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Curve Reconstruction in Inverse Problems: From Divergence-Measure Vector Fields to Density Lebesgue Measures

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Abstract

We introduce a new approach to perform curve reconstruction in microscopy images via vector field optimization. By formulating the problem in a suitable functional space, we establish equivalence in terms of solutions with the functional developed in a previous work within the divergence measure fields space. This allows for the use of traditional convex optimization algorithms for optimization.

Keywords: Inverse problem, curve reconstruction, divergence measure fields, imaging.

1. Introduction

In ill-posed inverse problems, finding a regularizer that promotes curve structures is of great interest in many applications, including the super-resolution of filament structures in fluorescence microscopy images.

Recent work by Laville et al. [7] introduced a new regularizer, defined by the norm:

$$\|\cdot\|_{\mathcal{V}(\mathcal{X})} = \|\cdot\|_{\mathcal{M}(\mathcal{X})^2} + \|\operatorname{div}(\cdot)\|_{\mathcal{M}(\mathcal{X})},$$

on the space of divergence-measure fields:

$$\mathcal{V}(\mathcal{X}) = \{m \in \mathcal{M}(\mathcal{X})^2 : \operatorname{div}(m) \in \mathcal{M}(\mathcal{X})\},$$

where \mathcal{X} is a locally compact Hausdorff topological space, and $\mathcal{M}(\mathcal{X})^2$ and $\mathcal{M}(\mathcal{X})$ denote the spaces of 2-dimensional and 1-dimensional finite signed Radon measures on \mathcal{X} , respectively. Elements of this space $\mathcal{V}(\mathcal{X})$ can be decomposed into curves, as established in [10, 2, 9]. They proposed an optimization functional called CROC (Curves Represented on Charges):

$$\arg \min_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Phi(m)\|_{\mathcal{H}}^2 + \lambda \|m\|_{\mathcal{V}(\mathcal{X})}, \quad (\text{CROC})$$

where y represents the observed data, $\Phi : \mathcal{V}(\mathcal{X}) \rightarrow \mathcal{H}$ is the linear and continuous forward operator, \mathcal{H} is the Hilbert space of observations, which can be $L^2(\mathcal{X})$ or \mathbb{R}^N (for discrete measurements, with $N \in \mathbb{N}^*$), and $\lambda \in \mathbb{R}^+$ is the regularization parameter. This functional has a solution that is a linear combination of measures supported by curves, making it suitable for performing curve reconstruction. Specifically, for a discrete measurement y of dimension $N \in \mathbb{N}^*$, a solution is given by:

$$\sum_{i=1}^N a_i \mu_{\gamma_i},$$

where $a_i \in \mathbb{R}$ and μ_{γ_i} is a vector measure supported by a parametrized curve γ_i . In the distributional sense, μ_{γ_i} is defined as:

$$\forall g \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2), \quad \langle \mu_{\gamma_i}, g \rangle_{\mathcal{M}(\mathcal{X})^2 \times \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)} = \int_0^1 g(\gamma_i(t)) \cdot \dot{\gamma}_i(t) dt.$$

While this theoretical framework is well-established, the challenge lies in developing an algorithm to effectively reconstruct these curve-based solutions. In [6], the authors introduced a Curve Sliding Frank-Wolfe algorithm, which is an extension of the Sliding Frank-Wolfe algorithm (a conditional gradient-based algorithm) developed in [5], to solve the CROC functional. The key step in this algorithm involves addressing the certificate of the functional by solving an optimization problem in the space of curves, formulated as:

$$\arg \max_{\gamma} \left(\frac{|\langle \mu_{\gamma}, \eta \rangle|}{\|\mu_{\gamma}\|_{\mathcal{V}(\mathcal{X})}} \right),$$

where η , the certificate, is defined as:

$$\eta = \frac{1}{\lambda} \Phi^*(\Phi m - y),$$

with $\Phi^* : \mathcal{H} \rightarrow \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$ being the adjoint operator of Φ , and $m \in \mathcal{V}(\mathcal{X})$ a solution of the CROC functional. This optimization problem in the space of curves is challenging. Furthermore, the algorithm incorporates a non-convex step over the set of curves and their amplitudes. These considerations raise an important question within the scientific community: is it possible to discretize the convex CROC functional to enable its implementation using classical convex optimization algorithms?

This work aims to provide a relaxation of the CROC functional within a more tractable space, than the measure space $\mathcal{M}(\mathcal{X})^2$ enabling the use of traditional convex optimization algorithms for implementation.

1.1 Contributions 1

We introduce a new space of vector-valued Lebesgue measures, whose densities are vector-valued functions in L^1 with divergence also in L^1 : 2
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$$\mathcal{W}(\mathcal{X}) = \{f \, dx \mid f \in L^1(\mathcal{X}, \mathbb{R}^2), \operatorname{div}(f) \in L^1(\mathcal{X}, \mathbb{R})\},$$

and prove that $\mathcal{W}(\mathcal{X})$ is dense in $\mathcal{V}(\mathcal{X})$ under the weak-* topology in Section 3. 4

Next, in Section 4, we reformulate the original CROC functional in the space $\mathcal{W}(\mathcal{X})$, leading to the definition of the relaxed functional CROC_R : 5
6

$$\arg \min_{f \, dx \in \mathcal{W}(\mathcal{X})} D_\lambda(f) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Phi(f \, dx)\|_{\mathcal{H}}^2 + \lambda \left(\|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} + \|\operatorname{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R})} \right), \quad (\text{CROC}_R)$$

and establish the equivalence of their infima (Theorem 4.3): 7

$$\inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) = \inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m).$$

This result ensures that every minimizing sequence of CROC_R converges to a solution of the original CROC functional (Proposition 4.4). This finding provides a theoretical foundation for the practical use of CROC_R . 8
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Finally, in Section 5, we perform numerical implementations based on this new approach and present and discuss experiments in the case of curve reconstruction on blurry and noisy fluorescent microscopy images. 11
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2. Preliminary Concepts and Notations 14

This section provides a brief overview of mathematical concepts relevant to this topic, which are detailed in [1, 8, 4]. 15
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Notation 2.1. We denote by \mathcal{X} an open bounded subset of \mathbb{R}^2 . 17

Definition 2.2. Let $n \in \mathbb{N}^*$. We denote by $\mathcal{C}_c(\mathcal{X}, \mathbb{R}^n)$ the space of continuous functions from \mathcal{X} to \mathbb{R}^n with compact support. The space $\mathcal{C}_0(\mathcal{X}, \mathbb{R}^n)$, called the space of evanescent continuous functions, consists of continuous functions $f : \mathcal{X} \rightarrow \mathbb{R}^n$ that vanish at infinity. That is, a function $f \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^n)$ if 18
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$$\forall \epsilon > 0, \quad \exists K \text{ compact subset of } \mathcal{X}, \quad \forall x \in \mathcal{X} \setminus K, \quad \|f(x)\|_{\mathbb{R}^n} \leq \epsilon.$$

This space is equipped with the supremum norm $\|\cdot\|_{\infty, \mathcal{X}}$, defined as: 22

$$\forall f \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^n), \quad \|f\|_{\infty, \mathcal{X}} := \sup_{x \in \mathcal{X}} \|f(x)\|_{\mathbb{R}^n}.$$

Moreover the closure of $\mathcal{C}_c(\mathcal{X}, \mathbb{R}^n)$ under the supremum norm $\|\cdot\|_{\infty, \mathcal{X}}$ is $\mathcal{C}_0(\mathcal{X}, \mathbb{R}^n)$. 23

Definition 2.3 (Topological dual space). Let E be a normed vector space. The dual space of E , denoted E^* , is the set of all continuous linear functionals from E into \mathbb{R} or \mathbb{C} . The weak-* topology on E^* , is defined by:

A sequence $(f_n)_{n \in \mathbb{N}}$ in E^* is said to weak-* converge to $f \in E^*$, denoted by $f_n \xrightarrow{*} f$, if:

$$\forall x \in E, \quad f_n(x) \xrightarrow{n \rightarrow +\infty} f(x) \quad (\text{pointwise convergence}).$$

Definition 2.4 (Finite Radon measure). Let $n \in \mathbb{N}^*$. We denote $\mathcal{M}(\mathcal{X})^n$ the set of n -dimensional vector-valued finite Radon measures, which is the topological dual of $\mathcal{C}_0(\mathcal{X}, \mathbb{R}^n)$ endowed with the supremum norm. The duality bracket $\langle \cdot, \cdot \rangle$ on $\mathcal{M}(\mathcal{X})^n$ is defined as:

$$\forall m \in \mathcal{M}(\mathcal{X})^n, \quad \forall f \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^n), \quad \langle m, f \rangle = \int_{\mathcal{X}} f \, dm.$$

The norm of the strong topology on $\mathcal{M}(\mathcal{X})^n$ is called total variation norm and is defined as:

$$\forall m \in \mathcal{M}(\mathcal{X})^n, \quad \|m\|_{TV^n} = \sup \{ \langle m, f \rangle, \quad f \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^n), \quad \|f\|_{\infty, \mathcal{X}} \leq 1 \}.$$

Definition 2.5 (Total variation measure). Let m be an n -dimensional vector-valued Radon measure on \mathcal{X} . Denote by $\mathcal{B}(\mathcal{X})$ the Borel σ -algebra of subsets of \mathcal{X} . The total variation measure of m , denoted $|m|$, is defined as:

$$\forall A \in \mathcal{B}(\mathcal{X}), \quad |m|(A) = \sup \{ \langle m, \varphi \rangle : \varphi \in \mathcal{C}_0(A, \mathbb{R}^n), \quad \|\varphi\|_{\infty} \leq 1 \}.$$

The total variation norm satisfies:

$$\|m\|_{TV^n} = |m|(\mathcal{X}).$$

Definition 2.6 (local weak*-convergence of Radon measures). Let $n \in \mathbb{N}^*$. We say that a sequence $(\mu_h)_h \subset \mathcal{M}(\mathcal{X})^n$ locally weak* converges to $\mu \in \mathcal{M}(\mathcal{X})^n$ ($\mu_h \xrightarrow{*} \mu$ locally) if

$$\lim_{h \rightarrow \infty} \int_{\mathcal{X}} u \, d\mu_h = \int_{\mathcal{X}} u \, d\mu, \quad \text{for every } u \in C_c(\mathcal{X}, \mathbb{R}^n).$$

Remark 2.7. We say that a sequence $(\mu_h)_h \subset \mathcal{V}(\mathcal{X})$ weak* converges to $\mu \in \mathcal{V}(\mathcal{X})$ if

$$\mu_h \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\mathcal{X})^2 \quad \text{and} \quad \operatorname{div}(\mu_h) \xrightarrow{*} \operatorname{div}(\mu) \quad \text{in } \mathcal{M}(\mathcal{X}).$$

Proposition 2.8 ((local weak* and weak*)-convergence). The weak*-convergence of a sequence $(\mu_h)_h \subset \mathcal{M}(\mathcal{X})^n$ is equivalent to the local weak*-convergence together with the condition $\sup_h |\mu_h|(\mathcal{X}) < \infty$.

