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1 On the speed of convergence of the pressure function
2 at zero temperature

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4 January 19, 2025

5 **Abstract**

6 We prove here that the pressure function cannot converge to the limit entropy at
7 zero temperature faster than some exponential rate. Furthermore, we characterize
8 this limit rate via an expression involving the Peierls barriers between the irreducible
9 components of the Aubry set. This extends and completes results from [8] and [7].
10 In the first one, an exact exponential speed of convergence was proved, under the
11 assumption that the Aubry set is a subshift of finite type. In the later one, a rate
12 was given but without interpretation in term of Thermodynamical quantities of the
13 system.

14 Keywords : Ergodic optimization, selection, zero temperature.
15 MSC2020: 37D35, 37A60

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1 Introduction

1.1 Background and settings

In this paper we show that the pressure function converges at most exponentially fast to the limit entropy as the temperature goes to 0. We remind that, given a dynamical system (X, T) and a potential $A : X \rightarrow \mathbb{R}$, the pressure function is defined by

$$P(\beta) := \sup_{\mu \text{ } T\text{-inv}} \left\{ h_{\mu}(T) + \beta \cdot \int A d\mu \right\},$$

where $h_{\mu}(T)$ is the Kolmogorov entropy and β is a real parameter. In statistical mechanics β represents the inverse of the temperature. It is known that $P(\beta)$ goes to a non-negative value h as β goes to $+\infty$. This value h is the topological entropy of the Aubry set Ω , and also of the Mather set for the potential A , \mathcal{M}_a . The Mather set is the union of the supports of the *maximizing measures*, that is, any T -invariant probability measure μ such that

$$\int A d\mu = m(A) =: \sup_{\nu \text{ } T\text{-inv}} \int A d\nu.$$

In the historical unpublished paper [5] it was proved that $P(\beta)$ goes to h faster than $1/\beta^2$. In [7], it is proved that the convergence cannot be faster than exponential, but there is no identification of the exponential rate (in term of quantities linked to the thermodynamics). In [8] it is proved, under the assumption that the Aubry set is a subshift of finite type, that the convergence occurs at exponential rate, and the rate is characterized (in term of quantities linked to the thermodynamics).

Here, we prove that in the general case, the convergence cannot be faster than exponential and, as in [8], we show that the minimum rate is linked to thermodynamic quantities.

1.2 Settings and results

In the following $X \subseteq \{1, \dots, D\}^{\mathbb{N}}$ is a subshift of finite type given by an aperiodic matrix. A point in X will be referred to as an infinite sequence or an infinite words $x = x_0 x_1 \dots$,

1 where the x_i 's are the *digits* (in $\{1, \dots, D\}$). A cylinder $[\omega_0 \dots \omega_n]$ is the set of words
 2 $x = x_0x_1 \dots$ such that $x_i = \omega_i$ for $i \leq n$. It will also be denoted as a $(n + 1)$ -cylinder.
 X equipped with the metric d defined by

$$d(x_0x_1x_2x_3 \dots, y_0y_1y_2y_3 \dots) = 2^{-\min\{i \mid x_i \neq y_i\}}$$

is a compact space. The shift map $\sigma : X \rightarrow X$ is defined by

$$x_0x_1x_2x_3 \dots = x_1x_2x_3x_4 \dots$$

We consider $A : X \rightarrow \mathbb{R}$, Lipschitz continuous. The set of σ -invariant probabilities is denoted by $\mathcal{M}(\sigma)$. We set

$$m(A) := \max_{\mu \in \mathcal{M}(\sigma)} \int A d\mu.$$

3 Any measure realizing this supremum is called A -maximizing or maximizing for A . We
 4 denote by $\mathcal{M}_{max}(A)$ the set of A -maximizing measures. We remind (see [1] and below)
 5 that we always may assume that A is non-positive and $m(A) = 0$. From now on, we
 6 assume that this holds.

7 We remind that $S_n(A)$ stands for the Birkhoff sum $A + \dots + A \circ \sigma^{n-1}$.

Definition 1. *The **Mañé potential** associated to A is defined by*

$$S(x, y) := \lim_{\epsilon \rightarrow 0} [\sup\{S_n(A)(z), \sigma^n(z) = y, d(x, z) < \epsilon\}],$$

the **Aubry set** of A is defined by

$$\Omega := \{x \in X \mid S(x, x) = 0\}$$

8 Note that $A \leq 0$ yields that $S(\cdot, \cdot)$ is non-positive.

9 We remind that the Mather set \mathcal{M}_a is included into the Aubry set Ω . We emphasize
 10 that the Mather set is never empty, since $\mu \mapsto \int A d\mu$ is continuous over the compact set
 11 $\mathcal{M}(\sigma)$. Hence the Aubry set is neither non-empty.

12 We remind (see also Lemma 5):

- 1 • $S(x, y) + S(y, z) \leq S(x, z)$ for every x, y, z in X ;
- 2 • for any fixed $x \in X$, the map $y \mapsto S(x, y)$ is Lipschitz continuous;
- 3 • $x \mapsto S(x, y)$ is upper semi-continuous.

4 The next definition aims to define irreducible components of the Aubry set. The proof
5 of the proposition part is done later (see Lemma 6).

6 **Definition 2 (and Proposition).** *The relation $S(x, y) + S(y, x) = 0$ defines an equiv-*
7 *alence relation. Each equivalence class is called an irreducible component of the Aubry*
8 *set.*

9 It is proved below (see Lemma 10) that irreducible components are closed sets. There-
10 fore, they are compact sets and two different irreducible components are at positive dis-
11 tance one from the other.

Definition 3. *Let Ω_i and Ω_j be irreducible components (possibly $i = j$). Then we set*

$$S^{ext}(j, i) := \inf_{x \in \Omega_i} \sup \{ S(y, z) + A(z), \sigma(z) = x, z \notin \Omega_i, y \in \Omega_j \}.$$

12 We claim (see Lemma 11) that each $S^{ext}(j, i)$ is a non-positive real number. Further-
13 more, Lemma 5 will show that the sup can be replaced by a max.

14 If $\Omega_{i_1}, \dots, \Omega_{i_n}$ are (different) irreducible components, then we denote by $\lambda(i_1, \dots, i_n)$ the
15 unique eigenvalue (see [4]) for the $n \times n$ matrix with entries $S^{ext}(j, i)$ and $i, j = i_1, \dots, i_n$
16 within the **max-plus formalism**.

