



HAL
open science

Legendre approximation-based stability test for distributed delay systems *

Alejandro Castaño, Mathieu Bajodek, Sabine Mondié

► **To cite this version:**

Alejandro Castaño, Mathieu Bajodek, Sabine Mondié. Legendre approximation-based stability test for distributed delay systems *. 2025. hal-04923013

HAL Id: hal-04923013

<https://hal.science/hal-04923013v1>

Preprint submitted on 31 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Legendre approximation-based stability test for distributed delay systems*

Alejandro Castaño, Mathieu Bajodek, Sabine Mondié

2024

Abstract

This contribution presents an exponential stability criterion for linear systems with multiple pointwise and distributed delays. This result is obtained in the Lyapunov-Krasovskii framework via the approximations of the argument of the functional by projection on the first Legendre polynomials. The reduction of the number of mathematical operations in the stability test is a benefit of the supergeometric convergence of Legendre polynomials approximation. For a single-delay linear system with a constant distributed kernel, a new computational procedure for the solution of the integrals involved in the stability test is developed considering the case of Jordan nilpotent blocks. This strategy is the basis for developing new procedures that allow the numerical construction of the stability test for different classes of kernels, such as polynomial, exponential, or γ distribution.

Keywords: linear functional differential equation, necessary and sufficient stability condition, Legendre polynomials approximation.

1 Introduction

Time delays inevitably arise in all systems that involve data and signals processing, propagation phenomena, measurement of variables, or due to the intrinsic nature of the system's behavior. Particularly, distributed delays are a special class of time delays that model the cumulative effect of past history on the dynamics of a system, and thus the interactions between its different components [16]. Consequently, systems with distributed delays are realistic models of various phenomena arising in physics [29], engineering [18], biology [24], ecology [11], and other disciplines. Motivated by its potential applications, the effect of distributed delay on the stability of time-delay systems has been a problem of general interest from both theoretical and practical perspectives.

The Lyapunov-Krasovskii framework which uses, instead of Lyapunov functions, functionals that capture the true state of the system is widely used to derive stability conditions. A first approach consists of the proposal of functionals of prescribed form leading to sufficient stability conditions formulated in terms of linear matrix inequalities (see for example [12, 23, 26], and the references therein). A second approach is based on converse results: it consists of determining the form of the functional with a given prescribed negative quadratic derivative along the solutions of the system (see [9, 28, 15, 25]).

In [17], Lyapunov-Krasovskii functionals of complete type are presented. These functionals are defined in terms of a matrix function called delay Lyapunov matrix which satisfies the continuity, dynamic, symmetric, and algebraic properties [16]. Moreover, the functional admits a quadratic lower bound if the system is exponentially stable. This approach opens the way to finding necessary and sufficient stability conditions.

A first progress was to present necessary stability conditions expressed in terms of the system delay Lyapunov matrix for several classes of delay systems [21]. The result is achieved by replacing the functional argument by an approximation depending on the system's fundamental matrix. The new stability/instability theorems on a special set of bounded functional arguments introduced in [20, 19] are crucial for estimating the approximation order required to ensure sufficiency. Unfortunately, the poor convergence properties of the fundamental matrix-based approximations resulted in high-order estimates that often exceed computer capacity. For retarded type systems, recent research revealed that the methodology presented in [13, 21] can be applied to other approximation techniques of the functional argument, significantly reducing the numerical complexity. In particular, new necessary and sufficient conditions were formulated using Legendre polynomials [4] and piecewise linear functions approximations [1]. Nevertheless, for systems with multiple pointwise and distributed delays, the problem remains open.

For the case of systems with distributed delays, through fundamental matrix-based approximations, stability criteria are presented in terms of point values of the delay Lyapunov matrix and the system's fundamental matrix [6]; nevertheless, limited by the discretization schema, the overlarge orders of approximation prompt that its numerical implementation demands a high computational effort. This contribution overcomes this dimensional issue by approximating the Lyapunov-Krasovskii functional's argument through projection on a Legendre polynomial basis. Therefore, a stability criterion that depends on integrals of the Lyapunov matrix, the first Legendre polynomials, and the kernel of distributed delay is derived. The reason for using Legendre polynomials approximation is to exploit

*This work was supported by project CONACYT A1-S-24796, Mexico.

the following underlying properties: 1) rapid convergence for smooth arguments, 2) orthogonality concerning the Lebesgue measure, and 3) second-order recurrence satisfied by the coefficients. These properties help to significantly reduce the dimension of the stability test, which is numerically tractable.

The criterion presented is deduced from three key elements: 1) generalization of the stability/instability theorems on a bounded set of the class of distributed delay systems, 2) determination of the quadratic form resulting from the substitution of the Legendre approximation of the argument, and estimation of the resulting functional approximation error, 3) use of the stability/instability theorems to determine the approximation order guaranteeing sufficiency. The numerical evaluation of the stability criterion depends on the delay Lyapunov matrix, the first Legendre polynomials, and the kernel class in the distributed delay; therefore, a particular recursive method must be developed for each kernel type. We present an efficient computational procedure based on the Jordan canonical form to solve the first Legendre coefficient delay Lyapunov matrix-based integrals involved in the stability test for a single-delay linear system with a constant distributed kernel delay. This procedure provides the methodology for obtaining recursive methods for different systems with pointwise delays multiple of a basic one and distributed terms with piecewise constant, polynomial, exponential, or combinations such as γ distribution.

The organization of the paper is as follows: 2 is devoted to some preliminaries on systems with distributed delays, and to the generalization of the fundamental stability theorems. Additionally, the approximation of functions by Legendre polynomials and their supergeometric convergence are presented. The main results and their proofs are presented in 3. In 4, recursive relations for the effective computation of the quadratic form resulting from the substitution of the Legendre approximation into the functional are presented, emphasizing the special case of Jordan nilpotent blocks. The paper ends with illustrative examples in 5, followed by concluding remarks.

Notation: We denote the space of piecewise continuous, continuous, continuously differentiable, and smooth functions defined from X with values in Y by $\mathcal{PC}(X, Y)$, $\mathcal{C}(X, Y)$, $\mathcal{C}_1(X, Y)$, $\mathcal{C}_\infty(X, Y)$, respectively and let $\mathcal{PC}_h = \mathcal{PC}([-h, 0], \mathbb{R}^{n_x})$. In the paper, the Euclidean 2-norm for vectors and the induced norm for matrices are denoted by $\|\cdot\|$. For functions φ , we use the uniform norm $\|\varphi\|_h = \sup_{\theta \in [-h, 0]} \|\varphi(\theta)\|$. The transpose of a matrix A is denoted

by A^T , and the $q \times q$ identity matrix by I_q . The minimum and maximum eigenvalues of a symmetric matrix Q are represented by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$, respectively. The notation $Q > 0$ ($Q \geq 0$, $Q \not\geq 0$) means that the symmetric square matrix Q is positive definite (positive semidefinite, not positive semidefinite). The symbol $\lceil \cdot \rceil$ represents the ceiling function, and δ_{jk} denotes the Kronecker delta function. The symbols \otimes , \oplus stand for the Kronecker product, and the direct sum operator of matrices, respectively, while vec and vec^{-1} indicate the vectorization and devectorization operations.

2 Preliminaries

This section recalls essential concepts for the time-domain stability analysis of linear systems with pointwise and distributed delays and for the numerical analysis based on Legendre polynomials approximation.

2.1 System and stability concept

Consider linear time-delay systems of the form

$$\begin{cases} \dot{x}(t) = \sum_{j=0}^m A_j x(t - h_j) + \int_{-h}^0 G(\theta) x(t + \theta) d\theta, \quad \forall t \geq 0, \\ x(t) = \varphi(t), \quad \forall t \in [-h, 0], \quad \varphi \in \mathcal{PC}_h. \end{cases} \quad (1)$$

where A_j , $j = 0, \dots, m$, are given real $n_x \times n_x$ matrices and $0 = h_0 < h_1 < \dots < h_m = h$ are time delays, and the function $G \in \mathcal{PC}([-h, 0], \mathbb{R}^{n_x \times n_x})$ represents the kernel of the distributed delay. The restriction of the solution $x(t)$ of initial-value problem (1) on the segment $[t - h, t]$ is defined by $x_t : \theta \rightarrow x(t + \theta)$, $\theta \in [-h, 0]$. When the initial condition φ must be indicated explicitly, we use the notations $x(t, \varphi)$ and $x_t(\varphi)$.

Definition 1 (see [5]) *System (1) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$ such that every solution of the system satisfies the inequality*

$$\|x(t, \varphi)\| \leq \gamma e^{\sigma t} \|\varphi\|_h, \quad t \geq 0.$$

Fundamental results concerning the Lyapunov-Krasovskii approach are presented hereafter.

2.2 Lyapunov-Krasovskii functional

In this contribution, we make use of special matrix-based functionals with a prescribed quadratic negative derivative. According to [17, 8], for any matrix $W > 0$, the unique functional $v(x_t)$ satisfying

$$\frac{d}{dt} v(x_t(\varphi)) = -x^T(t - h, \varphi) W x(t - h, \varphi),$$

along the trajectories of system (1) has the form

$$v(\varphi) = \varphi^T(0)U(0)\varphi(0) + \sum_{j=1}^6 I_j \quad (2)$$

where we defined the following integrals

$$\begin{aligned} I_1 &= 2\varphi^T(0) \sum_{j=1}^m \int_{-h_j}^0 U^T(\theta + h_j) A_j \varphi(\theta) d\theta, \\ I_2 &= \sum_{k=1}^m \sum_{j=1}^m \iint_{\mathcal{R}_1} \varphi^T(\theta_1) A_k^T \mathcal{F}_U(\theta_1, \theta_2, -h_k, -h_j) A_j \varphi(\theta_2) d\mathcal{R}_1, \\ I_3 &= 2\varphi^T(0) \int_{-h}^0 \int_{-h}^{\theta} U^T(\theta - \xi) G(\xi) d\xi \varphi(\theta) d\theta, \\ I_4 &= 2 \sum_{k=1}^m \iiint_{\mathcal{R}_2} \varphi^T(\theta_1) A_j^T \mathcal{F}_U(\theta_1, \theta_2, -h_k, \xi) G(\xi) \varphi(\theta_2) d\mathcal{R}_2, \\ I_5 &= \iiint \iiint_{\mathcal{R}_3} \varphi^T(\theta_1) G^T(\xi_1) \mathcal{F}_U(\theta_1, \theta_2, \xi_1, \xi_2) G(\xi_2) \varphi(\theta_2) d\mathcal{R}_3, \quad I_6 = \int_{-h}^0 \varphi^T(\theta) W \varphi(\theta) d\theta, \end{aligned}$$

with the function $\mathcal{F}_U(\theta_1, \theta_2, \xi_1, \xi_2) = U(\theta_1 - \theta_2 - \xi_1 + \xi_2)$, and the integration regions

$$\begin{aligned} \mathcal{R}_1 &:= \{\theta_1 \in [-h_k, 0], \theta_2 \in [-h_j, 0]\}, \\ \mathcal{R}_2 &:= \{\theta_1 \in [-h_k, 0], \theta_2 \in [-h, 0], \xi \in [-h, \theta_2]\}, \\ \mathcal{R}_3 &:= \{\theta_1 \in [-h, 0], \theta_2 \in [-h, 0], \xi_1 \in [-h, \theta_1], \xi_2 \in [-h, \theta_2]\}, \\ d\mathcal{R}_1 &= d\theta_2 d\theta_1, \quad d\mathcal{R}_2 = d\xi d\theta_2 d\theta_1, \quad d\mathcal{R}_3 = d\xi_2 d\xi_1 d\theta_2 d\theta_1. \end{aligned}$$

The matrix-valued function, $U(\tau)$, is known as the *delay Lyapunov matrix* of system (1), associated to W . It satisfies the following set of properties:

1. Dynamic property for $\tau > 0$

$$\frac{d}{d\tau} U(\tau) = \sum_{j=0}^m U(\tau - h_j) A_j + \int_{-h}^0 U(\tau + \theta) G(\theta) d\theta.$$

2. Symmetry property for $\tau \in \mathbb{R}$

$$U(\tau) = U^T(-\tau),$$

3. Algebraic property

$$-W = \sum_{j=0}^m \left[A_j^T U^T(-h_j) + U(-h_j) A_j \right] + \int_{-h}^0 \left[G^T(\theta) U^T(\theta) + U(\theta) G(\theta) \right] d\theta.$$

The characteristic equation of system (1) is [5]

$$\det \left[sI - \sum_{j=0}^m A_j e^{-sh_j} - \int_{-h}^0 e^{s\theta} G(\theta) d\theta \right] = 0.$$

Let $\tilde{\Lambda}$ be the spectrum of system (1), that is the set of its characteristic roots. It is said that system (1) satisfies the Lyapunov condition, if $s \in \tilde{\Lambda}$ implies $-s \notin \tilde{\Lambda}$. The Lyapunov condition guarantees the existence and uniqueness of the delay Lyapunov matrix [16].

