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On the first positive position of a random walker

Claude Godrèche* and Jean-Marc Luck

Université Paris-Saclay, CEA, CNRS, Institut de Physique Théorique,
91191 Gif-sur-Yvette, France.

*Corresponding author(s). E-mail(s): claude.godreche@ipht.fr;

Contributing authors: jean-marc.luck@ipht.fr;

Abstract

The distribution of the first positive position reached by a random walker starting from the origin is fundamental for understanding the statistics of extremes and records in one-dimensional random walks. We present a comprehensive study of this distribution, focusing particularly on its moments and asymptotic tail behaviour, in the case where the step distribution is continuous and symmetric, encompassing both diffusive random walks and Lévy flights.

Keywords: Random walks, Lévy flights, Records, Ladder variables, Renewal theory, Wiener-Hopf

Contents

| | | |
|----------|---|----------|
| 1 | Introduction and summary | 2 |
| 2 | The main outlines of the probabilistic formalism | 6 |
| 2.1 | Ladder variables, records and the first positive position | 7 |
| 2.2 | Three fundamental events | 8 |
| 2.3 | Event $\mathcal{A}_n(x)$ and definition of $F(x)$ | 8 |
| 2.4 | Event $\mathcal{B}_n(x)$ and definition of $G(x)$ | 9 |
| 2.5 | Event $\mathcal{C}_n(x)$ and renewal function $\Psi(x)$ | 11 |
| 2.6 | Duality and the equality of $G(x)$ and $\Psi(x)$ | 13 |
| 2.7 | Further results from renewal theory | 14 |
| 2.8 | Generating series | 16 |
| 2.9 | Recapitulation | 16 |

| | | |
|----------|---|-----------|
| 3 | Asymptotic tail behaviour of the distribution of the first positive position | 18 |
| 3.1 | Superexponentially decaying distributions | 18 |
| 3.2 | Exponentially decaying distributions | 19 |
| 3.3 | Subexponentially decaying distributions | 20 |
| 4 | Moments of the first positive position | 24 |
| 5 | Stable distributions: Gauss, Cauchy, Lévy | 28 |
| 5.1 | A reminder of definitions | 28 |
| 5.2 | Gaussian random walk | 29 |
| 5.3 | Cauchy flight | 32 |
| 5.4 | Lévy flights | 34 |
| 6 | A complementary analytical approach | 36 |
| 6.1 | The setting | 36 |
| 6.2 | Solutions of integral equations and factorisation formulas | 37 |
| 6.3 | Connection with the Pollaczek-Spitzer formula | 38 |
| 6.4 | Connection with Sparre Andersen theory | 39 |
| 6.5 | General results on the distribution of H | 40 |
| 6.6 | Connection with the solution of equation (1.2) | 41 |
| 6.7 | Moments of the first positive position | 42 |
| 6.8 | Symmetric exponential distribution | 42 |
| 6.9 | Double symmetric exponential distribution | 44 |
| 6.10 | Symmetric Erlang distributions | 47 |
| A | Details of derivations | 53 |
| A.1 | Derivation of equation (2.52) | 53 |
| A.2 | Proof that (2.72) implies the Pollaczek-Spitzer formula | 54 |
| A.3 | Derivation of equation (4.22) | 54 |
| A.4 | Derivation of equation (6.7) | 55 |
| A.5 | Derivation of equation (6.8) | 57 |

1 Introduction and summary

This paper revisits the classic problem of characterising the distribution of the first positive sum in a sequence of independent and identically distributed (iid) random variables, or equivalently of the first positive position reached by a one-dimensional random walker starting from the origin. If the walker's steps are denoted by $\eta_1, \eta_2, \dots, \eta_n$, its position x_n at time n is given by

$$x_n = x_{n-1} + \eta_n, \quad x_0 = 0. \tag{1.1}$$

Let $N \geq 1$ denote the first time when the walker's position is positive, and $H = x_N$ denote this first positive position (see figure 1). How can we characterise the joint distribution of the two random variables (N, H) ?

This problem, along with related questions, has been the focus of significant mathematical research since the 1950s, led by prominent figures such as Spitzer [1–4], Pollaczek [5], Feller [6], Blackwell [7] and Baxter [8]. By the end of the 1950s, the essential aspects of the problem were understood. The techniques employed were primarily probabilistic and combinatorial, though influenced by earlier analytical developments known as the Wiener-Hopf approach [9]¹. Notably, two of Spitzer’s papers on this topic are titled *The Wiener-Hopf equation whose kernel is a probability density* [2, 3]. The primary motivation of these two papers was to investigate the maximum position reached by a random walk after n steps. In the second of these papers, Spitzer provides a summary of his results, which we outline below in order to set the context for our study. The key equation of the whole approach is the homogeneous Wiener-Hopf integral equation,

$$G(x) = \int_0^\infty dy G(y)\rho(x-y) \quad (x > 0), \quad (1.2)$$

with boundary condition $G(0) = 1$, where the kernel $\rho(x)$ is the probability density of the walker’s steps,

$$\rho(x) = \frac{d}{dx} \mathbb{P}(\eta < x). \quad (1.3)$$

Apart from the assumption that the step distribution is symmetric (i.e., even) and continuous, no other restrictions apply to it. Thus, the process defined in (1.1) describes diffusive random walks if the diffusion constant D , defined as

$$2D = \langle \eta^2 \rangle = \int_{-\infty}^\infty dx x^2 \rho(x), \quad (1.4)$$

is finite, and Lévy flights if it is infinite. The integral equation (1.2) has a unique solution, whose derivative, $g(x) = G'(x)$, is given in Laplace space, for $\text{Re } p > 0$, by

$$\hat{g}(p) = \int_0^\infty dx e^{-px} g(x) = \exp\left(-\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln(1 - \tilde{\rho}(q))\right), \quad (1.5)$$

where

$$\tilde{\rho}(q) = \int_{-\infty}^\infty dx e^{iqx} \rho(x) \quad (1.6)$$

is the Fourier transform of the density $\rho(x)$. The integral representation (1.5) for $\hat{g}(p)$ in terms of $\tilde{\rho}(q)$ is known as the Pollaczek-Spitzer formula.

For large x , $G(x)$ is such that

$$\lim_{x \rightarrow \infty} \frac{G(x)}{x} = \frac{1}{\sqrt{D}}, \quad (1.7)$$

¹See [10] for an overview of the Wiener-Hopf approach and [11] for a historical account.

where D may be finite or infinite. In the former case, writing

$$G(x) = \frac{x + \lambda(x)}{\sqrt{D}}, \quad (1.8)$$

the function $\lambda(x)$ has a finite limit as $x \rightarrow \infty$, given by

$$\ell = -\frac{1}{\pi} \int_0^\infty \frac{dq}{q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}, \quad (1.9)$$

whenever $\langle |\eta|^3 \rangle$ is finite. Thus

$$G(x) \approx \frac{x + \ell}{\sqrt{D}}, \quad (1.10)$$

where ℓ is known as the extrapolation length.

Spitzer also provides a probabilistic interpretation of these results, which we will explore in detail. This interpretation relates the solution $G(x)$ of the homogeneous equation (1.2) to our object of study, the distribution of the first positive sum H (or the first positive position of the walker):

$$f(x) = \frac{d}{dx} \mathbb{P}(H < x). \quad (1.11)$$

In particular, he derives the relation

$$\hat{g}(p) = \frac{1}{1 - \hat{f}(p)} \quad (\text{Re } p > 0), \quad (1.12)$$

where

$$\hat{f}(p) = \langle e^{-pH} \rangle = \int_0^\infty dx e^{-px} f(x). \quad (1.13)$$

As a consequence of (1.12), the analysis of (1.5) to leading order at small p yields the mean first positive position

$$\langle H \rangle = \sqrt{D}. \quad (1.14)$$

All the results recalled above are central to the study we shall undertake. In particular, the identity (1.12), attributed to Baxter by Spitzer, plays a fundamental role in what follows.

As mentioned earlier, while most advances on the probabilistic side were achieved during the 1950s, articles on the subject have continued to appear regularly in the mathematical literature since then (see, e.g., [12–21]). More recently, the distribution of the pair (N, H) , and especially that of the first positive position H , has sparked renewed interest within the statistical physics community. This interest stems from the central role these random variables play in studying the statistics of extremes and records in random walks and Lévy flights (see [22], and [23, 24] for reviews). The theory of records for random walks represents a natural step in complexity beyond the classical theory of records, which is based on sequences of iid random variables [25–31]. Records for sequences of iid random variables are naturally encountered in statistical mechanics

models (see, e.g., [32–37]). More broadly, the study of records has attracted significant attention due to its wide-ranging applications in complex systems (see [23, 24] for references).

This work aims to shed new light, from various perspectives, on the distribution of the first positive position of a one-dimensional random walker, presenting an essentially self-contained exposition.

In section 2 we revisit the main outlines of the probabilistic formalism underlying the results sketched above. Our primary purpose is to clarify the probabilistic significance of quantities such as $G(x)$, $g(x)$ and $f(x)$, and of the relationships between them. This interpretation of the results mentioned above is enriched with concepts from the Wiener-Hopf method, particularly drawing on developments by Feller [6]. According to Feller, although the mathematical apparatus of the Wiener-Hopf method is not essential, its underlying concepts are, and they lend themselves to a probabilistic interpretation.

The following sections present new results. We begin, in section 3, with a systematic investigation of the asymptotic tail behaviour of the distribution of H near its upper edge, whether finite or infinite. The analysis, which is based on the Wiener-Hopf factorisation identity (2.70), addresses three classes of step distributions.

The first class encompasses all step distributions $\rho(x)$ whose Laplace transform $\hat{\rho}(p)$ is analytic over the entire complex p -plane. This includes, on the one hand, distributions with finite support (e.g., the uniform distribution) and, on the other hand, step distributions that extend to infinity and decay faster than any exponential function (e.g., the Gaussian distribution). We show that the distributions of η and H are asymptotically equivalent: $f(x) \approx \rho(x)$, as x approaches the upper edge of the support of $f(x)$, whether finite or infinite.

The second class includes step distributions whose decay is essentially exponential and for which the Laplace transform $\hat{\rho}(p)$ is analytic within a strip. There, the tails of both distributions satisfy $f(x) \approx K\rho(x)$, where the proportionality constant K depends on details of $\rho(x)$.

The third class consists of step distributions $\rho(x)$ with subexponential decay, i.e., those whose falloff is slower than any exponential function, so that only the Fourier transform $\tilde{\rho}(q)$ is well-defined. In the case where $\rho(x) \approx c|x|^{-(1+\theta)}$, we find $f(x) \approx ax^{-(1+\sigma)}$, where the tail exponent σ equals $\theta - 1$ for diffusive walks ($\theta > 2$), and $\theta/2$ for Lévy flights ($0 < \theta < 2$). Figure 5 illustrates the dependence of the tail exponent σ on θ . The regime where $\sigma = \theta/2$ was identified by Sinai [12] and revisited in [22], whereas the regime where $\sigma = \theta - 1$ is original to this work. The amplitude a has distinct expressions in these two regimes, as given in (3.15) and (3.18). We also consider step distributions falling off faster than any power law, such as those with stretched exponential tails. The asymptotic relationship between $f(x)$ and $\rho(x)$ is given in (3.33), (3.36).

Section 4 presents a detailed study of the moments of H . Several attempts have been made in the past to resolve this question (see, e.g., [16, 17, 20, 21]). However, none are entirely satisfactory, as they fail to provide simple and systematic expressions for the moments. The methods used in this work greatly simplify the matter and yield more elegant expressions. We assume that the step distribution $\rho(x)$ decreases

rapidly, ensuring that all moments of H are finite. The moments of H are expressed in terms of the cumulants of the excess length E , which is the stationary limit of the overshoot E_x of the random walk over the ‘barrier’ located at x (see figure 2). The even cumulants of E have explicit expressions in terms of the moments of the step distribution (see (4.14)), whereas the odd ones have more intricate integral representations (see (4.22)). The first cumulant, $c_1 = \langle E \rangle$, is identified with ℓ , the extrapolation length. Therefore, in some sense, the higher-order odd cumulants c_3, c_5, \dots are generalisations of the latter.

Section 5 is devoted to the class of stable step distributions, with an index in the range $0 < \alpha \leq 2$. We successively consider the Gaussian distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1$) and general Lévy stable distributions. For the Gaussian case, we obtain explicit expressions of the moments of H to arbitrary order, in terms of the Riemann zeta function (see (5.21)). We also derive a power-series expansion in x for the distribution $f(x)$, yielding an accurate convergent series representation of the latter (see figure 6). For the Cauchy flight, we derive a closed-form expression for the distribution $f(x)$ in Laplace space (see (5.31)). This allows the determination of the asymptotic behaviour of the distribution $f(x)$ at large x , as well as an accurate calculation of the distribution $f(x)$ (see figure 7). For Lévy flights, corresponding to an arbitrary index $0 < \alpha < 2$, the asymptotic form of the distribution $f(x)$ for large x , as well as its power-series expansion in x , can be derived following the same approach as in the Gaussian case.