Definition 2.9 (Space of distributions). The space of distributions $\mathcal{D}'(\mathcal{X})$ is the topological dual of the space of smooth functions with compact support denoted $\mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})$. Note that $\overline{\mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})}^{\|\cdot\|_\infty} = \mathcal{C}_0(\mathcal{X}, \mathbb{R})$ and $\mathcal{M}(\mathcal{X})$ is a subset of $\mathcal{D}'(\mathcal{X})$.

Definition 2.10 (Parametrized curve). *A curve is the image of an interval of \mathbb{R} to a topological space by a continuous function. This function is called a curve parametrization. In the following, we will often use the term "curve" to refer interchangeably to both the parameterization function and the curve itself.*

Lemma 2.11. *Let $E \subset \mathbb{R}^2$. Then $E = \bigcup_{\epsilon > 0} K_\epsilon^E$, where $K^E = \{x \in \mathbb{R}^2 \mid d(x, E^c) \geq 2\epsilon\}$ is a compact subset of E .*

Notation 2.12 (Standard mollifier). *We denote by $(\rho_\epsilon)_{\epsilon > 0}$ the standard symmetric mollifier satisfying $\rho_\epsilon \in C_c^\infty(\mathbb{R}^2)$, $\text{spt}(\rho_\epsilon) \subset \overline{B}(0, \epsilon)$, and $\int_{\mathbb{R}^2} \rho_\epsilon dx = 1$. This notation will be used throughout this work.*

Lemma 2.13. *Let $u \in L^\infty(\mathcal{X})$. Then, there exists a sequence $(u_n)_n$ in $C_c^\infty(\mathcal{X})$ such that: $\|u_n\|_\infty \leq \|u\|_\infty \forall n$, $u_n \rightarrow u$ almost everywhere in \mathcal{X} , and $u_n \xrightarrow{*} u$ in $L^\infty(\mathcal{X})$ under the weak-* topology.*

To develop a more tractable optimization algorithm for the CROC functional (CROC), we propose working within a subspace of $\mathcal{V}(\mathcal{X})$ consisting of Lebesgue measures with vector-valued densities. While one might initially consider using vectors of finite linear combinations of Dirac masses to approximate vector measures, this space, although weak*-dense in $\mathcal{M}(\mathcal{X})^2$ [3], does not belong to $\mathcal{V}(\mathcal{X})$ [7]. However, as also pointed out in [3], the space of Lebesgue measures with vector-valued densities in $L^2(\mathcal{X})^2$ is dense in $\mathcal{M}(\mathcal{X})^2$. This inspired us to define a new space $\mathcal{W}(\mathcal{X})$, which is weak*-dense in $\mathcal{V}(\mathcal{X})$. This will be the focus of the following section.

3. The Space \mathcal{W} : Definition and Analysis

Definition 3.1. *We denote by $\mathcal{W}(\mathcal{X})$ the space of Lebesgue measures on $\mathcal{X} \subseteq \mathbb{R}^2$ with densities that are vector functions in $L^1(\mathcal{X}, \mathbb{R}^2)$, whose divergence (in the distributional sense) belongs to $L^1(\mathcal{X}, \mathbb{R})$:*

$$\mathcal{W}(\mathcal{X}) = \{f dx \mid f \in L^1(\mathcal{X}, \mathbb{R}^2), \text{div}(f) \in L^1(\mathcal{X}, \mathbb{R})\}.$$

Here, $f dx$ represents a measure whose density f is defined with respect to the Lebesgue measure on \mathbb{R}^2 . We write $dx := dx_1 dx_2$, where $x = (x_1, x_2) \in \mathbb{R}^2$.

Proposition 3.2. *$\mathcal{W}(\mathcal{X})$ is a subset of the space of divergence measure fields $\mathcal{V}(\mathcal{X})$.*

Proof. Let $f dx \in \mathcal{W}(\mathcal{X})$. Denote by f_1 and f_2 the components of the vector function f . By definition of $\mathcal{W}(\mathcal{X})$, $f \in L^1(\mathcal{X}, \mathbb{R}^2)$.

- $f dx$ is a finite Radon measure since the map T defined by

$$T : \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2) \longrightarrow \mathbb{R}$$

$$\varphi \longmapsto \sum_{i=1}^2 \int_{\mathcal{X}} \varphi_i f_i dx.$$

is a linear continuous operator on $\mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$. In fact, for $\varphi = (\varphi_1, \varphi_2) \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$, 1

$$|\langle T, \varphi \rangle| \leq \max \left(\sup_{x \in \mathcal{X}} |\varphi_1(x)|, \sup_{x \in \mathcal{X}} |\varphi_2(x)| \right) \int_{\mathcal{X}} |f_1(x)| + |f_2(x)| dx.$$

implying 2

$$|\langle T, \varphi \rangle| \leq \|\varphi\|_{\mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)} \|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)}, \quad \forall \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2).$$

Moreover, since $f \in L^1(\mathcal{X}, \mathbb{R}^2)$, and based on the density of $\mathcal{C}_c^\infty(\mathcal{X})$ in $L^\infty(\mathcal{X})$ as 3
stated in Lemma 2.13, it follows that the total variation $\|f dx\|_{\mathcal{T}\mathcal{V}^2}$ of the measure 4
 $f dx$ given by $\sup \left\{ \int_{\mathcal{X}} f \cdot \varphi dx \mid \varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \right\}$ equals to $\|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)}$. 5

- Note that $\operatorname{div}(f dx) = \operatorname{div}(f)$ since for all $\varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})$, 6

$$\langle \operatorname{div}(f) dx, \varphi \rangle = -\langle f dx, \nabla \varphi \rangle_{\mathcal{M}(\mathcal{X}, \mathbb{R}^2) \times \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)} = -\int_{\mathcal{X}} f(x) \cdot \nabla \varphi(x) dx = \langle \operatorname{div}(f), \varphi \rangle_{\mathcal{D}'(\mathcal{X}) \times \mathcal{D}(\mathcal{X})}.$$

Since $\operatorname{div}(f) \in L^1(\mathcal{X}, \mathbb{R})$, $\sup \left\{ \int_{\mathcal{X}} \operatorname{div}(f) \varphi dx \mid \varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R}), \|\varphi\|_\infty \leq 1 \right\}$ equals to 7
 $\|\operatorname{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R})}$ and thus, the map 8

$$\begin{aligned} L : \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R}) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \int_{\mathcal{X}} \operatorname{div}(f) \varphi dx \end{aligned}$$

is a bounded linear functional on $\mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})$. As $\overline{\mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})}^{\|\cdot\|_\infty} = \mathcal{C}_0(\mathcal{X}, \mathbb{R})$, by the 9
Riesz Representation Theorem, L extends to a bounded continuous functional \tilde{L} on 10
 $\mathcal{C}_0(\mathcal{X}, \mathbb{R})$. Therefore, $\operatorname{div}(f dx)$ is a finite Radon measure with total variation equal 11
to $\|\operatorname{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R}^2)}$. 12

Hence, $\mathcal{W}(\mathcal{X})$ is included in $\mathcal{V}(\mathcal{X})$. 13

□ 14

Proposition 3.3. *The space $\mathcal{W}(\mathcal{X})$ is dense in $\mathcal{V}(\mathcal{X})$ for the weak- $*$ topology.* 15

Proof. The detailed proof of Proposition 3.3 is given in Appendix A. It is inspired by 16
the proof of the density of the space of elements $g dx$, where $g \in L^2(\mathcal{X})$, within $\mathcal{M}(\mathcal{X})$, 17
provided in the appendix of [3]. The main steps are as follows: 18

For all $\epsilon > 0$, we define a sequence $g_\epsilon = \mathbf{1}_{K_{\frac{2\epsilon}{3}}} * \rho_{\frac{2\epsilon}{3}}$, where $K_\epsilon = \{x \in \mathbb{R}^2 \mid d(x, \mathcal{X}^c) \geq 2\epsilon\}$. 19
This guarantees that $g_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ and that it has compact support within \mathcal{X} . 20

Next, we introduce the map 21

$$\begin{aligned} Z_\epsilon : L^1(\mathcal{X}, \mathbb{R}^2) &\longrightarrow \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2), \\ f &\longmapsto g_\epsilon f * \rho_{\frac{\epsilon}{3}}, \end{aligned} \tag{1}$$

which is well-defined, linear, and continuous. Its adjoint operator, denoted Z_ϵ^* , defined 22

by:

$$\begin{aligned} Z_\epsilon^* : \mathcal{M}(\mathcal{X})^2 &\longrightarrow L^\infty(\mathcal{X}, \mathbb{R}^2) \subset L^1(\mathcal{X}, \mathbb{R}^2), \\ \mu &\longmapsto Z_\epsilon^* \mu, \end{aligned} \tag{2}$$

is also linear and continuous. It helps to construct a family of L^1 -vector functions from a vector measure field.