Remark 1 *The delay Lyapunov matrix $U(\tau)$, $\tau \in [-h, h]$ can be computed using the dynamic, symmetric, and algebraic properties by a semi-analytic procedure when the delays are multiple of a basic one, and G is a piecewise continuous polynomial [16, 2]. In the numerical part, examples with a single delay and constant kernel G are considered.*

2.3 Fundamental stability theorems

The main attribute of the functional v is that it admits a quadratic lower bound on the set of functions \mathcal{PC}_h when the system is stable [10, 6]. This condition is established in the following result.

Theorem 1 [10, Theorem 1] *If system (1) is exponentially stable, then there exist positive numbers α_0 and α_1 such that*

$$v(\varphi) \geq \alpha_0 \|\varphi(0)\|^2 + \alpha_1 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \quad \varphi \in \mathcal{PC}_h. \quad (3)$$

We introduce the following compact set in the space of continuously differentiable functions parameterized by $\mu > 0$

$$\mathcal{S}_\mu := \left\{ \varphi \in \mathcal{C}_\infty([-h, 0], \mathbb{R}^{n_x}) \mid \|\varphi\|_h = \|\varphi(0)\| = 1, \|\varphi^{(k)}\|_h \leq \mu^k, \forall k \in \mathbb{N} \right\}. \quad (4)$$

The following theorem, given in the framework of the set \mathcal{S}_μ , establishes that if the system is unstable, the functional v does not admit a nonnegative lower bound.

Theorem 2 [10, Theorem 2] *Let $\mu = \sum_{j=0}^m \|A_j\| + h\|G\|_h$. If system (1) is unstable and the Lyapunov condition holds, then there exists $\hat{\varphi} \in \mathcal{S}_\mu$ such that*

$$v(\hat{\varphi}) \leq -\beta^* = -\frac{\lambda_{\min}(W)}{8\mu e^{2h\mu}}(1 + \cos b), \quad (5)$$

where $b \in (0, \pi)$ is a unique solution of the equation

$$((2h\mu)^2 + b^2)(1 - \cos b)^2 = 4(2h\mu)^2 \quad (6)$$

The preceding result reveals the crucial role of the set \mathcal{S}_μ . Indeed, by negation, 2 provides a sufficiency stability theorem, which is decisive for proving the main result of this contribution.

2.4 Legendre polynomials approximation

Key concepts about the Legendre polynomials and their use in the approximation of the initial function $\varphi \in \mathcal{PC}_h$ are introduced.

Legendre polynomials on $[-h, 0]$ are defined as

$$l_k(\theta) = \sum_{i=0}^k u_i^k \left(\frac{\theta+h}{h}\right)^i, \quad \forall k \in \mathbb{N}, \quad (7)$$

with $u_i^k = (-1)^{k+i} \binom{k}{i} \binom{k+i}{i}$, where $\binom{k}{i}$ stands for the binomial coefficient. Legendre polynomials are a system of complete and orthogonal polynomials that span the space of square-integrable functions.

For $\theta \in [-h, 0]$, we introduce the matrix $L_n(\theta) \in \mathbb{R}^{n \cdot n_x \times n_x}$ defined by

$$L_n(\theta) := \left[l_0(\theta) \ l_1(\theta) \ \cdots \ l_{n-1}(\theta) \right]^T \otimes I_{n_x}, \quad \forall \theta \in [-h, 0].$$

For any function $\varphi \in \mathcal{PC}_h$ and for any approximation order $n \in \mathbb{N}$, let us decompose $\forall \theta \in [-h, 0]$

$$\varphi(\theta) = \varphi_n(\theta) + \tilde{\varphi}_n(\theta)$$

where $\varphi_n(\theta) = L_n^T(\theta)\Phi_n$ is the polynomial approximation and $\tilde{\varphi}_n(\theta) = \varphi(\theta) - \varphi_n(\theta)$ is the residual error. The vector Φ_n represents the normalized n first polynomial coefficients of the function φ and is defined by

$$\Phi_n = \text{diag}\left(\frac{1}{h}, \dots, \frac{2n-1}{h}\right) \otimes I_{n_x} \int_{-h}^0 L_n(\theta)\varphi(\theta)d\theta \in \mathbb{R}^{n \cdot n_x}.$$

Next, we prove that the Legendre approximation φ_n converges uniformly towards φ with respect to θ and we also quantify its convergence rate on the set \mathcal{S}_μ defined in (4).

2.5 Convergence of the Legendre polynomials approximation

The following convergence lemma allows us to estimate the order that ensures that, for any $\varphi \in \mathcal{S}_\mu$, the approximation error $\tilde{\varphi}_n(\theta) = \varphi - \varphi_n$ is upper bounded by $\varepsilon > 0$.

Lemma 1 *Let $\mu > 0$ and consider $\varphi \in \mathcal{S}_\mu$. For any $\varepsilon > 0$, the Legendre truncated error function $\tilde{\varphi}_n$ satisfies the following inequality*

$$\|\tilde{\varphi}_n\|_h \leq \varepsilon, \quad \forall n \geq \mathcal{N}_\mu(\varepsilon)$$

where

$$\mathcal{N}_\mu(\varepsilon) = \left\lceil \bar{\mu} \exp \left[1 + \mathcal{W} \left((\varepsilon \bar{\mu})^{-1} \log \left(\frac{\varrho}{\varepsilon} \right) \right) \right] \right\rceil. \quad (8)$$

with $\varrho = \frac{5}{3 \min(1, (h\mu)^2)}$, $\bar{\mu} = \frac{2}{3}h\mu$ and $\mathcal{W} : \begin{cases} \mathbb{R}_+ & \rightarrow \mathbb{R}_+ \\ y = ze^z & \mapsto z \end{cases}$ Lambert's function [7].

Proof : For $n = \{1, 2\}$, we have roughly

$$\begin{aligned}\|\tilde{\varphi}_1\|_h &\leq 2\|\varphi\|_h = 2, \\ \|\tilde{\varphi}_2\|_h &\leq 5\|\varphi\|_h = 5.\end{aligned}\tag{9}$$

For $n \geq 3$ and $d \leq n-1$, according to [3, Lemma 2.2], for any $\varphi \in \mathcal{S}_\mu$ an upper bound of the Legendre approximation error is given by

$$\|\tilde{\varphi}_n\|_h \leq \frac{(h\mu)^{d+1}}{2^{d(d-1)} \binom{n-\frac{3}{2}}{d} \dots \binom{n-d+\frac{1}{2}}{d}},$$

and, with the largest possible value $d = n-1$, which takes advantage of a maximum number of bounded derivatives and leads to the smallest bound when h or μ are small, we have

$$\|\tilde{\varphi}_n\|_h \leq \frac{2\left(\frac{h\mu}{2}\right)^n}{n!} \frac{n(n-1)\dots 2}{(n-2)(n-\frac{3}{2})\dots\left(\frac{3}{2}\right)} \leq \frac{2\left(\frac{h\mu}{2}\right)^n 3\left(\frac{4}{3}\right)^{n-1}}{n!} = \frac{9}{2} \left(\frac{2h\mu}{3}\right)^n.\tag{10}$$

Merging (9)-(10) leads to

$$\|\tilde{\varphi}_n\|_h \leq \varrho \frac{\bar{\mu}^n}{n!},\tag{11}$$

where $\varrho = \frac{5}{3 \min(1, (h\mu)^2)}$ and $\bar{\mu} = \frac{2}{3}h\mu$ are set at the beginning. If we apply the logarithm function “log” to the foregoing inequality, then

$$\log(\|\tilde{\varphi}_n\|_h) \leq \log(\varrho) + n \log(\bar{\mu}) - \sum_{k=1}^n \log(k).$$

The Maclaurin integral test gives the following upper-bound

$$\log(\|\tilde{\varphi}_n\|_h) \leq \log(\varrho) + n \log(\bar{\mu}) - \int_0^n \log(s) ds = \log(\varrho) - n \log\left(\frac{n}{e\bar{\mu}}\right).$$

Defining $z_n := \log\left(\frac{n}{e\bar{\mu}}\right)$, imply that $e^{z_n} = \frac{n}{e\bar{\mu}}$ and

$$\log(\|\tilde{\varphi}_n\|_h) \leq \log(\varrho) - e\bar{\mu}(z_n e^{z_n}).\tag{12}$$

Therefore, for that (12) is bounded by $\log(\varepsilon)$, the following inequality need to be satisfied

$$z_n e^{z_n} \geq (e\bar{\mu})^{-1} \log\left(\frac{\varrho}{\varepsilon}\right).$$

By definition of \mathcal{W} Lambert’s function [7] and its increasing behavior on \mathbb{R}_+ , we have

$$z_n = \log\left(\frac{n}{e\bar{\mu}}\right) \geq \mathcal{W}\left((e\bar{\mu})^{-1} \log\left(\frac{\varrho}{\varepsilon}\right)\right),$$

or equivalently that the order n is sufficiently large to satisfy

$$n \geq \bar{\mu} \exp\left[1 + \mathcal{W}\left((e\bar{\mu})^{-1} \log\left(\frac{\varrho}{\varepsilon}\right)\right)\right].$$

Hence the proof is completed. \square

3 A new stability condition for systems with distributed delays

The main result of this contribution is obtained through the approximation of the Lyapunov - Krasovskii functional v in (2) by the n first Legendre polynomials for particular functions φ from the subsets of \mathcal{PC}_h . Based on this idea, we present a necessary and sufficient stability criterion for system (1), whose test requires a finite number of mathematical operations.

3.1 Approximated Lyapunov-Krasovskii functional

For any $\varphi \in \mathcal{PC}_h$, the approximated Lyapunov-Krasovskii functional at order n is given by

$$v_n(\varphi) = \gamma_n^T \begin{bmatrix} U(0) & \mathbf{Q}_n + \mathbf{R}_n \\ \mathbf{Q}_n^T + \mathbf{R}_n^T & \mathbf{T}_n + \mathbf{S}_n + \mathbf{S}_n^T + \mathbf{D}_n + \mathbf{I}_n^{-1} \end{bmatrix} \gamma_n = \gamma_n^T \mathbf{P}_n \gamma_n,\tag{13}$$

with $\gamma_n = [\varphi^T(0) \quad \Phi_n^T]^T \in \mathbb{R}^{(n+1)n_x}$, and $\mathbf{P}_n \in \mathbb{R}^{(n+1)n_x \times (n+1)n_x}$.

In the foregoing expression, we have

$$\begin{aligned}
\mathbf{Q}_n &:= \sum_{j=1}^m \int_{-h_j}^0 U^T(\theta + h_j) A_j L_n^T(\theta) d\tau, \quad \mathbf{I}_n^{-1} := \int_{-h}^0 L_n(\tau) W L_n^T(\tau) d\tau, \\
\mathbf{R}_n &:= \int_{-h}^0 \int_{-h}^{\theta} U^T(\theta - \xi) G(\xi) L_n^T(\theta) d\xi d\theta, \\
\mathbf{T}_n &:= \sum_{k=1}^m \sum_{j=1}^m \iint_{\mathcal{R}_1} L_n(\theta_1) A_k^T \mathcal{F}_U(\theta_1, \theta_2, -h_k, -h_j) A_j L_n^T(\theta_2) d\mathcal{R}_1, \\
\mathbf{S}_n &:= \sum_{k=1}^m \iiint_{\mathcal{R}_2} L_n(\theta_1) A_k^T \mathcal{F}_U(\theta_1, \theta_2, -h_k, \xi) G(\xi) L_n^T(\theta_2) d\mathcal{R}_2, \\
\mathbf{D}_n &:= \iiint \iiint_{\mathcal{R}_3} L_n(\theta_1) G^T(\xi_1) \mathcal{F}_U(\theta_1, \theta_2, \xi_1, \xi_2) G(\xi_2) L_n^T(\theta_2) d\mathcal{R}_3,
\end{aligned} \tag{14}$$

where the function \mathcal{F}_U and the regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are defined in 2.2.

