \in L^2(\hat{S}, \mathcal{H})$, $(a, b) \in S$, and $(\chi_1, \chi_2) \in \hat{S}$.

The pair $(U^{(m)}, V^{(m)})$ is realized on $L^2(\widehat{S}, \mathcal{H}) \cong \mathcal{H}^{(m)}$ by

$$(**) \begin{cases} U^m(a)(H)(\chi_1, \chi_2) = U(a)(H(\chi_1, \chi_2)) \\ V^m(b)(H)(\chi_1, \chi_2) = V(b)(H(\chi_1, \chi_2)). \end{cases}$$

Recall that \widehat{S} , equipped with the topology of uniform convergence on compact subsets of S , is a locally compact group. Moreover, \widehat{S} is second-countable, since R is assumed to be second-countable (see e.g. [BdlH20, A.7.3]).

We claim that the image of $\nabla_\lambda^{(2d)} : S \mapsto \widehat{S}$ is dense in \widehat{S} . Indeed, let $(x, y) \in S$ be such that $\chi(x, y) = 1$ for all $\chi \in \nabla_\lambda^{(2d)}(S)$, that is, such that $\lambda(x \cdot a)\lambda(y \cdot b) = 1$ and hence $\lambda(x \cdot a) = \lambda(y \cdot b) = 1$ for all $a, b \in R^d$. By Conditions (Sym) and (Faith), it follows that $x = y = 0$. Pontrjagin's duality implies then that the subgroup $\nabla_\lambda^{(2d)}(S)$ is dense in \widehat{S} .

We can now proceed with the proof of Theorem C. Fix a compact subset K of R^d and a real number $\varepsilon > 0$. We are going to construct a unitary isomorphism $\Phi : L^2(\widehat{S}, \mathcal{H}) \rightarrow L^2(\widehat{S}, \mathcal{H})$ such that

$$\sup_{a \in K} \|\Phi \mathcal{F}(\widetilde{U})(a) \Phi^{-1} - U^m(a)\| \leq \varepsilon \quad \text{and} \quad \sup_{b \in K} \|\Phi \mathcal{F}(\widetilde{V})(b) \Phi^{-1} - V^m(b)\| \leq \varepsilon.$$

Set $C := K \times K$ and consider the compact neighbourhood U of 1_S (the identity element in \widehat{S}) defined by

$$U = \{\chi \in \widehat{S} : \sup_{s \in C} |\chi(s) - 1| \leq \varepsilon\}.$$

Let $\widehat{S} = \coprod_{i \geq 1} \Omega_i$ be a partition of \widehat{S} in countably many Borel subsets Ω_i satisfying Conditions (P1) and (P2) from Lemma 3.

Since $\nabla_\lambda^{(2d)}(S)$ is dense in \widehat{S} , it follows from Condition (P2) that we can find, for every $i \geq 1$, an element $s_i \in S$ with $\nabla_\lambda^{(2d)}(s_i) \in \Omega_i$. Let $a_i, b_i \in R^d$ be such that $s_i = (a_i, b_i)$.

We define an operator

$$\Phi : L^2(\widehat{S}, \mathcal{H}) \rightarrow L^2(\widehat{S}, \mathcal{H})$$

as follows. For $F \in L^2(\widehat{S}, \mathcal{H})$, set

$$\Phi(F)(\chi) = V(-a_i)U(b_i)(F(\chi)) \quad \text{for all } i \geq 1 \quad \text{and} \quad \chi \in \Omega_i.$$

We have

$$\begin{aligned} \|\Phi(F)\|^2 &= \sum_{i \geq 1} \int_{\Omega_i} \|V(-a_i)U(b_i)(F(\chi))\|^2 d\nu(\chi) \\ &= \sum_{i \geq 1} \int_{\Omega_i} \|F(\chi)\|^2 d\nu(\chi) \\ &= \|F\|^2. \end{aligned}$$

Therefore, Φ is an isometry. Moreover, Φ is bijective with inverse defined by

$$\Phi^{-1}(F)(\chi) = U(-b_i)V(a_i)(F(\chi)) \quad \text{for all } i \geq 1 \quad \text{and} \quad \chi \in \Omega_i.$$

So, $\Phi : L^2(\widehat{S}, \mathcal{H}) \rightarrow L^2(\widehat{S}, \mathcal{H})$ is a unitary isomorphism.