We remind that for the $n \times n$ matrix with entries $a_{ij} > -\infty$, the eigenvalue λ within
the max-plus formalism is the maximal mean value of the entries along the cycles:

$$\lambda = \max_{i_1, i_2, \dots, i_k \text{ all different}} \frac{a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k}.$$

1 We remind that the pressure function $\beta \mapsto P(\beta)$ is convex¹ with $P'(\beta) = \int A d\mu_\beta$,
 2 where μ_β is the unique equilibrium state for $\beta.A$. Therefore, $P(\beta)$ is non-increasing (in β)
 3 and is bounded from below by h_μ where μ is any maximizing measure. A classical argu-
 4 ment shows that $P(\beta)$ converges to h , which is the maximal entropy among maximizing
 5 measures.

6 The main question we are interested in is at which speed does the convergence occur.

Theorem A. *We get a minimal exponential rate for the convergence of the pressure:*

$$\liminf_{\beta \rightarrow +\infty} \frac{1}{\beta} \log(P(\beta) - h) \geq \sup\{\lambda(i_1, \dots, i_n), n \in \mathbb{N}^*, \Omega_{i_k} \text{ are irreducible components with maximal entropy}\}.$$

7 1.3 On the structure of the Aubry set

8 Subshifts of finite type are perfectly well understood. We refer to [6, Ex. 1.9.4-] for their
 9 decompositions in irreducible components. This has been used in [8].

10 We emphasize here at least two important points where the assumption “the Aubry
 11 set is a subshift of finite type” is relevant:

- 12 1. Equality or inequality on the speed of convergence. Equality comes from the exis-
 13 tence of conformal measures associated to the measure of maximal entropy in each
 14 irreducible component with maximal entropy. On the other hand, for general shift,
 15 there is no general theory. This explains why we only get² inequalities (see below)
 16 in Theorem A.
- 17 2. Influence of the components with minor entropy. In the case $h > 0$, it is proved³
 18 in [8] that components of the Aubry with entropy strictly less than h do not play
 19 any role in the speed of convergence. The proof deeply uses the fact that in the

¹this immediately follows from definition.

²at least for the moment.

³still under the assumption that the Aubry set is a subshift of finite type.

1 case the Aubry set is a subshift of finite type, it only has finitely irreducible compo-
2 nents. In the case Ω is a general subshift, it may have uncountably many irreducible
3 components. We provide an example below.

4 We warmly thank Sebastien Ferenczy for having given us the references we needed to
5 construct the example.

6 We consider some small closed interval $[a, b]$ and then consider the union of Sturmian
7 shifts with angle $\alpha \in [a, b]$. We refer to [10] for a general description of these Sturmian
8 trajectories.

9 More precisely, Ω_α is the subshift whose language correspond to all the words that
10 appear, considering a line passing to $(0, 0)$ in \mathbb{R}^2 , with slope α , and writing 0 when the
11 line crosses an horizontal line in the net defined by \mathbb{Z}^2 , and 1 if it crosses a vertical line.

12 It is known that each Ω_α is uniquely ergodic which implies that $\Omega := \bigcup_{\alpha \in [a, b]} \Omega_\alpha$ has
13 uncountably many irreducible components, some of them being periodic orbit (iff $\alpha \in \mathbb{Q}$).

14 It is known (see [9, Corollary 18]) that the number A_n of all the words of length
15 n appearing in some Sturmian subshift is equivalent to $\frac{n^3}{\pi^2}$ as n goes to $+\infty$. This
16 immediately yields that Ω has zero entropy.

17 Now consider $A := -d(\cdot, \Omega)$. Then Ω is the Aubry set for A and it has infinitely many
18 irreducible components.

19 1.4 Plan of the paper

20 In Section 2 we give technical results. In particular we remind basic results on the cali-
21 brated subactions and the Mañé Potential. In Section 3 we do the proof of the Theorem
22 A. The main point is Proposition 20 where we prove a key inequality.

1 2 Technical results

2 2.1 On the Mañé potential

3 2.1.1 Technics to compute/estimate it

For $\varepsilon > 0$ we set

$$S^\varepsilon(x, y) := \sup \{S_k(A)(z), \sigma^k(z) = y, d(x, z) < \varepsilon\}.$$

4 Then $S(x, y) = \lim_{\varepsilon \rightarrow 0} S^\varepsilon(x, y)$.

Lemma 4. *Let x and y be in X . Let $\bar{\xi}_n$ be points such that $\bar{\xi}_n \rightarrow x$ as n goes to $+\infty$, and for any n there exists k_n with $k_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that $\sigma^{k_n}(\bar{\xi}_n) = y$. Then,*

$$S(x, y) \geq \limsup_{n \rightarrow +\infty} S_{k_n}(A)(\bar{\xi}_n).$$

Proof. Pick $\varepsilon > 0$. Let N be such that for every $n > N$, $d(x, \bar{\xi}_n) < \varepsilon$. Then

$$S^\varepsilon(x, y) := \sup \{S_k(A)(z), \sigma^k(z) = y, d(x, z) < \varepsilon\} \geq S_{k_n}(A)(\bar{\xi}_n)$$

5 holds as soon as $n > N$ holds. This yields, for every ε , $S^\varepsilon(x, y) \geq \limsup_n S_{k_n}(A)(\bar{\xi}_n)$.

6 Hence doing $\varepsilon \downarrow 0$ we get the result. □

7 2.1.2 Mains properties and one immediate consequence

8 Let us first recall a classical result for the Mañé potential:

9 **Lemma 5** (See [8]). *Let S be the Mañé potential. We have:*

- 10 1. $S(x, y) + S(y, z) \leq S(x, z)$ for every x, y, z in X ;
- 11 2. for any fixed $x \in X$, the map $y \mapsto S(x, y)$ is Lipschitz continuous;
- 12 3. if $x \sim y$, then $S(x, \cdot) = S(y, \cdot)$;

1 4. if $\sigma(x) = y$, then $S(x, y) = A(x)$.

2 **Lemma 6.** $S(x, y) + S(y, x) = 0$ defines an equivalence relation.