Remark 2 Functional v_n is a Legendre polynomials approximation of the Lyapunov - Krasovskii functional v defined by (2). It is a quadratic form composed of a finite number of coefficients stored in matrix \mathbf{P}_n . The difference between v_n and v depends on the Legendre residual error $\tilde{\varphi}_n$.

3.2 Convergence of the approximated Lyapunov - Krasovskii functional

The convergence of the functional v_n towards the Lyapunov-Krasovskii functional v is now established. The result follows from Legendre polynomials approximation presented in the previous section.

For $\varphi \in \mathcal{PC}_h$, we define the Lyapunov-Krasovskii functional remainder as

$$\tilde{v}_n(\varphi) := v(\varphi) - v_n(\varphi) = \sum_{j=1}^6 \tilde{I}_j, \tag{15}$$

where the remained integrals are expressed using symmetry properties in terms of the original function φ and the residual error $\tilde{\varphi}_n$ as follows

$$\begin{aligned}
\tilde{I}_1 &= 2\varphi^T(0) \sum_{j=1}^m \int_{-h_j}^0 U^T(\theta + h_j) A_j \tilde{\varphi}_n(\theta) d\theta, \quad \tilde{I}_6 = - \int_{-h}^0 \tilde{\varphi}_n^T(\theta) W \tilde{\varphi}_n(\theta) d\theta, \\
\tilde{I}_2 &= \sum_{k=1}^m \sum_{j=1}^m \iint_{\mathcal{R}_1} \left(2\varphi^T(\theta_1) A_k^T - \tilde{\varphi}_n^T(\theta_1) A_k^T \right) \mathcal{F}_U(\theta_1, \theta_2, -h_k, -h_j) A_j \tilde{\varphi}_n(\theta_2) d\mathcal{R}_1, \\
\tilde{I}_3 &= 2\varphi^T(0) \int_{-h}^0 \int_{-h}^{\theta} U^T(\theta - \xi) G(\xi) d\xi \tilde{\varphi}_n(\theta) d\theta, \\
\tilde{I}_4 &= 2 \sum_{k=1}^m \iiint_{\mathcal{R}_2} \left(2\varphi(\theta_1) A_j^T - \tilde{\varphi}_n(\theta_1) A_j^T \right) \mathcal{F}_U(\theta_1, \theta_2, -h_k, \xi) G(\xi) \tilde{\varphi}_n(\theta_2) d\mathcal{R}_2, \\
\tilde{I}_5 &= \iiint \iiint_{\mathcal{R}_3} \left(2\varphi^T(\theta_1) G^T(\xi_1) - \tilde{\varphi}_n^T(\theta_1) G^T(\xi_1) \right) \mathcal{F}_U(\theta_1, \theta_2, \xi_1, \xi_2) G(\xi_2) \tilde{\varphi}_n(\theta_2) d\mathcal{R}_3,
\end{aligned}$$

where the function $\mathcal{F}_U(\cdot)$ and the integration regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are defined in 2.2. The next step is to bound the functional approximation error. We prove that in the compact set \mathcal{S}_μ , \tilde{v}_n converges to zero with a guaranteed and quantified convergence rate.

Lemma 2 Let $\mu > 0$. For any $\varphi \in \mathcal{S}_\mu$, we have

$$\|\tilde{v}_n\| \leq \beta, \quad \forall n \geq \mathcal{N}_\mu(\mathcal{E}(\beta))$$

where \tilde{v}_n is given by (15), \mathcal{N}_μ is given by (8), and

$$\mathcal{E}(\beta) := -\frac{\kappa_1}{\kappa_2} + \sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)^2 + \frac{\beta}{\kappa_2}},$$

with scalars

$$\begin{aligned}\kappa_1 &:= \|U\|_h \left(h \sum_{j=1}^m \|A_j\| + h^2 \|G\|_h \right) \left(1 + h \sum_{j=1}^m \|A_j\| + h^2 \|G\|_h \right), \\ \kappa_2 &:= \|U\|_h \left(h \sum_{j=1}^m \|A_j\| + h^2 \|G\|_h \right)^2 + h \|W\|.\end{aligned}$$

Proof : We estimate an upper bound of each summand of \tilde{v}_n by the use of Cauchy-Schwarz inequality

$$\begin{aligned}\tilde{I}_1 &\leq 2\|U\|_h h \sum_{j=1}^m \|A_j\| \|\tilde{\varphi}_n\|_h \|\varphi\|_h, & \tilde{I}_2 &\leq \|U\|_h \left(h \sum_{j=1}^m \|A_j\| \right)^2 (2\|\varphi\|_h + \|\tilde{\varphi}_n\|_h) \|\tilde{\varphi}_n\|_h, \\ \tilde{I}_3 &\leq 2\|U\|_h h^2 \|G\|_h \|\tilde{\varphi}_n\|_h \|\varphi\|_h, & \tilde{I}_4 &\leq 2\|U\|_h h^3 \|G\|_h \sum_{j=1}^m \|A_j\| (2\|\varphi\|_h + \|\tilde{\varphi}_n\|_h) \|\tilde{\varphi}_n\|_h, \\ \tilde{I}_5 &\leq \|U\|_h h^4 \|G\|_h^2 (2\|\varphi\|_h + \|\tilde{\varphi}_n\|_h) \|\tilde{\varphi}_n\|_h, & \tilde{I}_6 &\leq h \|W\| \|\tilde{\varphi}_n\|_h^2,\end{aligned}$$

yielding

$$\|\tilde{v}_n(\varphi)\| \leq \kappa_2 \|\tilde{\varphi}_n\|_h^2 + 2\kappa_1 \|\tilde{\varphi}_n\|_h \|\varphi\|_h. \quad (16)$$

Having $\|\varphi\|_h \leq 1$, we assess that $\|\tilde{v}_n\| \leq \beta$ holds under the following quadratic constraint

$$\|\tilde{\varphi}_n\|_h^2 + 2\frac{\kappa_1}{\kappa_2} \|\tilde{\varphi}_n\|_h - \frac{\beta}{\kappa_2} \leq 0$$

which is satisfied for $\|\tilde{\varphi}_n\| \leq \mathcal{E}(\beta)$. Finally, applying 1 and considering $\varphi \in \mathcal{S}_\mu$, we have $\|\tilde{\varphi}_n\|_h \leq \mathcal{E}(\beta)$ for $n \geq \mathcal{N}_\mu(\mathcal{E}(\beta))$ which concludes the proof. \square

We are now in a position to present the main contribution of this research. This result, inspired by [13, 3, 1], provides necessary and sufficient stability conditions expressed in terms of the positive definiteness of the matrix \mathbf{P}_n defined in (13). The resulting criterion requires the verification of just one condition evaluated at a fixed and explicitly given approximation order n^* .

3.3 Necessary stability condition

It is evident that the substitution of any particular initial function into the functional (2) combined with the quadratic lower bound on the functional will provide a set of necessary stability conditions.

Theorem 3 *If system (1) is exponentially stable, then*

$$\mathbf{P}_n > 0 \quad \forall n \in \mathbb{N},$$

where matrix \mathbf{P}_n is given by (13).

Proof : For any vector $\gamma_n = (x^T \quad \Phi_n^T)^T \in \mathbb{R}^{(n+1)n_x}$ and $n \in \mathbb{N}$, we define the initial function $\bar{\varphi} \in \mathcal{PC}_h$ as

$$\bar{\varphi}(\theta) = \begin{cases} L_n^T(\theta)\Phi_n, & \forall \theta \in [-h, 0), \\ x, & \text{if } \theta = 0. \end{cases}$$

The evaluation of the functional $v(\varphi)$ on the special function $\bar{\varphi}$ reduces to $v(\bar{\varphi}) = \gamma_n^T \mathbf{P}_n \gamma_n$. As system (1) is assumed to be exponentially stable, inequality (3) in 1 holds, therefore $v(\bar{\varphi}) > 0$, which yields $\mathbf{P}_n > 0$, for all $n \in \mathbb{N}$, since x and Φ_n are arbitrary vectors. Hence the theorem is proven. \square

Remark 3 *The necessary stability condition in 3 can be used as a sufficient condition of instability: if $\mathbf{P}_n \not> 0$ then system (1) is unstable. In practice, it leads to an over estimation of the stability regions in the state of parameters.*

3.4 Sufficient stability condition

It is interesting and novel to see that a positive quadratic bound for v_n that depends on the approximation order n can be used as a sufficient condition for exponential stability.

Theorem 4 Let $\mu = \sum_{j=0}^m \|A_j\| + h\|G\|_h$ and order $n \in \mathbb{N}$. If the Lyapunov condition holds and if the positiveness semi-definite test $\mathbf{P}_n - \mathcal{P}_n \geq 0$ is true where \mathbf{P}_n is given in (13), β^* in (5),

$$\mathcal{P}_n := \begin{bmatrix} (\delta_n - \beta^*) I_{n_x} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\delta_n := \left(\|U\|_h (h\mu)^2 + h\|W\| \right) \left(\frac{5\bar{\mu}^n}{n!} \right)^2 + 2\|U\|_h (h\mu) (1 + h\mu) \left(\frac{5\bar{\mu}^n}{n!} \right),$$

and $\bar{\mu} = \max(1, \frac{2}{3}\mu)$, then system (1) is exponentially stable.

Proof: The proof is done by contradiction. We assume that system (1) is not exponentially stable, but $\mathbf{P}_n - \mathcal{P}_n > 0$ and the Lyapunov condition holds. The application of 2 for this particular value of the parameter μ implies that there exists a function $\varphi \in \mathcal{S}_\mu$ such that

$$v(\varphi) \leq -\beta^* = -\beta^* \|\varphi(0)\|^2. \quad (17)$$

Next, it follows from (13) that

$$v(\varphi) = \gamma_n^T \mathbf{P}_n \gamma_n + \tilde{v}_n(\varphi) \geq \gamma_n^T \mathbf{P}_n \gamma_n - \|\tilde{v}_n(\varphi)\|.$$

Owing to (16) allows us to obtain

$$v(\varphi) \geq \gamma_n^T \mathbf{P}_n \gamma_n - \kappa_2 \|\tilde{\varphi}_n\|_h^2 - 2\kappa_1 \|\tilde{\varphi}_n\|_h \|\varphi\|_h,$$

where $\kappa_1 = \|U\|_h (h\mu) (1 + h\mu)$, $\kappa_2 = \|U\|_h (h\mu)^2 + h\|W\|$ and $\|\varphi\|_h = 1$. Then, the application of (11) gives

$$v(\varphi) \geq \gamma_n^T \mathbf{P}_n \gamma_n - \delta_n \|\varphi(0)\|^2. \quad (18)$$

Lastly, merging (17) and (18) yields

$$\gamma_n^T (\mathbf{P}_n - \mathcal{P}_n) \gamma_n < 0,$$

and contradicts the positiveness semi-definite assumption on $\mathbf{P}_n - \mathcal{P}_n$. \square

Remark 4 The sufficient stability condition in Theorem 4 can be used to ensure stability, even for low order n . In practice, it leads to an under estimation of the stability regions in the state of parameters. Note that, when the order n is sufficiently large to obtain $\delta_n < \beta^*$, the condition amounts to a positivity test on the matrix \mathbf{P}_n .

3.5 Necessary and sufficient stability condition

The following result is a necessary and sufficient stability condition for systems with pointwise and distributed delays, that can be verified in a finite number of mathematical operations.

Theorem 5 Let $\mu = \sum_{j=0}^m \|A_j\| + h\|G\|_h$. System (1) is exponentially stable if and only if the Lyapunov condition and

$$\mathbf{P}_{n^*} > 0, \quad (19)$$

with $n^* = \mathcal{N}_\mu(\mathcal{E}(\beta^*))$, where β^* , \mathcal{N}_μ , \mathcal{E} are determined by ??, respectively.