Let $s = (a, b) \in S$ and $F \in L^2(\widehat{S}, \mathcal{H})$. For $i \geq 1$ and $\chi = (\chi_1, \chi_2) \in \Omega_i$, on the one hand, we have, in view of Formula (*) for the pair $(\mathcal{F}(\widetilde{U}), \mathcal{F}(\widetilde{V}))$:

$$\begin{aligned} \mathcal{F}(\widetilde{U})(a)(\Phi(F))(\chi) &= \chi_1(a)U(a)(\Phi(F)(\chi)) \\ &= \chi_1(a)U(a)(F(\chi)) \end{aligned}$$

and similarly

$$\begin{aligned}\mathcal{F}(\tilde{V})(b)(\Phi(F))(\chi) &= \chi_2(b)V(b)(\Phi(F)(\chi)) \\ &= \chi_2(b)V(b)(\Phi(F)(\chi));\end{aligned}$$

on the other hand, we have, in view of Formula (**) for the pair $(U^{(m)}, V^{(m)})$:

$$\begin{aligned}(\Phi(U^{(m)}(a)F))(\chi) &= \Phi(U(a)(F(\chi))) \\ &= V(-a_i)U(b_i)U(a)(F(\chi)) \\ &= V(-a_i)U(a)U(b_i)(F(x, y)) \\ &= \lambda(a \cdot a_i)U(a)(V(-a_i)U(b_i)(F(\chi))) \\ &= \lambda(a_i \cdot a)U(a)(\Phi(F(\chi)))\end{aligned}$$

and similarly

$$\begin{aligned}(\Phi(V^{(m)}(b)F))(\chi) &= \Phi(V(b)(F(\chi))) \\ &= V(-a_i)U(b_i)V(b)(F(\chi)) \\ &= \lambda(b_i \cdot b)V(b)\Phi(F(\chi)).\end{aligned}$$

Now, by the choice of $s_i = (a_i, b_i)$, we have $\nabla_\lambda^{(2d)}(s_i) \in \Omega_i$; hence, by Condition (P1), we have $\chi \in \nabla_\lambda^{(2)}(s_i)U$, that is,

$$\sup_{(a,b) \in C} |\chi(a, b) - \nabla_\lambda^{(2)}(s_i)(a, b)| \leq \varepsilon$$

and consequently, using Condition (Sym),

$$\sup_{a \in K} |\chi_1(a) - \lambda(a_i \cdot a)| \leq \varepsilon \quad \text{and} \quad \sup_{b \in K} |\chi_2(b) - \lambda(b_i \cdot b)| \leq \varepsilon.$$

We have therefore, for $a \in K$,

$$\begin{aligned}\|\mathcal{F}(\tilde{U})(a)\Phi(F) - \Phi(U^{(m)}(a)F)\|^2 &= \sum_{i \geq 1} \int_{\Omega_i} |\chi_1(a) - \lambda(a_i \cdot a)|^2 \|U(a)(\Phi(F)(\chi))\|^2 d\nu(\chi) \\ &= \sum_{i \geq 1} \int_{\Omega_i} |\chi_1(a) - \lambda(a_i \cdot a)|^2 \|\Phi(F)(\chi)\|^2 d\nu(\chi) \\ &\leq \varepsilon^2 \sum_{i \geq 1} \int_{\Omega_i} \|\Phi(F)(\chi)\|^2 d\nu(\chi) = \varepsilon^2 \|\Phi(F)\|^2 \\ &= \varepsilon^2 \|F\|^2\end{aligned}$$

and, similarly, for $b \in K$,

$$\|\mathcal{F}(\tilde{V})(b)\Phi(F) - \Phi(V^{(m)}(b)F)\|^2 \leq \varepsilon^2 \|F\|^2.$$

So, $(U^{(m)}, V^{(m)})$ is approximately equivalent to $(\mathcal{F}(\tilde{U}), \mathcal{F}(\tilde{V}))$, which is obviously equivalent to (\tilde{U}, \tilde{V}) ; since, by Proposition 2, (\tilde{U}, \tilde{V}) is equivalent to $(U_{\text{reg}}^{(m)}, V_{\text{reg}}^{(m)})$, this concludes the proof of Theorem C.