Proof. By definition $x \in \Omega$ if and only if $S(x, x) = 0$, which yields $x \sim x$. Furthermore

$$S(x, y) + S(y, x) = S(y, x) + S(x, y),$$

3 hence $x \sim y$ yields $y \sim x$. Finally, if $x \sim y$ and $y \sim z$ hold,

$$\begin{aligned} 0 \geq S(x, z) + S(z, x) &\geq S(x, y) + S(y, z) + S(z, y) + S(y, x) \\ &= S(x, y) + S(y, x) + S(y, z) + S(z, y) = 0 + 0 + 0 + 0 = 0. \end{aligned}$$

4 Hence, $x \sim z$ holds. □

5 We remind that irreducible components of Ω are the classes of equivalences in Ω .

6 **Notation 7.** For x in Ω_i we shall set $S(\Omega_i, \cdot)$ for $S(x, \cdot)$.

7 An immediate consequence is that in the definition of $S^{ext}(j, i)$ we can replace sup by
8 max, since any x admits finitely preimages (of order 1) and $S(y, z)$ does not depend on
9 the choice of $y \in \Omega_j$.

10 2.1.3 Irreducible components and costs

11 **Lemma 8.** The potential A is equal to 0 on Ω .

Proof. By definition x belongs to Ω if and only if $S(x, x) = 0$. Hence we get

$$0 \geq A(x) \geq S(x, x) = 0.$$

12 □

13 **Lemma 9.** If Ω_i is an irreducible component, then $S^{ext}(i, i) < 0$.

1 *Proof.* By our assumption on A , $S^{ext}(i, i)$ is non-positive. If x is in Ω_i and $\sigma(z) = x$ with
 2 $z \notin \Omega_i$, then two cases occur:

- 3 1. either $A(z) < 0$, and $S(\Omega_i, z) + A(z) < 0$,
- 4 2. or $A(z) = 0$ and $S(\Omega_i, z) < 0$.

Indeed, in the later case if $S(\Omega_i, z) = 0$, we get $S(x, z) = 0$ (since $x \in \Omega_i$) and $A(z) = S(z, x) = 0$. This yields

$$S(x, z) + S(z, x) = 0,$$

hence $z \sim x$ which does not hold. As x has finitely many preimages

$$S^{ext}(i, i) \leq \max\{S(\Omega_i, z) + A(z), \sigma(z) = x, z \notin \Omega_i\} < 0.$$

5

□

6 **Lemma 10.** *Irreducible components of the Aubry set are closed sets.*

7 *Proof.* Let us pick x and y_n such that $x \sim y_n$ for every n . We assume that (y_n) converges
 8 to y . We prove that $y \sim x$.

We remind that $S(x, \cdot)$ is Lipschitz continuous. Furthermore, $\sim y_n$ means $S(x, y_n) + S(y_n, x) = 0$, and since S is non-positive it yields (for every n)

$$S(x, y_n) = 0 = S(y_n, x).$$

9 Then, doing $n \rightarrow +\infty$ in $S(x, y_n) = 0$ yields $S(x, y) = 0$.

10 Let us now prove that $S(y, x) = 0$ holds.

Let α be positive. Let ε be positive. We choose N such that $d(y_N, y) < \varepsilon/2$. As $S(y_N, x) = 0$, we get

$$\limsup_{\rho \rightarrow 0} \{S_n(A)(z), \sigma^n(z) = x, d(z, y_N) < \rho\} = 0.$$

We can find $\rho_0 > 0$ such that for every $0 < \rho < \rho_0$,

$$S^\rho(y_N, x) = \sup\{S_n(A)(z), \sigma^n(z) = x, d(z, y_N) < \rho\} > -\frac{\alpha}{2}.$$

Let us pick $\rho > 0$, such that $\rho < \rho_0$ and $\rho < \frac{\varepsilon}{2}$ hold. One can find z and n such that $\sigma^n(z) = x$, $d(z, y_N) < \rho$ and

$$S_n(A)(z) \geq -\alpha.$$

Then $d(z, y) < \varepsilon$, $\sigma^n(z) = x$ and $S_n(A)(z) > -\alpha$. This yields

$$S^\varepsilon(x, y) = \sup\{S_n(A)(z), \sigma^n(z) = x, d(z, y) < \varepsilon\} > -\alpha.$$

1 This holds for every ε , hence $S(y, x) := \lim_{\varepsilon \rightarrow 0} S^\varepsilon(y, x)$ is greater than $-\alpha$. This also
 2 holds for every α , and since $S(y, x)$ is non-positive, we get $S(y, x) = 0$. \square

3 **Lemma 11.** For any i and j , $S^{ext}(i, j)$ is a finite non-positive real number.

Proof. By our assumption on A , $S^{ext}(i, j)$ is non-positive. Then, for any $x \in \Omega_i$, and for any y ,

$$\sup_{z \in \Omega_i} S(z, y) \geq S(x, y).$$

The map $y \mapsto S(x, y)$ is Lipschitz continuous hence bounded on X . Therefore

$$S^{ext}(i, j) = \inf_{y' \in \Omega_j} \sup_{z \in \Omega_i} \{S(z, y), y \notin \Omega_i, \sigma(y) = y'\} \geq -\|S(x, \cdot)\|_\infty > -\infty.$$

4

\square

5 **Lemma 12.** Let y be in Ω_i and x be in X . There exists a sequence of points (ξ^n) and
 6 integers k_n such that

- 7 1. $\xi^n \rightarrow y$ as $n \rightarrow +\infty$,
- 8 2. $\sigma^{k_n}(\xi^n) = x$,
- 9 3. $S(y, x) = \lim_{n \rightarrow +\infty} S_{k_n}(A)(\xi^n)$.