Proof: We prove separately the necessity and the sufficiency.

Necessity: The necessity directly follows from 3, since (19) holds for every integer n , and in particular n^* .

Sufficiency: The sufficiency directly follows from 4 since inequality $\delta_n \leq \beta^*$ holds for all integer $n \geq n^*$. \square

In the next section, we introduce an algorithm to solve the problem of the numerical computation of matrix \mathbf{P}_n .

4 Legendre projection computations

It is evident that the availability of a computational scheme for the numerical test of the stability conditions presented in 3 is required. However, analytically constructing the matrix \mathbf{P}_n is a numerically complicated task due to the presence of polynomial sequences, integral terms, and especially the distributed delay. To overcome this difficulty, we can develop effective inductive relations for the computation of \mathbf{P}_n using the following process. First, we introduce, depending on kernel type, the semianalytic expression for the computation of the delay Lyapunov matrix $U(\tau)$, $\tau \in [-h, h]$ associated with $W = I_{n_x}$ [2]. Second, we decompose the matrix \mathbf{P}_n given by (13) in terms of integrals of quasi-polynomials kernels. Third, we develop a recursive method for computing these integrals using the properties of Legendre polynomials (orthogonality, point-wise values, and differentiation properties [27]). This procedure provides the methodology for obtaining recursive methods for different systems with point-wise delays multiple of a basic one and distributed terms with piecewise constant, polynomial, exponential, or combinations such as γ distribution. For the case of systems with constant distributed kernel delay and a single point-wise delay ($G(\theta) = G$, $h_1 = h$), we present the corresponding recursive relations.

4.1 Delay Lyapunov matrix: semianalytic method

The delay Lyapunov matrix $U \in \mathbb{R}^{n_x \times n_x}$ is given by $U = \text{vec}^{-1}(\mathcal{U})$, where $\mathcal{U} = \text{vec}(U)$ is defined by [2]

$$\mathcal{U}(\tau) = \mathcal{E}_1 e^{\tau L} \mathcal{X}_0, \quad \tau \in [0, h]$$

with $\mathcal{W}_1 = \text{vec}(W)$, and

$$\begin{aligned} \mathcal{X}_0 &= \left(M + [-\varepsilon^T \mathcal{N}_1] e^{\begin{bmatrix} 0_{n_x^2} & \mathcal{E} \\ 0_{3n_x^2} & L \end{bmatrix} h} \begin{bmatrix} 0_{n_x^2 \times 3n_x^2} \\ I_{3n_x^2} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0_{n_x^2} \\ 0_{n_x^2} \\ -\mathcal{W}_1 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 0_{n_x^2} & I_{n_x^2} & 0_{n_x^2} \end{bmatrix}, \\ L &= \begin{bmatrix} A_0^T \otimes I_{n_x} & A_1^T \otimes I_{n_x} & G^T \otimes I_{n_x} \\ -I_{n_x} \otimes A_1^T & -I_{n_x} \otimes A_0^T & -I_{n_x} \otimes G^T \\ I_{n_x} \otimes I_{n_x} & -I_{n_x} \otimes I_{n_x} & 0_n \otimes 0_{n_x} \end{bmatrix}, \quad M = \begin{bmatrix} I_{n_x^2} & 0_{n_x^2} & 0_{n_x^2} \\ 0_{n_x^2} & 0_{n_x^2} & I_{n_x^2} \\ A_0^T \otimes I_{n_x} & A_1^T \otimes I_{n_x} & G^T \otimes I_{n_x} \end{bmatrix}, \\ \mathcal{N}_1 &= \begin{bmatrix} 0_{n_x^2} & -I_{n_x^2} & 0_{n_x^2} \\ 0_{n_x^2} & 0_{n_x^2} & 0_{n_x^2} \\ I_{n_x} \otimes A_1^T & I_{n_x} \otimes A_0^T & I_{n_x} \otimes G^T \end{bmatrix}, \quad \mathcal{E}_1 = \begin{bmatrix} I_{n_x^2} & 0_{n_x^2} & 0_{n_x^2} \end{bmatrix}. \end{aligned} \quad (20)$$

For the solution of the integrals defined in (14) there exist recursive relations only for the case where L is nonsingular [3]. A technical challenge in the case of distributed delay systems is that matrix L in (20) is, for most systems parameters, a singular matrix. By introducing its canonical Jordan form, we guarantee that the analytical-recursive solution of the integrals in (14) can be computed for the invertible and non-invertible parts of matrix L .

Lemma 3 (See [14]) *Let $L \in \mathbb{R}^{n_x \times n_x}$ defined by (20). There exist a nonsingular $P_L \in \mathbb{C}^{n_x \times n_x}$, positive integers q, m_1, \dots, m_q with $m_1 + m_2 + \dots + m_q = n_x$, and distinct eigenvalues $s_1, \dots, s_q \in \mathbb{C}$ of L , such that*

$$L = P_L \underbrace{\left(J_{m_1}(s_1) \oplus J_{m_2}(s_2) \oplus \dots \oplus J_{m_q}(s_q) \right)}_{J_L} P_L^{-1}.$$

A Jordan block $J_{m_i}(s_i)$ is a m_i -by- m_i upper triangular matrix where the scalar s_i appears m_i times on the main diagonal (algebraic multiplicity); if $m_i > 1$, there are $m_i - 1$ entries “+1” in the super-diagonal; all other entries are zero.

Remark 5 *Notice that the matrix J_L defined in 4 can be expressed by*

$$J_L = \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix},$$

where B is a direct sum of all the nonsingular Jordan blocks and N is a direct sum of all the nilpotent blocks. The direct summand B is absent if L is nilpotent; N is absent if L is nonsingular. Then, the exponential matrix of $L = P_L J_L P_L^{-1}$ is defined by

$$e^{L\tau} = P_L e^{\tau J_L} P_L^{-1} = P_L \begin{bmatrix} e^{B\tau} & 0 \\ 0 & e^{N\tau} \end{bmatrix} P_L^{-1}.$$

The exponential matrix of the nilpotent Jordan block $J_{m_i}(0)$, contained in matrix \mathbf{P}_n , is given by

$$e^{J_{m_i}(0)\tau} = \begin{bmatrix} \frac{1}{0!} \tau^0 & \frac{1}{1!} \tau^1 & \dots & \frac{1}{(m_i-1)!} \tau^{m_i-1} \\ 0 & \frac{1}{0!} \tau^0 & \dots & \frac{1}{(m_i-2)!} \tau^{m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{0!} \tau^0 \end{bmatrix}. \quad (21)$$

where m_i represents the algebraic multiplicity.

4.2 Components of the matrix \mathbf{P}_n

Each coefficient of matrix \mathbf{P}_n given by (13) needs to be evaluated numerically, which is more complicated since matrices $\mathbf{Q}_n, \mathbf{R}_n, \mathbf{T}_n, \mathbf{S}_n$ and \mathbf{D}_n contain the first Legendre coefficients of the delay Lyapunov matrix. Therefore, substituting the semianalytic expression of U in each component of the matrix \mathbf{P}_n , and considering the Jordan

canonical form of L described in 4, we obtain

$$\begin{aligned} \mathbf{Q}_n &= [Q_0^T A_1 \quad Q_1^T A_1 \quad \cdots \quad Q_{n-1}^T A_1], \quad \mathbf{R}_n = [R_0^T G \quad R_1^T G \quad \cdots \quad R_{n-1}^T G], \\ \mathbf{T}_n &= \begin{bmatrix} A_1^T \mathbf{T}_{00} A_1 & A_1^T \mathbf{T}_{01} A_1 & \cdots & A_1^T \mathbf{T}_{0n-1} A_1 \\ \star & A_1^T \mathbf{T}_{11} A_1 & \cdots & A_1^T \mathbf{T}_{1n-1} A_1 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & A_1^T \mathbf{T}_{n-1, n-1} A_1 \end{bmatrix}, \\ \mathbf{S}_n &= \begin{bmatrix} A_1^T \mathbf{S}_{00} G & A_1^T \mathbf{S}_{01} G & \cdots & A_1^T \mathbf{S}_{0, n-1} G \\ A_1^T \mathbf{S}_{10} G & A_1^T \mathbf{S}_{11} G & \cdots & A_1^T \mathbf{S}_{1, n-1} G \\ \vdots & \vdots & \ddots & \vdots \\ A_1^T \mathbf{S}_{n-1, 0} G & A_1^T \mathbf{S}_{n-1, 1} G & \cdots & A_1^T \mathbf{S}_{n-1, n-1} G \end{bmatrix}, \\ \mathbf{I}_n^{-1} &= h \cdot \text{diag} \left(1, \frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2n-1} \right) \otimes I_{n_x}, \\ \mathbf{D}_n &= \begin{bmatrix} G^T \mathbf{D}_{00} G & G^T \mathbf{D}_{01} G & \cdots & G^T \mathbf{D}_{0, n-1} G \\ \star & G^T \mathbf{D}_{11} G & \cdots & G^T \mathbf{D}_{1, n-1} G \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & G^T \mathbf{D}_{n-1, n-1} G \end{bmatrix}, \end{aligned}$$

where $Q_k, R_k, \mathbf{T}_{jk} = T_{jk} + T_{jk}^{\flat T}, \mathbf{S}_{jk} = S_{jk} + S_{jk}^{\flat T}$ and $\mathbf{D}_{jk} = D_{jk} + D_{jk}^{\flat T}$ are the Legendre coefficients of the Lyapunov matrix U given in the vector form as follows

$$\begin{cases} \mathcal{Q}_k = \text{vec}(Q_k) = \mathcal{E}_1 P_L \Gamma_k P_L^{-1} \mathcal{X}_0, \\ \mathcal{R}_k = \text{vec}(R_k) = \mathcal{E}_1 P_L \Lambda_k P_L^{-1} \mathcal{X}_0, \\ \mathcal{T}_{jk} = \text{vec}(T_{jk}) = \mathcal{E}_1 P_L \bar{\Gamma}_{jk} P_L^{-1} \mathcal{X}_0, \\ \mathcal{T}_{jk}^{\flat} = \text{vec}(T_{jk}^{\flat}) = \mathcal{E}_1 P_L \bar{\Gamma}_{jk}^{\flat} P_L^{-1} \mathcal{X}_0, \\ \mathcal{S}_{jk} = \text{vec}(S_{jk}) = \mathcal{E}_1 P_L \bar{\Lambda}_{jk} P_L^{-1} \mathcal{X}_0, \\ \mathcal{S}_{jk}^{\flat} = \text{vec}(S_{jk}^{\flat}) = \mathcal{E}_1 P_L \bar{\Lambda}_{jk}^{\flat} P_L^{-1} \mathcal{X}_0, \\ \mathcal{D}_{jk} = \text{vec}(D_{jk}) = \mathcal{E}_1 P_L \bar{\Delta}_{jk} P_L^{-1} \mathcal{X}_0, \\ \mathcal{D}_{jk}^{\flat} = \text{vec}(D_{jk}^{\flat}) = \mathcal{E}_1 P_L \bar{\Delta}_{jk}^{\flat} P_L^{-1} \mathcal{X}_0, \end{cases}$$

with

$$\begin{aligned} \Gamma_k &= \int_{-h}^0 e^{(h+\theta)J_L} l_k(\theta) d\theta, & \Lambda_k &= \int_{-h}^0 \int_{-h}^{\theta} e^{(\theta-\xi)J_L} d\xi l_k(\theta) d\theta, \\ \bar{\Gamma}_{jk} &= \int_{-h}^0 \int_{-h}^{\theta_1} l_j(\theta_1) e^{(\theta_1-\theta_2)J_L} l_k(\theta_2) d\theta_2 d\theta_1, & \bar{\Gamma}_{jk}^{\flat} &= \int_{-h}^0 \int_{\theta_1}^0 l_j(\theta_1) e^{(\theta_2-\theta_1)J_L} l_k(\theta_2) d\theta_2 d\theta_1, \\ \bar{\Lambda}_{jk} &= \int_{-h}^0 \int_{-h}^0 \int_0^{\theta_1+h} l_j(\theta_1) e^{\xi J_L} l_k(\theta_2) d\xi d\theta_2 d\theta_1, \\ \bar{\Lambda}_{jk}^{\flat} &= \int_{-h}^0 \int_{-h}^0 \int_{\theta_1-\theta_2}^0 l_j(\theta_1) e^{-\xi J_L} l_k(\theta_2) d\xi d\theta_2 d\theta_1, \\ \bar{\Delta}_{jk} &= \int_{-h}^0 l_j(\theta_1) \int_{-h}^0 l_k(\theta_2) \int_0^{\theta_1+h} \int_0^{\xi_1} e^{(\xi_1-\xi_2)J_L} d\xi_2 d\xi_1 d\theta_2 d\theta_1, \\ \bar{\Delta}_{jk}^{\flat} &= \int_{-h}^0 l_j(\theta_1) \int_{-h}^0 l_k(\theta_2) \int_0^{\theta_1+h} \int_{\xi_1}^{\theta_2+h} e^{(\xi_2-\xi_1)J_L} d\xi_2 d\xi_1 d\theta_2 d\theta_1, \end{aligned} \tag{22}$$

where P_L, J_L are defined by 4. In the following section, by the properties of the Legendre polynomials, we present a recursive numerical scheme for the calculation of the integrals involved in the determination of the matrices defined in (22). The reduction of the computational burden makes the stability conditions numerically tractable.