3. EXAMPLES

We give a few examples of locally compact rings R and unitary characters λ in connection with the various Conditions (Isom), (Sym), and (Faith).

- (1) Let \mathbf{k} be a locally compact field, that is, a non-discrete locally compact topological field; it is well-known (see e.g. [Bou64, Chap VI, §9]) that \mathbf{k} is topologically isomorphic to one of the following fields and that its topology is defined by the corresponding absolute value:

- \mathbf{R} or \mathbf{C} with the usual absolute value $|\cdot|$.

- a finite extension of the field of p -adic numbers \mathbf{Q}_p with an extension of the p -adic absolute value $|\cdot|_p$.
- the field $\mathbf{F}((X))$ of Laurent series over a finite field \mathbf{F} with absolute value $|\sum_{i=m}^{\infty} a_i X^i| = e^{-m}$ with $a_m \in \mathbf{F} \setminus \{0\}$.

Let $\lambda \in \widehat{\mathbf{k}}$ be a non trivial character of the additive group of \mathbf{k} . The map

$$\nabla_\lambda : \mathbf{k} \mapsto \widehat{\mathbf{k}}, a \mapsto (x \mapsto \lambda(ax))$$

is an isomorphism of topological groups (see [Wei74, Chap. II, Theorem 3] for a more general result); for the convenience of the reader, here is a direct proof. The kernel of ∇_λ is an ideal of \mathbf{k} and hence is trivial, since $\lambda \neq 1_{\mathbf{k}}$. By Pontrjagin duality, the image of ∇_λ is a dense subgroup H of $\widehat{\mathbf{k}^d}$ (see the argument at the beginning of Subsection 2.4). We claim that the map ∇_λ , which is obviously continuous, is in fact a homeomorphism between \mathbf{k} and H . Indeed, assume, by contradiction, that $\nabla_\lambda^{-1} : H \rightarrow \mathbf{k}$ is not continuous. Then, there exists a sequence $(a_n)_n$ in \mathbf{k} and a number $\delta > 0$ with $|a_n| > \delta$ for all n such that $\lim_n \nabla_\lambda(a_n)(a) = 1$ uniformly on bounded subsets of \mathbf{k} . Since λ is non trivial, we can find $a \in \mathbf{k}$ with $\lambda(a) \neq 1$. However, since $|aa_n^{-1}| \leq \frac{|a|}{\delta}$, we have

$$\lambda(a) = \lambda(aa_n^{-1}a_n) = \lim_n \lambda(aa_n^{-1}a_n) = \lim_n \nabla_\lambda(a_n)(aa_n^{-1}) = 1$$

and this is a contradiction. So, ∇_λ^{-1} is continuous and hence H is locally compact; this implies that it is a closed subgroup of $\widehat{\mathbf{k}}$ and, since it is also dense, we have $H = \widehat{\mathbf{k}}$.

- (2) Let $R = \mathbf{A}$ be the ring of adèles of \mathbf{Q} . Recall that \mathbf{A} is the restricted product $\mathbf{A} = \mathbf{R} \times \prod_{p \in \mathcal{P}} (\mathbf{Q}_p, \mathbf{Z}_p)$ relative to the subgroups \mathbf{Z}_p ; thus,

$$\mathbf{A} = \left\{ (x_\infty, (x_p)_{p \in \mathcal{P}}) \in \mathbf{R} \times \prod_{p \in \mathcal{P}} \mathbf{Q}_p : x_p \in \mathbf{Z}_p \text{ for almost every } p \in \mathcal{P} \right\},$$

where \mathcal{P} be the set of primes numbers, \mathbf{Q}_p is the field of p -adic numbers, and $\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \leq 1\}$ is the compact open subring of p -adic integers.

For every finite subset $F \subset \mathcal{P}$,

$$\mathbf{A}(F) = \mathbf{R} \times \prod_{p \in F} \mathbf{Q}_p \times \prod_{p \notin F} \mathbf{Z}_p,$$

equipped with the product topology, is a locally compact ring. A ring topology is defined on \mathbf{A} on prescribing that every $\mathbf{A}(F)$ is an open subring of \mathbf{A} . With this topology, \mathbf{A} is a locally compact ring.