Section 6 presents a complementary analytical approach to the problem, building on the longstanding observation that Wiener-Hopf equations can be solved using elementary methods when the Laplace transform $\hat{\rho}(p)$ of the step density is a rational function of p [38, 39]. This approach bypasses the formal mathematical framework of the Wiener-Hopf method, while explicitly reproducing its outcomes. We are thus able to establish a range of general results, including Wiener-Hopf factorisation properties, general expressions for $G(x)$, $g(x)$, and $f(x)$ in Laplace space, as well as an independent derivation of (1.12). We then proceed with a detailed analysis of three representative examples: the symmetric exponential distribution, the double symmetric exponential distribution, and symmetric Erlang distributions.

The details of some derivations are deferred to an appendix.

2 The main outlines of the probabilistic formalism

In this section, we introduce the foundational concepts necessary for our study. The objective is, starting from the definitions of a few basic events, to give a probabilistic meaning to the quantities introduced above, such as $G(x)$ and $g(x)$, and to explain the origin of (1.7) and (1.8). In section 2.9, we will summarise the progress made and highlight what remains to be addressed. This will be supplemented by the introduction of the Wiener-Hopf factorisation identity and completed from another perspective in section 6. Most of the material in this exposition is inspired by the works of Feller [6] and Spitzer [2, 3].

2.1 Ladder variables, records and the first positive position

Let us come back to the definition of the random walk given in (1.1). The steps η_1, η_2, \dots are iid random variables with common probability density $\rho(x)$. The induced random walk is

$$x_n = \eta_1 + \dots + \eta_n, \quad x_0 = 0. \quad (2.1)$$

As noted above, $\rho(x)$ being symmetric and continuous, but otherwise arbitrary, this random walk may be either diffusive or a Lévy flight.

As emphasised by Feller in [6, chapter XII], which is devoted to the one-dimensional random walk, *Looking at the graph of a random walk, one notices as a striking feature the points where x_n reaches a record value, that is, where x_n exceeds all previously attained values x_0, \dots, x_{n-1} . These are the ladder points [...]. The theoretical importance of ladder points derives from the fact that the sections between them are probabilistic replicas of each other, and therefore important conclusions concerning the random walk can be derived from a study of the first ladder point.* These observations are illustrated in figure 1.

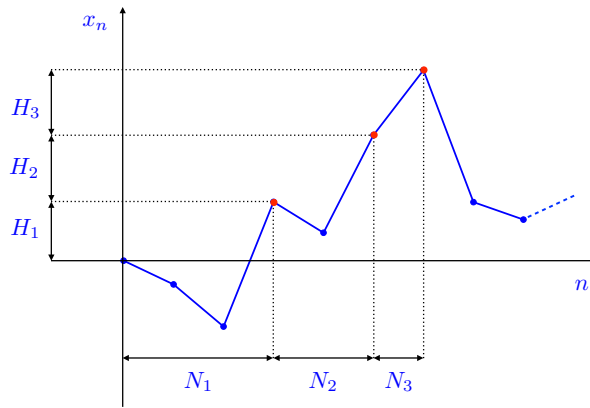


Fig. 1 The red dots represent the ladder points, or records, of the random walk. The coordinates of the first ladder point, or first record, are denoted by (N_1, H_1) . The successive ladder times, or record times, are denoted by N_1, N_2, N_3 . The successive ladder heights, or record heights, are denoted by H_1, H_2, H_3 . The ladder points form a two-dimensional renewal process, meaning that the walk starts afresh from each red dot, taken as the new origin of the walk.

Let (N, H) represent the coordinates of the first ladder point, or first record. The first ladder epoch, or record time (that is, the time of the first entry into the positive half-axis), is defined by the event²

$$\{N = n\} = \{x_1 < 0, \dots, x_{n-1} < 0, x_n > 0\}. \quad (2.2)$$

The first ladder height, or first record value (that is, the first positive position of the walker), is given by $H = x_N$. These quantities were originally introduced by Blackwell [7].

²Since $\rho(x)$ is continuous, no distinction is made between strict and non-strict inequalities.

2.2 Three fundamental events

We introduce the following three events, defined for $n \geq 1$ and $x > 0$,

$$\begin{aligned}\mathcal{A}_n(x) &= \{x_1 < 0, \dots, x_{n-1} < 0, 0 < x_n < x\}, \\ \mathcal{B}_n(x) &= \{x_1 > 0, \dots, x_{n-1} > 0, 0 < x_n < x\}, \\ \mathcal{C}_n(x) &= \{x_n > x_0, x_n > x_1, \dots, x_n > x_{n-1}, 0 < x_n < x\}.\end{aligned}\tag{2.3}$$

In essence, these events can be interpreted as follows:

1. $\mathcal{A}_n(x)$ describes a first-passage event, specifically the first entry into the interval $(0, x)$ at epoch n .
2. $\mathcal{B}_n(x)$ describes a survival event, specifically when the interval $(0, x)$ is visited at epoch n without prior passage through negative values.
3. $\mathcal{C}_n(x)$ describes the occurrence of a record at time n within the interval $(0, x)$.

All quantities considered below are based on these three events. The corresponding probabilities are analysed successively in what follows.

2.3 Event $\mathcal{A}_n(x)$ and definition of $F(x)$

The probability of event $\mathcal{A}_n(x)$, that is the probability of first entry in $(0, x)$ at epoch n , represents the joint distribution of N and H ,

$$F_n(x) = \mathbb{P}(\mathcal{A}_n(x)) = \mathbb{P}(N = n, H < x) \quad (n \geq 1, x > 0).\tag{2.4}$$

The marginal distribution of H is deduced by summing over n ,

$$F(x) = \mathbb{P}(H < x) = \sum_{n \geq 1} F_n(x).\tag{2.5}$$

The density associated to $F_n(x)$ is given by

$$f_n(x) = \frac{d}{dx} F_n(x),\tag{2.6}$$

i.e.,³

$$f_n(x)dx = \mathbb{P}(x_1 < 0, \dots, x_{n-1} < 0, x < x_n < x + dx),\tag{2.7}$$

where $x > 0$. This represents the probability that the walker, starting at the origin, stays below the origin up to epoch $n - 1$ and then makes a jump to the positive side, reaching $x > 0$ (up to dx) at epoch n . The density associated to $F(x)$,

$$f(x) = \sum_{n \geq 1} f_n(x) = \frac{d}{dx} F(x),\tag{2.8}$$

³This density is referred to as $J(n, x)$ in [22]. It also appears in the study of the order statistics of random walks [40, 41].

is the central object of our study, that is the distribution of the first positive sum H (or the first positive position of the walker) (see (1.11)). The marginal distribution of N , or first-passage probability, is obtained by summation on x :

$$f_n = \mathbb{P}(N = n) = \int_0^\infty dx f_n(x) = \lim_{x \rightarrow \infty} F_n(x). \quad (2.9)$$

For symmetric continuous step distributions, the first-passage probability f_n and the survival probability g_n defined below (see (2.26)) have simple universal expressions, given by Sparre Andersen theory, as will be recalled in section 6.4.

2.4 Event $\mathcal{B}_n(x)$ and definition of $G(x)$

Following the same approach as above, we define the following quantities. The probability of event $\mathcal{B}_n(x)$ is denoted by

$$G_n(x) = \mathbb{P}(\mathcal{B}_n(x)) \quad (n \geq 1, x > 0). \quad (2.10)$$

This represents the probability that the random walk remains positive up to time $n-1$ and reaches a position between 0 and x at time n . For $n=0$, we set

$$G_0(x) = \mathbb{I}(x \geq 0), \quad (2.11)$$

where $\mathbb{I}(x \geq 0)$ is equal to 1 if $x \geq 0$ and to 0 otherwise. The sum

$$G(x) = \sum_{n \geq 0} G_n(x) = \mathbb{I}(x \geq 0) + \sum_{n \geq 1} G_n(x) \quad (2.12)$$

represents the expected number of visits to the interval $(0, x)$ before entering the region $(-\infty, 0)$ (this is further discussed in section 2.6). The equivalence between the quantity defined in (2.12) and the solution of the homogeneous equation (1.2) will also be addressed later (see sections 2.9 and 6.6).

By differentiation of (2.10) and (2.12), we obtain respectively the density

$$g_n(x) = \frac{d}{dx} G_n(x), \quad (2.13)$$

such that

$$g_n(x) dx = \mathbb{P}(x_1 > 0, \dots, x_{n-1} > 0, x < x_n < x + dx), \quad (2.14)$$

and the density

$$g(x) = \frac{d}{dx} G(x) = \sum_{n \geq 0} g_n(x) = \delta(x) + g_{\text{reg}}(x), \quad (2.15)$$

with regular part

$$g_{\text{reg}}(x) = \sum_{n \geq 1} g_n(x). \quad (2.16)$$

The density $g_n(x)$ satisfies the following recurrence, which is a forward equation derived by conditioning on the last step,

$$g_n(x) = \int_0^\infty dy g_{n-1}(y) \rho(y-x) \quad (n \geq 1), \quad (2.17)$$

with $g_0(x) = \delta(x)$. As a consequence,

$$g(x) = \delta(x) + \int_0^\infty dy g(y) \rho(x-y). \quad (2.18)$$

By the very definitions of $f_n(x)$ and $g_n(x)$, we have

$$f_n(0) = g_n(0) \quad (n \geq 1). \quad (2.19)$$

The quantity

$$\omega = f(0) = g_{\text{reg}}(0) = G'(0) \quad (2.20)$$

can be evaluated as the limit of the product $p\hat{f}(p)$ as $p \rightarrow +\infty$. Using (1.5) and (1.12) (or (2.66) and (2.67)), we obtain the integral representation

$$\omega = -\frac{1}{\pi} \int_0^\infty dq \ln(1 - \tilde{\rho}(q)), \quad (2.21)$$

which is similar to the expression (1.9) for the extrapolation length ℓ . Since $G(x)$ is a probability, the product $\omega\sqrt{D}$ is dimensionless. Moreover, in view of (1.10), the same applies to the ratio ℓ/\sqrt{D} . We denote these two dimensionless quantities by

$$\mathbf{A} = \frac{\ell}{\sqrt{D}}, \quad \mathbf{B} = \omega\sqrt{D} = \frac{g_{\text{reg}}(0)}{g_{\text{reg}}(\infty)}. \quad (2.22)$$

An alternative expression for ω can be obtained by expanding the logarithm in (2.21) in powers of $\tilde{\rho}(q)$, which yields

$$\omega = \sum_{n \geq 1} \frac{f_{x_n}(0)}{n}, \quad (2.23)$$

where

$$f_{x_n}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dq \tilde{\rho}(q)^n e^{-iqx} \quad (2.24)$$

is the distribution of the walker's position x_n at time n . The identity

$$g_n(0) = \frac{f_{x_n}(0)}{n} \quad (2.25)$$

actually holds for any discrete time n and any continuous symmetric step distribution.

Integrating (2.14) on x yields the survival probability of the walk⁴,

$$g_n = \mathbb{P}(x_1 > 0, \dots, x_n > 0) = \int_0^\infty dx g_n(x) = \lim_{x \rightarrow \infty} G_n(x), \quad (2.26)$$

which is related to f_n by $f_n = g_{n-1} - g_n$ for $n \geq 1$, and so

$$\sum_{n \geq 1} f_n = g_0 = 1, \quad (2.27)$$

ensuring the normalisation of the probabilities (2.9).

Finally, the following relation holds [22–24], as is evident upon inspection:

$$f_n(x) = \int_0^\infty dy g_{n-1}(y) \rho(y+x) \quad (n \geq 1), \quad (2.28)$$

hence

$$f(x) = \int_0^\infty dy g(y) \rho(y+x), \quad (2.29)$$

from which we deduce (see (2.15) and (2.18))

$$f(0) = \int_0^\infty dy g(y) \rho(y) = g_{\text{reg}}(0), \quad (2.30)$$

thus recovering (2.20).

2.5 Event $\mathcal{C}_n(x)$ and renewal function $\Psi(x)$

Consider the renewal function for the height process defined as [6]

$$\Psi(x) = \Psi_0(x) + \sum_{m \geq 1} \mathbb{P}(\Sigma_m < x), \quad (2.31)$$

where

$$\Sigma_m = H_1 + \dots + H_m \quad (2.32)$$

and

$$\Psi_0(x) = \mathbb{I}(x \geq 0). \quad (2.33)$$

The renewal function $\Psi(x)$ has three distinct interpretations:

1. As a consequence of its very definition, it is given by the sum

$$\Psi(x) = \Psi_0(x) + \sum_{n \geq 1} \Psi_n(x), \quad (2.34)$$

where

$$\Psi_n(x) = \mathbb{P}(\mathcal{C}_n(x)) \quad (n \geq 1, x > 0). \quad (2.35)$$

⁴By duality (see section 2.6), g_n is also the probability that n is a ladder epoch.