4.3 Iterative calculation of Legendre exponential coefficients

The following propositions address the recursive computation of the integrals defined in (22). The highly technical proofs are given as Supplementary material.

Proposition 1 *The matrices Γ_k and Λ_k can be computed by*

$$\Gamma_k = \Gamma_k(B) \oplus \Gamma_k(N), \quad \Lambda_k = \Lambda_k(B) \oplus \Lambda_k(N),$$

where the matrices B, N are defined in 7.

For the case of B , the following recursive relation holds

$$\begin{aligned}\Gamma_k(B) &= \Gamma_{k-2}(B) - \frac{2(2k-1)}{h} B^{-1} \Gamma_{k-1}(B), \quad \forall k \geq 2, \\ \Lambda_k(B) &= B^{-1} \Gamma_k(B), \quad \forall k \geq 1,\end{aligned}$$

initialized with

$$\begin{bmatrix} \Gamma_0(B) \\ \Gamma_1(B) \\ \Lambda_0(B) \end{bmatrix} = \begin{bmatrix} B^{-1}(e^{hB} - I_{m_i}) \\ B^{-1}(e^{hB} + I_{m_i}) - \frac{2}{h} B^{-1} \Gamma_0(B) \\ B^{-1}(\Gamma_0(B) - hI_{m_i}) \end{bmatrix},$$

For the case of the nilpotent matrix N , we have that

$$\begin{aligned}\Gamma_k(N) &= \Gamma_{k,1}(0) \oplus \Gamma_{k,2}(0) \oplus \cdots \oplus \Gamma_{k,b}(0), \\ \Lambda_k(N) &= \Lambda_{k,1}(0) \oplus \Lambda_{k,2}(0) \oplus \cdots \oplus \Lambda_{k,b}(0),\end{aligned}$$

with

$$\begin{aligned}\Gamma_{k,i}(0) &= \begin{bmatrix} \gamma_{k,0} & \gamma_{k,1} & \cdots & \gamma_{k,m_i-1} \\ 0 & \gamma_{k,0} & \cdots & \gamma_{k,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{k,0} \end{bmatrix}, & \Lambda_{k,i}(0) &= \begin{bmatrix} \lambda_{k,0} & \lambda_{k,1} & \cdots & \lambda_{k,m_i-1} \\ 0 & \lambda_{k,0} & \cdots & \lambda_{k,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k,0} \end{bmatrix}, \\ \gamma_{k,p} &= \sum_{i_1=0}^k \frac{u_{i_1}^k h^{p+1}}{(i_1 + p + 1)p!}, & \lambda_{k,p} &= \gamma_{k,p+1},\end{aligned}$$

where $u_{i_1}^k$ is given by (7), $p = 0, \dots, m_i$, m_i is the algebraic multiplicity of nilpotent blocks i , $i = 1, 2, \dots, b$, b the number of nilpotent blocks of L , and $k = 0, \dots, n-1$.

Proposition 2 For any matrix J_L , and for matrices $\bar{\Gamma}_{jk}, \bar{\Gamma}_{jk}^b$ the following equality holds

$$\bar{\Gamma}_{jk} = (-1)^{j+k} \bar{\Gamma}_{jk}^b \quad \forall (j, k) \in \{0, 1, \dots, n-1\}^2.$$

Proposition 3 The matrices $\bar{\Gamma}_{jk}$ and $\bar{\Gamma}_{jk}^b$ are calculated by

$$\bar{\Gamma}_{jk} = \bar{\Gamma}_{jk}(B) \oplus \bar{\Gamma}_{jk}(N), \quad \bar{\Gamma}_{jk}^b = \bar{\Gamma}_{jk}^b(B) \oplus \bar{\Gamma}_{jk}^b(N),$$

where the matrices B, N are defined in 7.

The matrices $\bar{\Gamma}_{jk}(B)$ are computed as follows relations

$$\bar{\Gamma}_{jk}(B) = \begin{cases} (-1)^{j+k} \bar{\Gamma}_{kj}(B), & k < j, \\ \left(\bar{\Gamma}_{jk-2}(B) + \frac{2(2k+1)}{h} B^{-1} \bar{\Gamma}_{jk-1}(B) \right) \\ \quad - \frac{h}{2k+1} B^{-1} (\delta_{jk} - \delta_{jk-2}), & k \geq \max(2, j), \end{cases}$$

initialized with

$$\begin{bmatrix} \bar{\Gamma}_{00}(B) \\ \bar{\Gamma}_{01}(B) \\ \bar{\Gamma}_{11}(B) \end{bmatrix} = \begin{bmatrix} B^{-1}(\Gamma_0(B) - hI_{m_i}) \\ -B^{-1} \Gamma_1(B) \\ B^{-1} \left(\left(\frac{2}{h} B^{-1} - I_{m_i} \right) \Gamma_1(B) - \frac{h}{3} I_{m_i} \right) \end{bmatrix}.$$

For the case of a nilpotent N , we have that

$$\bar{\Gamma}_{jk}(N) = \bar{\Gamma}_{jk,1}(0) \oplus \bar{\Gamma}_{jk,2}(0) \oplus \cdots \oplus \bar{\Gamma}_{jk,b}(0),$$

with

$$\begin{aligned}\bar{\Gamma}_{jk,i}(0) &= \begin{bmatrix} \tilde{\gamma}_{jk,0} & \tilde{\gamma}_{jk,1} & \cdots & \tilde{\gamma}_{jk,m_i-1} \\ 0 & \tilde{\gamma}_{jk,0} & \cdots & \tilde{\gamma}_{jk,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\gamma}_{jk,0} \end{bmatrix}, \\ \tilde{\gamma}_{jk,p} &= \frac{1}{p!} \sum_{i_1=0}^j \sum_{i_2=0}^k \sum_{i_3=0}^p \binom{p}{i_3} \frac{(-1)^{i_3} u_{i_1}^j u_{i_2}^k h^{p+2}}{i_1 + i_2 + p + 2},\end{aligned}$$

where $u_{i_1}^j, u_{i_2}^k$ are given by (7), $p = 0, \dots, m_i - 1$, m_i is the algebraic multiplicity of nilpotent blocks i , $i = 1, 2, \dots, b$, b the number of nilpotent blocks of L , and $k = 0, \dots, n-1$.

Proposition 4 The matrices $\bar{\Lambda}_{jk}$ and $\bar{\Lambda}_{jk}^b$ are given by

$$\bar{\Lambda}_{jk} = \bar{\Lambda}_{jk}(B) \oplus \bar{\Lambda}_{jk}(N), \quad \bar{\Lambda}_{jk}^b = \bar{\Lambda}_{jk}^b(B) \oplus \bar{\Lambda}_{jk}^b(N),$$

where

$$\bar{\Lambda}_{jk}(B) = \begin{cases} hB^{-1}\Gamma_j(B) & \text{if } k = 0, \quad j \geq 1, \\ 0 & \text{if } k \geq 1, \end{cases}$$

$$\bar{\Lambda}_{jk}^b(B) = (-1)^j B^{-1}\Gamma_k(B)\Gamma_j(B)e^{-hB}, \quad j \geq 1, k \geq 1,$$

and

$$\begin{bmatrix} \bar{\Lambda}_{00}(B) \\ \bar{\Lambda}_{00}^b(B) \end{bmatrix} = \begin{bmatrix} hB^{-1}\Gamma_0(B) - h^2B^{-1} \\ \Gamma_0(B)\Gamma_0(B)e^{-hB} - h^2B^{-1} \end{bmatrix}.$$

For the case of nilpotent matrix N , we have

$$\bar{\Lambda}_{jk}(N) = \bar{\Lambda}_{jk,1}(0) \oplus \bar{\Lambda}_{jk,2}(0) \oplus \cdots \oplus \bar{\Lambda}_{jk,b}(0)$$

$$\bar{\Lambda}_{jk}^b(N) = \bar{\Lambda}_{jk,1}^b(0) \oplus \bar{\Lambda}_{jk,2}^b(0) \oplus \cdots \oplus \bar{\Lambda}_{jk,b}^b(0)$$

with

$$\bar{\Lambda}_{jk,i}(0) = \begin{bmatrix} \bar{\lambda}_{jk,0} & \bar{\lambda}_{jk,1} & \cdots & \bar{\lambda}_{jk,m_i-1} \\ 0 & \bar{\lambda}_{jk,0} & \cdots & \bar{\lambda}_{jk,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\lambda}_{jk,0} \end{bmatrix}, \quad \bar{\Lambda}_{jk,i}^b(0) = \begin{bmatrix} \bar{\lambda}_{jk,0}^b & \bar{\lambda}_{jk,1} & \cdots & \bar{\lambda}_{jk,m_i-1} \\ 0 & \bar{\lambda}_{jk,0} & \cdots & \bar{\lambda}_{jk,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\lambda}_{jk,0} \end{bmatrix},$$

$$\bar{\lambda}_{jk,p} = \begin{cases} h\gamma_{j,p+1} & \text{if } k = 0, \\ 0 & \text{if } k \geq 1, \end{cases}$$

$$\bar{\lambda}_{jk,p}^b = \frac{(-1)^{p+1}}{(p+1)!} \sum_{i_1=0}^j \sum_{i_2=0}^k \sum_{i_3=0}^p \frac{\binom{p}{i_3} (-1)^{i_3} u_{i_1}^j u_{i_2}^k h^{p+2}}{(i_1 - i_3 + p + 1)(i_2 + i_3 + 1)},$$

where $u_{i_1}^j, u_{i_2}^k$ are given by (7), $p = 0, \dots, m_i - 1$, m_i is the algebraic multiplicity of nilpotent blocks i , $i = 1, 2, \dots, b$, b the number of nilpotent blocks of L , and $k = 0, \dots, n - 1$.