Proof. We get $S(y, x) = \lim_{\varepsilon \rightarrow 0} S^\varepsilon(y, x)$. Set $\varepsilon := \frac{1}{n}$ and consider ξ^n and k_n such that

$$d(y, \xi^n) \leq \frac{1}{n}, \quad S^{\frac{1}{n}}(y, x) \geq S_{k_n}(A)(\xi^n) \geq S^{\frac{1}{n}}(y, x) - \frac{1}{n}, \quad x = \sigma^{k_n}(\xi^n).$$

10 Then, $\xi^n \rightarrow y$ as $n \rightarrow +\infty$

\square

1 2.2 Transfer Operator, eigenfunctions, calibrated subactions

2 For an irreducible subshift of finite type (not necessarily X), say Y , and $B : Y \rightarrow \mathbb{R}$
 3 Lipschitz-continuous, The transfer operator L_B is defined by

$$L_B(\psi)(x) = \sum_{\sigma(z)=x} e^{B(z)}\psi(z).$$

4 We refer to [3] for complete study of this operator. It acts on $\mathcal{C}(X)$ and on the space
 5 $\mathcal{C}^{0+1}(X)$ of Lipschitz continuous functions. Its spectral radius λ_B (for $\|\cdot\|_\infty$ -norm) is a
 6 single dominating eigenvalue on $\mathcal{C}^{0+1}(X)$. It also turns out to be equal to e^P , where
 7 P is the pressure associated to B and for the dynamical system (Y, σ) . The associated
 8 1-dimensional eigen-space is $\text{span}(H_B)$, where H_B is Lipschitz continuous, positive and
 9 uniquely determined up to some normalization. The dual operator L_B^* for the $\|\cdot\|_\infty$ -norm
 10 acts on the set of measures and ν_B is the unique probability satisfying $L_B^*(\nu_B) = \lambda_B \nu_B$.
 11 It is referred to as the *eigenmeasure* or the *conformal measure*.

In the case $Y = X$ and $B := \beta.A$, then the family of functions $\frac{1}{\beta} \log H_{\beta.A}$ is equicon-
 tinuous (actually Lipschitz continuous). Any accumulation point is a *calibrated subaction*.
 We remind that a calibrated subaction for A is a Lipschitz continuous function $V : X \rightarrow \mathbb{R}$
 satisfying for every $x \in X$

$$\max_{\sigma(z)=x} [A(z) + V(z)] = V(x).$$

12 Furthermore we have

Lemma 13 (See [2, 8]). *For any calibrated subaction V we have $\forall x_0 \in X$*

$$V(x_0) = \sup_{y \in \mathcal{M}_a} \{V(y) + S(y, x_0)\} = \sup_{a \in \Omega} \{V(y) + S(y, x_0)\}.$$

13 **Corollary 14.** *If V is a calibrated subaction for A then V is constant on each set irre-*
 14 *ducible component Ω_i of Ω .*

15 *Proof.* Let $x, y \in \Omega_i$ and V be a calibrated subaction. As $S(x, y) = S(y, x) = 0$, applying
 16 Lemma 13 we have

$$V(x) = \sup_{z \in \Omega} [S(z, x) + V(z)] \geq S(y, x) + V(y) = V(y).$$

1 and

$$V(y) = \sup_{z \in \Omega} [S(z, y) + V(z)] \geq S(x, y) + V(x) = V(x).$$

2

□

3 **Notation 15.** We shall write $V(\Omega_i)$.

4 **2.2.1 On the hypothesis on A**

For a general A one get

$$e^{P(\beta, A)} H_\beta(A)(x) = \sum_{y, \sigma(y)=x} e^{\beta \cdot A(y)} H_\beta(y).$$

Doing $\frac{1}{\beta} \log$ in this equality and then $\beta \rightarrow +\infty$, yields

$$m(A) + V(x) = \max_y \{A(y) + V(y)\}.$$

Thus for any y we get

$$A(y) + V(y) - V \circ \sigma(y) - m(A) \leq 0.$$

5 Hence, exchanging the general A by $A + V - V \circ \sigma - m(A)$, we get a new potential, equal
6 to A up to a co-boundary plus a constant, that is always non-positive and which maximal
7 integral with respect to any invariant measure in 0.

8 **2.3 A kind of Laplace method result**

9 **Lemma 16.** Let (E, δ) be a metric space. Let $\varphi_1, \dots, \varphi_k$ be continuous from E to \mathbb{R} . Let
10 $(\phi_{n,i})$ be a sequences of continuous functions from E to \mathbb{R} which uniformly converge to ϕ_i .

11 Let (x_n) be a sequence which converges to x in E . Then, $\frac{1}{n} \log \left(\sum_{i=1}^k e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} \right)$

12 converges to $\max_i \{\varphi_i(x) + \phi_i(x)\}$.

1 *Proof.* Let us set $\lambda_i := \varphi_i(x) + \phi_i(x)$, $\lambda := \max_i \lambda_i$. Let $I := \{i, \lambda_i = \lambda\}$.
2 Let $0 < \varepsilon$. Let N be such that for any $n \geq N$, for any i , $\|\phi_{n,i} - \phi_i\|_\infty < \varepsilon$.
3 Let N' be such that for every $n \geq N'$ and for any i , $|\varphi_i(x_n) - \varphi_i(x)| < \varepsilon$, and
4 $|\phi_i(x_n) - \phi_i(x)| < \varepsilon$,
5 For $n \geq \max(N, N')$ we get

$$\begin{aligned}
\sum_{i=1}^k e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} &\geq \sum_{i \in I} e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} \\
&\geq \sum_{i \in I} e^{n \cdot (\varphi_i(x) - \varepsilon) + n(\phi_i(x_n) - \varepsilon)} \\
&\geq \sum_{i \in I} e^{n \cdot (\varphi_i(x) - \varepsilon) + n(\phi_i(x) - 2\varepsilon)} \\
&\geq e^{n(\lambda - 3\varepsilon)}.
\end{aligned}$$

6 This yields for any ε , $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \left(\sum_{i=1}^k e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} \right) \geq \lambda - 3\varepsilon$.

7 On the other hand we get

$$\begin{aligned}
\sum_{i=1}^k e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} &\leq \sum_{i=1}^k e^{n \cdot (\varphi_i(x) + \varepsilon) + n(\phi_i(x_n) + \varepsilon)} \\
&\leq \sum_{i=1}^k e^{n \cdot (\varphi_i(x) + \varepsilon) + n(\phi_i(x) + 2\varepsilon)} \\
&\leq k \cdot e^{n(\lambda + 3\varepsilon)}.
\end{aligned}$$

8 This yields for any ε , $\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left(\sum_{i=1}^k e^{n \cdot \varphi_i(x_n) + n \phi_{n,i}(x_n)} \right) \leq \lambda + 3\varepsilon$.

9

□

10 **3 Proof**

11 In all that part, we shall consider a sequence (β_n) such that $\frac{1}{\beta_n} \log(P(\beta_n) - h)$ converges
12 to γ . For simplicity we shall always write $\beta \rightarrow +\infty$ even if we actually only consider

1 $\beta_n \rightarrow +\infty$.