Finally, we show that for all $\mu \in \mathcal{V}(\mathcal{X})$, the sequence $(Z_\epsilon^* \mu)_{\epsilon > 0}$ lies in $\mathcal{W}(\mathcal{X})$ and converges to μ in the weak-* topology of $\mathcal{V}(\mathcal{X})$, thereby concluding the proof of density.

Remark 3.4. *More generally, for all $p \in \mathbb{N}$, the space $\mathcal{W}^p(\mathcal{X})$, defined by*

$$\mathcal{W}^p(\mathcal{X}) = \{f \, dx \mid f \in L^p(\mathcal{X}, \mathbb{R}^2), \operatorname{div}(f) \in L^p(\mathcal{X}, \mathbb{R})\},$$

is dense in $\mathcal{V}(\mathcal{X})$ for the weak-topology.*

The proof is obtained by following the same process as above, since $(Z_\epsilon^ \mu)_{\epsilon > 0}$ also lies in $\mathcal{W}^p(\mathcal{X}) \subset \mathcal{W}(\mathcal{X})$, as \mathcal{X} is a bounded subset of \mathbb{R}^2 .*

In the following section, we define a relaxation of the CROC functional on $\mathcal{W}(\mathcal{X})$ and provide a proof of its connection with the original CROC functional. This allows to explore new optimization strategies while ensuring consistency with the established functional framework.

4. CROC Relaxation Functional

Before defining the relaxation of the CROC functional in $\mathcal{W}(\mathcal{X})$, we first revisit its definition.

Definition 4.1 (CROC functional). *Let $y \in \mathcal{H} := L^2(\mathcal{X})$ and let $\Phi : \mathcal{V}(\mathcal{X}) \rightarrow \mathcal{H}$ be a linear map. The CROC (Curves Represented on Charges) functional is defined as:*

$$\arg \min_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Phi(m)\|_{\mathcal{H}}^2 + \lambda \|m\|_{\mathcal{V}(\mathcal{X})},$$

where $m \in \mathcal{V}(\mathcal{X})$, and $\lambda \in \mathbb{R}^+$ is the regularization parameter. The operator $\Phi(m)$ is defined as: $\Phi(m) := \int_{\mathcal{X}} \phi(x) \, dm(x)$, where $\phi \in L^1(\mathcal{X}, \mathcal{H})$ is the kernel of the operator Φ . Specifically, ϕ is often given by:

$$\forall x \in \mathcal{X}, \quad \phi(x) : \mathcal{X} \rightarrow \mathbb{R}, \quad \text{with } \phi(x)(y) = \varphi(x - y), \quad \forall y \in \mathcal{X}.$$

For Φ to be well-defined and weak-to-weak continuous, φ must be in $L^2(\mathcal{X})^2$ and, for any $q \in L^2(\mathcal{X})^2$, $\varphi * q$ must belong to $C_0(\mathcal{X}, \mathbb{R}^2)$. Note that $\varphi \in C_0(\mathcal{X}, \mathbb{R}^2)$ is a sufficient condition. Under these conditions, as justified in the introduction section, the CROC functional admits a solution that is a linear combination of measures supported on curves lying in \mathcal{X} .*

Definition 4.2 (The CROC_R functional). *The CROC functional in the space $\mathcal{W}(\mathcal{X})$, denoted as CROC_R , is given by:*

$$\arg \min_{f dx \in \mathcal{W}(\mathcal{X})} D_\lambda(f) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Phi(f dx)\|_{\mathcal{H}}^2 + \lambda \left(\|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} + \|\text{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R})} \right) \quad (3)$$

The following theorem establishes that the infimum of the CROC functional coincides with the infimum of its relaxed version, CROC_R .

Theorem 4.3 (Infimum Equivalence between CROC and CROC_R).

$$\inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) = \inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m). \quad (4)$$

Proof. We will consider a given kernel ϕ and its associated $\varphi \in C_0(\mathcal{X}, \mathbb{R}^2)$, as an *acquisition kernel* for the CROC problem.

Since $\mathcal{W}(\mathcal{X}) \subset \mathcal{V}(\mathcal{X})$, it follows that

$$\inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) \leq \inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m). \quad (5)$$

To establish equality, it remains to demonstrate the reverse inequality. The remainder of the proof will therefore focus on this.

Let $\bar{m} \in \mathcal{V}(\mathcal{X})$ be a solution of the CROC problem, represented as a linear combination of measures supported on curves within \mathcal{X} . Since the curves involved in this decomposition are also naturally defined in \mathbb{R}^2 , it follows that \bar{m} also belongs to $\mathcal{V}(\mathbb{R}^2)$.

Indeed: Consider the measure $\bar{m} = \sum_{i=1}^N a_i \mu_{\gamma_i}$, where each $\gamma_i : [a, b] \rightarrow \mathcal{X}$ is a parameterized curve with support in \mathcal{X} . Since $\mathcal{X} \subset \mathbb{R}^2$, each γ_i can also be viewed as a parameterized curve in \mathbb{R}^2 , with its support remaining within \mathcal{X} .

Additionally, the divergence of \bar{m} can be expressed in the sense of distributions as $\text{div}(\bar{m}) = \sum_{i=1}^N a_i (\delta_{\gamma_i(a)} - \delta_{\gamma_i(b)})$, where $\delta_{\gamma_i(a)}$ and $\delta_{\gamma_i(b)}$ are Dirac measures supported at the endpoints of the curves γ_i . This is a finite Radon measure on \mathbb{R}^2 . Hence, $\bar{m} \in \mathcal{V}(\mathbb{R}^2)$.

1. Step 1: Mollification of \bar{m}

Let $\bar{m}_\epsilon = \bar{m} * \rho_\epsilon$ for all $\epsilon > 0$. Since $\bar{m} \in \mathcal{M}(\mathbb{R}^2)^2$, this convolution is well defined on \mathbb{R}^2 , $\bar{m}_\epsilon \in C^\infty(\mathbb{R}^2)$, and $\bar{m}_\epsilon \xrightarrow{*} \bar{m}$ locally on \mathbb{R}^2 .

Additionally, $\text{div}(\bar{m}_\epsilon) = \text{div}(\bar{m}) * \rho_{\frac{\epsilon}{3}}$, since $\text{div}(\bar{m}) \in \mathcal{M}(\mathbb{R}^2)$ and $\text{div}(\bar{m}) * \rho_{\frac{\epsilon}{3}} \xrightarrow{*} \text{div}(\bar{m})$ locally on \mathbb{R}^2 . These properties are discussed in [1].

Note also that:

1

$$\begin{aligned}
|\overline{m}_\epsilon|(\mathcal{X}) &= \int_{\mathcal{X}} |\overline{m}_\epsilon(x)| \, dx \\
&= \int_{\mathcal{X}} \left| \int_{\mathbb{R}^2} \rho_\epsilon(x-y) \, d\overline{m}(y) \right| \, dx \\
&\leq \int_{\mathcal{X}} \int_{\mathbb{R}^2} \rho_\epsilon(x-y) \, d|\overline{m}|(y) \, dx \\
&= \int_{\mathcal{X}} \int_{\mathcal{X}} \rho_\epsilon(x-y) \, d|\overline{m}|(y) \, dx \quad (\text{since } \text{supp}(\overline{m}) \subset \mathcal{X}) \\
&= \int_{\mathcal{X}} \left(\int_{\mathcal{X}} \rho_\epsilon(x-y) \, dx \right) d|\overline{m}|(y) \\
&\leq |\overline{m}|(\mathcal{X}) \quad (\text{since } \int_{\mathcal{X}} \rho_\epsilon(x-y) \, dx \leq \int_{\mathbb{R}^2} \rho_\epsilon(z) \, dz = 1).
\end{aligned}$$

Thus,

2

$$|\overline{m}_\epsilon|(\mathcal{X}) \leq |\overline{m}|(\mathcal{X}), \quad \text{for all } \epsilon > 0. \quad (6)$$

Applying the same argument to $|\text{div}(\overline{m}_\epsilon)|(\mathcal{X})$, we obtain:

3

$$|\text{div}(\overline{m}_\epsilon)|(\mathcal{X}) \leq |\text{div}(\overline{m})|(\mathcal{X}), \quad \text{for all } \epsilon > 0. \quad (7)$$

2. *Step 2: Convergence of $T_\lambda(\overline{m}_\epsilon|_{\mathcal{X}})$ to $T_\lambda(\overline{m})$*

4

Since $\overline{m}_\epsilon \xrightarrow{*} \overline{m}$ locally on \mathbb{R}^2 , it follows that

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$$\liminf_{\epsilon \rightarrow 0} |\overline{m}_\epsilon|(\mathcal{X}) \geq |\overline{m}|(\mathcal{X}),$$

due to the lower semicontinuity of the total variation norm with respect to the local weak-* convergence on every open set. Consequently, we have

6

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$$|\overline{m}|(\mathcal{X}) \leq \liminf_{\epsilon \rightarrow 0} |\overline{m}_\epsilon|(\mathcal{X}) \leq \limsup_{\epsilon \rightarrow 0} |\overline{m}_\epsilon|(\mathcal{X}) \leq |\overline{m}|(\mathcal{X}),$$

where the last inequality follows from equation (6). Therefore, we conclude that

8

$$\lim_{\epsilon \rightarrow 0} |\overline{m}_\epsilon|(\mathcal{X}) = |\overline{m}|(\mathcal{X}). \quad (8)$$