Let λ be a nontrivial unitary character of \mathbf{A} . An example of such a character can be obtained as follows. Let $\lambda_\infty \in \widehat{\mathbf{R}}$ be given by $\lambda_\infty(x) = e^{-2\pi i x}$; for every $p \in \mathcal{P}$, let $\lambda_p \in \widehat{\mathbf{Q}_p}$ be given by $\lambda_p(x) = \exp(2\pi i \{x\})$, where $\{x\} = \sum_{j=m}^{-1} a_j p^j$ denotes the ‘‘fractional part’’ of a p -adic number $x = \sum_{j=m}^{\infty} a_j p^j$ for integers $m \in \mathbf{Z}$ and $a_j \in \{0, \dots, p-1\}$. Define now $\lambda \in \widehat{\mathbf{A}}$ by

$$\lambda(x_\infty, (x_p)_{p \in \mathcal{P}}) = \lambda_\infty(x_\infty) \prod_{p \in \mathcal{P}} \lambda_p(x_p) \quad \text{for all } (x_\infty, (x_p)_{p \in \mathcal{P}}) \in \mathbf{A}$$

(the product of the right hand side of the formula is well-defined, since $x_p \in \mathbf{Z}_p$ for almost every p and since λ_p is trivial on \mathbf{Z}_p for every p). Then

λ satisfies Condition (Isom), that is,

$$\mathbf{A} \mapsto \widehat{\mathbf{A}}, a \mapsto (b \mapsto \lambda(ba))$$

is a topological isomorphism (see [Wei74, Chap. IV, Theorem 3]).

- (3) Let \mathbf{k} be a local field. For an integer $n \geq 1$, let $R = M_n(\mathbf{k})$ be the ring of $n \times n$ -matrices over \mathbf{k} ; it is a locally compact ring for the topology obtained by identifying the additive group of $M_n(\mathbf{k})$ with \mathbf{k}^{n^2} . Let $\lambda_0 \in \widehat{\mathbf{k}}$ be a non trivial character of \mathbf{k} . Let $\lambda \in \widehat{M_n(\mathbf{k})}$ be defined by

$$\lambda(A) = \lambda_0(\text{Tr}(A)) \quad \text{for all } A \in M_n(\mathbf{k}),$$

where $\text{Tr}(A) \in \mathbf{k}$ is the trace of A . Since $\text{Tr}(AB) = \text{Tr}(BA)$ for all A, B , the character λ satisfies the symmetry Condition (Sym). It satisfies Condition (Isom) as well. Indeed, in view of Condition (Sym), the kernel of the map

$$\nabla_\lambda : M_n(\mathbf{k}) \mapsto \widehat{M_n(\mathbf{k})}, A \mapsto (X \mapsto \lambda(XA))$$

is a two-sided ideal of $M_n(\mathbf{k})$; since $M_n(\mathbf{k})$ is a simple ring and λ is non trivial, it follows that ∇_λ is injective.

On the one hand, viewing the additive group of $M_n(\mathbf{k})$ as \mathbf{k}^{n^2} , Item (1) above shows that every $\chi \in \widehat{M_n(\mathbf{k})}$ is of the form $X \mapsto \lambda(\varphi(X))$ for some linear form $\varphi \in \text{Hom}_{\mathbf{k}}(M_n(\mathbf{k}), \mathbf{k})$ on $M_n(\mathbf{k})$. On the other hand, it is well-known that every such φ can be written as the map $X \mapsto \text{Tr}(XA)$ for some $A \in M_n(\mathbf{k})$. This shows that the continuous injective map ∇_λ is surjective and is hence a topological isomorphism.

- (4) The previous example can be generalized as follows to simple central algebras over local fields; for general facts concerning these algebras, see [Wei74, Chap IX].