2 **3.1 Neighborhoods**

3 We pick some irreducible component, say Ω_i , with entropy h_i equal to the maximal entropy
4 h . We fix some small positive real number ε_0 and then, consider a small neighborhood
5 $B(\Omega_i, \varepsilon_0)$, that is the points at distance to Ω_i lower than ε_0 .

6 Then we denote by $\Sigma_{i, \varepsilon_0}$ the maximal invariant set in $B(\Omega_i, \varepsilon_0)$. It is non-empty
7 since it contains Ω_i and is a subshift of finite type. It can be decomposed in finitely many
8 irreducible components, and for simplicity we shall assume that $\Sigma_{i, \varepsilon_0}$ is indeed irreducible.
9 We shall discuss this assumption later.

10 Then, we denote by ν_{ε_0} the unique conformal measure for the potential $B \equiv 0$ in $\Sigma_{i, \varepsilon_0}$.
11 In the case $\Sigma_{i, \varepsilon_0}$ is not irreducible any irreducible component (they are finitely many)
12 contains such a conformal measure.

13 We then do the same work for a smaller neighborhood $B(\Omega_i, \varepsilon)$ with $\varepsilon < \varepsilon_0$, with
14 associated subshift of finite type $\Sigma_{i, \varepsilon}$ and conformal measure ν_ε (for the potential constant
15 equal to zero).

16 **Lemma 17.** *Any accumulation point for the weak* topology $\nu_{1/\infty}$ for ν_ε as ε goes to 0*
17 *has support in Ω_i .*

18 *Proof.* For $x \notin \Omega_i$, $d(\{x\}, \Omega_i)$ is positive, thus bigger than some $2\varepsilon_1$. For $\varepsilon < \varepsilon_1$, $x \notin$
19 $B(\Omega_i, \varepsilon)$, then x is not in $\Sigma_{i, \varepsilon}$. Furthermore $\Sigma_{i, \varepsilon}$ contains the support for ν_ε .

As $B(x, \varepsilon_1)$ is a clopen set, $\mathbb{1}_{B(x, \varepsilon_1)}$ is a continuous function. Hence for any $\varepsilon < \varepsilon_1$,

$$\nu_\varepsilon(B(x, \varepsilon_1)) = \int \mathbb{1}_{B(x, \varepsilon_1)} d\nu_\varepsilon = 0.$$

20 This yields $\nu_{1/\infty}(B(x, \varepsilon_1)) = 0$, which means that x does not belong to $\text{supp } \nu_{1/\infty}$. \square

1 **Remark 18.** *We point out one of the main problems we shall deal with. There is no way*
 2 *to know if $\nu_{1/\infty}$ has full support or no. ■*

3 In particular, we will need estimates on points $x \in \Omega_i$ with preimages outside Ω_i . As
 4 we cannot ensure such a set with positive $\nu_{1/\infty}$ -measure exists, we need to get round this
 5 difficulty.

6 **Lemma 19.** *There exists N_0 such that for every $x \in \Sigma_{i,\varepsilon_0}$, there exists $x' \in \sigma^{-N_0}(x)$ and*
 7 *$x' \notin \Sigma_{i,\varepsilon_0}$.*

Proof. If $x = x_0x_1\dots$ is a point in Σ_{i,ε_0} and $[\omega] = [\omega_0\dots\omega_k]$ a cylinder with empty
 intersection with $B(\Omega_i, \varepsilon_0)$. Then mixing shows that there exists $z_0\dots z_r$ with $z_0 = \omega_k$
 and $z_r = x_0$ such that the word $\omega_0\dots\omega_kz_1\dots z_{r-1}x_0$ is admissible in X . Setting

$$y := \omega_0\dots\omega_kz_1\dots z_{r-1}x,$$

8 we get $\sigma^{k+r}(y) = x$ and $y \notin \Sigma_{i,\varepsilon_0}$.

9 Note that, since Σ_{i,ε_0} is invariant, no preimage of y can be in Σ_{i,ε_0} . We also emphasize
 10 that r does only depends on x_0 , and the construction is rigid⁴.

11 Doing the same for any $[x_0]$ appearing in Σ_{i,ε_0} , we get N_0 . □

12 3.2 Computations

13 The key result is the following:

Proposition 20. *For any irreducible component Ω_j ,*

$$\gamma + V(\Omega_i) \geq S^{ext}(j, i) + V(j)$$

14 The rest of the subsection is devoted to the proof of the proposition.

⁴in the sense it holds for any point sufficiently close to x .

1 **3.2.1 First step: an inequality involving Ω_i , N_0 and $\nu_{1/\infty}$**

Let us now consider the Ruelle operator for βA , given by

$$L_{\beta A}(\varphi)(x) = \sum_{y \in \sigma^{-1}(x)} e^{\beta A(y)} \varphi(y).$$

2 Since $A_{\Omega_i} \equiv 0$, $v(\varepsilon) := \sup_{x \in \Sigma_{i,\varepsilon}} |A(x)|$ goes to 0 as ε goes to 0.

We can also consider the Ruelle operator for the zero potential acting over $\mathcal{C}(\Sigma_{i,\varepsilon})$:

$$\mathcal{L}_{i,\varepsilon}(\varphi)(x) = \sum_{y \in \sigma^{-1}(x), y \in \Sigma_{i,\varepsilon}} \varphi(y)$$

3 where $x \in \Sigma_{i,\varepsilon}$. Furthermore for the conformal measure⁵ ν_ε satisfies

$$e^{h_{i,\varepsilon}} \int \varphi(x) d\nu_\varepsilon(x) = \int \mathcal{L}_{i,\varepsilon}(\varphi)(x) d\nu_\varepsilon(x), \quad (1)$$

4 where $h_{i,\varepsilon}$ is the topological entropy of $\Sigma_{i,\varepsilon}$, which corresponds to the pressure of the zero
5 function on $\Sigma_{i,\varepsilon}$.

6 For $\varepsilon < \varepsilon_0$, $B(\Omega_i, \varepsilon) \subset B(\Omega_i, \varepsilon_0)$ and then $\Sigma_{i,\varepsilon} \subset \Sigma_{i,\varepsilon_0}$. Therefore, Lemma 19 yields
7 that any point in $\Sigma_{i,\varepsilon}$ contains an N_0 -preimage outside $B(\Omega_i, \varepsilon_0)$.