By applying the same argument and using equation (7), we also deduce that

9

$$\lim_{\epsilon \rightarrow 0} |\text{div}(\overline{m}_\epsilon)|(\mathcal{X}) = |\text{div}(\overline{m})|(\mathcal{X}). \quad (9)$$

Remark that

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$$\|y - \Phi(\overline{m}_\epsilon|_{\mathcal{X}})\|_{\mathcal{L}^2(\mathcal{X})}^2 = \|y\|_{\mathcal{L}^2(\mathcal{X})}^2 - 2\langle y, \Phi(\overline{m}_\epsilon|_{\mathcal{X}}) \rangle_{\mathcal{L}^2(\mathcal{X})} + \|\Phi(\overline{m}_\epsilon|_{\mathcal{X}})\|_{\mathcal{L}^2(\mathcal{X})}^2. \quad (10)$$

Since $\bar{m}_\epsilon \xrightarrow{*} \bar{m}$ locally in $\mathcal{M}(\mathbb{R}^2)^2$, the restriction $\bar{m}_\epsilon|_{\mathcal{X}}$ also satisfies $\bar{m}_\epsilon|_{\mathcal{X}} \xrightarrow{*} \bar{m}|_{\mathcal{X}} = \bar{m}$ locally in $\mathcal{M}(\mathcal{X})^2$, due to the inclusion $\mathcal{C}_c(\mathcal{X}; \mathbb{R}^2) \subset \mathcal{C}_c(\mathbb{R}^2; \mathbb{R}^2)$. 1
2

Moreover, the uniform bound 3

$$\sup_{\epsilon > 0} |\bar{m}_\epsilon|_{\mathcal{X}}(\mathcal{X}) = \sup_{\epsilon > 0} |\bar{m}_\epsilon|(\mathcal{X}) \leq |\bar{m}|(\mathcal{X}) < \infty$$

guarantees that the conditions of Proposition 2.8 are satisfied. Consequently, $\bar{m}_\epsilon|_{\mathcal{X}} \xrightarrow{*} \bar{m}$ in $\mathcal{M}(\mathcal{X})^2$. This implies 4
5

$$\langle y, \Phi(\bar{m}_\epsilon|_{\mathcal{X}}) \rangle_{\mathcal{L}^2(\mathcal{X})} \xrightarrow{\epsilon \rightarrow 0} \langle y, \Phi(\bar{m}) \rangle_{\mathcal{L}^2(\mathcal{X})}. \quad (11)$$

since Φ is weak*-to-weak continuous from $\mathcal{M}(\mathcal{X})^2$ to $L^2(\mathcal{X})$. 6

Furthermore, 7

$$\|\Phi(\bar{m}_\epsilon|_{\mathcal{X}})\|_{\mathcal{L}^2(\mathcal{X})}^2 = \int_{\mathcal{X}} (\Phi(\bar{m}_\epsilon|_{\mathcal{X}})(x))^2 dx = \int_{\mathcal{X}} |\langle \bar{m}_\epsilon|_{\mathcal{X}}, \varphi(\cdot - x) \rangle|^2 dx. \quad (12)$$

Since $\forall x \in \mathcal{X}$, $\varphi(\cdot - x) \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$, and $\bar{m}_\epsilon|_{\mathcal{X}} \xrightarrow{*} \bar{m}$ in $\mathcal{M}(\mathcal{X})^2$, then 8

$$\forall x \in \mathcal{X}, \langle \bar{m}_\epsilon|_{\mathcal{X}}, \varphi(\cdot - x) \rangle \xrightarrow{\epsilon \rightarrow 0} \langle \bar{m}, \varphi(\cdot - x) \rangle. \quad (13)$$

Moreover, for all $x \in \mathcal{X}$, 9

$$\begin{aligned} |\langle \bar{m}_\epsilon|_{\mathcal{X}}, \varphi(\cdot - x) \rangle| &= \left| \int_{\mathcal{X}} \varphi(y - x) d\bar{m}_\epsilon(y) \right| \\ &\leq \|\varphi\|_{\infty} |\bar{m}_\epsilon|(\mathcal{X}) \\ &\leq \|\varphi\|_{\infty} |\bar{m}|(\mathcal{X}). \end{aligned} \quad (14)$$

Due to equation (13) and equation (14), we can apply the Lebesgue Dominated Convergence Theorem to the integral in equation (12), to get: 10
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$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{X}} \langle \bar{m}_\epsilon|_{\mathcal{X}}, \varphi(\cdot - x) \rangle^2 dx = \int_{\mathcal{X}} \langle \bar{m}, \varphi(\cdot - x) \rangle^2 dx, \quad (15)$$

meaning that 12

$$\|\Phi(\bar{m}_\epsilon|_{\mathcal{X}})\|_{\mathcal{L}^2(\mathcal{X})}^2 \xrightarrow{\epsilon \rightarrow 0} \|\Phi(\bar{m})\|_{\mathcal{L}^2(\mathcal{X})}^2. \quad (16)$$

From the equations (10), (11) and (16), we get 13

$$\frac{1}{2} \|y - \Phi(\bar{m}_\epsilon)\|_{\mathcal{L}^2(\mathcal{X})}^2 \xrightarrow{\epsilon \rightarrow 0} \frac{1}{2} \|y - \Phi(\bar{m})\|_{\mathcal{L}^2(\mathcal{X})}^2. \quad (17)$$

Hence, from equations (6) and (7), we conclude that:

$$T_\lambda(\bar{m}_\epsilon|_{\mathcal{X}}) = \frac{1}{2} \|y - \Phi(\bar{m}_\epsilon|_{\mathcal{X}})\|_{\mathcal{L}^2(\mathcal{X})}^2 + \lambda |\bar{m}_\epsilon|(\mathcal{X}) + \lambda |\operatorname{div}(\bar{m}_\epsilon)|(\mathcal{X})$$

converges as $\epsilon \rightarrow 0$ to

$$T_\lambda(\bar{m}) = \frac{1}{2} \|y - \Phi(\bar{m})\|_{\mathcal{L}^2(\mathcal{X})}^2 + \lambda |\bar{m}|(\mathcal{X}) + \lambda |\operatorname{div}(\bar{m})|(\mathcal{X}).$$

3. Step 3: Inequality between infima

By the definition of convergence, for any $\delta > 0$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$T_\lambda(\bar{m}_\epsilon|_{\mathcal{X}}) \leq T_\lambda(\bar{m}) + \delta.$$

Since $\bar{m}_\epsilon|_{\mathcal{X}} \in \mathcal{W}(\mathcal{X})$ (as $\bar{m}_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}^2)$), we deduce that for any $\delta > 0$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the following holds:

$$\inf_{m \in \mathcal{W}(\mathcal{X})} T_\lambda(m) \leq T_\lambda(\bar{m}) + \delta = \inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m) + \delta.$$

By letting $\delta \rightarrow 0$, we obtain:

$$\inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m) \leq \inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m),$$

which completes the proof of Theorem 4.3.

□

Proposition 4.4. *Any minimizing sequence for the functional CROC_R in $\mathcal{W}(\mathcal{X})$ has a subsequence that converges to a minimizer of the functional CROC in $\mathcal{V}(\mathcal{X})$.*

Proof. Let (m_n) be a minimizing sequence for D_λ in $\mathcal{W}(\mathcal{X})$, meaning

$$D_\lambda(m_n) \rightarrow \inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m) \quad \text{as } n \rightarrow \infty.$$

Since D_λ is coercive, the sequence $(m_n)_n$ is bounded. Thus, we can extract a subsequence (m_{n_k}) that converges to a limit point \bar{m} in $\mathcal{V}(\mathcal{X})$.

Since T_λ is lower semicontinuous with respect to the weak-* topology, we have

$$T_\lambda(\bar{m}) \leq \liminf_{k \rightarrow \infty} T_\lambda(m_{n_k}) = \liminf_{k \rightarrow \infty} D_\lambda(m_{n_k}) = \inf_{m \in \mathcal{W}(\mathcal{X})} D_\lambda(m) = \inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m).$$

This implies $T_\lambda(\bar{m}) = \inf_{m \in \mathcal{V}(\mathcal{X})} T_\lambda(m)$, meaning that \bar{m} is a minimizer of CROC .

In conclusion, every minimizing sequence of CROC_R has a subsequence that converges to a minimizer of CROC .

□

Having defined CROC_R and established some of its properties, we now proceed with numerical simulations.

5. Numerical Illustration

Data term Modeling

To optimize the functional CROC_R , we first need to define the observed quantity y and the forward model Φ .

Note that, while this functional is designed to recover curves in images, the optimization process does not directly identify the curve itself. Instead, it optimizes vector field functions within $\mathcal{W}(\mathcal{X})$. Once the optimal vector field is determined, the curve must be derived from it. Therefore, it is essential to design a model that specifies how the curve can be extracted from the recovered vector field.

We assume that curves we are looking for are the integral curves of the optimal vector field \mathbf{f} . These curves are defined as solutions to the ordinary differential equation:

$$\frac{d\gamma(t)}{dt} = \mathbf{f}(\gamma(t)), \quad \gamma(0) = x_0,$$

where $\gamma(t)$ represents the trajectory of a point $x_0 \in \mathcal{X}$ under the flow induced by \mathbf{f} . Thus, the vector field that we aim to recover should represent the velocity vector field along the curve we wish to reconstruct, so that by integrating, we can obtain the desired curve. As a velocity vector field involves the derivative operator, we consider a data term based on the gradient of the observed image.