2. Using the fact that

$$\frac{d}{dx} \mathbb{P}(\Sigma_m < x) = (f^*)^m(x), \quad (2.36)$$

is an m -fold convolution, it follows by differentiating (2.31) that

$$\psi(x) = \frac{d}{dx} \Psi(x) = \delta(x) + \sum_{m \geq 1} (f^*)^m(x). \quad (2.37)$$

This function is the renewal density of the height process, such that $\psi(x)dx$ is the mean number of events (here, records) in the interval $(x, x + dx)$ (see (2.42)). In Laplace space, we obtain

$$\hat{\psi}(p) = \frac{1}{1 - \hat{f}(p)}, \quad (2.38)$$

where we denote the usual Laplace transforms as

$$\begin{aligned} \hat{\psi}(p) &= \mathcal{L}_x \psi(x) = \int_0^\infty dx e^{-px} \psi(x), \\ \hat{f}(p) &= \mathcal{L}_x f(x) = \int_0^\infty dx e^{-px} f(x). \end{aligned} \quad (2.39)$$

We shall also later require the bilateral Laplace transform of the density $\rho(x)$, denoted by

$$\hat{\rho}(p) = \langle e^{-p\eta} \rangle = \int_{-\infty}^\infty dx e^{-px} \rho(x). \quad (2.40)$$

3. Finally, we have

$$\Psi(x) = 1 + \langle R_x \rangle, \quad (2.41)$$

where R_x is the number of records in the interval $(0, x)$, not taking into account the record at the origin (see figure 2). This can be understood as follows. We can first check (2.41) by taking the Laplace transform of its differentiated form and using the expression for $\mathcal{L}_x \langle R_x \rangle$ known from renewal theory (see (A.3)). This yields

$$\hat{\psi}(p) = p \mathcal{L}_x (1 + \langle R_x \rangle) = 1 + p \mathcal{L}_x \langle R_x \rangle = \frac{1}{1 - \hat{f}(p)}, \quad (2.42)$$

which is precisely (2.38). We can also observe that the following identity holds:

$$\langle R_x \rangle = \sum_{m \geq 1} \mathbb{P}(R_x \geq m) = \sum_{m \geq 1} \mathbb{P}(\Sigma_m < x), \quad (2.43)$$

implying that the right-hand sides of (2.31) and (2.41) coincide.

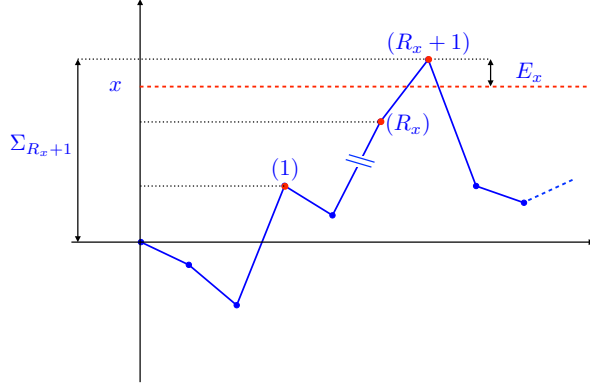


Fig. 2 The number of records of the random walk in the interval $(0, x)$ (not taking into account the record at the origin) is denoted by R_x . The excess length E_x represents the overshoot of the walk over the ‘barrier’ at x . The corresponding height of the walk is denoted by Σ_{R_x+1} . For a diffusive random walk, the mean of this quantity is equal to $\sqrt{D} G(x)$, see (2.52) and (2.53). At large x , $\langle E_x \rangle \rightarrow \ell$, the extrapolation length.

2.6 Duality and the equality of $G(x)$ and $\Psi(x)$

Introduce the dual steps $\eta_1^* = \eta_n$, $\eta_2^* = \eta_{n-1}$, \dots , $\eta_n^* = \eta_1$. Their partial sums are

$$x_0^* = 0, \quad x_1^* = x_n - x_{n-1}, \quad x_k^* = x_n - x_{n-k}, \quad x_n^* = x_n. \quad (2.44)$$

The dual walk, as defined, is obtained from the original by fixing the endpoint at the origin, and then performing a 180° rotation. The joint distributions of (x_1, \dots, x_n) and its dual are identical.

Consider the event

$$\{\text{a record occurs at } n\} = \{x_n > x_0, x_n > x_1, \dots, x_n > x_{n-1}\}, \quad (2.45)$$

which corresponds by duality to the event $\{x_1^* > 0, \dots, x_n^* > 0\}$. For example $\eta_1 = 1$, $\eta_2 = -3$, $\eta_3 = 6$ generates the path $\{x_1 = 1, x_2 = -2, x_3 = 4\}$. By duality, we have $\eta_1^* = 6$, $\eta_2^* = -3$, $\eta_3^* = 1$, which generates the path $\{x_1^* = 6, x_2^* = 3, x_3^* = 4\}$, as illustrated in figure 3.

The probabilities of the following two events are thus equal [3, 6]

$$\mathbb{P}(x_n > x_0, \dots, x_n > x_{n-1}, x_n < x) = \mathbb{P}(x_1 > 0, \dots, x_n > 0, x_n < x). \quad (2.46)$$

This reads

$$\mathbb{P}(\mathcal{C}_n(x)) = \mathbb{P}(\mathcal{B}_n(x)), \quad (2.47)$$

or else

$$\Psi_n(x) = G_n(x) \quad (n \geq 1). \quad (2.48)$$

Summing over $n \geq 1$ and adding the contribution of $\mathbb{I}(x \geq 0)$ for $n = 0$ on both sides results in

$$\Psi(x) = G(x). \quad (2.49)$$

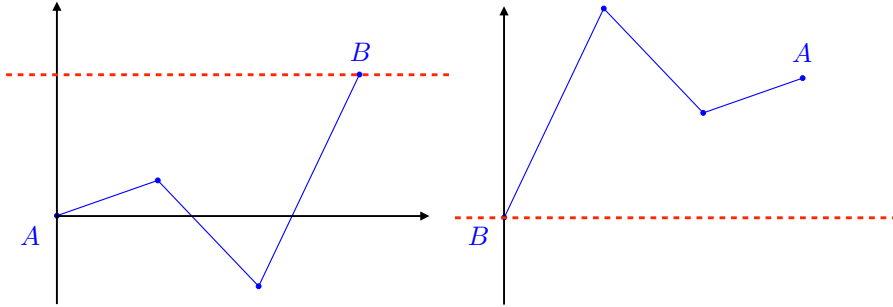


Fig. 3 The dual walk on the right is obtained from the original walk on the left by fixing the endpoint B at the origin, then performing a 180° rotation.

The renewal function is thus equal to [3, 6]

1. the average number of records in the interval $(0, x)$ (counting the record at the origin),
2. the average number of visits to the same interval such that $x_k > 0$ for all $k = 1, \dots, n$.

By combining the above results (2.38) and (2.49), we have established the fundamental identity, announced in (1.12), namely

$$\hat{g}(p) = \hat{\psi}(p) = \frac{1}{1 - \hat{f}(p)}, \quad (2.50)$$

where $g(x) = \psi(x)$ is the renewal density of the height process, such that $g(x)dx$ equals

1. the average number of records in $(x, x + dx)$ (counting the record at the origin),
2. the average number of visits to the same interval such that $x_k > 0$ for all $k = 1, \dots, n$.

2.7 Further results from renewal theory

We consider again the renewal process formed by the successive heights H_1, H_2, \dots . Assume that $\langle H \rangle$ and $\langle H^2 \rangle$ are finite (which holds whenever $\langle |\eta|^3 \rangle$ is finite). Define the forward recurrence length (or excess length) E_x by (see (2.32))

$$\Sigma_{R_x+1} = H_1 + H_2 + \dots + H_{R_x+1} = x + E_x. \quad (2.51)$$

Using the classical result stating that [42] (see Appendix A.1)

$$\langle \Sigma_{R_x+1} \rangle = \langle H \rangle (1 + \langle R_x \rangle), \quad (2.52)$$

we conclude, using (1.14), (2.41) and (2.49) that

$$G(x) = \frac{x + \langle E_x \rangle}{\sqrt{D}}, \quad (2.53)$$

which provides, as illustrated in figure 2, a pictorial representation of $G(x)$, the homogeneous solution of the Wiener-Hopf equation (1.2), which is also the renewal function defined in (2.10), (2.12) and (2.31). Moreover, this demonstrates that $\lambda(x)$, as defined in (1.8), identifies with $\langle E_x \rangle$. Another consequence of (2.53) is that $G(x)$ cannot grow faster than x at large x , since $\langle E_x \rangle$ tends to a constant, as we now show.

Since $\langle H \rangle$ is assumed to be finite, the renewal process reaches an equilibrium as $x \rightarrow \infty$. Specifically, E_x converges to a random variable E with distribution [43, 44]

$$f_E(y) = \frac{d}{dy} \mathbb{P}(E < y) = \frac{1}{\langle H \rangle} \int_y^\infty dx f(x), \quad (2.54)$$

which implies

$$\langle E \rangle = \int_0^\infty dy y f_E(y) = \frac{\langle H^2 \rangle}{2\langle H \rangle}. \quad (2.55)$$

As a result, asymptotically,

$$G(x) \approx \frac{x + \langle E \rangle}{\sqrt{D}}, \quad (2.56)$$

recovering the asymptotic form (1.10), where the extrapolation length is given by

$$\ell = \langle E \rangle = \frac{\langle H^2 \rangle}{2\sqrt{D}}, \quad (2.57)$$

provided this quantity is finite [3]. The first two moments of H thus read

$$\langle H \rangle = \sqrt{D}, \quad \langle H^2 \rangle = 2\sqrt{D} \ell. \quad (2.58)$$

One can also derive (2.57) by a more elementary method, as follows. The mean number of renewals occurring in $(0, x)$ (excluding the event at the origin) is given for large x by [42, 44]

$$\langle R_x \rangle \approx \frac{x}{\langle H \rangle} + \left(\frac{\langle H^2 \rangle}{2\langle H \rangle^2} - 1 \right). \quad (2.59)$$

This equation, along with (1.14), (2.41) and (2.49), again leads to (2.57).

Finally, (2.54) reads, in Laplace space,

$$\hat{f}_E(p) = \frac{1 - \hat{f}(p)}{p\langle H \rangle}, \quad (2.60)$$

which, expanding both sides into power series in p , leads to the following relationship between the moments of H and of E :

$$\langle E^k \rangle = \frac{1}{k+1} \frac{\langle H^{k+1} \rangle}{\langle H \rangle}. \quad (2.61)$$

A systematic investigation of the moments of H and E is presented in section 4. Finally, consider the backward recurrence length B_x , defined by $\Sigma_{R_x} + B_x = x$. At

equilibrium, due to time reversibility, the limiting random variable B shares the same distribution as E , as given by (2.54). In particular, the extrapolation length ℓ also equals $\langle B \rangle$.

2.8 Generating series

Thus far, quantities dependent on the discrete time n have been summed over n to define ‘stationary’ quantities such as $G(x)$, $F(x)$, $\Psi(x)$ and their derivatives. A more general perspective consists in introducing the following generating series involving an additional variable s , conjugate to n .

For s complex with $|s| < 1$, we define

$$f(s, x) = \sum_{n \geq 1} f_n(x) s^n, \quad g(s, x) = \sum_{n \geq 0} g_n(x) s^n. \quad (2.62)$$

This leads to the decomposition, inherited from (2.15)

$$g(s, x) = \delta(x) + g_{\text{reg}}(s, x), \quad g_{\text{reg}}(s, x) = \sum_{n \geq 1} g_n(x) s^n. \quad (2.63)$$

We have in particular $f(1, x) = f(x)$, $g(1, x) = g(x)$ and $g_{\text{reg}}(1, x) = g_{\text{reg}}(x)$. Equation (2.17) translates to the inhomogeneous Wiener-Hopf integral equation

$$g(s, x) = \delta(x) + s \int_0^\infty dy g(s, y) \rho(x - y) \quad (x \geq 0), \quad (2.64)$$

whereas (2.29) translates to

$$f(s, x) = s \int_0^\infty dy g(s, y) \rho(x + y) \quad (x > 0). \quad (2.65)$$

2.9 Recapitulation

In recapitulation, this section has provided a probabilistic interpretation of Spitzer’s results, as outlined in the introduction, along with the associated quantities introduced therein. This approach, notably, enabled the derivation of (1.12) (or (2.50)), which will play a central role in the forthcoming analysis. This derivation suffices to establish that the renewal density $g(x)$ introduced in (2.15) identifies with the derivative of $G(x)$, the solution of the homogeneous equation (1.2)⁵. Another derivation of this identification will be presented in section 6.

So far, the Pollaczek-Spitzer formula (1.5) has been introduced without proof. A derivation of this formula will be given in section 6 (see (6.14)), using an alternative analytic approach based on a class of step distributions of the form (6.1).

By inserting (1.5) into (1.12) (derived in (2.50) or (6.21)), we obtain

$$\hat{f}(p) = 1 - \exp(-I(p)) \quad (\text{Re } p > 0), \quad (2.66)$$

⁵Further developments on this topic can be found in [2–4].

with

$$I(p) = -\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln(1 - \tilde{\rho}(q)). \quad (2.67)$$

An alternative form of (2.67), obtained through integration by parts, reads

$$I(p) = -\frac{1}{\pi} \int_0^\infty dq \frac{\tilde{\rho}'(q)}{1 - \tilde{\rho}(q)} \arctan \frac{q}{p}. \quad (2.68)$$

For diffusive random walks, where $1 - \tilde{\rho}(q) \approx Dq^2$, it is useful to rewrite (2.66) as

$$\hat{f}(p) = 1 - p\sqrt{D} \exp\left(\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}\right). \quad (2.69)$$

We will use either of these forms of $\hat{f}(p)$, depending on the circumstances.