Proposition 5 The matrices $\bar{\Delta}_{jk}$ and $\bar{\Delta}_{jk}^b$ are computed as

$$\bar{\Delta}_{jk} = \bar{\Delta}_{jk}(B) \oplus \bar{\Delta}_{jk}(N), \quad \bar{\Delta}_{jk}^b = \bar{\Delta}_{jk}^b(B) \oplus \bar{\Delta}_{jk}^b(N),$$

where

$$\bar{\Delta}_{jk}(B) = \begin{cases} hB^{-2}\Gamma_j(B) & \text{if } k = 0, \quad j \geq 2, \\ 0 & \text{if } k \geq 1, \end{cases}$$

$$\bar{\Delta}_{jk}^b(B) = \begin{cases} -\bar{\Lambda}_{jk}^b(B) & \text{if } j > 0, k \geq 1, \\ -\bar{\Lambda}_{jk}^b(B) + hB^{-2}\Gamma_k(B) & \text{if } j = 0, k \geq 1, \end{cases}$$

with

$$\begin{bmatrix} \Omega_{00}(B) \\ \Omega_{10}(B) \\ \Omega_{00}^b(B) \\ \Omega_{10}^b(B) \end{bmatrix} = \begin{bmatrix} hB^{-1} \left(B^{-1}\Gamma_0(B) - \frac{h^2}{2} I_{m_i} - hB^{-1} \right) \\ -\frac{h^3}{6} B^{-2} + hB^{-2}\Gamma_1(B) \\ -B^{-2} \left(\Gamma_0^2(B)e^{-hB} - h\Gamma_0(B) + \frac{h^3}{2} B \right) \\ B^{-2}\Gamma_0(B)\Gamma_1(B)e^{-hB} - \frac{h^3}{6} B^{-1} \end{bmatrix}.$$

For the case of a nilpotent matrix N , we have

$$\bar{\Delta}_{jk}(N) = \bar{\Delta}_{jk,1}(0) \oplus \bar{\Delta}_{jk,2}(0) \oplus \cdots \oplus \bar{\Delta}_{jk,b}(0),$$

$$\bar{\Delta}_{jk}^b(N) = \bar{\Delta}_{jk,1}^b(0) \oplus \bar{\Delta}_{jk,2}^b(0) \oplus \cdots \oplus \bar{\Delta}_{jk,b}^b(0),$$

where

$$\bar{\Delta}_{jk,i}(0) = \begin{bmatrix} \bar{\delta}_{jk,0} & \bar{\delta}_{jk,1} & \cdots & \bar{\delta}_{jk,m_i-1} \\ 0 & \bar{\delta}_{jk,0} & \cdots & \bar{\delta}_{jk,m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\delta}_{jk,0} \end{bmatrix}, \quad \bar{\Delta}_{jk,i}^b(0) = \begin{bmatrix} \bar{\delta}_{jk,0}^b & \bar{\delta}_{jk,1}^b & \cdots & \bar{\delta}_{jk,m_i-1}^b \\ 0 & \bar{\delta}_{jk,0}^b & \cdots & \bar{\delta}_{jk,m_i-2}^b \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\delta}_{jk,0}^b \end{bmatrix},$$

$$\bar{\delta}_{jk,p} = \bar{\lambda}_{jk,p+1}, \quad \bar{\delta}_{jk,p}^b = \bar{\lambda}_{jk,p+1} - \bar{\lambda}_{jk,p+1}^b,$$

with $p = 0, \dots, m_i - 1$, m_i the algebraic multiplicity of nilpotent block i , $i = 1, 2, \dots, b$, b the number of nilpotent blocks of L , and $k = 0, \dots, n - 1$.

Remark 6 A worthy of mention fact is that the numerical scheme developed in this contribution for the calculation of the integrals (14) can be applied to compute the integrals required in [3, Theorem 1] when the matrix M of the semianalytic method is singular, a case that has not been treated before in the literature.

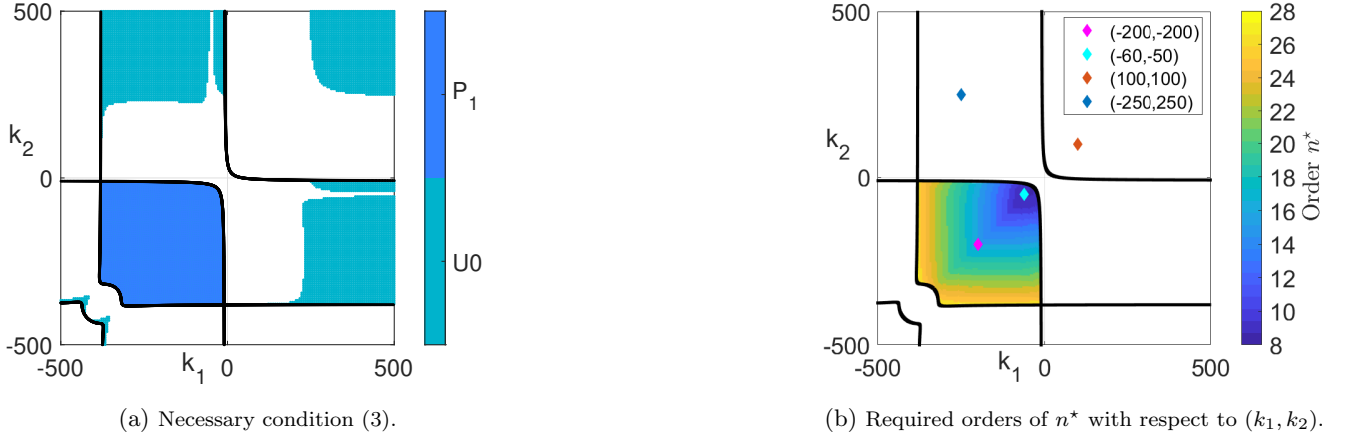


Figure 1: Necessary and sufficient stability condition of 5 for 1.

Table 1: Stability criteria of 5, and Theorem 7 and 8 in [6], for some parameters (k_1, k_2) (highlighted in 1b), 1.

(k_1, k_2)	n^* Thm 5	r^* [6, Thm 8]	\hat{r} [6, Thm 7]	Result
$(-200, -200)$ \blacklozenge	15	3473	16664261	Stable
$(-60, -50)$ \blacklozenge	9	107	6102	Stable
$(100, 100)$ \blacklozenge	10	599	119286	Unstable
$(-250, 250)$ \blacklozenge	18	185498	$\simeq 10^9$	Unstable

5 Numerical Examples

In this section, we illustrate how the stability conditions of ?? can be applied to determine the stability region in the space of parameters, including delays. The numerical implementation of the recursive scheme presented in 4 is carried out in MATLAB. The numerical errors in the recurrence relations, accumulated as n increases, are reduced by using the *vpa* function. The positivity of \mathbf{P}_n is verified with the *chol* function, while the solution of (6) is computed with the *fzero* function. In each figure, the system parameters that satisfy the stability conditions are represented by isolated points, while the solid lines indicate the stability/instability boundaries obtained by the \mathcal{D} -Partition method [22]. It is essential to clarify that, unlike in frequency-domain methods, Theorem 5 allows the independent analysis of the stability of any isolated point of system (1). The numerical computations are performed on a desktop computer with Intel Core i9-9900, 3.10 GHz, 6 cores, and 32 GB RAM processor. For comparison purposes, we also present the number \hat{r}, r^* of the stability test for systems with distributed delays introduced in [6].

Example 1 Consider the following system with an exponential kernel

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & -2.5 \\ -2.5 & 0 \end{pmatrix} x(t-h) + \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \int_{-h}^0 e^{\gamma\theta} x(t+\theta) d\theta$$

with $h = 0.1$, $\gamma = -2.5$, and $k_1, k_2 \in \mathbb{R}$ are free parameters.

The points where the conditions of 3 are satisfied for different values of n are depicted in 1a. As the conditions are necessary for any n , an outer estimate of the stability region is obtained. Consequently, points that do not satisfy the condition for some n are not evaluated for greater values of n . 1b represents the map of the order n^* for which the conditions of 5 become sufficient, considering only the parameter region obtained from 3 with $n = 2$. The estimation order n^* and the estimates of \hat{r}, r^* found in [6] are given in 1 for several parameter pairs (k_1, k_2) . The reduction of the estimation order and the gain in numerical complexity is clear.

Example 2 Consider the following two-dimensional example:

$$\dot{x}(t) = \begin{pmatrix} -1 & 0.5 \\ 0 & p \end{pmatrix} x(t-h) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \int_{-h}^0 x(t+\theta) d\theta,$$

where $p \in \mathbb{R}$ and $h \geq 0$ are free parameters.

1a illustrates the stability regions obtained through the necessary condition of 5. The number n^* defined in 5 is depicted in 2b for pairs (h, p) in the space of parameters. 2 presents a comparison of the orders required to

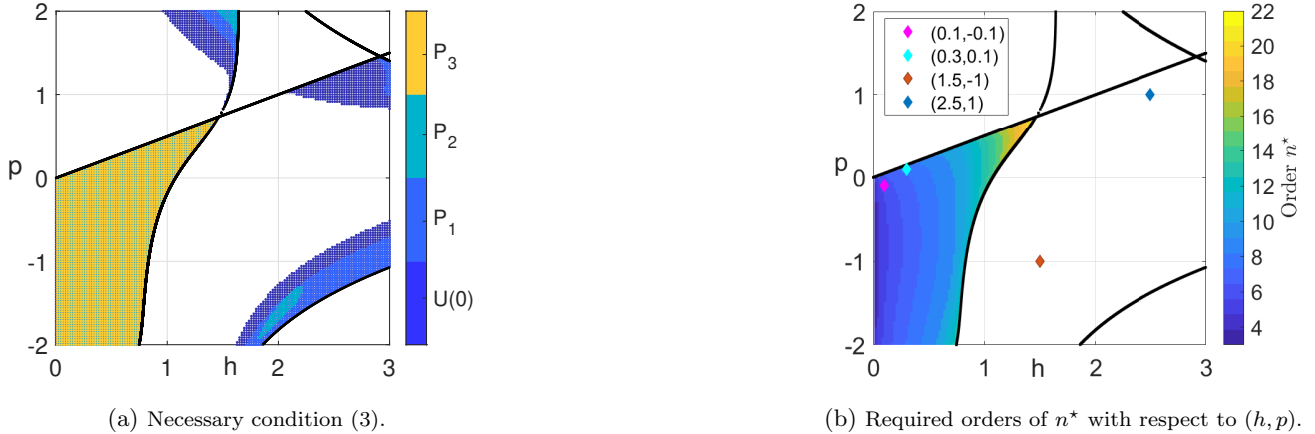


Figure 2: Necessary and sufficient stability condition of 5 for 23.

Table 2: Stability criteria of 5, and Theorem 7 and 8 in [6], for some parameters (h, p) , highlighted in 2b, 23.

(h, p)	n^* Thm 5	r^* [6, Thm 8]	\hat{r} [6, Thm 7]	Result
$(0.1, -0.1)$ ♦	6	4	13	Stable
$(0.3, 0.1)$ ♦	8	90	873	Stable
$(1.5, -1)$ ♦	20	15819	2188460189	Unstable
$(2.5, 1)$ ♦	41	144859603	$\simeq 10^{18}$	Unstable

guarantee the sufficiency of 5 and [6, Theorem 7, Theorem 8] for selected pairs of parameters (h, p) . We can see that our stability criterion is much more efficient for low orders. We also note that, as in other stability criteria, the order increases along the stability boundaries.

Example 3 Now, we consider a liquid monopropellant rocket motor with a pressure feeding system [29]. A linearized version of the dynamic equation of its feeding system and combustion chamber, assuming nonsteady flow and taking nonuniform lag consideration, is given by

$$\dot{x}(t) = \begin{pmatrix} \gamma-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ -\kappa & 0 & -\kappa & \kappa \\ 0 & 1 & -1 & 0 \end{pmatrix} x(t) - \begin{pmatrix} \frac{\gamma}{h} & 0 & \frac{1}{h} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int_{-h}^0 x(t+\theta) d\theta \quad (23)$$

where $\kappa = 0.5556$, $\gamma \in \mathbb{R}$, and $h > 0$ are system parameters.

The test of the conditions of 3 allows inferring that the stability region in the parameter space $\{(h, \gamma) : 0 < h < 6.5, 0 \leq \gamma \leq 6.5\}$ can be obtained with $n = 1$ (see 3a). In addition, the estimated order n^* , as expressed in 5, is illustrated in 3b. It is worth mentioning that although n^* can be large in some cases where the Lyapunov condition is almost violated, as shown in 3, it is still finite and less conservative compared to the values of \hat{r} and r^* reported in [6].

6 Conclusion

This contribution presents a new necessary and sufficient stability condition for linear systems with pointwise and distributed delays. The formulated criterion links the system's stability with the positivity of a particular matrix, and its verification only requires n^* finite number of operations. This result is based on the approximation by the

Table 3: Stability criteria of 5, and Theorem 7 and 8 in [6], for several parameters (h, γ) , highlighted in 3b, 3.