Therefore, we define Φ as the vector acquisition operator, which maps a vector field $f = (f_1, f_2)$ from the space $\mathcal{W}(\mathcal{X})$ to the Hilbert space \mathcal{H}^2 , as follows:

$$\Phi : \mathcal{W}(\mathcal{X}) \longrightarrow \mathcal{H}^2, \quad f = (f_1, f_2) \longmapsto \Phi(f) := (f_1 * h, f_2 * h),$$

where $*$ denotes the convolution operation, and h is a blur kernel.

The optimization functional is then given by:

$$\arg \min_{f \in \mathcal{W}(\mathcal{X})} \frac{1}{2} \|y_1 - f_1 * h\|_{\mathcal{H}}^2 + \frac{1}{2} \|y_2 - f_2 * h\|_{\mathcal{H}}^2 + \alpha \|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} + \alpha \|\text{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R})},$$

where y_1 and y_2 represent the derivatives of the observed image O with respect to the first and second directions, respectively.

Implementation

We perform simulations on simulated noisy images that present blurry curves according to the fluorescent microscopy principles. We consider the blur function h as a two-dimensional Gaussian kernel, and Gaussian noise, denoted η .

To implement this functional, we discretize the space of positions \mathcal{X} into a grid of $N = N_x \times N_y$ pixels, denoted $(P_i)_{1 \leq i \leq N}$, where N_x and N_y represent the number of pixels in the x - and y -directions, respectively.

Assuming the function f is constant within each pixel, we express it as:

$$f(x) = \sum_{i=1}^N f_i \mathbb{1}_{P_i}(x),$$

where $\mathbb{1}_{P_i}(x)$ is the indicator function of the pixel domain P_i . The divergence of the vector field f is then discretized using finite differences. To make the functional convex and differentiable, we relax the absolute value function using a smooth approximation given by:

$$x \mapsto \sqrt{x^2 + \epsilon} - \sqrt{\epsilon},$$

where ϵ is a small positive constant.

We use the L-BFGS (Limited-memory Broyden–Fletcher–Goldfarb–Shanno) for optimization.

5.0.1 Image Simulation

In fluorescence microscopy, the observed curve is the convolution of the ideal filament curve with a blur kernel h , which accounts for the thickening of the curve in the image due to diffraction. This can be modeled as follows:

Definition 5.1 (Blurry curve). *Let $\gamma : [0, 1] \rightarrow \mathcal{X}$ be a parametrized, simple, Lipschitz curve with support Γ . We define a blurry curve as $\delta_\Gamma * h$, given by:*

$$\delta_\Gamma * h : x \in \mathcal{X} \mapsto \int_0^1 h(\gamma(t) - x) \cdot \|\dot{\gamma}(t)\| dt,$$

where δ_Γ is the scalar Radon measure defined by:

$$\begin{aligned} \delta_\Gamma : \mathcal{C}_0(\mathcal{X}, \mathbb{R}) &\rightarrow \mathbb{R} \\ \varphi &\mapsto \int_\Gamma \varphi(y) dy = \int_0^1 \varphi(\gamma(t)) \cdot \|\dot{\gamma}(t)\| dt. \end{aligned} \tag{18}$$

In practice, we consider the polygonal discretization of $\gamma : [0, 1] \rightarrow \mathcal{X}$. That is, γ is regarded as a polygonal curve, characterized by its vertices $(\gamma(t_j))_{j=0}^n$, with $t_0 = 0$ and $t_n = 1$. A simulated blurry and noisy image O presenting the curve structure γ is then given by $O = (O_i)_{1 \leq i \leq N}$, where:

$$O_i = \sum_{j=1}^N \|\gamma(t_{j+1}) - \gamma(t_j)\| h \left(\frac{\gamma(t_{j+1}) + \gamma(t_j)}{2} - c_i \right) + \eta_i, \tag{19}$$

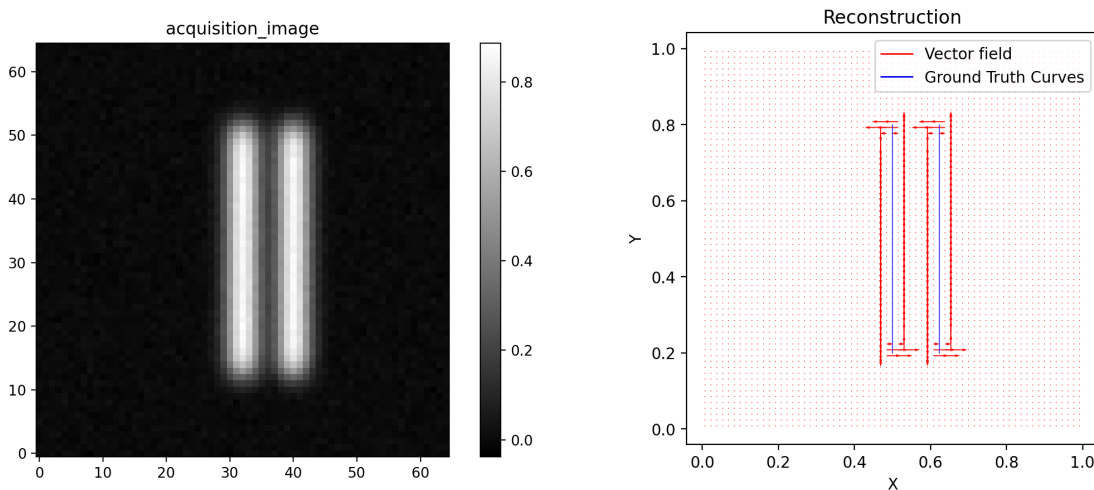
with c_i being the center of the pixel domain P_i , and $(\eta_i)_{1 \leq i \leq N}$ representing a random

sample of Gaussian noise η .

To simulate a noisy image y that presents a blurry set of simple curves $(\gamma_k)_{k=1}^K$ with varying amplitudes $(a_k)_{k=1}^K$, $a_k \in \mathbb{R}$, we compute each pixel O_i as follows:

$$O_i = \sum_{k=1}^K a_k \left(\sum_{j=1}^N \|\gamma_k(t_{j+1}) - \gamma_k(t_j)\| h \left(\frac{\gamma_k(t_{j+1}) + \gamma_k(t_j)}{2} - c_i \right) \right) + \eta_i.$$

5.1.1 Simulation Results



a) Simulated Observed Scalar Image

b) Reconstructed vector field

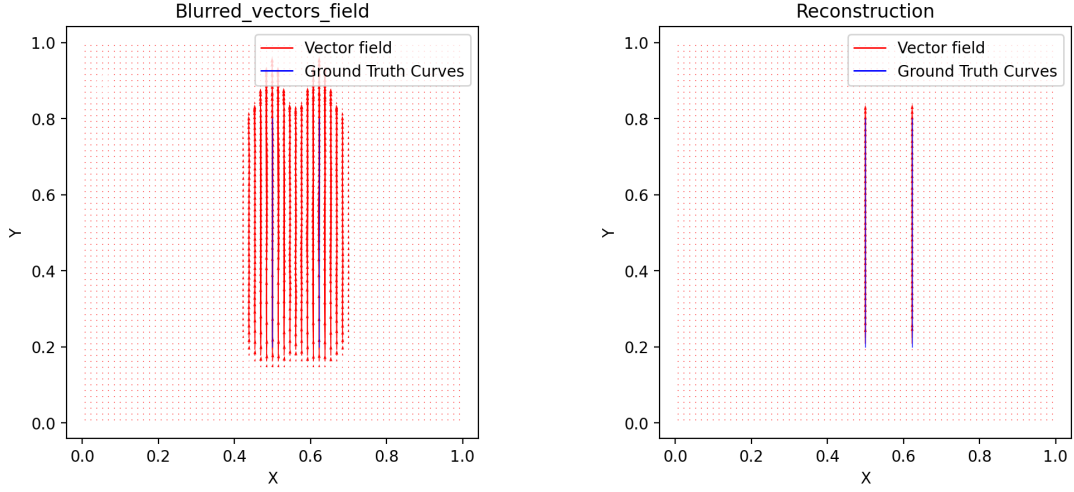
Figure 1: Simulation results using the CROC_R functional with vector observation modeled as the orthogonal of the image gradient. The standard deviations of the Gaussian blur kernel and noise are $\sigma_{\text{PSF}} = 3 \times 10^{-2}$ and $\sigma = 1 \times 10^{-2}$, respectively.

In Figure 1, it can be observed that the reconstruction, obtained by taking the observed vector field as the orthogonal of the gradient of the observed image, results in a velocity vector field associated with closed curves. This velocity vector field corresponds to the contour of the region characterized by the blurred curves in the image, rather than representing the velocity vector field of the curves themselves.

This is due to the fact that the orthogonal of the gradient of the observed scalar image does not characterize the blurred velocity field associated with the curves. It is important to note that this does not undermine our functional, which is well-suited for the super-resolution of blurred and noisy vector fields, as shown in Figure 2. Instead, it highlights the necessity of being able to effectively extract an observed vector field from the image that represents the blurred and noisy velocity field of the curves present in the image.

Since extracting appropriate vector information from a scalar image is challenging, we define a new data-term model based directly on the observed scalar image. The new functional is then given by:

$$\arg \min_{m \in \mathcal{V}(\mathcal{X})} \mathbb{T}_\lambda^{\text{scalar}}(m) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Lambda(m)\|_{\mathcal{H}}^2 + \lambda \|m\|_{\mathcal{V}(\mathcal{X})}, \quad (20)$$



a) Blurry and noisy velocity vector field b) Reconstructed vector field

Figure 2: Simulation results using the $CROC_R$ functional, assuming a blurry and noisy velocity vector field for the curves in Figure 1 a) .