Another fundamental result for the forthcoming developments is the Wiener-Hopf factorisation identity [6]

$$(1 - \tilde{f}(s, q))(1 - \tilde{f}(s, -q)) = 1 - s\tilde{\rho}(q), \quad (2.70)$$

where

$$\tilde{f}(s, q) = \sum_{n \geq 1} s^n \int_0^\infty dx e^{iqx} f_n(x) = \langle s^N e^{iqH} \rangle \quad (2.71)$$

is the Fourier transform of $f(s, x)$. In the context of random walks and Lévy flights, Fourier methods are particularly well-suited, especially when the step distribution exhibits a fat tail (see [6, chapter XVIII]). As shown there, $\tilde{f}(s, q)$ satisfies

$$\ln \frac{1}{1 - \tilde{f}(s, q)} = \sum_{n \geq 1} \frac{s^n}{n} \int_0^\infty dx e^{iqx} f_{x_n}(x), \quad (2.72)$$

where $f_{x_n}(x)$ is the probability density of the position x_n of the walker at time n . As a consequence, $\tilde{f}(s, q)$ satisfies (2.70). Indeed, observing that

$$\sum_{n \geq 1} \frac{s^n}{n} \int_{-\infty}^\infty dx e^{iqx} f_{x_n}(x) = \sum_{n \geq 1} \frac{s^n}{n} \tilde{\rho}(q)^n = \ln \frac{1}{1 - s\tilde{\rho}(q)}, \quad (2.73)$$

and assuming that (2.72) holds, we directly obtain (2.70). It can be easily shown that (2.72) implies the Pollaczek-Spitzer formula (1.5) or (6.14) (see Appendix A.2). Let us also highlight the equivalent of (2.72) in Laplace space, a result of Spitzer [4], credited by him to Baxter:

$$\ln \frac{1}{1 - \langle s^N e^{-pH} \rangle} = \sum_{n \geq 1} \frac{s^n}{n} \int_0^\infty dx e^{-px} f_{x_n}(x). \quad (2.74)$$

A final comment is in order. One might hope to use (2.29) or (2.65) to evaluate $f(x)$ for an arbitrary step distribution $\rho(x)$. However, apart from simple step distributions

such as the symmetric exponential or the symmetric Erlang distribution, this approach proves impracticable (see [23]). Similarly, extracting the asymptotic tail behaviour of $f(x)$ for $\rho(x)$ given by (3.10), with $0 < \theta < 2$, requires considerable effort and complex calculations (see [40, 41]).

In contrast, as shown in sections 3, 4, and 5, much greater progress in the analysis can be achieved, and with considerably simpler methods, by primarily relying on the formula (2.66) which relates $\hat{f}(p)$ to the Pollaczek-Spitzer formula (1.5), where $I(p)$ is expressed using one of the forms (2.67) or (2.68), or alternatively on formula (2.69), along with the Wiener-Hopf factorisation identity (2.70). The probabilistic interpretation presented in section 2 will also prove to be of fundamental importance.

3 Asymptotic tail behaviour of the distribution of the first positive position

This section is devoted to the asymptotic behaviour of the distribution $f(x)$ of the first positive position H in the vicinity of its upper edge H_{\max} , which may be either finite or infinite. The following analysis is entirely based on the Wiener-Hopf factorisation identity (2.70), evaluated at $s = 1$,

$$(1 - \tilde{f}(q))(1 - \tilde{f}(-q)) = 1 - \tilde{\rho}(q), \quad (3.1)$$

which holds for any continuous and symmetric step distribution $\rho(x)$. When the latter distribution decays at least exponentially, the Fourier transforms in (3.1) can be continued to Laplace transforms that are analytic in a strip of the complex p -plane, including the imaginary axis. The identity (3.1) thus becomes

$$(1 - \hat{f}(p))(1 - \hat{f}(-p)) = 1 - \hat{\rho}(p). \quad (3.2)$$

A systematic analysis of the asymptotic tail behaviour of $f(x)$ leads to distinguishing three classes of step distributions, shown in different colours in figure 4.

3.1 Superexponentially decaying distributions

This first class encompasses all step distributions $\rho(x)$ for which $\hat{\rho}(p)$ is analytic throughout the entire complex p -plane. A first situation is where the step distribution has a finite support $(-a, a)$ (such as, e.g., the uniform distribution). The distribution $f(x)$ then has the same upper edge, i.e., $H_{\max} = a$. A second situation is where the step distribution extends to infinity and falls off more rapidly than any exponential (such as, e.g., the Gaussian distribution). The distribution $f(x)$ then also extends to infinity, i.e., $H_{\max} = \infty$.

Let us consider the factorisation identity (3.2) in the $p \rightarrow -\infty$ limit. The second factor in the left-hand side goes to unity, and so

$$\hat{f}(p) \approx \hat{\rho}(p) \quad (p \rightarrow -\infty). \quad (3.3)$$

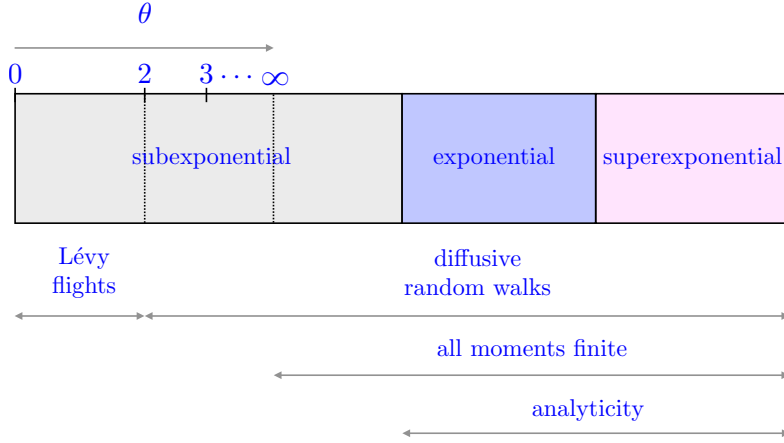


Fig. 4 Schematic representation of the three classes of step distributions involved in the analysis of the asymptotic tail behaviour of $f(x)$, in order of increasing regularity, along with their main characteristics.

This implies the asymptotic equivalence

$$f(x) \approx \rho(x) \quad (3.4)$$

as x goes to the upper edge H_{\max} of the support of $f(x)$, whether it is finite or infinite.

3.2 Exponentially decaying distributions

This second class consists of the step distributions whose falloff is essentially given by a decaying exponential, namely

$$\rho(x) \sim e^{-b|x|} \quad (3.5)$$

for some $b > 0$. The symbol \sim means that the leading exponential decay may be multiplied by any prefactor having a less steep dependence on x , such as, e.g., a power of $|x|$. The Laplace transform $\hat{\rho}(p)$ is then analytic in the strip $|\operatorname{Re} p| < b$.

Let us consider the factorisation identity (3.2) in the $p \rightarrow -b$ limit. The second factor in the left-hand side goes to a constant, namely $1 - \hat{f}(b)$. We thus obtain

$$\hat{f}(p) \approx K \hat{\rho}(p) \quad (p \rightarrow -b), \quad (3.6)$$

with

$$K = \frac{1}{1 - \hat{f}(b)} = \hat{g}(b) \quad (3.7)$$

(see (1.12), (2.50)). This implies the asymptotic proportionality

$$f(x) \approx K \rho(x) \quad (x \rightarrow \infty). \quad (3.8)$$

The proportionality constant K depends on details of the step distribution. For the symmetric exponential distribution (see section 6.8), we have $K = 2$. For the double

symmetric exponential distribution (see section 6.9),

$$K = \frac{2(p_1 + p_2)}{p_1 + z} \quad (3.9)$$

depends on all model parameters.

3.3 Subexponentially decaying distributions

This third class encompasses all step distributions $\rho(x)$ whose falloff is slower than any exponential. In such a situation, only the Fourier transform $\tilde{\rho}(q)$ is well-defined. This situation formally amounts to taking the $b \rightarrow 0$ limit of the previous one, where the constant K diverges. This suggests that $f(x)$ should fall off (slightly) less rapidly than $\rho(x)$ at large x . This heuristic expectation is corroborated by the quantitative results derived below.

Power-law decaying distributions

We consider first the situation where the step distribution has a power-law decay, namely

$$\rho(x) \approx \frac{c}{|x|^{1+\theta}} \quad (x \rightarrow \pm\infty), \quad (3.10)$$

with an arbitrary tail exponent $\theta > 0$. The Fourier transform $\tilde{\rho}(q)$ of the step distribution then behaves at small q as

$$\tilde{\rho}(q) = 1 - Dq^2 + \dots + \tilde{\rho}_{\text{sing}}(q). \quad (3.11)$$

The regular part consists of even powers of q , while the singular part is given by

$$\tilde{\rho}_{\text{sing}}(q) \approx 2c\Gamma(-\theta) \cos(\pi\theta/2)q^\theta \quad (q > 0), \quad (3.12)$$

whenever θ is not an even integer.

For diffusive random walks, i.e., for $\theta > 2$, implying a finite diffusion coefficient D , the term in Dq^2 dominates in the expansion (3.11). Conversely, for superdiffusive Lévy flights, where $\theta < 2$ and D diverges, the singular part (3.12) becomes dominant. This leads to the following classification.

1. For diffusive random walks, i.e., for $\theta > 2$, the consistency of the regular and singular parts of all factors entering the identity (3.1) implies

$$f(x) \approx \frac{a}{x^\theta} \quad (x \rightarrow \infty). \quad (3.13)$$

We have then

$$\tilde{f}(q) = 1 - iq\langle H \rangle + \dots + a\Gamma(1 - \theta)(iq)^{\theta-1} + \dots \quad (3.14)$$

Inserting the expansions (3.11), (3.12) and (3.14) into (3.1) and identifying terms, we recover $\langle H \rangle = \sqrt{D}$ (see (1.14)), and predict the amplitude

$$a = \frac{c}{\theta\sqrt{D}}. \quad (3.15)$$

So, for $\theta > 2$, the tail exponent of $f(x)$ is one unit below that of $\rho(x)$, and the amplitudes a and c satisfy the linear relationship (3.15).

2. For superdiffusive walks (Lévy flights), i.e., for $\theta < 2$, the consistency of the leading singular parts of all factors entering the identity (3.1) implies

$$f(x) \approx \frac{a}{x^{1+\theta/2}} \quad (x \rightarrow \infty). \quad (3.16)$$

We have then

$$\tilde{f}(q) = 1 + a\Gamma(-\theta/2)(iq)^{\theta/2} + \dots \quad (3.17)$$

Inserting the expansions (3.11) and (3.17) in (3.1) yields

$$a = R(\theta)\sqrt{c}, \quad (3.18)$$

with

$$R(\theta) = \Gamma(1 + \theta/2) \left(\frac{\sin(\pi\theta/2)}{\pi\Gamma(1 + \theta)} \right)^{1/2} \quad (0 < \theta < 2). \quad (3.19)$$

So, for $\theta < 2$, the tail exponent of $f(x)$ is half that of $\rho(x)$, confirming a result of Sinai [12], and the amplitudes a and c satisfy the nonlinear relationship (3.18). An equivalent result is given in [22–24]. The function $R(\theta)$ vanishes at both endpoints, as

$$R(\theta) \approx \sqrt{\frac{\theta}{2}} \quad (\theta \rightarrow 0), \quad R(\theta) \approx \frac{\sqrt{2-\theta}}{2} \quad (\theta \rightarrow 2). \quad (3.20)$$

Its maximum $R_{\max} = 0.507018$ is reached for $\theta = 0.857060$. The value $R(1) = 1/2$ is hardly below R_{\max} .

3. In the marginal case where $\theta = 2$, i.e.,

$$\rho(x) \approx \frac{c}{|x|^3}, \quad (3.21)$$

so that the diffusion coefficient D is logarithmically divergent, we mention, skipping details, that the same line of reasoning yields a logarithmic correction of the form

$$f(x) \approx \frac{1}{2x^2} \left(\frac{c}{\ln x} \right)^{1/2}. \quad (3.22)$$

To sum up, whenever the step distribution has a power-law decay of the form (3.10), with tail exponent θ , the distribution of the first positive position also has a power-law

decay,

$$f(x) \approx \frac{a}{x^{1+\sigma}} \quad (x \rightarrow \infty), \quad (3.23)$$

with a tail exponent σ given by (see (3.13), (3.16))

$$\sigma = \begin{cases} \theta/2 & (\theta < 2), \\ \theta - 1 & (\theta > 2), \end{cases} \quad (3.24)$$

and an amplitude a given by (3.15) or (3.18). Figure 5 illustrates the dependence of the tail exponent σ on θ . The inequality $\sigma < \theta$ corroborates the heuristic expectation that the distribution $f(x)$ falls off less rapidly than the step distribution $\rho(x)$. The break point at $\theta = 2$ corresponds to the logarithmic correction given by (3.22).