Let \mathbf{k} be a local field. Let \mathcal{A} be a finite-dimensional central simple algebra over \mathbf{k} . Then \mathcal{A} is isomorphic to a matrix algebra $M_n(\mathbf{D})$ over some central division algebra \mathbf{D} over \mathbf{k} for some $n \geq 1$. For instance, if $\mathbf{k} = \mathbf{R}$, then \mathbf{D} is necessarily isomorphic to \mathbf{R} or to \mathbf{H} , where \mathbf{H} is the 4-dimensional division algebra of the usual real quaternions. Let $\tau \in \text{Hom}_{\mathbf{k}}(\mathcal{A}, \mathbf{k})$ be the so-called *reduced trace* on \mathcal{A} , which is defined as follows: let \mathbf{K} be the algebraic closure of \mathbf{k} ; then there exists an algebra isomorphism $F : \mathcal{A} \otimes_{\mathbf{k}} \mathbf{K} \rightarrow M_r(\mathbf{K})$, for some r and one defines $\tau(a) = \text{Tr}(F(a))$ for $a \in \mathcal{A}$.

Let λ_0 be a non trivial unitary character of \mathbf{k} and define $\lambda \in \widehat{\mathcal{A}}$ by

$$\lambda(a) = \lambda_0(\tau(a)) \quad \text{for all } a \in \mathcal{A}.$$

Then λ satisfies Conditions (Sym) and (Isom). For instance, when \mathcal{A} is the algebra of real quaternions

$$\mathbf{H} = \{t + ui + vj + wk : t, u, v, w \in \mathbf{R}\},$$

we have $\tau(a) = 2t$ for $a = t + ui + vj + wk$ and so, taking $\lambda_0 = e^{2\pi \cdot}$, we have

$$\lambda(a) = e^{4\pi it} \quad \text{for all } a = t + ui + vj + wk \in \mathbf{H}.$$

- (5) Let R be a finite field or a direct sum of finite fields. It can easily be checked that every non trivial unitary character λ of R satisfies Condition (Isom). Then same holds when R is the ring $\mathbf{Z}/n\mathbf{Z}$ of integers modulo $n \in \mathbf{N}^*$.
- (6) For $R = \mathbf{Z}$, there exists *no* $\lambda \in \widehat{\mathbf{Z}}$ which satisfies Condition (Isom). Indeed, every $\lambda \in \widehat{\mathbf{Z}}$ is of the form

$$\lambda_\theta(k) = e^{2\pi i\theta k} \quad \text{for all } k \in \mathbf{Z},$$

for a uniquely determined real number $\theta \in [0, 1)$. In particular, $\widehat{\mathbf{Z}}$ is uncountable and, since \mathbf{Z} is countable, $\nabla_\lambda : \mathbf{Z} \rightarrow \widehat{\mathbf{Z}}$ cannot be surjective. Observe, however, that Condition (Faith) is satisfied by λ_θ for every irrational θ .

- (7) Similarly to the previous Item, if R any countable infinite ring (equipped with the discrete topology), then there exists no $\lambda \in \widehat{R}$ which satisfies Condition (Isom).
- (8) Let $R = \mathbf{Z}_p$ be the ring of p -adic integers for a prime number p . Then, since \mathbf{Z}_p is compact and infinite, its dual group $\widehat{\mathbf{Z}_p}$ is a discrete infinite group; in fact, the unitary character $\lambda_p \in \widehat{\mathbf{Q}_p}$ from Item (2) induces an isomorphism $\mathbf{Q}_p/\mathbf{Z}_p \cong \widehat{\mathbf{Z}_p}$, given by $a + \mathbf{Z}_p \mapsto \nabla_{\lambda_p}(a)|_{\mathbf{Z}_p}$. So, there exists no $\lambda \in \widehat{\mathbf{Z}_p}$ which satisfies Condition (Faith) and, let alone, Condition (Isom). However, one can easily determine the pairs of unitary representations (U, V) of \mathbf{Z}_p on a Hilbert space \mathcal{H} satisfying $(\text{CCR})_\lambda$ for any $\lambda \in \widehat{\mathbf{Z}_p}$; indeed, assume by simplicity that (U, V) is irreducible. The sequence of ideals $J_n = p^n \mathbf{Z}_p$ ($n \geq 1$) is a basis of neighbourhoods in 0 in \mathbf{Z}_p . So, there exists $N \geq 1$ such that $\lambda(J_N) = 1$ and such that the space \mathcal{H}^{J_N} of invariant vectors under $U(J_N)$ and $V(J_N)$ is non zero (see [BdlHV08, 1.1.5]); by irreducibility, we have then $\mathcal{H} = \mathcal{H}^{J_N}$. This shows that λ and (U, V) factorize to a character $\bar{\lambda}$ and a pair (\bar{U}, \bar{V}) of unitary representations of the finite quotient ring $\mathbf{Z}_p/J_N \cong \mathbf{Z}/p^N \mathbf{Z}$ satisfying $(\text{CCR})_{\bar{\lambda}}$.
- (9) The conclusion of the previous Item can be generalized to any compact ring. Indeed, let R be such a ring (commutative or not). Then R is a profinite ring, that is, R is topologically isomorphic to a projective limit $\varprojlim_i R_i$ of finite rings R_i (see [RZ00, Proposition 5.1.2]). As in the previous item, the study of pairs of unitary representations (U, V) of R satisfying $(\text{CCR})_\lambda$ for any $\lambda \in \widehat{R}$ can be reduced to the study of such pairs for the finite rings R_i .