For $x \in \Sigma_{i,\varepsilon}$ we have

$$e^{N_0 P(\beta A)} H_{\beta A}(x) = \sum_{y \in \sigma^{-N_0}(x), y \in \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) + \sum_{y \in \sigma^{-N_0}(x), y \notin \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y).$$

Let $r(\varepsilon)$ denote $\sup_{z \in B(\Omega_i, \varepsilon)} |A(z)|$. Note that $A|_{\Omega_i} \equiv 0$ (by Lemma 8), thus $r(\varepsilon)$ goes to 0 as ε goes to 0. Then we get

$$e^{N_0 P(\beta A)} H_{\beta A}(x) \geq e^{-\beta \cdot N_0 r(\varepsilon)} \sum_{y \in \sigma^{-N_0}(x), y \in \Sigma_{i,\varepsilon}} H_{\beta A}(y) + \sum_{y \in \sigma^{-N_0}(x), y \notin \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y).$$

8 This later inequality can be rewritten as

$$e^{n_0 P(\beta A)} H_{\beta A}(x) \geq e^{-\beta \cdot N_0 r(\varepsilon)} \mathcal{L}_{i,\varepsilon}^{N_0}(H_{\beta A})(x) + \sum_{y \in \sigma^{-N_0}(x), y \notin \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y).$$

⁵or any convex combination of the conformal measures on each irreducible piece if $\Sigma_{i,\varepsilon}$ is not irreducible.

Integrating both sides with respect to the eigenmeasure ν_ε of $\mathcal{L}_{i,\varepsilon}$ we have⁶

$$e^{N_0 P(\beta A)} \int H_{\beta A}(x) d\nu_\varepsilon(x) \geq e^{-\beta \cdot N_0 r(\varepsilon)} \int \mathcal{L}_{i,\varepsilon}^{N_0}(H_{\beta A})(x) d\nu_\varepsilon(x) + \int \sum_{y \in \sigma^{-N_0}(x), y \notin \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) d\nu_\varepsilon(x)$$

which yields

$$e^{N_0 P(\beta A)} \int H_{\beta A}(x) d\nu_\varepsilon(x) \geq e^{N_0 h_{i,\varepsilon} - \beta \cdot N_0 r(\varepsilon)} \int H_{\beta A}(x) d\nu_\varepsilon(x) + \int \sum_{y \in \sigma^{-N_0}(x), y \notin \Sigma_{i,\varepsilon}} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) d\nu_\varepsilon(x).$$

1 If $z \notin B(\Omega_i, \varepsilon_0)$, then $z \notin \Sigma_{i,\varepsilon}$, and therefore we get

$$e^{N_0 P(\beta A)} \int H_{\beta A}(x) d\nu_\varepsilon(x) \geq e^{N_0 h_{i,\varepsilon} - \beta \cdot N_0 r(\varepsilon)} \int H_{\beta A}(x) d\nu_\varepsilon(x) + \int \sum_{y \in \sigma^{-N_0}(x), y \notin B(\Omega_i, \varepsilon_0)} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) d\nu_\varepsilon(x). \quad (2)$$

2 Note that $h_{i,\varepsilon}$ decreases as ε decreases. Every $\Sigma_{i,\varepsilon}$ contains Ω_i , hence $h_{i,\varepsilon} \geq h_i$ for
3 every ε . On the other side, if μ_ε is a measure of maximal entropy for $\Sigma_{i,\varepsilon}$, its support
4 is contained in $B(\Omega_i, \varepsilon)$, thus any accumulation point has support in Ω_i (see Lemma
5 17). Consequently, the limit for $h_{i,\varepsilon}$ as ε goes to 0 is lower or equal to h_i . This yields
6 $h_{i,\varepsilon} \rightarrow_{\varepsilon \rightarrow 0} h_i$.

7 Then, considering any accumulation point $\nu_{1/\infty}$ for ν_ε for the weak* topology, (2)
8 yields

$$e^{N_0 P(\beta A)} \int H_{\beta A}(x) d\nu_{1/\infty}(x) \geq e^{N_0 h_i} \int H_{\beta A}(x) d\nu_{1/\infty}(x) + \int \sum_{y \in \sigma^{-N_0}(x), y \notin B(\Omega_i, \varepsilon_0)} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) d\nu_{1/\infty}(x). \quad (3)$$

9 or equivalently,

$$(e^{N_0(P(\beta A) - h_i)} - 1) e^{N_0 h_i} \int H_{\beta A}(x) d\nu_{1/\infty}(x) \geq \int \sum_{y \in \sigma^{-N_0}(x), y \notin B(\Omega_i, \varepsilon_0)} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) d\nu_{1/\infty}(x). \quad (4)$$

⁶if $\Sigma_{i,\varepsilon}$ has several irreducible components, the computation can be done on each of these components.

1 **3.2.2 Limit for the left hand side term at exponential scale**

2 If we do $\frac{1}{\beta} \log$ on the left hand side in (4) and then do $\beta \rightarrow +\infty$, then

- 3 • $\frac{1}{\beta} \log(e^{N_0(P(\beta A) - h_i)} - 1)$ goes to γ by assumption on “ $\beta \rightarrow +\infty$ ”,
- 4 • $\frac{1}{\beta} \log(e^{N_0 h_i})$ goes to 0 and
- 5 • $\frac{1}{\beta} \log\left(\int H_{\beta A}(x) d\nu_{1/\infty}(x)\right)$ goes to $V(\Omega_i)$ since $\frac{1}{\beta} \log H_\beta$ uniformly (in X) goes to
- 6 V and $\nu_{1/\infty}$ has support in Ω_i .

7 **3.2.3 Limit for the right hand side term at exponential scale in the case $j \neq i$**

8 Inequality (4) holds for any choice of ε_0 . We can thus choose ε_0 such that $d(\Omega_j, \Omega_i)$ is

9 larger than $10 \cdot \varepsilon_0$.