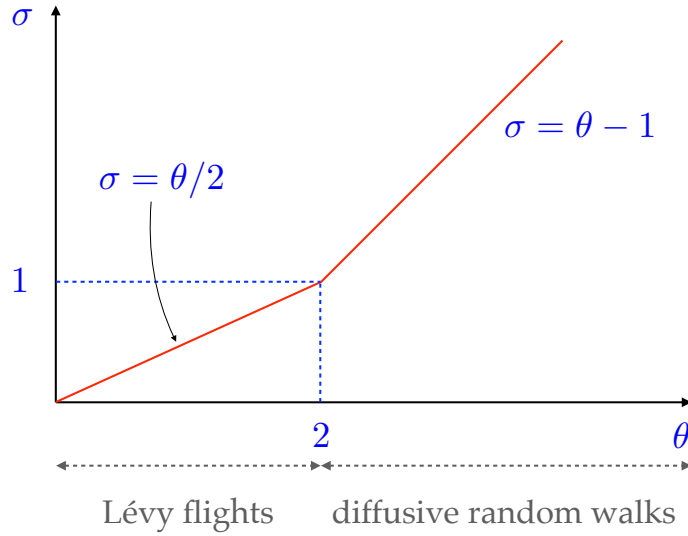


Fig. 5 Dependence of the tail exponent σ characterising the decay of the distribution $f(x)$ of the first positive position on the tail exponent θ of the step distribution, as given by (3.24).

The preceding results have the following consequences on the moments of H (see section 4). For superdiffusive walks, such that D is divergent (i.e., $\theta < 2$), the mean value of H diverges, in agreement with (1.14). For diffusive walks, such as D is finite (i.e., $\theta > 2$), the moments $\langle H^k \rangle$ are finite for $k = 1, \dots, k_{\max}$, with

$$k_{\max} = \text{Int}(\sigma) = \text{Int}(\theta) - 1. \quad (3.25)$$

Other subexponentially decaying distributions

We now turn to the analysis of step distributions $\rho(x)$ whose decay, while subexponential, is faster than that of any negative power of $|x|$, so that the tail exponent θ is formally infinite. In such a situation, only the Fourier transform $\tilde{\rho}(q)$ is well-defined, even though all moments of $|\eta|$ are finite.

We consider first, for definiteness, the prototypical example of such distributions, the stretched exponential

$$\rho(x) = A e^{-|x|^a}, \quad A = \frac{a}{2\Gamma(1/a)}, \quad (3.26)$$

with arbitrary stretching exponent $0 < a < 1$. The corresponding moments read

$$\langle |\eta|^m \rangle = \frac{\Gamma((m+1)/a)}{\Gamma(1/a)}. \quad (3.27)$$

The Fourier transform

$$\tilde{\rho}(q) = A \int_{-\infty}^{\infty} dx e^{iqx - |x|^a} \quad (3.28)$$

therefore has the formal power-series expansion

$$\tilde{\rho}(q) = \sum_{k \geq 0} \frac{(-q^2)^k}{(2k)!} \langle \eta^{2k} \rangle = \sum_{k \geq 0} \frac{(-q^2)^k}{(2k)!} \frac{\Gamma((2k+1)/a)}{\Gamma(1/a)}. \quad (3.29)$$

The above expansion is only a divergent asymptotic one, because the moments $\langle \eta^{2k} \rangle$ grow faster than $(2k)!$. In other terms, $\tilde{\rho}(q)$ is indefinitely differentiable, but not analytic. Its singular part $\tilde{\rho}_{\text{sing}}(q)$ is thus expected to be smaller than any power of q . In line with the usual analysis of the divergent perturbation series met in quantum mechanics and quantum field theory (see, e.g., [45]), it is natural to estimate $\tilde{\rho}_{\text{sing}}(q)$ as the contribution of the non-trivial complex saddle point x_* to the integral (3.28), given by

$$iq = ax_*^{a-1}. \quad (3.30)$$

Skipping the subleading prefactor, we thus obtain

$$\tilde{\rho}_{\text{sing}}(q) \sim \exp\left(- (1-a) \left(\frac{a}{|q|}\right)^{a/(1-a)}\right). \quad (3.31)$$

This is indeed an exponentially small essential singularity.

Inserting (3.11) and its counterpart for $\tilde{f}(q)$ into (3.1), and identifying regular and singular parts, we obtain

$$iq\sqrt{D} \tilde{f}_{\text{sing}}(q) \approx \tilde{\rho}_{\text{sing}}(q). \quad (3.32)$$

with $D = \langle \eta^2 \rangle / 2 = \Gamma(3/a) / (2\Gamma(1/a))$ (see (3.27)). The relationship between the tails of $f(x)$ and $\rho(x)$ can be derived, to leading order at large x , by invoking again the saddle-point approximation, but without having to perform any explicit derivation. In line with (3.30), it is indeed sufficient to replace in (3.32) iq by ax^{a-1} . We thus obtain

$$f(x) \approx \frac{x^{1-a}}{a\sqrt{D}} \rho(x). \quad (3.33)$$

So, for the stretched exponential step distribution (3.26), the tail of the distribution $f(x)$ is nearly identical to that of the step distribution itself, differing only by the

prefactor given explicitly in (3.33), involving a power of x . This result matches previous ones at both endpoints of the range of the stretching exponent a . As $a \rightarrow 0$, disregarding amplitudes, (3.33) involves a factor x , consistent with the relation $\sigma = \theta - 1$ between the two tail exponents (see (3.24)). As $a \rightarrow 1$, (3.33) predicts an asymptotic proportionality between the two distributions, consistent with (3.8).

The above analysis can be readily extended to any step distribution with a tail of the form

$$\rho(x) \sim \exp(-\varphi(x)), \quad (3.34)$$

where $\varphi(x)$ is a smoothly increasing function whose growth at large x is much faster than $\ln x$ and much slower than x . In such a circumstance, the key formulas (3.30) and (3.33) respectively generalise to

$$iq = \varphi'(x^*) \quad (3.35)$$

and

$$f(x) \approx \frac{\rho(x)}{\sqrt{D} \varphi'(x)}. \quad (3.36)$$

The derivative $\varphi'(x)$ slowly goes to zero at large x . This behaviour has two consequences. First, x^* becomes large as $q \rightarrow 0$, thus validating the saddle-point approach. Second, the expression (3.36) corroborates our heuristic expectation that $f(x)$ falls off slightly less rapidly than $\rho(x)$.

4 Moments of the first positive position

The purpose of this section is to determine the moments $\langle H^k \rangle$ of the first positive position H . We assume that the step distribution $\rho(x)$ decreases more rapidly than any power of x , so that all moments of H are finite. If this is not the case, the expressions derived below are only valid for the lowest-order moments, whose order k is at most k_{\max} , given by (3.25).

The analysis begins with the following observation. A comparison between (2.60) and (2.69) yields

$$\hat{f}_E(p) = \langle e^{-pE} \rangle = \exp\left(\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}\right). \quad (4.1)$$

Let us denote the cumulants of the excess length E by c_k and the corresponding generating series by

$$K(p) = \sum_{k \geq 1} \frac{c_k}{k!} (-p)^k = \ln \langle e^{-pE} \rangle = \ln \hat{f}_E(p). \quad (4.2)$$

In view of (4.1), we have

$$K(p) = \frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}, \quad (4.3)$$

which is related to the integral $I(p)$ defined in (2.67) as

$$K(p) = -I(p) - \ln(p\sqrt{D}). \quad (4.4)$$

The integral expression (4.3) allows to determine all the cumulants c_k of E . This is analysed in detail below. Even cumulants are given by the series (4.13), yielding the polynomial expressions (4.14) in terms of even moments of the step distribution. Odd cumulants will be given more intricate expressions involving subtracted integrals (see (4.22)). The first of them identifies with the extrapolation length, i.e.,

$$c_1 = \langle E \rangle = \ell, \quad (4.5)$$

in agreement with (2.57). Its expression (4.22) agrees with (1.9), as should be.

The moments of the excess length E are expressed in terms of its cumulants through the Bell polynomials,

$$\langle E^k \rangle = B_k(c_1, c_2, \dots, c_k), \quad (4.6)$$

that is, explicitly,

$$\begin{aligned} \langle E \rangle &= c_1, \\ \langle E^2 \rangle &= c_2 + c_1^2, \\ \langle E^3 \rangle &= c_3 + 3c_1c_2 + c_1^3, \\ \langle E^4 \rangle &= c_4 + 4c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4, \\ \langle E^5 \rangle &= c_5 + 5c_1c_4 + 10c_2c_3 + 10c_1^2c_3 + 15c_1c_2^2 + 10c_1^3c_2 + c_1^5, \end{aligned} \quad (4.7)$$

and so forth.

The moments of the first positive position H are related to those of E by (2.61). We thus obtain the moments of H in terms of the cumulants of E ,

$$\langle H^{k+1} \rangle = (k+1)\sqrt{D} B_k(c_1, c_2, \dots, c_k), \quad (4.8)$$

that is, explicitly⁶,

$$\begin{aligned} \langle H \rangle &= \sqrt{D}, \\ \langle H^2 \rangle &= 2\sqrt{D} c_1, \\ \langle H^3 \rangle &= 3\sqrt{D} (c_2 + c_1^2), \\ \langle H^4 \rangle &= 4\sqrt{D} (c_3 + 3c_1c_2 + c_1^3), \\ \langle H^5 \rangle &= 5\sqrt{D} (c_4 + 4c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4), \\ \langle H^6 \rangle &= 6\sqrt{D} (c_5 + 5c_1c_4 + 10c_2c_3 + 10c_1^2c_3 + 15c_1c_2^2 + 10c_1^3c_2 + c_1^5), \end{aligned} \quad (4.9)$$

and so forth.

⁶Note that $B_0 = 1$.

The universal expression of the first moment $\langle H \rangle$ has been known for long [3, 6] (see (1.14)). The expression of the second moment $\langle H^2 \rangle$ agrees with (2.57), (2.58). The variance of H reads

$$\langle H^2 \rangle - \langle H \rangle^2 = (2\mathbf{A} - 1)D \quad (4.10)$$

in terms of the dimensionless quantity \mathbf{A} introduced in (2.22), which therefore satisfies the inequality $\mathbf{A} > 1/2$.

We now turn to the derivation of explicit expressions for the cumulants c_k of E . As already said above, even and odd values of k have to be dealt with separately.

Even cumulants of E

The derivation of the even cumulants c_{2m} relies on the Wiener-Hopf factorisation identity (3.2). By substituting (2.60) into (3.2), we obtain:

$$\hat{f}_E(p)\hat{f}_E(-p) = \frac{\hat{\rho}(p) - 1}{Dp^2}, \quad (4.11)$$

i.e.,

$$K(p) + K(-p) = \ln \frac{\hat{\rho}(p) - 1}{Dp^2}. \quad (4.12)$$

Using (4.2), this yields

$$\sum_{m \geq 1} \frac{c_{2m}}{(2m)!} p^{2m} = \frac{1}{2} \ln \frac{\hat{\rho}(p) - 1}{Dp^2}. \quad (4.13)$$

The even cumulants follow by expanding the right-hand side, which results in

$$\begin{aligned} c_2 &= \frac{\langle \eta^4 \rangle}{24D}, & c_4 &= \frac{\langle \eta^6 \rangle}{60D} - \frac{\langle \eta^4 \rangle^2}{96D^2}, & c_6 &= \frac{\langle \eta^8 \rangle}{112D} - \frac{\langle \eta^4 \rangle \langle \eta^6 \rangle}{48D^2} + \frac{5\langle \eta^4 \rangle^3}{576D^3}, \\ c_8 &= \frac{\langle \eta^{10} \rangle}{180D} - \frac{\langle \eta^4 \rangle \langle \eta^8 \rangle}{48D^2} - \frac{7\langle \eta^6 \rangle^2}{360D^2} + \frac{7\langle \eta^4 \rangle^2 \langle \eta^6 \rangle}{144D^3} - \frac{35\langle \eta^4 \rangle^4}{2304D^4}, \end{aligned} \quad (4.14)$$

and so forth. We recall that $\langle \eta^2 \rangle = 2D$.

Odd cumulants of E

The evaluation of the odd cumulants is more intricate. An efficient approach consists in introducing the Mellin transform $M_K(s)$ of the function $K(p)$ defined in (4.3),

$$M_K(s) = \int_0^\infty dp p^{s-1} K(p) = \frac{1}{\pi} \int_0^\infty dp p^s \int_0^\infty \frac{dq}{p^2 + q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}. \quad (4.15)$$

Interchanging the order of integrations and using the identity

$$\int_0^\infty dp \frac{p^s}{p^2 + q^2} = \frac{\pi q^{s-1}}{2 \cos(\pi s/2)} \quad (-1 < \operatorname{Re} s < 1), \quad (4.16)$$

we obtain

$$M_K(s) = \frac{\mu_K(s)}{2 \cos(\pi s/2)}, \quad (4.17)$$

with

$$\mu_K(s) = \int_0^\infty dq q^{s-1} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2} \quad (-2 < \operatorname{Re} s < 0). \quad (4.18)$$

The expression (4.17) holds for $-1 < \operatorname{Re} s < 0$. The inverse Mellin formula reads

$$K(p) = \int \frac{ds}{2\pi i} p^{-s} M_K(s), \quad (4.19)$$

where the integral runs along a vertical contour in the strip $-1 < \operatorname{Re} s < 0$. The power-series expansion of $K(p)$ is obtained by collecting the contributions of all the poles of $M_K(s)$ in the left-hand half-plane ($\operatorname{Re} s < 0$).