4. HEISENBERG GROUPS

We rephrase our results in terms of representations of Heisenberg groups. Let R be a (not necessarily commutative) locally compact ring with unit and let $d \geq 1$ be an integer. The *Heisenberg group* of dimension $2d + 1$ over R is defined as the group $H_{2d+1}(R)$ of $(2d + 1) \times (2d + 1)$ -matrices with coefficients in R of the form

$$m(a, b, c) = m(a_1, \dots, a_d, b_1, \dots, b_d, c) := \begin{pmatrix} 1 & a_1 & \dots & a_d & c \\ 0 & 1 & \dots & 0 & b_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_d \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

The group law on $H_{2d+1}(R)$ is given by

$$m(a, b, c)m(a', b', c') = m(a + a', b + b', c + c' + a \cdot b'),$$

for $a, b, a', b' \in R^d$ and $c, c' \in R$. Equipped with the topology induced from $M_{2d+1}(R)$, the group $H_{2d+1}(R)$ is a locally compact group. Its center is

$$C = \{m(0, 0, c) : c \in R\} \cong R.$$

Let $\pi : G \rightarrow U(\mathcal{H})$ and $\pi' : G \rightarrow U(\mathcal{H}')$ be two unitary representations of a locally compact group G on Hilbert spaces \mathcal{H} and \mathcal{H}' . Recall that π and π' are equivalent if there exists a unitary isomorphism $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$\Phi\pi(g)\Phi^* = \pi'(g) \quad \text{for all } g \in G.$$

We say that π and π' are *approximately equivalent* if there exists a sequence $(\Phi_n)_n$ of unitary isomorphisms $\Phi_n : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that, for every compact subset K of G , we have

$$\lim_{n \rightarrow \infty} \sup_{g \in K} \|\Phi_n \pi(g) \Phi_n^* - \pi'(g)\| = 0.$$

Approximate equivalence is an equivalence relation on, say, the set of unitary representations of G on separable Hilbert spaces.

Fix $\lambda \in \widehat{R}$. Let (U, V) be a pair of unitary representations of R^d on a Hilbert space \mathcal{H} satisfying $(\text{CCR})_\lambda$. For $m(a, b, c) \in H_{2d+1}(R)$, define a unitary operator $\pi(m(a, b, c))$ on \mathcal{H} by

$$\pi(m(a, b, c)) = \lambda(c)V(b)U(a).$$

One checks immediately that $\pi : H_{2d+1}(R) \rightarrow U(\mathcal{H})$ is a unitary representation of $H_{2d+1}(R)$; observe that π has λ as central character, that is, $\pi(m(0, 0, c)) = \lambda(c)\text{Id}_{\mathcal{H}}$ for $c \in R$. If (U', V') is another pair which is equivalent (respectively, approximately equivalent) to (U, V) , then the corresponding representation π' is equivalent (respectively, approximately equivalent) to π .