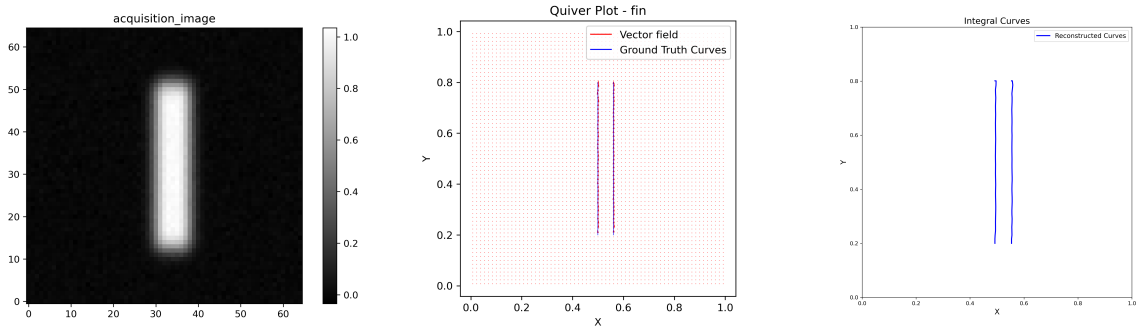
where $\Lambda(m)$ is defined as $|m| * h$, y the observed image and h the blur kernel. 1

The corresponding relaxed functional is given by: 2

$$\arg \min_{f \lambda_2 \in \mathcal{W}(\mathcal{X})} D_\lambda^{scalar}(f) \stackrel{\text{def.}}{=} \frac{1}{2} \|y - \Lambda(f \lambda_2)\|_{\mathcal{H}}^2 + \lambda \left(\|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} + \|\text{div}(f)\|_{L^1(\mathcal{X}, \mathbb{R})} \right) \quad (21)$$

Although the new functional is no longer linear and convex like the previously defined 3
one, simulation results, as shown in Figure 3 and Figure 4, demonstrate its potential and 4
promising performance in handling blurred and noisy scalar image data presenting curve 5
structures. Further theoretical investigations into its properties will be presented in our 6
upcoming article. 7

The implementation code presented in this work is available in the repository [https://
gitlab.inria.fr/atsafack/implementation-of-croc_relaxed-functional-on-w](https://gitlab.inria.fr/atsafack/implementation-of-croc_relaxed-functional-on-w). 8
9



a) Simulated observed image b) Reconstructed vector field c) Integral curves from the reconstructed vector field

Figure 3: Simulation results for two curves with a distance smaller than the diffraction limit, using the new scalar-based relaxed functional. The standard deviations of the Gaussian blur kernel and noise are $\sigma_{PSF} = 3 \times 10^{-2}$ and $\sigma = 1 \times 10^{-2}$, respectively.

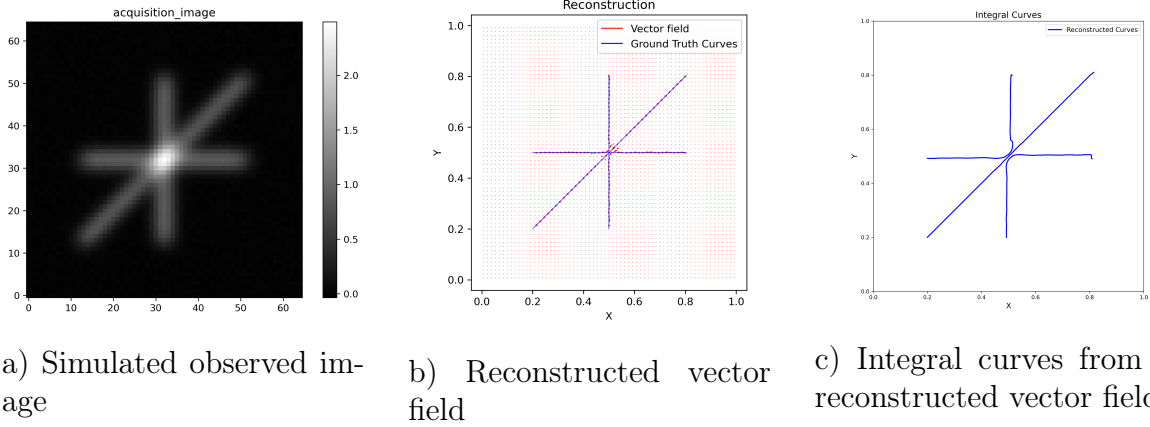


Figure 4: Simulation results for three crossed curves using the new scalar-based relaxed functional. The standard deviations of the Gaussian blur kernel and noise are $\sigma_{\text{PSF}} = 3 \times 10^{-2}$ and $\sigma = 1 \times 10^{-2}$, respectively.

6. Conclusion

This work aimed to design a relaxed convex functional, associated with the original CROC functional in the divergence measure field $\mathcal{V}(\mathcal{X})$, which can be implemented using classical optimization techniques. We successfully achieved this by introducing the space $\mathcal{W}(\mathcal{X})$, dense in $\mathcal{V}(\mathcal{X})$. As the acquisition operator is linear, given that we are working with vector measures, the main challenge was to derive a vector acquisition from the scalar observed image that accurately represents the blurry curves in images. We proposed using the orthogonal of the image gradient. Though it reveals limitations, as it leads to vector field solutions that characterize the contour of the blurry region rather than the curves themselves. Future work will investigate how to extract a more accurate observed vector field that truly represents the blurry and noisy velocity vector fields from the images as the functional is suitable to perform super-resolution of blurry and noisy vector fields. Additionally, we present a new optimization functional with a data term based on the scalar image. This new functional shows significant potential, and further theoretical investigations will be the focus of an upcoming paper.

A. Proof of Proposition 3.3

Proof. We first recall by Lemma 2.11 that, \mathcal{X} can be written as a union of compact subsets:

$$\mathcal{X} = \bigcup_{\epsilon > 0} K_\epsilon, \quad \text{where} \quad K_\epsilon = \{x \in \mathbb{R}^2 \mid d(x, \mathcal{X}^c) \geq 2\epsilon\}.$$

Let $\epsilon > 0$. Since the function $x \mapsto d(x, \mathcal{X}^c)$ is continuous and K_ϵ is compact, then $\exists t_\epsilon \in K_\epsilon$, such that

$$\begin{aligned} \min_{x \in K_\epsilon} d(x, \mathcal{X}^c) &= d(t_\epsilon, \mathcal{X}^c) \\ &= 2\epsilon. \end{aligned}$$

We define $g_\epsilon = \mathbb{1}_{K_{\frac{2\epsilon}{3}}} * \rho_{\frac{2\epsilon}{3}}$. Then, $g_\epsilon \in C^\infty(\mathbb{R}^2, \mathbb{R})$, and $\text{spt}(g_\epsilon) \subset K_{\frac{2\epsilon}{3}} + B(0, \frac{2\epsilon}{3})$, since the support of the convolution is contained in the sum of the supports of $\mathbb{1}_{K_{\frac{2\epsilon}{3}}}$ and $\rho_{\frac{2\epsilon}{3}}$. Moreover, $K_{\frac{2\epsilon}{3}} + B(0, \frac{2\epsilon}{3}) \subset K_{\frac{\epsilon}{3}}$. *Indeed:* Let $z = x + y$ with $x \in K_{\frac{2\epsilon}{3}}$ and $y \in B(0, \frac{2\epsilon}{3})$. We have:

$$\begin{aligned} d(z, \mathcal{X}^c) &= d(x + y, \mathcal{X}^c) \geq d(x, \mathcal{X}^c) - d(x + y, x) \\ &\geq \frac{4\epsilon}{3} - \|y\| \\ &\geq \frac{4\epsilon}{3} - \frac{2\epsilon}{3} = 2\left(\frac{\epsilon}{3}\right). \end{aligned}$$

Thus, $z \in K_{\frac{\epsilon}{3}}$, implying $\text{spt}(g_\epsilon) \subset K_{\frac{\epsilon}{3}} \subset \mathcal{X}$. Consequently, $\text{spt}(g_\epsilon) \subset K_{\frac{\epsilon}{3}}$.

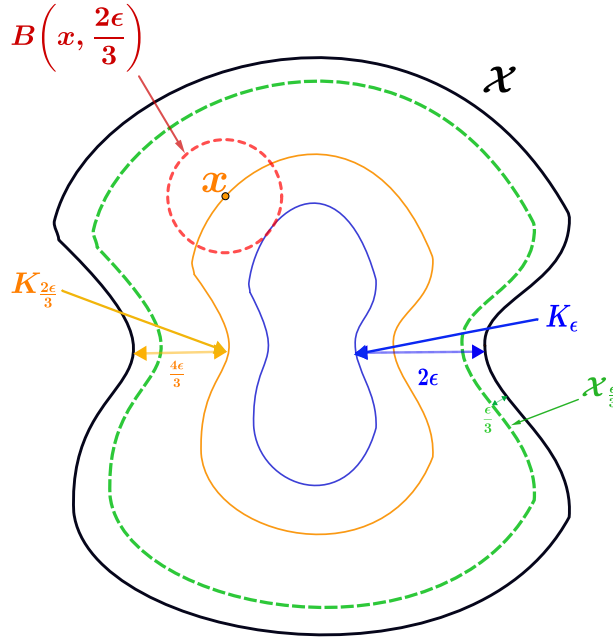


Figure 5: Illustration of the set-theoretic considerations used in the proof of Proposition 3.3 in Appendix A.