10 Let x be any point in Ω_i . Lemma 16 yields⁷

$$\lim_{\beta \rightarrow +\infty} \frac{1}{\beta} \log \left(\sum_{y \in \sigma^{-N_0}(x), y \notin B(\Omega_i, \varepsilon_0)} e^{\beta S_{N_0}(A)(y)} H_{\beta A}(y) \right) = \max_{y, \sigma^{N_0}(y)=x, y \notin B(\Omega_i, \varepsilon_0)} \{S_{N_0}(A)(y) + V(y)\}. \quad (5)$$

11 Let us pick such a y . By definition, $y \notin \Omega_i$ but $\sigma^{N_0}(y) = x$ belongs to Ω_i . Therefore

12 one can consider the smallest $1 \leq k_y \leq N_0$ such that $x' := \sigma^{k_y}(y)$ belongs to Ω_i . We

13 denote this operation by $y \rightsquigarrow x'$.

14 **Remark 21.** *In the following we shall always use k_y but we let the reader check that k_y*

15 *actually only depends on x' . ■*

⁷We emphasize we use a weaker version of the Lemma, with a constant point x .

We get

$$S_{N_0}(A)(z) = S_k(A)(y) + S_{N_0-k_y}(A)(x') = S_{k_y}(A)(y),$$

1 since $A \equiv 0$ on Ω_i . Furthermore Lemma 13 yields $V(y) \geq V(\Omega_j) + S(\Omega_j, y)$, which yields

$$S_{N_0}(A)(y) + V(y) \geq V(\Omega_j) + S(\Omega_j, y) + S_{k_y}(A)(y), \quad (6)$$

2 and $\sigma^{k_y}(y) = x' \in \Omega_i$.

3 If z is a preimage of x' and $z \notin \Omega_i$, then z is also a preimage of x and any preimage
4 of z of order $k_y - 1$ is a preimage of x of order N_0 and is outside Ω_i . By construction, if
5 y' is one of these preimages, we also get $y' \rightsquigarrow x'$.

6 We also remind the immediate inequality:

$$\max_{y, \sigma^{N_0}(y)=x, y \notin \Omega_i} \{S(\Omega_j, y) + S_{N_0}(y)\} \geq \max \{S(\Omega_j, y) + S_{k_y}(A)(y), y \rightsquigarrow x', \sigma^{k_y-1}(y) \notin \Omega_i\}. \quad (7)$$

7 Note that $y \rightsquigarrow x'$ and $\sigma^{k_y-1}(y) \notin \Omega_i$ exactly means $\sigma^{N_0}(y) = x$, $\sigma^{k_y}(y) = x'$ and
8 $\sigma^{k_y-1}(y') \notin \Omega_i$. Therefore (7) is equivalent to

$$\max_{y, \sigma^{N_0}(y)=x, y \notin \Omega_i} \{S(\Omega_j, y) + S_{N_0}(y)\} \geq \max \{S(\Omega_j, y) + S_{k_y-1}(A)(y) + A(z), \sigma(z) = x', z \notin \Omega_i, y \in \sigma^{-k_y+1}(z)\}. \quad (8)$$

Now, let us pick some z satisfying $z \notin \Omega_i$ and $\sigma(z) = x'$. We consider a sequence
(ξ^n) of points such as in Lemma 12 such that $S(\Omega_j, z) = \lim_{n \rightarrow +\infty} S_{k_n}(A)(\xi^n)$ holds. As
 $\sigma^{-k_y+1}(\{z\})$ is finite, infinitely manies ξ^n satisfy $\sigma^{k_n-k_y}(\xi^n) = y$ where y is some preimage
of order $k_y - 1$ of z . Furthermore, for such ξ^n , Lemma 4 yields⁸

$$S(\Omega_j, y) \geq \limsup_{n \rightarrow +\infty} S_{k_n-k_y}(A)(\xi^n).$$

9 This yields that $\max \{S(\Omega_j, y) + S_{k_y-1}(A)(y) + A(z), \sigma(z) = x', z \notin \Omega_i, y \in \sigma^{-k_y+1}(z)\}$ is
10 greater or equal to $\max \{S(\Omega_j, z) + A(z), \sigma(z) = x', z \notin \Omega_i\}$ which is greater or equal
11 to $S^{ext}(j, i)$.

This actually yields

$$\gamma + V(\Omega_i) \geq S^{ext}(j, i) + V(\Omega_i).$$

⁸Note that $k_n \rightarrow +\infty$ since $x \notin \Omega_j$ and ξ^n converges to some point in Ω_j .

1 **3.2.4 Proof for the case $j = i$**

2 For the case $j \neq i$, one of the key points in the computation of $S^{ext}(j, i)$ was that relevant
 3 backward orbits have to be uniformly far from Ω_i (that is at distance greater than ε_0).
 4 This does not hold anymore for the case $j = i$. However, relevant backward orbits have
 5 to get out Ω_i , hence to be at positive distance of Ω_i . This is the fact we want to use to
 6 get the proof of Proposition 20.

7 We first emphasize a by-product result from 3.2.3:

8 **Claim 1.** *Any $x \in \Omega_i$ admits a preimage $x' \in \Omega_i$ of order less than $N_0 - 1$ such that x'
 9 admits preimages of order 1 outside Ω_i .*

10 We can thus consider the set $\Omega_{i,k,\varepsilon}$ of points $x \in \Omega_i$ such that:

- 11 1. $\exists x' \in \Omega_i, \sigma^k(x') = x, k \leq N_0 - 1,$
- 12 2. for all these x' 's, $\exists z \in \sigma^{-1}(x'), z \notin \Omega_i,$
- 13 3. for all these z 's, $d(z, \Omega_i) > \varepsilon.$

14 Letting $\varepsilon \downarrow 0$, one can find ε_1 and $k = N_1 - 1 \leq N_0 - 1$ such that $\Omega_{i,k,\varepsilon_1}$ has positive $\nu_{1/\infty}$
 15 measure.