Conversely, let $\pi : H_{2d+1}(R) \rightarrow U(\mathcal{H})$ be a unitary representation of $H_{2d+1}(R)$ which has λ as central character. Define

$$U(a) = \pi(m(a, 0, 0)) \quad \text{and} \quad V(b) = \pi(m(0, b, 0))$$

for $a, b \in R^d$. Then (U, V) is a pair of unitary representations of R^d on \mathcal{H} satisfying $(\text{CCR})_\lambda$. Moreover, if π' is another representation of $H_{2d+1}(R)$ which is equivalent (respectively, approximately equivalent) to π , then π' has λ as central character and the corresponding pair (U', V') is equivalent (respectively, approximately equivalent) to (U, V) .

Summarizing, the map $(U, V) \mapsto \pi$ described above induces a bijection between

- the set of equivalence (respectively, approximate equivalence) pairs of unitary representations of R^d satisfying $(\text{CCR})_\lambda$, and
- the set of equivalence (respectively, approximate equivalence) classes of unitary representations of $H_{2d+1}(R)$ which have λ as central character,

The representation π_{Schr} of $H_{2d+1}(R)$ corresponding to the Schrödinger pair $(U_{\text{Schr}}, V_{\text{Schr}})$ with central character λ is given on $L^2(R^d, \mu)$ by

$$\pi_{\text{Schr}}(m(a, b, c))f(x) = \lambda(c)\lambda(x \cdot b)f(x + a) \quad f \in L^2(R^d, \mu),$$

where μ is a Haar measure on $(R^d, +)$.

We can now reformulate Theorems A and C in terms of representations of $H_{2d+1}(R)$.

Theorem D. (Stone-von Neumann Theorem) *Let R be a unital second-countable locally compact ring and let $\lambda \in \widehat{R}$ satisfying Conditions (Sym) and (Isom). For $d \geq 1$, let π be a unitary representation of $H_{2d+1}(R)$ on a separable Hilbert space with central character λ . Then $\pi^{(\infty)}$ is equivalent to $\pi_{\text{Schr}}^{(\infty)}$.*

Theorem E. *Let R be as Theorem D, and let $\lambda \in \widehat{R}$ satisfying Conditions (Sym) and (Faith). For $d \geq 1$, let π be a unitary representation of $H_{2d+1}(R)$ on a separable Hilbert space with central character λ . Then $\pi^{(\infty)}$ is approximately equivalent to $\pi_{\text{Schr}}^{(\infty)}$.*

Remark 4. A crucial role in the proofs of our results was played by the regular pair $(U_{\text{reg}}, V_{\text{reg}})$ introduced just before Proposition 2. The corresponding representation π_{reg} of $H_{2d+1}(R)$ is defined on the Hilbert space $L^2(R^d \times R^d, \mu \otimes \mu)$, by

$$(\pi_{\text{reg}}(m(a, b, c))\xi)(x, y) = \lambda(c)\lambda(x \cdot b)\xi(x + a, y + b),$$

for all $\xi \in L^2(R^d \times R^d)$ and $x, y \in R^d$. It can be checked that π_{reg} is equivalent to the representation $\text{Ind}_C^{H_{2d+1}(R)} \lambda$ induced by the character λ of the centre C .

To a pair (U, V) of unitary representations of R^d on \mathcal{H} satisfying $(\text{CCR})_\lambda$, we associated a new pair (\tilde{U}, \tilde{V}) on $L^2(R^d \times R^d, \mathcal{H})$. If π is the representation of $H_{2d+1}(R)$ corresponding to (U, V) , the representation $\tilde{\pi}$ of $H_{2d+1}(R)$ corresponding to (\tilde{U}, \tilde{V}) is given by defined by

$$(\tilde{\pi}(m(a, b, c)F)(x, y) = \lambda(c)V(b)U(a)(F(x + a, y + b))$$

for all $F \in L^2(R^d \times R^d, \mathcal{H})$ and $x, y \in R^d$. It can be checked that $\tilde{\pi}$ is equivalent to the tensor product representation $\rho \otimes \pi$, where $\rho = \text{Ind}_C^{H_{2d+1}(R)} 1_C$ is the regular representation of $H_{2d+1}(R)/C \cong R^d \times R^d$ lifted to $H_{2d+1}(R)$. Fell's absorption principle alluded to before Proposition 2 is that $\rho \otimes \pi$ is equivalent to a multiple of $\text{Ind}_C^{H_{2d+1}(R)} \lambda$.

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