(h, γ)	n^* Thm 5	r^* [6, Thm 8]	\hat{r} [6, Thm 7]	Result
$(0.1, 0.3)$ ♦	8	1935	32440	Stable
$(0.7, 1.5)$ ♦	24	253553	167248147506	Stable
$(0.5, 5)$ ♦	25	1018773	1033202905518	Unstable
$(1.2, 3)$ ♦	43	173260910	$\simeq 10^{18}$	Unstable

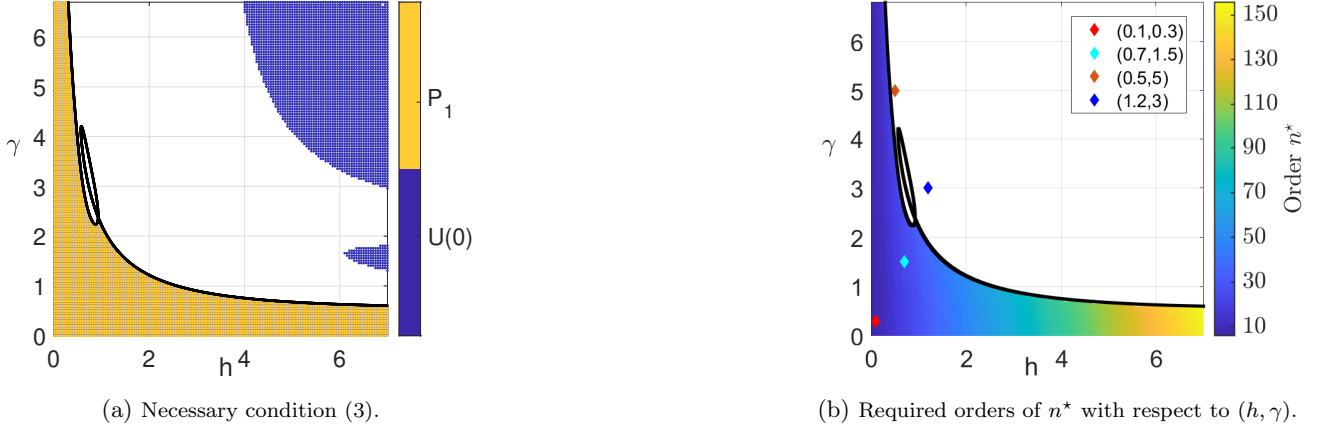


Figure 3: Necessary and sufficient stability condition of 5 for 3.

first n Legendre polynomials of the initial function argument in the Lyapunov-Krasovskii functional with prescribed derivative and recent fundamental stability theorems. The supergeometric convergence rate satisfied by the Legendre polynomials demonstrates that Theorem 5 requires fewer operations than the results in [6]. Furthermore, the recursive numerical scheme guarantees an efficient implementation of the stability conditions, which is numerically tractable. Research perspectives consist of reducing the computational effort of the stability test by improving the estimation of n^* whose conservatism derives from the overestimations of the bounds of the fundamental stability/instability theorems. An ongoing objective is to develop numerical schemes for systems with distributed delays with non-constant kernels and multiple-point delays. Future work also consists of studying the computational complexity of the recursive relationships of the stability test to improve the efficiency of the developed algorithms. Another main line of research consists of finding estimates of the exponential decay rates of these systems. Finally, this contribution is a significant step in continuously improving stability theory for linear systems with distributed delays.

A Proofs of the new recursive method for projection computation: Propositions 4.3 - 4.7

In this section, we present the technical details corresponding to the proofs of the recursive relations of the numerical scheme for the calculation of the integrals involved in the construction of the matrix \mathbf{P}_n . These results are obtained using the properties of the Legendre polynomials and the Jordan canonical form of the matrix L of the semi-analytical method for systems with distributed delays.

In order to present the results clearly, we recall the following results.

Lemma 4 (See [14]) *Let $L \in \mathbb{R}^{n_x \times n_x}$. There exist a nonsingular $P_L \in \mathbb{C}^{n_x \times n_x}$, positive integers q, m_1, \dots, m_q with $m_1 + m_2 + \dots + m_q = n_x$, and distinct eigenvalues $s_1, \dots, s_q \in \mathbb{C}$ of L , such that*

$$L = P_L \underbrace{\left(J_{m_1}(s_1) \oplus J_{m_2}(s_2) \oplus \dots \oplus J_{m_q}(s_q) \right)}_{J_L} P_L^{-1}.$$

A Jordan block $J_{m_i}(s_i)$ is a m_i -by- m_i upper triangular matrix where the scalar s_i appears m_i times on the main diagonal (algebraic multiplicity); if $i > 1$, there are $i - 1$ entries “+1” in the super-diagonal; all other entries are zero.

Remark 7 *Notice that the matrix J_L defined in 4 can be expressed by*

$$J_L = \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix},$$

where B is a direct sum of all the nonsingular Jordan blocks and N is a direct sum of all the nilpotent blocks. The direct summand B is absent if L is nilpotent; N is absent if L is nonsingular. Then, the exponential matrix of $L = P_L J_L P_L^{-1}$ is defined by

$$e^{L\tau} = P_L e^{\tau J_L} P_L^{-1} = P_L \begin{bmatrix} e^{B\tau} & 0 \\ 0 & e^{N\tau} \end{bmatrix} P_L^{-1}$$

The exponential matrix of the nilpotent Jordan block $J_{m_i}(0)$, contained in matrix \mathbf{P}_n , is given by

$$e^{J_{m_i}(0)\tau} = \begin{bmatrix} \frac{1}{0!}\tau^0 & \frac{1}{1!}\tau^1 & \cdots & \frac{1}{(m_i-1)!}\tau^{m_i-1} \\ 0 & \frac{1}{0!}\tau^0 & \cdots & \frac{1}{(m_i-2)!}\tau^{m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{0!}\tau^0 \end{bmatrix}. \quad (24)$$

where m_i represent the algebraic multiplicity.

The proofs of the recursive relations of the numerical scheme for the construction of \mathbf{P}_n are presented below.

Proof : [Proposition 4.3] The key point of the proof is the relation

$$l'_k - l'_{k-2} = \frac{2(2k-1)}{h} l_{k-1}, \quad \forall k \geq 2, \quad (25)$$

satisfied by Legendre polynomials. To compute $\Gamma_k(B)$, partial integration shows that

$$\begin{aligned} \Gamma_k(B) - \Gamma_{k-2}(B) &= \int_{-h}^0 e^{(h+\theta)B} (l_k(\theta) - l_{k-2}(\theta)) d\theta \\ &= -\frac{2(2k-1)}{h} B^{-1} \int_{-h}^0 e^{(h+\theta)B} l_{k-1}(\theta) d\theta + B^{-1} \left[e^{(h+\theta)B} (l_k(\theta) - l_{k-2}(\theta)) \right] \Big|_{-h}^0, \end{aligned}$$

and using the boundary conditions $l_k(-h) = l_{k-2}(-h) = (-1)^k$ and $l_k(0) = l_{k-2}(0) = 1$, the last term vanishes. Now, for $k = \{0, 1\}$, we have that

$$\begin{aligned} \Gamma_0(B) &= \int_{-h}^0 e^{(h+\theta)M} d\theta = B^{-1} (e^{hB} - I) \\ \Gamma_1(B) &= \int_{-h}^0 e^{(h+\theta)M} \left(\frac{2\theta+h}{h} \right) d\theta = B^{-1} (e^{hB} + I) - \frac{2}{h} B^{-1} \Gamma_0(B) \end{aligned}$$

The solution of $\Lambda_k(B)$ leads to the following equality

$$\Lambda_k(B) = -B^{-1} \int_{-h}^0 \left(e^{(\theta-\xi)B} \Big|_{-h}^{\theta} \right) l_k(\theta) d\theta = B^{-1} \Gamma_k(B) - B^{-1} \int_{-h}^0 l_k(\theta) d\theta,$$

where the second term vanishes for $k \geq 1$, therefore

$$\Lambda_0(B) = B^{-1} \left(\Gamma_0(B) - \int_{-h}^0 d\theta \right) = B^{-1} \Gamma_0(B) - hB^{-1}.$$

Finally, for $\Gamma_{k,i}(0)$, $\Lambda_{k,i}(0)$ associated with the nilpotent matrix N , we have that

$$\begin{aligned} \Gamma_{k,i}(0) &= \int_{-h}^0 e^{(h+\theta)J_{m_i}(0)} l_k(\theta) d\theta \\ \Lambda_{k,i}(0) &= \int_{-h}^0 \int_{-h}^{\theta} e^{(\theta-\xi)J_{m_i}(0)} l_k(\theta) d\xi d\theta. \end{aligned}$$

Replacing the exponential matrix $J_{m_i}(0)$ by (24) leads to (4.4). Each component of $\Gamma_{k,i}(0)$, $\Lambda_{k,i}(0)$ is defined by

$$\gamma_{k,p} = \frac{1}{p!} \int_{-h}^0 (h+\theta)^p l_k(\theta) d\theta,$$

and substituting $l_k(\theta)$ given by (2.7) yields the solution of $\gamma_{k,p}$. \square

Proof : [Proposition 4.4] Using the symmetry property of Legendre Polynomials, i.e. $l_k(-\theta-h) = (-1)^k l_k(\theta)$, $\theta \in [-h, 0]$, and the successive changes of variables $\tau_1 = -(\theta_1 + h)$ and $\tau_2 = -(\theta_2 + h)$, we have that

$$\begin{aligned} \bar{\Gamma}_{jk} &= \int_0^{-h} \int_0^{\tau_1} l_j(-\tau_1 - h) e^{(\tau_2 - \tau_1)J_L} l_k(-\tau_2 - h) d\tau_2 d\tau_1 \\ &= \int_{-h}^0 \int_{\tau_1}^0 (-1)^j l_j(\tau_1) e^{(\tau_2 - \tau_1)J_L} (-1)^k l_k(\tau_2) d\tau_2 d\tau_1 = (-1)^{j+k} \bar{\Gamma}_{jk}. \end{aligned}$$

\square

Proof : [Proposition 4.5] As in the proof of Proposition 4.3, an integration by parts and (25) ensure that $\bar{\Gamma}_{jk}(B)$ satisfies the recursive relation

$$\begin{aligned}\bar{\Gamma}_{jk}(B) - \bar{\Gamma}_{jk-2}(B) &= \int_{-h}^0 \int_{-h}^{\theta_1} l_j(\theta_1) e^{(\theta_1 - \theta_2)B} (l_k(\theta_2) - l_{k-2}(\theta_2)) d\theta_2 d\theta_1 \\ &= -B^{-1} \int_{-h}^0 l_j(\theta_1) \left((l_k(\theta_2) - l_{k-2}(\theta_2)) e^{(\theta_1 - \theta_2)B} \Big|_{-h}^{\theta_1} \right) d\theta_1 \\ &\quad + B^{-1} \int_{-h}^0 l_j(\theta_1) \int_{-h}^{\tau_1} e^{(\theta_1 - \theta_2)B} (l'_k(\theta_2) - l'_{k-2}(\theta_2)) d\theta_2 d\theta_1 \\ &= -\frac{h}{2k+1} B^{-1} (\delta_{jk} - \delta_{jk-2}) + \frac{2(2k+1)}{h} B^{-1} \bar{\Gamma}_{jk-1}(B).\end{aligned}$$

Note that the region of integration of $\bar{\Gamma}_{jk}$ is defined by

$$\mathcal{R}_4 = \left\{ \theta_1 \in [-h, 0], \theta_2 \in [-h, \theta_1] \right\} = \left\{ \theta_2 \in [-h, 0], \theta_1 \in [\theta_2, 0] \right\},$$

which means that using the change of coordinates $\tau_2 = \theta_1$, $\tau_1 = \theta_2$, implies

$$\bar{\Gamma}_{jk}(B) = \int_{-h}^0 \int_{\tau_1}^0 l_k(\tau_1) e^{(\tau_2 - \tau_1)B} l_j(\tau_2) d\tau_2 d\tau_1 = \bar{\Gamma}_{kj}^b(B) = (-1)^{j+k} \bar{\Gamma}_{kj}(B)$$

An integration by parts yields the initial values. The expression (4.5) is obtained by applying the procedure described in the proof of Proposition 4.3 for the case of the nilpotent matrix N , where

$$\bar{\gamma}_{jk,p} = \frac{1}{p!} \int_{-h}^0 \int_{-h}^{\theta_1} l_j(\theta_1) (\theta_1 - \theta_2)^p l_k(\theta_2) d\theta_2 d\theta_1.$$