Furthermore, $g_\epsilon = 1$ on K_ϵ . *Indeed,* for any $x \in K_\epsilon$,

$$g_\epsilon(x) = \int_{\mathbb{R}^2} \mathbb{1}_{K_{\frac{2\epsilon}{3}}}(x - y) \rho_{\frac{2\epsilon}{3}}(y) dy = \int_{x - K_{\frac{2\epsilon}{3}}} \rho_{\frac{2\epsilon}{3}}(y) dy = \int_{B(0, \frac{2\epsilon}{3})} \rho_{\frac{2\epsilon}{3}}(y) dy = 1.$$

This holds because the support of $\rho_{\frac{2\epsilon}{3}}$, that is $B(0, \frac{2\epsilon}{3})$, is included in $x - K_{\frac{2\epsilon}{3}}$: for any $u \in B(0, \frac{2\epsilon}{3})$, $u = x - (x - u)$, and $d(x - u, \mathcal{X}^c) \geq d(x, \mathcal{X}^c) - d(x - u, x) \geq 2\epsilon - \|u\| \geq \frac{4\epsilon}{3} = 2\left(\frac{2\epsilon}{3}\right)$.

Note also that g_ϵ is non-negative and $\|g_\epsilon\|_\infty \leq \|\mathbf{1}_{K_{\frac{2\epsilon}{3}}}\|_\infty \times \|\rho_{\frac{2\epsilon}{3}}\|_1 = 1$. 1

Consider the map 2

$$\begin{aligned} Z_\epsilon : L^1(\mathcal{X}, \mathbb{R}^2) &\longrightarrow \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2) \\ f &\longmapsto g_\epsilon f * \rho_{\frac{\epsilon}{3}} \end{aligned} \quad (22)$$

1. Z_ϵ is well-defined. 3

Since $\rho_{\frac{\epsilon}{3}} \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R})$ and $g_\epsilon f \in L^1(\mathbb{R}^2, \mathbb{R}^2)$ as $\|g_\epsilon\|_\infty \leq 1$, it follows that $g_\epsilon f * \rho_{\frac{\epsilon}{3}}$ is well-defined and continuous on \mathbb{R}^2 . Moreover, 4
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$$\text{spt}(g_\epsilon f * \rho_{\frac{\epsilon}{3}}) \subset \text{spt}(g_\epsilon f) + \text{spt}(\rho_{\frac{\epsilon}{3}}) \subset \text{spt}(g_\epsilon) + \text{spt}(\rho_{\frac{\epsilon}{3}}) \subset K_{\frac{\epsilon}{3}} + B\left(0, \frac{\epsilon}{3}\right),$$

the latter being compact and included in \mathcal{X} . 6

Indeed: Let $z = x + y$ with $x \in K_{\frac{\epsilon}{3}}$ and $y \in \overline{B}(0, \frac{\epsilon}{3})$. We have: 7

$$\begin{aligned} d(z, \mathcal{X}^c) &\geq d(x, \mathcal{X}^c) - d(x, z) \\ &\geq \frac{2\epsilon}{3} - \frac{\epsilon}{3} = 2\left(\frac{\epsilon}{6}\right) \end{aligned}$$

So, $z \in K_{\frac{\epsilon}{6}}$. Then $\text{spt}(g_\epsilon f * \rho_{\frac{\epsilon}{3}}) \subset K_{\frac{\epsilon}{6}} \subset \mathcal{X}$. Thus, $g_\epsilon f * \rho_{\frac{\epsilon}{3}}$ is continuous with compact support in \mathcal{X} . Therefore, Z_ϵ is well-defined. 8
9

2. Z_ϵ is a continuous linear operator. 10

It is easily seen that Z_ϵ is linear. Let $f \in L^1(\mathcal{X}, \mathbb{R}^2)$. 11

$$\begin{aligned} \text{We have: } \sup_{x \in \mathcal{X}} |(Z_\epsilon f)_1(x)| &= \sup_{x \in \mathcal{X}} |g_\epsilon f_1 * \rho_{\frac{\epsilon}{3}}(x)| \\ &= \sup_{x \in \mathcal{X}} \left| \int_{\mathbb{R}^2} g_\epsilon(y) f_1(y) \rho_{\frac{\epsilon}{3}}(x-y) \, dy \right| \\ &\leq \sup_{x \in \mathcal{X}} \int_{K_{\frac{\epsilon}{3}}} g_\epsilon(y) |f_1(y)| \rho_{\frac{\epsilon}{3}}(x-y) \, dy \\ &\leq \sup_{y \in K_{\frac{\epsilon}{3}}} g_\epsilon(y) \times \sup_{x \in \mathcal{X}} \left(\sup_{y \in K_{\frac{\epsilon}{3}}} \rho_{\frac{\epsilon}{3}}(x-y) \right) \times \int_{K_{\frac{\epsilon}{3}}} |f_1(y)| \, dy \\ &\leq \|g_\epsilon\|_\infty \times \|\rho_{\frac{\epsilon}{3}}\|_\infty \times \|f_1\|_{L^1(\mathcal{X}, \mathbb{R})} \\ &\leq \|\rho_{\frac{\epsilon}{3}}\|_\infty \times \|f_1\|_{L^1(\mathcal{X}, \mathbb{R})}. \end{aligned}$$

Thus, 12

$$\begin{aligned} \sup_{x \in \mathcal{X}} |(Z_\epsilon f)(x)| &= \max \left(\sup_{x \in \mathcal{X}} |(Z_\epsilon f)_1(x)|, \sup_{x \in \mathcal{X}} |(Z_\epsilon f)_2(x)| \right) \\ &\leq \sup_{x \in \mathcal{X}} |(Z_\epsilon f)_1(x)| + \sup_{x \in \mathcal{X}} |(Z_\epsilon f)_2(x)| \\ &\leq (\|f_1\|_{L^1(\mathcal{X}, \mathbb{R})} + \|f_2\|_{L^1(\mathcal{X}, \mathbb{R})}) \times \|\rho_{\frac{\epsilon}{3}}\|_\infty \\ &\leq \|f\|_{L^1(\mathcal{X}, \mathbb{R}^2)} \times \|\rho_{\frac{\epsilon}{3}}\|_\infty. \end{aligned}$$

Therefore, Z_ϵ is a continuous linear operator. 1

We then consider its adjoint operator denoted Z_ϵ^* defined by: 2

$$\begin{aligned} Z_\epsilon^* : \mathcal{M}(\mathcal{X})^2 &\longrightarrow L^\infty(\mathcal{X}, \mathbb{R}^2) \subset L^1(\mathcal{X}, \mathbb{R}^2) \\ \mu &\longmapsto Z_\epsilon^* \mu. \end{aligned} \quad (23)$$

which is a continuous linear map. It helps to construct a family of L^1 vector functions from a vector measure field. 3
4

3. Let μ in $\mathcal{V}(\mathcal{X})$. Then $(Z_\epsilon^* \mu)dx \in \mathcal{W}(\mathcal{X})$. 5

We already know that $(Z_\epsilon^* \mu)dx \in L^1(\mathcal{X}, \mathbb{R}^2)$. 6

Let us denote by μ_1 and μ_2 the different components of the vector μ . Let $\varphi \in C_c^\infty(\mathcal{X})$. We have 7
8

$$\begin{aligned} \langle \operatorname{div}(Z_\epsilon^* \mu), \varphi \rangle &= - \langle \mu, Z_\epsilon \nabla \varphi \rangle \\ &= - \left\langle \mu_1, Z_\epsilon \frac{\partial \varphi}{\partial x_1} \right\rangle - \left\langle \mu_2, Z_\epsilon \frac{\partial \varphi}{\partial x_2} \right\rangle \end{aligned}$$

and for $i \in \{1, 2\}$, 9

$$\begin{aligned} \left\langle \mu_i, Z_\epsilon \frac{\partial \varphi}{\partial x_i} \right\rangle &= \int_{\mathcal{X}} \left(g_\epsilon \frac{\partial \varphi}{\partial x_i} * \rho_{\frac{\epsilon}{3}} \right) (x) d\mu_i(x) \\ &= \int_{\mathcal{X}} \int_{\mathcal{X}} g_\epsilon(y) \frac{\partial \varphi}{\partial x_i}(y) \rho_{\frac{\epsilon}{3}}(x-y) dy d\mu_i(x) \\ &= \int_{\mathcal{X}} g_\epsilon(y) \frac{\partial \varphi}{\partial x_i}(y) \left(\int_{\mathcal{X}} \rho_{\frac{\epsilon}{3}}(x-y) d\mu_i(x) \right) dy \\ &= \int_{\mathcal{X}} g_\epsilon(y) \frac{\partial \varphi}{\partial x_i}(y) (\hat{\rho}_{\frac{\epsilon}{3}} * \mu_i)(y) dy \quad \text{where } \hat{\rho}_{\frac{\epsilon}{3}}(x) = \rho_{\frac{\epsilon}{3}}(-x) \\ &= \int_{\mathcal{X}} g_\epsilon(y) \frac{\partial \varphi}{\partial x_i}(y) (\rho_{\frac{\epsilon}{3}} * \mu_i)(y) dy \quad \text{because } \rho_{\frac{\epsilon}{3}} \text{ is symmetric} \\ &= \left\langle g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_i), \frac{\partial \varphi}{\partial x_i} \right\rangle \\ &= - \left\langle \frac{\partial}{\partial x_i} (g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_i)), \varphi \right\rangle \end{aligned}$$

Thus, 10

$$\begin{aligned} \langle \operatorname{div}(Z_\epsilon^* \mu), \varphi \rangle &= \left\langle \frac{\partial}{\partial x_1} (g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_1)), \varphi \right\rangle + \left\langle \frac{\partial}{\partial x_2} (g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_2)), \varphi \right\rangle \\ &= \left\langle \frac{\partial}{\partial x_1} (g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_1)) + \frac{\partial}{\partial x_2} (g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_2)), \varphi \right\rangle \quad \forall \varphi \in C_c^\infty(\mathcal{X}). \end{aligned}$$

Hence, 11

$$\operatorname{div}(Z_\epsilon^* \mu) = \operatorname{div}(g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu)). \quad (24)$$

It is known that $\rho_{\frac{\epsilon}{3}} * \mu_i$ is C^∞ on $\mathcal{X}_{\frac{\epsilon}{3}}$, where

$$\mathcal{X}_{\frac{\epsilon}{3}} := \left\{ x \in \mathcal{X} \mid d(x, \partial\mathcal{X}) > \frac{\epsilon}{3} \right\} = K_{\frac{\epsilon}{6}}^\circ.$$

Additionally, $K_{\frac{\epsilon}{3}} \subset K_{\frac{\epsilon}{6}}^\circ$.