1. Even cumulants are in correspondence with the poles of $\mu_K(s)$ at $s = -2m$ with $m = 1, 2, \dots$. We have⁷

$$[p^{2m}]K(p) = \frac{c_{2m}}{(2m)!} = \frac{(-1)^m}{2} [q^{2m}] \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}. \quad (4.20)$$

This expression is equivalent to (4.13).

2. Odd cumulants are in correspondence with the poles of $1/(2 \cos(\pi s/2))$ at $s = -2m - 1$ with $m = 0, 1, \dots$. We have

$$[p^{2m+1}]K(p) = -\frac{c_{2m+1}}{(2m+1)!} = \frac{(-1)^m}{\pi} \mu_K(-2m-1). \quad (4.21)$$

The integral expression (A.18) for $\mu_K(-2m-1)$, whose proof is provided in Appendix A.3, yields

$$c_{2m+1} = (-1)^{m+1} \frac{(2m+1)!}{\pi} \int_0^\infty \frac{dq}{q^{2m+2}} \left(\ln \frac{1 - \tilde{\rho}(q)}{Dq^2} - 2 \sum_{n=1}^m \frac{c_{2n}}{(2n)!} (-q^2)^n \right). \quad (4.22)$$

This integral formula generalises the expression (1.9) for the extrapolation length ℓ . The latter is recovered for $m = 0$, the only case where no subtraction is involved. Unlike (4.14), which are fully explicit, the expressions (1.9) and (4.22) can be evaluated in closed form for only a few step distributions—essentially the Gaussian distribution (see (5.19)) and the class of distributions of the form (6.1) (see (6.41)). For generic step distributions, even the simplest ones, such as the uniform distribution, the formulas (1.9) and (4.22) only lend themselves to numerical evaluation.

The expressions (4.14) and (4.22) demonstrate that c_k is finite whenever $\langle |\eta|^{k+2} \rangle$ converges, which essentially amounts to $\rho(x)$ falling off faster than $1/|x|^{k+3}$. In terms of the tail exponent θ (see (3.10)), this reads $\theta > k + 2$. In particular, we recover

⁷The symbol $[x^n]f(x)$ denotes the coefficient of x^n in the power-series expansion of $f(x)$.

that the extrapolation length ℓ is finite for $\theta > 3$. Hence the moment $\langle H^k \rangle$ is finite whenever $\rho(x)$ falls off faster than $1/|x|^{k+2}$, i.e., $\theta > k + 1$, in agreement with (3.25).

To close, we recall that the Wiener-Hopf factorisation (3.1) yields an infinite sequence of identities relating the moments of η and H [20]. For a continuous symmetric step distribution these relations take the form:

$$\langle \eta^{2n} \rangle = \sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n}{k} \langle H^k \rangle \langle H^{2n-k} \rangle, \quad (4.23)$$

with the following first examples:

$$\begin{aligned} \langle \eta^2 \rangle &= 2\langle H \rangle^2, \\ \langle \eta^4 \rangle &= 8\langle H \rangle \langle H^3 \rangle - 6\langle H^2 \rangle^2, \\ \langle \eta^6 \rangle &= 12\langle H \rangle \langle H^5 \rangle - 30\langle H^2 \rangle \langle H^4 \rangle + 20\langle H^3 \rangle^2, \\ \langle \eta^8 \rangle &= 16\langle H \rangle \langle H^7 \rangle - 56\langle H^2 \rangle \langle H^6 \rangle + 112\langle H^3 \rangle \langle H^5 \rangle - 70\langle H^4 \rangle^2. \end{aligned} \quad (4.24)$$

The expressions (4.9) of the moments of H satisfy these relations, as expected. The key element in verifying this lies in the expressions (4.14) for the even cumulants of E .

5 Stable distributions: Gauss, Cauchy, Lévy

5.1 A reminder of definitions

Lévy stable distributions naturally arise in the study of random walks due to the generalised central limit theorem (see, e.g., [46–49]). This theorem states that for a sequence of iid random variables X_1, X_2, \dots , if the distribution of the sum $S_n = \sum_{i=1}^n X_i$, appropriately normalised, converges to a limiting distribution as $n \rightarrow \infty$, then this limiting distribution is stable. Equivalently, the distribution of X is said to belong to the domain of attraction of a stable distribution.

Among the classes of symmetric distributions considered in section 3, superexponentially decaying distributions, exponentially decaying distributions, and subexponentially decaying distributions with $\theta \geq 2$, belong to the domain of attraction of the stable law with $\alpha = 2$. In contrast, subexponentially decaying distributions with $\theta < 2$ belong to the domain of attraction of the stable law with $\alpha = \theta$.

The distribution of the random variable X is (strictly) stable⁸ if there exist constants $c_n > 0$ such that, for any $n \geq 1$,

$$X_1 + \dots + X_n \stackrel{d}{=} c_n X. \quad (5.1)$$

The short-hand notation $\stackrel{d}{=}$ indicates that the random variables on either side of the equal sign have the same distribution. The norming constants c_n are necessarily of

⁸The general definition of a stable distribution requires the existence of constants c_n and b_n such that $X_1 + \dots + X_n \stackrel{d}{=} c_n X + b_n$. Here we restrict ourselves to the case where the distribution of X is symmetric. A symmetric stable distribution is necessarily strictly stable [6].

the form $c_n = n^{1/\alpha}$ for some $0 < \alpha \leq 2$ [6]. The constant α is called the *index* or *characteristic exponent* of the distribution. In Fourier space, the symmetric stable distribution of index α is given by

$$\tilde{\rho}(q) = e^{-|q|^\alpha}, \quad (5.2)$$

and so

$$\rho(x) = \frac{1}{\pi} \int_0^\infty dq \cos qx e^{-q^\alpha}. \quad (5.3)$$

All symmetric stable distributions fall off monotonically on either side of their maximum

$$\rho(0) = \frac{\Gamma(1 + 1/\alpha)}{\pi}. \quad (5.4)$$

The Gaussian (or normal) distribution is stable with $\alpha = 2$, while the Cauchy distribution corresponds to $\alpha = 1$. For $0 < \alpha < 2$, the stable distribution has a fat tail with exponent α : $\rho(x)$ decays as a power law of the form (3.10) with $\theta = \alpha$, namely

$$\rho(x) \approx \frac{c}{|x|^{1+\alpha}}, \quad c = \frac{\Gamma(1 + \alpha) \sin(\pi\alpha/2)}{\pi}. \quad (5.5)$$

Let us now apply these general considerations to the problem at hand. Hereafter, we shall consider random walks with step distributions that are stable with $\alpha = 2$ (see section 5.2), $\alpha = 1$ (see section 5.3), and $0 < \alpha < 2$ (see section 5.4). The definition (5.1) (with $X = \eta$ and $c_n = n^{1/\alpha}$) implies that the distribution of the position x_n of the walker at time n is given in terms of the symmetric stable step distribution as

$$f_{x_n}(x) = n^{-1/\alpha} \rho(n^{-1/\alpha} x). \quad (5.6)$$

Note that this suffices to determine the quantity ω defined in (2.20). Since $f_{x_n}(0) = n^{-1/\alpha} \rho(0)$, where $\rho(0)$ is given in (5.4), (2.23) yields

$$\omega = \frac{\Gamma(1 + 1/\alpha) \zeta(1 + 1/\alpha)}{\pi}, \quad (5.7)$$

where ζ denotes the Riemann zeta function.

5.2 Gaussian random walk

The stable distribution (5.3) with $\alpha = 2$ is the Gaussian (or normal) distribution

$$\rho(x) = \frac{e^{-x^2/4}}{2\sqrt{\pi}}, \quad (5.8)$$

so that $\langle \eta^2 \rangle = 2$, i.e., $D = 1$. The Fourier (see (5.2)) and Laplace transforms of this density read

$$\tilde{\rho}(q) = e^{-q^2}, \quad \hat{\rho}(p) = e^{p^2}. \quad (5.9)$$

Thus, according to (2.66), $\hat{f}(p) = 1 - \exp(-I(p))$, with (see (2.67))

$$I(p) = -\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln(1 - e^{-q^2}). \quad (5.10)$$

The Gaussian density (5.8) decays superexponentially (see section 3.1), which implies that $\hat{\rho}(p)$ and $\hat{f}(p)$ are entire functions, analytic throughout the entire complex p -plane. Systematic expansions of $\hat{f}(p)$ for $p \rightarrow 0$ and $p \rightarrow +\infty$ can be obtained by using again the Mellin transformation, along the lines of section 4. In the present situation, it is preferable to consider the Mellin transform $M_I(s)$ of $I(p)$. The main reason is that $I(p)$ and $M_I(s)$ still make sense for the Lévy laws considered hereafter (see sections 5.3 and 5.4), where $K(p)$ is not defined. The Mellin transform

$$M_I(s) = \int_0^\infty dp p^{s-1} I(p), \quad (5.11)$$

reads, in analogy with (4.17),

$$M_I(s) = -\frac{\mu_I(s)}{2 \cos(\pi s/2)}, \quad (5.12)$$

where

$$\mu_I(s) = \int_0^\infty dq q^{s-1} \ln(1 - e^{-q^2}) = -\frac{\Gamma(s/2)\zeta(1+s/2)}{2}. \quad (5.13)$$

Substituting this expression for $\mu_I(s)$ into (5.12), we obtain

$$M_I(s) = \frac{\Gamma(s/2)\zeta(1+s/2)}{4 \cos(\pi s/2)} = -\frac{(2\pi)^{1+s/2} \zeta(-s/2)}{4s \sin(\pi s/4) \cos(\pi s/2)}. \quad (5.14)$$

Both expressions are valid for $0 < \operatorname{Re} s < 1$. They are related through the reflection formula for the Riemann zeta function. The inverse Mellin transform reads

$$I(p) = \int \frac{ds}{2\pi i} p^{-s} M_I(s), \quad (5.15)$$

where the integral runs along a vertical contour in the strip $0 < \operatorname{Re} s < 1$.

The moments of H can be determined by means of the power-series expansion of $I(p)$, which is obtained by inserting the second expression of (5.14) into (5.15), and summing the contributions of all the poles of the integrand to the left of the contour, namely a double pole at $s = 0$, and simple poles at $s = -2$, $s = -4n$ for $n = 1, 2, \dots$, and $s = -(2n + 1)$ for $n = 0, 1, \dots$. We thus obtain

$$I(p) = -\ln p + I_{\text{even}}(p) + I_{\text{odd}}(p), \quad (5.16)$$

with

$$I_{\text{even}}(p) = -\frac{p^2}{4} - \sum_{n \geq 1} \frac{B_{2n}}{4n(2n)!} p^{4n} = -\frac{1}{2} \ln \frac{e^{p^2} - 1}{p^2}, \quad (5.17)$$

where the B_{2n} denote the Bernoulli numbers, and

$$I_{\text{odd}}(p) = \sum_{n \geq 0} \frac{(-1)^{n+1} \zeta(n+1/2)}{(2n+1) \sin((2n+1)\pi/4) (2\pi)^{n+1/2}} p^{2n+1}. \quad (5.18)$$

In terms of the cumulants c_k of E (see (4.2), (4.4)), the above results translate to

$$\begin{aligned} c_2 &= \frac{1}{2}, & c_{2m} &= \frac{(2m-1)!}{m!} B_m \quad (m \text{ even}), & c_{2m} &= 0 \quad (m \text{ odd} \neq 1), \\ c_{2m+1} &= \frac{(-1)^{m+1} (2m)! \zeta(m+1/2)}{\sin((2m+1)\pi/4) (2\pi)^{m+1/2}}. \end{aligned} \quad (5.19)$$

In particular, the extrapolation length reads

$$\ell = -\frac{\zeta(1/2)}{\sqrt{\pi}} \approx 0.823916. \quad (5.20)$$

Inserting (5.19) into (4.9), we obtain the following expressions for the moments of H :

$$\begin{aligned} \langle H \rangle &= 1, \\ \langle H^2 \rangle &= -\frac{2\zeta(1/2)}{\sqrt{\pi}} \approx 1.647833, \\ \langle H^3 \rangle &= \frac{3}{2\pi} (2\zeta(1/2)^2 + \pi) \approx 3.536516, \\ \langle H^4 \rangle &= \frac{2}{\sqrt{\pi^3}} (2\zeta(3/2) - 2\zeta(1/2)^3 - 3\pi\zeta(1/2)) \approx 9.057323, \\ \langle H^5 \rangle &= \frac{5}{4\pi^2} (-16\zeta(1/2)\zeta(3/2) + 4\zeta(1/2)^4 + 12\pi\zeta(1/2)^2 + 5\pi^2) \approx 26.467489, \\ \langle H^6 \rangle &= \frac{3}{\sqrt{4\pi^5}} (24\zeta(5/2) + 40\zeta(1/2)^2\zeta(3/2) + 20\pi\zeta(3/2) - 4\zeta(1/2)^5 \\ &\quad - 20\pi\zeta(1/2)^3 - 25\pi^2\zeta(1/2)) \approx 85.897890, \end{aligned} \quad (5.21)$$

and so forth. Equivalent expressions up to the fourth moment can be found in [19].