16 Then, one can redo what has been done above but replacing (ε_0, N_0) with (ε_1, N_1) .
 17 This yields

$$(e^{N_1(P(\beta A) - h_i)} - 1)e^{N_1 h_i} \int H_{\beta A}(x) d\nu_{1/\infty}(x) \geq \int \sum_{y \in \sigma^{-N_1}(x), y \notin B(\Omega_i, \varepsilon_1)} e^{\beta S_{N_1}(A)(y)} H_{\beta A}(y) d\nu_{1/\infty}(x). \quad (9)$$

The result for the left hand holds as previously. We thus just need to study the right
 hand side term. Note that we get

$$\int \sum_{y \in \sigma^{-N_1}(x), y \notin B(\Omega_i, \varepsilon_1)} e^{\beta S_{N_1}(A)(y)} H_{\beta A}(y) d\nu_{1/\infty}(x) \geq \int \sum_{\Omega_{i, N_1-1, \varepsilon_1}} \sum_{y \in \sigma^{-N_1}(x), y \notin B(\Omega_i, \varepsilon_1)} e^{\beta S_{N_1}(A)(y)} H_{\beta A}(y) d\nu_{1/\infty}(x)$$

and this last term can we rewritten as

$$\int_{\Omega_{i,N_1-1,\varepsilon_1}} \sum_z e^{\beta A(z)} H_{\beta A}(z) d\nu_{1/\infty}(x),$$

where we reuse notations from the set $\Omega_{i,N_1-1,\varepsilon_1}$. We emphasize that for these z 's $S_{N_1}(A)(z) = A(z)$ since $\sigma(z) = x' \in \Omega_i$. By construction they satisfy $d(z', \Omega_i) > \varepsilon_1$. For all of these x ,

$$\frac{1}{\beta} \log \left(\sum_z e^{\beta A(z)} H_{\beta A}(z) \right) \rightarrow_{\beta \rightarrow +\infty} \max_z \{A(z) + V(z)\},$$

and $V(z) \geq V(\Omega_i) + S(\Omega_i, z)$. This yields,

$$\lim_{\beta \rightarrow +\infty} \frac{1}{\beta} \log \left(\sum_z e^{\beta A(z)} H_{\beta A}(z) \right) \geq \max_z \{S(\Omega_i, z) + A(z)\} + V(\Omega_i).$$

1 As $\Omega_{i,N_1-1,\varepsilon_1}$ has positive $\nu_{1/\infty}$ -measure,

$$\liminf_{\beta \rightarrow +\infty} \frac{1}{\beta} \log \left(\int_{\Omega_{i,N_1-1,\varepsilon_1}} \sum_z e^{\beta A(z)} H_{\beta A}(z) d\nu_{1/\infty} \right) \geq \inf_{x \in \Omega_{i,N_1-1,\varepsilon_1}} \max \{S(\Omega_i, z) + A(z), \sigma(z) = x'\} + V(\Omega_i).$$

2 Now, $\inf_{x \in \Omega_{i,N_1-1,\varepsilon_1}} \dots$ is bigger than $\inf_{x \in \Omega_i} \dots$, and the result holds.

3.3 End of the proof of Theorem A

4 Let us pick n different irreducible components with maximal entropy h , Ω_{i_j} , $1 \leq j \leq n$,
 5 with $i_j \neq i_{j'}$ if $j \neq j'$. Proposition 20 yields

$$\begin{aligned} V(\Omega_{i_1}) + \gamma &\geq S^{ext}(i_2, i_1) + V(\Omega_{i_2}), \\ V(\Omega_{i_2}) + \gamma &\geq S^{ext}(i_3, i_2) + V(\Omega_{i_3}), \\ &\vdots \\ V(\Omega_{i_n}) + \gamma &\geq S^{ext}(i_1, i_n) + V(\Omega_{i_1}). \end{aligned}$$

Doing the sum, we get

$$\gamma \geq \frac{S^{ext}(i_2, i_1) + S^{ext}(i_3, i_2) + \dots + S^{ext}(i_1, i_n)}{n}.$$

1 This holds for any finite sub-family obtained from $\{i_1, \dots, i_n\}$, which exactly means that
2 γ is bigger than the eigenvalue for the matrix whose entries are the $S^{ext}(i_k, i_l)$'s.

3 This holds for any accumulation point γ for $\frac{1}{\beta} \log(P(\beta) - h)$, which concludes the
4 proof of Theorem A.

5 Note that we immediatly get:

Theorem B. *With the same assumptions, if Ω has only one irreducible component with maximal entropy, Ω' , then*

$$\liminf_{\beta \rightarrow +\infty} \frac{1}{\beta} \log(P(\beta) - h) \geq S^{ext}(\Omega', \Omega').$$

6 Note that by Lemma 9, $S^{ext}(\Omega', \Omega')$ is a negative real number.

7 References

- 8 [1] A. Baraviera, R. Leplaideur and A. Lopes, Ergodic optimization, zero temperature
9 limits and the max-plus algebra. IMPA, Rio de Janeiro (2013) (arXiv:1305.2396v2).
- 10 [2] A. Baraviera, A. Lopes and Ph. Thieullen, A large deviation principle for equilibrium
11 states of Hölder potentials: the zero temperature case, *Stochastics and Dynamics* **6**
12 (2006) 77-96.
- 13 [3] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.
14 2nd Edition. Edited by Jean-René Chazottes. Springer-Verlag, Berlin (2008).
- 15 [4] Synchronization and linearity. Baccelli, François Louis and Cohen, Guy and Olsder,
16 Geert Jan and Quadrat, Jean-Pierre. John Wiley & Sons, Ltd., Chichester, 1992,
17 xx+489 pp. ISBN: 0-471-93609-X

- 1 [5] J. P. Conze and Y. Guivarc'h, Croissance des sommes ergodiques et principe varia-
2 tionnel, manuscript circa (1993).
- 3 [6] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*,
4 Cambridge University Press (2010)
- 5 [7] T. Kucherenko, A. Quas. Asymptotic behavior of the pressure function for Hölder
6 potentials. preprint arXiv:2302.14839.
- 7 [8] R. Leplaideur, J. Mengue. On the selection of subaction and measure for perturbed
8 potentials . preprint arXiv:2404.02182.
- 9 [9] F.Mignosi, On the number of factors of Sturmian words, *Theoretical Computer Sci-*
10 *ence* Volume 82, Issue 1, 22 May 1991, Pages 71-84. [https://doi.org/10.1016/0304-](https://doi.org/10.1016/0304-3975(91)90172-X)
11 [3975\(91\)90172-X](https://doi.org/10.1016/0304-3975(91)90172-X)
- 12 [10] M. Morse, G. Hedlund, Symbolic Dynamics II: Sturmian Trajectories. *American*
13 *Journal of Mathematics* Vol. 62, No. 1 (1940), pp. 1-42.