So, $\rho_{\frac{\epsilon}{3}} * \mu_i \in C^\infty(K_{\frac{\epsilon}{3}})$. It follows that $g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_i) \in C^\infty(K_{\frac{\epsilon}{3}})$, for all $i \in \{1, 2\}$.

For all $i \in \{1, 2\}$, $\frac{\partial}{\partial x_i}(g_\epsilon(\rho_{\frac{\epsilon}{3}} * \mu_i))$ is continuous with support in \mathcal{X} , the latter having finite Lebesgue measure. Therefore, $\text{div}(Z_\epsilon^* \mu) \in L^1(\mathcal{X}, \mathbb{R})$. In conclusion, $(Z_\epsilon^* \mu) dx \in \mathcal{W}(\mathcal{X})$.

4. $Z_\epsilon^* \mu$ converges to μ for the weak-* topology in $\mathcal{M}(\mathcal{X})^2$.

Let $\varphi \in \mathcal{C}_c(\mathcal{X}, \mathbb{R}^2)$. Assume ϵ is sufficiently small such that $\text{spt}(\varphi) \subset K_\epsilon$. Then $g_\epsilon \varphi = \varphi$, as $g_\epsilon = 1$ on K_ϵ . Therefore, $Z_\epsilon \varphi = g_\epsilon \varphi * \rho_{\frac{\epsilon}{3}} = \varphi * \rho_{\frac{\epsilon}{3}}$.

Given $\delta > 0$, we are looking for $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$,

$$|(Z_\epsilon \varphi - \varphi)(x)| \leq \delta, \quad \forall x \in \mathcal{X}.$$

We have:

$$\begin{aligned} |(Z_\epsilon \varphi - \varphi)(x)| &= |(\varphi * \rho_{\frac{\epsilon}{3}})(x) - \varphi(x)| \\ &= \left| \int_{\mathbb{R}^2} \varphi(y) \rho_{\frac{\epsilon}{3}}(x - y) dy - \varphi(x) \right| \\ &= \left| \int_{\{|x-y| \leq \frac{\epsilon}{3}\}} \rho_{\frac{\epsilon}{3}}(x - y) (\varphi(y) - \varphi(x)) dy \right| \\ &\leq \sup_{\{|x-y| \leq \frac{\epsilon}{3}\}} |\varphi(y) - \varphi(x)| \int_{\{|x-y| \leq \frac{\epsilon}{3}\}} \rho_{\frac{\epsilon}{3}}(x - y) dy \\ &\leq \sup_{\{|x-y| \leq \frac{\epsilon}{3}\}} |\varphi(y) - \varphi(x)| \quad \text{because} \int_{\{|x-y| \leq \frac{\epsilon}{3}\}} \rho_{\frac{\epsilon}{3}}(x - y) dy = 1. \end{aligned}$$

Since φ is continuous with compact support, it is uniformly continuous. Therefore, there exists $\eta_\delta > 0$ such that

$$\forall x, y \in \mathcal{X}, |x - y| \leq \eta_\delta \implies |\varphi(y) - \varphi(x)| \leq \delta.$$

To ensure

$$\sup_{\{|x-y| \leq \frac{\epsilon}{3}\}} |\varphi(y) - \varphi(x)| \leq \delta,$$

it suffices to have $\frac{\epsilon}{3} \leq \eta_\delta$. Set $\epsilon_0 = 3\eta_\delta$.

Thus, the sequence $(Z_\epsilon \varphi)_{\epsilon > 0}$ converges to φ in $\mathcal{C}_0(\mathcal{X}, \mathbb{R}^2)$ for all $\varphi \in \mathcal{C}_c(\mathcal{X}, \mathbb{R}^2)$.

Consequently,

$$\langle \mu, Z_\epsilon \varphi \rangle \xrightarrow{\epsilon \rightarrow 0} \langle \mu, \varphi \rangle, \quad \forall \varphi \in \mathcal{C}_c(\mathcal{X}, \mathbb{R}^2). \quad (25)$$

This implies that $Z_\epsilon^* \mu$ locally weakly* converges to μ . 1

To prove that $Z_\epsilon^* \mu$ weakly* converges to μ , it suffices (Proposition 2.8) to show that 2

$$\sup_\epsilon |Z_\epsilon^* \mu|(\mathcal{X}) < \infty.$$

We have: 3

$$\begin{aligned} |Z_\epsilon^* \mu|(\mathcal{X}) &= \sup \{ \langle Z_\epsilon^* \mu, \varphi \rangle \mid \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \} \\ &= \sup \{ \langle \mu, Z_\epsilon \varphi \rangle \mid \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \}. \end{aligned}$$

$$\begin{aligned} \forall \epsilon > 0, \quad \langle \mu, Z_\epsilon \varphi \rangle &\leq |\langle \mu, Z_\epsilon \varphi \rangle| \\ &\leq \|\mu\|_{\text{TV}^2} \times \|Z_\epsilon \varphi\|_\infty \\ &= \|\mu\|_{\text{TV}^2} \times \max \left(\sup_{x \in \mathcal{X}} |(Z_\epsilon \varphi)_1(x)|, \sup_{x \in \mathcal{X}} |(Z_\epsilon \varphi)_2(x)| \right), \quad \forall \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2). \end{aligned}$$

Denoting $\varphi = (\varphi_1, \varphi_2)$, 5

$$\begin{aligned} \sup_{x \in \mathcal{X}} |(Z_\epsilon \varphi)_1(x)| &= \sup_{x \in \mathcal{X}} \left| \int_{\mathcal{X}} g_\epsilon(y) \varphi_1(y) \rho_{\frac{\epsilon}{3}}(x-y) dy \right| \\ &\leq \|\varphi_1\|_\infty \times \|g_\epsilon\|_\infty \times \sup_{x \in \mathcal{X}} \int_{\mathcal{X}} |\rho_{\frac{\epsilon}{3}}(x-y)| dy \\ &\leq \|\varphi_1\|_\infty \times \|g_\epsilon\|_\infty \times \sup_{x \in \mathcal{X}} \int_{\mathbb{R}^2} |\rho_{\frac{\epsilon}{3}}(x-y)| dy \\ &= \|\varphi_1\|_\infty \times \|g_\epsilon\|_\infty \times \sup_{x \in \mathcal{X}} \int_{\mathbb{R}^2} |\rho_{\frac{\epsilon}{3}}(z)| dz \\ &\leq \|\varphi_1\|_\infty \times \|g_\epsilon\|_\infty \times \|\rho_{\frac{\epsilon}{3}}\|_1 \\ &\leq \|\varphi_1\|_\infty \quad \text{since } \|g_\epsilon\|_\infty \leq 1 \text{ and } \|\rho_{\frac{\epsilon}{3}}\|_1 = 1. \end{aligned}$$

Applying the same majoration for $\sup_{x \in \mathcal{X}} |(Z_\epsilon \varphi)_2(x)|$, we get 6

$$\forall \epsilon > 0, \quad \langle \mu, Z_\epsilon \varphi \rangle \leq \|\mu\|_{\text{TV}^2} \times \|\varphi\|_\infty, \quad \forall \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2).$$

So, $\forall \epsilon > 0$, $|Z_\epsilon^* \mu|(\mathcal{X}) = \sup \{ \langle \mu, Z_\epsilon \varphi \rangle \mid \varphi \in \mathcal{C}_0(\mathcal{X}, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \} \leq \|\mu\|_{\text{TV}^2}$ and 7
therefore 8

$$\sup_\epsilon |Z_\epsilon^* \mu|(\mathcal{X}) \leq \|\mu\|_{\text{TV}^2} < \infty.$$

Thus, $Z_\epsilon^* \mu$ converges to μ for the weak-* topology in $\mathcal{M}(\mathcal{X})^2$. 9

5. $\text{div}(Z_\epsilon^* \mu)$ converges to $\text{div}(\mu)$ in the weak-* topology in $\mathcal{M}(\mathcal{X})$. 10

We have $\langle \mu, Z_\epsilon \nabla \varphi \rangle \xrightarrow{\epsilon \rightarrow 0} \langle \mu, \nabla \varphi \rangle$, $\forall \varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R})$ from equation (25), since 11

$\forall \varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R}), \nabla \varphi \in \mathcal{C}_c^\infty(\mathcal{X}, \mathbb{R}^2) \subset \mathcal{C}_c(\mathcal{X}, \mathbb{R})$. We can also extend this result by 12

density, to have

$$\langle \mu, Z_\epsilon \nabla \varphi \rangle \xrightarrow{\epsilon \rightarrow 0} \langle \mu, \nabla \varphi \rangle, \forall \varphi \in \mathcal{C}_0^\infty(\mathcal{X}, \mathbb{R}).$$

So,

$$\langle \operatorname{div}(Z_\epsilon^* \mu), \varphi \rangle \xrightarrow{\epsilon \rightarrow 0} \langle \operatorname{div}(\mu), \varphi \rangle, \forall \varphi \in \mathcal{C}_0^\infty(\mathcal{X}, \mathbb{R}).$$

From $\mu \in \mathcal{V}(\mathcal{X})$, we have constructed a sequence $(Z_\epsilon^* \mu)_{\epsilon > 0}$ in $\mathcal{W}(\mathcal{X})$, which converges to μ in the weak-* topology in $\mathcal{V}(\mathcal{X})$. \square

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