The distribution $f(x)$ has a power-series expansion in x , which is obtained through the asymptotic expansion of $I(p)$ as $p \rightarrow \infty$. The latter is derived by inserting the first expression of (5.14) into (5.15), and summing (minus) the contributions of the poles of the integrand at $s = 2m + 1$ for $m = 0, 1, \dots$. This reads

$$I(p) = \sum_{m \geq 0} \frac{(-1)^m a_m}{p^{2m+1}}, \quad a_m = \frac{\Gamma(m+1/2) \zeta(m+3/2)}{2\pi}. \quad (5.22)$$

Inserting this expansion into (2.66), we obtain

$$\hat{f}(p) = \frac{\omega}{p} - \frac{\omega^2}{2p^2} + \dots, \quad (5.23)$$

with

$$\omega = \frac{\zeta(3/2)}{2\sqrt{\pi}} \approx 0.736937, \quad (5.24)$$

in agreement with (5.7). The distribution of H therefore reads

$$f(x) = \omega - \frac{\omega^2}{2}x + \dots \quad (x \rightarrow 0). \quad (5.25)$$

A few more terms will be given in (5.46) for an arbitrary index α .

Pushing the expansion (5.25) to all orders yields a convergent series representation of the distribution $f(x)$. The numerical evaluation of 60 terms suffices to reach an accuracy of 10^{-10} for $x = 5$, much better than the accuracy of standard numerical inverse Laplace techniques. Figure 6 shows the distribution $f(x)$ thus obtained. The black dashed curve shows the Gaussian step distribution (5.8), which provides an asymptotic equivalent to the tail of $f(x)$ at large x (see (3.4)).

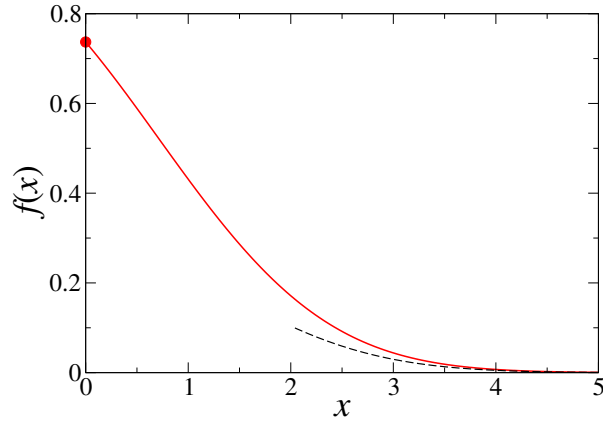


Fig. 6 Distribution $f(x)$ of the first positive position for the Gaussian step distribution (5.8), obtained by means of the numerical evaluation of 60 terms of the power-series representation (5.25). Black dashed curve: Gaussian step distribution (5.8). Symbol: $f(0) = \omega$ (see (5.24)).

5.3 Cauchy flight

The stable distribution (5.3) with $\alpha = 1$ is the Cauchy distribution

$$\rho(x) = \frac{1}{\pi(1+x^2)}. \quad (5.26)$$

In the notation of (3.10), we have $\theta = 1$ and $c = 1/\pi$. The Fourier transform of (5.26) is (see (5.2))

$$\tilde{\rho}(q) = e^{-|q|}, \quad (5.27)$$

so that the expression (2.68) of the integral $I(p)$ in (2.66) reads

$$I(p) = \frac{1}{\pi} \int_0^\infty \frac{dq}{e^q - 1} \arctan \frac{q}{p}. \quad (5.28)$$

This integral appears in the second of Binet's expressions for the logarithm of Euler's gamma function [50, 51],

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + 2 \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \arctan \frac{t}{z}. \quad (5.29)$$

Setting $q = 2\pi t$ and $p = 2\pi z$, we obtain

$$I(p) = \ln \Gamma\left(\frac{p}{2\pi}\right) - \frac{p}{2\pi} \left(\ln \frac{p}{2\pi} - 1\right) + \frac{1}{2} \ln p - \ln(2\pi). \quad (5.30)$$

Inserting this expression into (2.66), we obtain the following remarkable closed-form expression

$$\hat{f}(p) = 1 - \frac{\sqrt{p}}{\Gamma\left(\frac{p}{2\pi} + 1\right)} \left(\frac{p}{2\pi e}\right)^{p/(2\pi)}. \quad (5.31)$$

The reflection formula for the Gamma function ensures that this expression satisfies the factorisation identity (3.1).

The asymptotic behaviour of the distribution $f(x)$ at large x is obtained by expanding (5.31) for $p \rightarrow 0$ as

$$\hat{f}(p) = 1 - \sqrt{p} + \frac{\sqrt{p^3}}{2\pi} \left(\ln \frac{2\pi}{p} + 1 - \gamma\right) + \dots, \quad (5.32)$$

where γ denotes Euler's constant, which implies the following expansion:

$$f(x) = \frac{1}{2\sqrt{\pi x^3}} + \frac{3 \ln(8\pi x) - 5}{8\sqrt{\pi^3 x^5}} + \dots \quad (5.33)$$

The leading power-law decay with exponent $3/2$ agrees with the general results (3.16) and (3.18) with $c = 1/\pi$ and $R(1) = 1/2$. Higher-order terms involve polynomials in $\ln x$ with increasing degrees.

A systematic power-series expansion of the distribution $f(x)$ can be obtained by expanding $\hat{f}(p)$ for $p \rightarrow \infty$. The integral $I(p)$, defined in (5.28), has the following asymptotic expansion to all orders:

$$I(p) = \sum_{m \geq 0} \frac{B_{2m+2}}{2(m+1)(2m+1)} \left(\frac{2\pi}{p}\right)^{2m+1}, \quad (5.34)$$

where B_n are again the Bernoulli numbers. This expansion is derived by expressing the arctan function in (5.28) as a power series and integrating term by term. Inserting (5.34) into (2.66), we obtain

$$\hat{f}(p) = \frac{\pi}{6p} - \frac{\pi^2}{72p^2} - \frac{139\pi^3}{6480p^3} + \frac{571\pi^4}{155520p^4} + \dots, \quad (5.35)$$

hence

$$f(x) = \frac{\pi}{6} - \frac{\pi^2 x}{72} - \frac{139\pi^3 x^2}{12960} + \frac{571\pi^4 x^3}{933120} + \dots \quad (5.36)$$

A few more terms will be given in (5.46) for an arbitrary index α . The value

$$\omega = f(0) = \frac{\pi}{6} \approx 0.523598 \quad (5.37)$$

agrees with (5.7).

The expression (5.31) is also well-suited for a numerical Laplace inversion, enabling thus an accurate calculation of the distribution $f(x)$ (see figure 7).

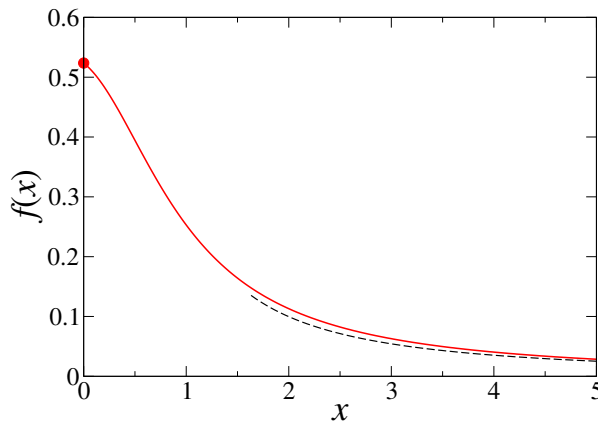


Fig. 7 Distribution $f(x)$ of the first positive position for the Cauchy step distribution (5.26), obtained by means of a numerical Laplace inversion of (5.31). Black dashed curve: leading power-law decay given in (5.33). Symbol: $f(0) = \omega$ (see (5.37)).

5.4 Lévy flights

In this section, we examine the stable distributions (5.3) with an arbitrary index $0 < \alpha < 2$. All these distributions result in superdiffusive Lévy flights. The forthcoming analysis closely follows that of the Gaussian distribution presented in section 5.2. Many details will therefore be omitted.

In the notation of (3.10), we have $\theta = \alpha$, whereas c is given by (5.5). Equation (2.66) reads $\hat{f}(p) = 1 - \exp(-I(p))$, with

$$I(p) = -\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln(1 - e^{-q^\alpha}). \quad (5.38)$$

In close analogy with (5.14), the Mellin transform of $I(p)$ reads

$$M_I(s) = \frac{\Gamma(s/\alpha) \zeta(1 + s/\alpha)}{2\alpha \cos(\pi s/2)} = -\frac{(2\pi)^{1+s/\alpha} \zeta(-s/\alpha)}{4s \sin(\pi s/(2\alpha)) \cos(\pi s/2)}. \quad (5.39)$$

Both expressions are valid for $0 < \operatorname{Re} s < 1$ and have a meromorphic continuation.

The asymptotic form of the distribution $f(x)$ at large x can be obtained by expanding $I(p)$ at small p . This task is carried out systematically by substituting the second expression of (5.39) into (5.15), and summing the contributions of all the poles of the integrand to the left of the contour. Assuming temporarily that the index α is an irrational number, $M_I(s)$ has a double pole at $s = 0$, and simple ones at $s = -\alpha$, at $s = -(2n + 1)$ for $n = 0, 1, \dots$ and at $s = -2n\alpha$ for $n = 1, 2, \dots$. We thus obtain

$$I(p) = -\frac{\alpha}{2} \ln p + \frac{p^\alpha}{4 \cos(\pi\alpha/2)} + I_1(p) + I_2(p), \quad (5.40)$$

with

$$\begin{aligned} I_1(p) &= \sum_{n \geq 0} \frac{(-1)^{n+1} \zeta((2n+1)/\alpha)}{(2n+1) (2\pi)^{(2n+1)/\alpha} \sin((2n+1)\pi/(2\alpha))} p^{2n+1}, \\ I_2(p) &= -\sum_{n \geq 1} \frac{B_{2n}}{4n(2n)!} \frac{p^{2n\alpha}}{\cos(n\pi\alpha)}. \end{aligned} \quad (5.41)$$

Equation (2.66) therefore reads

$$\hat{f}(p) = 1 - p^{\alpha/2} \exp\left(-\frac{p^\alpha}{4 \cos(\pi\alpha/2)} - I_1(p) - I_2(p)\right). \quad (5.42)$$

The leading singular term in $p^{\alpha/2}$ yields

$$f(x) \approx \frac{a}{x^{1+\alpha/2}}, \quad a = \frac{\alpha}{2\Gamma(1-\alpha/2)}, \quad (5.43)$$

in agreement with the general result (3.18), where c is given by (5.5). Expanding the exponential in the right-hand side of (5.42) gives rise to terms proportional to $p^{m+n\alpha}$ for all $m, n = 0, 1, \dots$. Taking the first two correction terms, namely $(m, n) = (1, 0)$ and $(0, 1)$, into account yields

$$f(x) = \frac{\alpha}{2\Gamma(1-\alpha/2)} x^{-1-\alpha/2}$$

$$\begin{aligned}
& + \frac{1}{4 \cos(\pi\alpha/2) \Gamma(-3\alpha/2)} x^{-1-3\alpha/2} \\
& - \frac{\zeta(1/\alpha)}{(2\pi)^{1/\alpha} \Gamma(-1-\alpha/2) \sin(\pi/(2\alpha))} x^{-2-\alpha/2} + \dots
\end{aligned} \tag{5.44}$$

The last two lines coalesce for $\alpha = 1$, i.e., for the Cauchy distribution studied in section 5.3. This degeneracy induces the logarithmic correction term entering (5.33). Higher-order coalescences affect higher-order correction terms whenever the index α is a rational number.

The distribution $f(x)$ has a power-series expansion in x , that can be obtained by means of the asymptotic expansion of $I(p)$ as $p \rightarrow \infty$. The latter can be derived by inserting the first expression of (5.39) into (5.15), and summing (minus) the contributions of the poles of the integrand at $s = 2m + 1$ for $m = 0, 1, \dots$. This reads

$$I(p) = \sum_{m \geq 0} \frac{(-1)^m a_m}{p^{2m+1}}, \quad a_m = \frac{\Gamma((2m+1)/\alpha) \zeta(1 + (2m+1)/\alpha)}{\pi\alpha}. \tag{5.45}$$

The expansions (5.34) and (5.22) are respectively recovered for $\alpha = 1$ and $\alpha = 2$. Inserting (5.45) into (2.66), and performing the inverse Laplace transform term by term, we obtain

$$\begin{aligned}
f(x) = & a_0 - \frac{a_0^2}{2} x + \left(\frac{a_0^3}{12} - \frac{a_1}{2} \right) x^2 + \left(\frac{a_0 a_1}{6} - \frac{a_0^4}{144} \right) x^3 \\
& + \left(\frac{a_2}{24} - \frac{a_0^2 a_1}{48} + \frac{a_0^5}{2880} \right) x^4 + \left(\frac{a_0^3 a_1}{720} - \frac{a_0 a_2}{120} - \frac{a_1^2}{240} - \frac{a_0^6}{86400} \right) x^5 + \dots
\end{aligned} \tag{5.46}$$

The value of $\omega = a_0$ agrees with (5.7), whereas the expansions (5.36) and (5.25) are respectively recovered for $\alpha = 1$ and $\alpha = 2$.