□

Proof : [Proposition 4.6] By integration, we have

$$\begin{aligned}\bar{\Lambda}_{jk} &= B^{-1} \int_{-h}^0 l_j(\theta_1) e^{(\theta_1+h)B} d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 - B^{-1} \int_{-h}^0 l_j(\theta_1) d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 \\ &= \left(B^{-1} \Gamma_j(B) - B^{-1} \int_{-h}^0 l_j(\theta_1) d\theta_1 \right) \int_{-h}^0 l_k(\theta_2) d\theta_2.\end{aligned}$$

Therefore, the first expression of (4.6) is obtained knowing that

$$\int_{-h}^0 l_k(\theta) d\theta = \begin{cases} h & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}. \quad (26)$$

For the term $\bar{\Lambda}_{jk}^b(B)$, we have, for all $(j, k) \in \{0, 1, \dots, n-1\}^2$

$$\begin{aligned}\bar{\Lambda}_{jk}^b &= -B^{-1} \int_{-h}^0 \int_{-h}^0 l_j(\theta_1) \left(e^{-\xi B} \Big|_{\theta_1 - \theta_2}^0 \right) l_k(\theta_2) d\theta_2 d\theta_1 \\ \bar{\Lambda}_{jk}^b &= -B^{-1} \int_{-h}^0 l_j(\theta_1) d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 + B^{-1} \int_{-h}^0 \int_{-h}^0 l_j(\theta_1) e^{-(\theta_1 - \theta_2)B} l_k(\theta_2) d\theta_2 d\theta_1\end{aligned}$$

Then for $j \geq 1$, $k \geq 1$, the change of variable $\tau_1 = -\theta_1 - h$, and by the symmetry property of Legendre polynomials implies that $\bar{\Lambda}_{jk}^b$ satisfies the following equality

$$\begin{aligned}\bar{\Lambda}_{jk}^b &= B^{-1} \int_{-h}^0 l_k(\theta_2) e^{(\theta_2+h)B} d\theta_2 \int_{-h}^0 e^{-\theta_1 B} l_k(\theta_1) d\theta_1 e^{-hB} \\ &= B^{-1} \Gamma_k(B) \int_{-h}^0 e^{(\tau_1+h)B} l_k(-\tau_1 - h) d\theta_2 e^{-hB} \\ &= (-1)^j B^{-1} \Gamma_k(B) \Gamma_j(B) e^{-hB}\end{aligned}$$

Finally, the procedure described in the proof of Proposition 4.3 is applied to nilpotent matrix N , where each component of $\bar{\Gamma}_{jk,i}(0)$ is given by

$$\begin{aligned}\bar{\lambda}_{jk,p} &= \frac{1}{p!} \int_{-h}^0 \int_{-h}^0 \int_0^{\theta_1+h} l_j(\theta_1) \xi^p l_k(\theta_2) d\xi d\theta_2 d\theta_1 \\ &= \frac{1}{(p+1)!} \int_{-h}^0 l_j(\theta_1) (\theta_1 + h)^{p+1} d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 = \gamma_{p+1,j} \int_{-h}^0 l_k(\theta_2) d\theta_2,\end{aligned}$$

and for $\bar{\Gamma}_{jk,i}^b(0)$ we have

$$\begin{aligned}\bar{\lambda}_{jk,i,p}^b &= \frac{1}{p!} \int_{-h}^0 \int_{-h}^0 \int_{\theta_1-\theta_2}^0 l_j(\theta_1)(-\xi)^p l_k(\theta_2) d\xi d\theta_2 d\theta_1 \\ &= \frac{(-1)^{p+1}}{(p+1)!} \int_{-h}^0 \int_{-h}^0 l_j(\theta_1)(\theta_1 - \theta_2)^{p+1} l_k(\theta_2) d\theta_2 d\theta_1\end{aligned}$$

whose solution is obtained by integration, replace l_k given by expression (2.7). \square

Proof : [Proposition 4.7] The next result follows immediately from of solution of $\bar{\Delta}_{jk}(B)$

$$\begin{aligned}\bar{\Delta}_{jk}(B) &= \int_{-h}^0 \int_{-h}^0 \int_0^{\theta_1+h} \int_0^{\xi_1} l_j(\theta_1) e^{(\xi_1-\xi_2)B} l_k(\theta_2) d\xi_2 d\xi_1 d\theta_2 d\theta_1 \\ &= -B^{-1} \int_{-h}^0 l_j(\theta_1) \left(\frac{h}{2} l_1(\theta_1) + \frac{h}{2} l_0(\theta_1) \right) d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 \\ &\quad + \left(B^{-2} \Gamma_j(B) - B^{-2} \int_{-h}^0 l_j(\theta_1) d\theta_1 \right) \int_{-h}^0 l_k(\theta_2) d\theta_2.\end{aligned}$$

By the property expressed in (26) we obtain directly the first expression of (4.7) for $k = 0, j \geq 2$. Otherwise, $\bar{\Delta}_{jk}(B) = 0$. The same procedure is applied to $\bar{\Delta}_{jk}^b(B)$ using Legendre polynomials properties, thus by integration by parts we obtain the initial values expressed in (4.8).

Finally, replace the exponential matrix of N by (24) into $\bar{\Delta}_{jk}(N)$, leads to the expression (4.9), where each component is defined by

$$\begin{aligned}\bar{\delta}_{jk,i,p} &= \frac{1}{p!} \int_{-h}^0 \int_{-h}^0 \int_0^{\theta_1+h} \int_0^{\xi_1} l_j(\theta_1)(\xi_1 - \xi_2)^p l_k(\theta_2) d\xi_2 d\xi_1 d\theta_2 d\theta_1 \\ &= \frac{1}{(p+2)!} \int_{-h}^0 l_j(\theta_1)(\theta_1 + h)^{p+2} d\theta_1 \int_{-h}^0 l_k(\theta_2) d\theta_2 = \bar{\lambda}_{p+1,jk}\end{aligned}$$

and for $\Omega_{j,k,i}^b(0)$ we have

$$\begin{aligned}\bar{\delta}_{jk,i,p}^b &= \frac{1}{p!} \int_{-h}^0 \int_{-h}^0 \int_0^{\theta_1+h} \int_{\xi_1}^{\theta_2+h} l_j(\theta_1)(\xi_2 - \xi_1)^p l_k(\theta_2) d\xi_2 d\xi_1 d\theta_2 d\theta_1 \\ &= -\frac{(-1)^{p+2}}{(p+2)!} \int_{-h}^0 \int_{-h}^0 l_j(\theta_1)(\theta_1 - \theta_2)^{p+2} l_k(\theta_2) d\theta_2 d\theta_1 \\ &\quad + \frac{1}{(p+2)!} \int_{-h}^0 l_j(\theta_1) d\theta_1 \int_{-h}^0 (\theta_2 + h)^{p+2} l_k(\theta_2) d\theta_2 \\ &= \bar{\lambda}_{p+1,kj} - \bar{\lambda}_{p+1,jk}^b.\end{aligned}$$

\square

Remark 8 *The authors have created a repository on the Git-Hub platform where we include the created codes (systems with distributed delays with constant and exponential kernel.): https://github.com/AlexCasHer/TDS_Test_Stability*

References

- [1] Irina V Alexandrova. “A finite Lyapunov matrix-based stability criterion for linear delay systems via piecewise linear approximation”. In: *Systems & Control Letters* 181 (2023), p. 105624.
- [2] AN Aliseyko. “Lyapunov matrices for a class of time-delay systems with piecewise-constant kernel”. In: *International Journal of Control* 92.6 (2019), pp. 1298–1305.
- [3] Mathieu Bajodek. “Stability analysis of linear ODE-PDE interconnected systems”. PhD thesis. Université de Toulouse 3 Paul Sabatier, 2022.
- [4] Mathieu Bajodek, Frédéric Gouaisbaut, and Alexandre Seuret. “Necessary and sufficient stability condition for time-delay systems arising from Legendre approximation”. In: *IEEE Transactions on Automatic Control* (2022).
- [5] R. E. Bellman and K. L. Cooke. “Differential-difference equations”. In: *Press, New York* (1963).

- [6] Alejandro Castaño et al. “Finite Lyapunov Stability Test for a Class of Linear Systems With Pointwise and Distributed Delays”. In: *IEEE Transactions on Automatic Control* (2024).
- [7] Robert M Corless et al. “On the Lambert W function”. In: *Advances in Computational mathematics* 5 (1996), pp. 329–359.
- [8] C. Cuvas and S. Mondié. “Necessary stability conditions for delay systems with multiple pointwise and distributed delays”. In: *IEEE Transactions on Automatic Control* 61.7 (2016), pp. 1987–1994.
- [9] Richard Datko. “An algorithm for computing Liapunov functionals for some differential-difference equations”. In: *Ordinary differential equations*. Elsevier, 1972, pp. 387–398.
- [10] Alexey V Egorov. “A finite necessary and sufficient stability condition for linear retarded type systems”. In: *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE. 2016, pp. 3155–3160.
- [11] Christian W Eurich, Andreas Thiel, and Lorenz Fahse. “Distributed delays stabilize ecological feedback systems”. In: *Physical review letters* 94.15 (2005), p. 158104.
- [12] Emilia Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.
- [13] Marco Gomez, Alexey V Egorov, and Sabine Mondié. “Lyapunov matrix based necessary and sufficient stability condition by finite number of mathematical operations for retarded type systems”. In: *Automatica* 108 (2019), p. 108475.
- [14] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge University Press, 2012.
- [15] EF Infante and WB Castelan. “A Lyapunov functional for a matrix difference-differential equation”. In: *Journal of Differential Equations* 29.3 (1978), pp. 439–451.
- [16] Vladimir L Kharitonov. *Time-delay systems: Lyapunov functionals and matrices*. Basel: Birkhäuser, 2013.
- [17] Vladimir L Kharitonov and Alexey P Zhabko. “Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems”. In: *Automatica* 39.1 (2003), pp. 15–20.
- [18] Vladimir Kolmanovskii and Anatolii Myshkis. *Applied theory of functional differential equations*. Vol. 85. Springer Science & Business Media, 2012.
- [19] I. V. Medvedeva and A. P. Zhabko. “Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems”. In: *Automatica* 51 (2015), pp. 372–377.
- [20] Irina V Medvedeva and Alexey P Zhabko. “Constructive method of linear systems with delay stability analysis”. In: *IFAC Proceedings Volumes* 46.3 (2013), pp. 1–6.
- [21] Sabine Mondié, Alexey Egorov, and Marco A Gomez. “Lyapunov stability tests for linear time-delay systems”. In: *Annual Reviews in Control* (2022).
- [22] Ju Neimark. “D-subdivisions and spaces of quasi-polynomials”. In: *Prikladnaya Matematika i Mekhanika* 13.5 (1949), pp. 349–380.
- [23] Silviu-Iulian Niculescu. *Delay effects on stability: a robust control approach*. Vol. 269. Springer Science & Business Media, 2001.
- [24] Hitay Ozbay, Catherine Bonnet, and Jean Clairambault. “Stability analysis of systems with distributed delays and application to hematopoietic cell maturation dynamics”. In: *2008 47th IEEE conference on decision and control*. IEEE. 2008, pp. 2050–2055.
- [25] Iu M Repin. “Quadratic Lyapunov functionals for systems with delay”. In: *Journal of Applied Mathematics and Mechanics* 29.3 (1965), pp. 669–672.
- [26] Alexandre Seuret, Frédéric Gouaisbaut, and Yassine Ariba. “Complete quadratic Lyapunov functionals for distributed delay systems”. In: *Automatica* 62 (2015), pp. 168–176.
- [27] Haiyong Wang and Shuhuang Xiang. “On the convergence rates of Legendre approximation”. In: *Mathematics of computation* 81.278 (2012), pp. 861–877.
- [28] Huang Wenzhang. “Generalization of Liapunov’s theorem in a linear delay system”. In: *Journal of mathematical analysis and applications* 142.1 (1989), pp. 83–94.
- [29] Feng Zheng and Paul M Frank. “Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers”. In: *Automatica* 38.3 (2002), pp. 487–497.