To conclude, we note that extending the expansion (5.46) to all orders results in a convergent series representation of the distribution $f(x)$ for $1 < \alpha \leq 2$.

6 A complementary analytical approach

6.1 The setting

This section examines a complementary analytical approach to the distribution of the pair (N, H) and, more specifically, the distribution $f(x)$ of the first positive position H reached by a walker starting from the origin. This approach deals with the class of step distributions $\rho(x)$ such that the bilateral Laplace transform $\hat{\rho}(p)$ (see (2.40)) is a rational function of p . The observation that Wiener-Hopf linear integral equations, such as (1.2) or (2.64), are more easily solvable for such distributions dates back at least to the works of Wick [38] and Chandrasekhar [39] (see [52] for a review). For this class of step distributions, most of the general results presented earlier will be rederived in a self-contained manner. These results will be recast into a form which strongly suggests their validity for arbitrary step distributions.

The step distributions considered here are finite superpositions of symmetric exponentials, of the form

$$\rho(x) = \frac{1}{2} \sum_a w_a p_a e^{-p_a |x|}, \quad (6.1)$$

where the index a runs over $a = 1, \dots, M$. The decay rates are assumed to be distinct and ordered as $0 < p_1 < \dots < p_M$. The normalisation of $\rho(x)$ imposes the sum rule

$$\sum_a w_a = 1. \quad (6.2)$$

In the case where all the weights w_a are positive, the distribution (6.1) is a mixture, i.e., a convex combination, of symmetric exponential (or Laplace) distributions. In general, the weights w_a are real, albeit not necessarily positive. They only satisfy the constraint that the density (6.1) remains positive for all values of x . This point will be illustrated in detail for the case $M = 2$ in section 6.9. As stated above, the key property of the step distribution (6.1) is that its bilateral Laplace transform is a rational function of p :

$$\hat{\rho}(p) = \int_{-\infty}^{\infty} dx e^{-px} \rho(x) = \sum_a \frac{w_a p_a^2}{p_a^2 - p^2}. \quad (6.3)$$

This expression is valid for $|\operatorname{Re} p| < p_1$, where p_1 is the smallest decay rate appearing in (6.1). We introduce for further reference the quantity

$$\phi(s, p) = 1 - s \hat{\rho}(p), \quad (6.4)$$

for s complex with $|s| < 1$. The expression (6.4) is an even rational function of p going to unity as $|p| \rightarrow \infty$. It can therefore be factorised over its $2M$ poles $\pm p_a$ and its $2M$ zeros $\pm z_b$ as

$$\phi(s, p) = \frac{\prod_b (p^2 - z_b^2)}{\prod_a (p^2 - p_a^2)}. \quad (6.5)$$

The M poles p_a are real and positive, and independent of s , as they coincide with the decay rates appearing in (6.1). The zeros $\pm z_b$ depend on s and are therefore not real in general. It can be checked that no zero can sit on the imaginary axis. One has indeed $|1 - \phi(s, iq)| = |s| |\tilde{\rho}(q)| < 1$, since each factor is less than unity in modulus. We denote by z_b the M zeros of $\phi(s, p)$ with positive real parts.

6.2 Solutions of integral equations and factorisation formulas

For the step distributions of the form (6.1), the solutions to the key equations (2.64) and (2.65) can be derived by elementary means. It is shown in full detail in Appendix A.4 and Appendix A.5 that the Laplace transforms

$$\hat{g}(s, p) = \int_0^{\infty} dx e^{-px} g(s, x), \quad \hat{f}(s, p) = \int_0^{\infty} dx e^{-px} f(s, x) \quad (6.6)$$

of the solutions to (2.64) and (2.65) are respectively given by the product formula

$$\hat{g}(s, p) = \frac{\prod_a (p + p_a)}{\prod_b (p + z_b)} \quad (6.7)$$

and by

$$\hat{f}(s, p) = 1 - \frac{1}{\hat{g}(s, p)}. \quad (6.8)$$

The result (1.12) (or (2.50)) is recovered for $s = 1$.

The expressions (6.7) and (6.8) imply that the quantities $\hat{g}(s, p)$ and $\hat{f}(s, p)$ satisfy the factorisation formulas

$$\hat{g}(s, p)\hat{g}(s, -p) = \frac{1}{\phi(s, p)} \quad (6.9)$$

and

$$(1 - \hat{f}(s, p))(1 - \hat{f}(s, -p)) = \phi(s, p), \quad (6.10)$$

that are central to the Wiener-Hopf approach. Equation (6.10) is the equivalent of the Wiener-Hopf factorisation identity (2.70) in Laplace space. Setting $s = 1$ in (6.10), we recover (3.2).

6.3 Connection with the Pollaczek-Spitzer formula

The formula (6.7) obtained for $\hat{g}(s, p)$ can be rewritten as

$$\hat{g}(s, p) = \exp\left(\sum_a \ln(p + p_a) - \sum_b \ln(p + z_b)\right). \quad (6.11)$$

The product formula (6.5) for $\phi(s, p)$ implies that its logarithmic derivative has the partial fraction expansion

$$\frac{\phi'(s, p)}{\phi(s, p)} = \sum_b \left(\frac{1}{p + z_b} + \frac{1}{p - z_b}\right) - \sum_a \left(\frac{1}{p + p_a} + \frac{1}{p - p_a}\right), \quad (6.12)$$

with simple poles with residue $+1$ at all zeros $\pm z_b$ of $\phi(s, p)$, and simple poles with residue -1 at all its poles $\pm p_a$. We have therefore

$$\hat{g}(s, p) = \exp\left(\int \frac{dr}{2\pi i} \frac{\phi'(s, r)}{\phi(s, r)} \ln(p + r)\right). \quad (6.13)$$

Equation (6.13) holds (at least) for $\text{Re } p > 0$ and $0 < \text{Re } r < p_1$, where p_1 is the smallest decay rate in (6.1). Using (6.4) and integration by parts, this equation becomes

$$\begin{aligned} \hat{g}(s, p) &= \exp\left(-\int \frac{dr}{2\pi i} \frac{\ln(1 - s\hat{\rho}(r))}{p + r}\right) \\ &= \exp\left(-\frac{p}{\pi} \int_0^\infty \frac{dq}{p^2 + q^2} \ln(1 - s\tilde{\rho}(q))\right). \end{aligned} \quad (6.14)$$

The second expression is obtained by moving the integration contour to the imaginary axis, setting $r = iq$, and using the fact that the Fourier transform (1.6), namely

$$\tilde{\rho}(q) = \hat{\rho}(iq), \quad (6.15)$$

is an even real function of q . The expression (6.14) identifies with the standard integral form of the Pollaczek-Spitzer formula [1, 5] (see also [2, 3]), which holds for any continuous symmetric step distribution $\rho(x)$, irrespective of its decay. For $s = 1$, (6.14) reproduces (1.5).

6.4 Connection with Sparre Andersen theory

Within the present formalism, the probabilities f_n and g_n introduced in section 2 can be evaluated by setting $p = 0$ in (6.8) and (6.9). Given that $\hat{\rho}(0) = 1$, it follows that $\phi(s, 0) = 1 - s$, which leads to

$$\hat{g}(s, 0) = \sum_{n \geq 0} g_n s^n = \frac{1}{\sqrt{1-s}}, \quad \hat{f}(s, 0) = \sum_{n \geq 1} f_n s^n = 1 - \sqrt{1-s}. \quad (6.16)$$

Hence

$$g_n = b_n, \quad f_n = b_{n-1} - b_n = \frac{b_n}{2n-1}, \quad (6.17)$$

where b_n is the binomial probability

$$b_n = \frac{(2n)!}{(2^n n!)^2} = \frac{\binom{2n}{n}}{2^{2n}}. \quad (6.18)$$

Thus

$$\begin{aligned} g_0 &= 1, & g_1 &= \frac{1}{2}, & g_2 &= \frac{3}{8}, & g_3 &= \frac{5}{16}, & g_4 &= \frac{35}{128}, \\ f_1 &= \frac{1}{2}, & f_2 &= \frac{1}{8}, & f_3 &= \frac{1}{16}, & f_4 &= \frac{5}{128}, \end{aligned} \quad (6.19)$$

and so on. At large times, we have

$$g_n \approx \frac{1}{\sqrt{\pi n}}, \quad f_n \approx \frac{1}{2\sqrt{\pi n^3}}. \quad (6.20)$$

These results are part of Sparre Andersen theory [53, 54]. They are universal, in the sense that they hold irrespective of the step distribution, provided it is symmetric and continuous, regardless of whether we are dealing with diffusive random walks (D finite) or Lévy flights (D infinite).

6.5 General results on the distribution of H

The general results concerning $f(x)$ that have been derived in section 2 can be recovered within the present formalism by taking the $s \rightarrow 1$ limit. First of all, setting $s = 1$ in (6.8), we recover (1.12) or (2.50), that is,

$$\hat{f}(p) = 1 - \frac{1}{\hat{g}(p)}, \quad (6.21)$$

where $\hat{f}(p) = \hat{f}(1, p)$, $\hat{g}(p) = \hat{g}(1, p)$. We have

$$\phi(p) = \phi(1, p) = 1 - \hat{\rho}(p), \quad (6.22)$$

and so the factorisation formulas (6.9) and (6.10) read

$$\hat{g}(p)\hat{g}(-p) = \frac{1}{1 - \hat{\rho}(p)}, \quad (6.23)$$

and

$$(1 - \hat{f}(p))(1 - \hat{f}(-p)) = 1 - \hat{\rho}(p), \quad (6.24)$$

which is (3.2).

The peculiar nature of the $s \rightarrow 1$ limit becomes more apparent in the case of the step distributions of the form (6.1). The difference $\phi(p) = 1 - \hat{\rho}(p)$ vanishes for $p = 0$. In view of (6.5), this implies that one of the zeros z_b , say z_M , goes to zero as $s \rightarrow 1$, whereas the other zeros approach fixed positions with positive real parts, still denoted by z_b , for $b = 1, \dots, M - 1$. The expression (6.5) thus becomes

$$\phi(p) = 1 - \hat{\rho}(p) = -p^2 \frac{\prod'_b (z_b^2 - p^2)}{\prod_a (p_a^2 - p^2)}. \quad (6.25)$$

Here and throughout the following, accents denote sums and products over the $M - 1$ remaining zeros z_b . The diffusion coefficient reads

$$D = - \lim_{p \rightarrow 0} \frac{\phi(p)}{p^2} = \frac{\prod'_b z_b^2}{\prod_a p_a^2}. \quad (6.26)$$

The expressions (6.7) and (6.8) respectively become

$$\hat{g}(p) = \frac{\prod_a (p + p_a)}{p \prod'_b (p + z_b)} \quad (6.27)$$

and

$$\hat{f}(p) = 1 - \frac{p \prod'_b (p + z_b)}{\prod_a (p + p_a)}. \quad (6.28)$$

These formulas show that the distribution $f(x)$ of the first positive position H has exactly the same spectrum of decay rates (p_a) as the step distribution $\rho(x)$ itself, whereas $g(x)$ is characterised by an entirely different spectrum of decay rates (z_b).

6.6 Connection with the solution of equation (1.2)

Within the present framework, the solution $G(x)$ of the homogeneous Wiener-Hopf equation (1.2), repeated here for convenience,

$$G(x) = \int_0^\infty dy G(y) \rho(x-y) \quad (x > 0), \quad (6.29)$$

with $G(0) = 1$, can be obtained along the lines of Appendix A.4. The Laplace transform $\hat{G}(p)$ of $G(x)$ reads

$$\hat{G}(p) = \frac{\prod_a (p + p_a)}{p^2 \prod_b' (p + z_b)} = \frac{\hat{g}(p)}{p}, \quad (6.30)$$

which implies

$$g_{\text{reg}}(x) = \frac{d}{dx} G(x) \quad (x > 0), \quad (6.31)$$

as it should (see (2.15)).

Expanding (6.30) for small p , we obtain

$$\hat{G}(p) = \frac{1}{\sqrt{D}} \left(\frac{1}{p^2} + \frac{\ell}{p} + \dots \right), \quad (6.32)$$

which implies the asymptotic behaviour

$$G(x) \approx \frac{x + \ell}{\sqrt{D}}, \quad (6.33)$$

in agreement with (1.10) and (2.56), and where the extrapolation length ℓ is given by

$$\ell = \sum_a \frac{1}{p_a} - \sum_b' \frac{1}{z_b}. \quad (6.34)$$

Using the identity (6.12) for $s = 1$, this expression can be recast as

$$\ell = \int \frac{dp}{2\pi i p} \frac{\phi'(p)}{\phi(p)} = -\frac{1}{\pi} \int_0^\infty \frac{dq}{q^2} \ln \frac{1 - \tilde{\rho}(q)}{Dq^2}, \quad (6.35)$$

thus recovering (1.9).

Another quantity of interest is $\omega = f(0) = g_{\text{reg}}(0) = G'(0)$, introduced in (2.20). It can be derived by expanding (6.28) as $p \rightarrow +\infty$, obtaining thus $\hat{f}(p) \approx \omega/p$, with

$$\omega = \sum_a p_a - \sum_b' z_b. \quad (6.36)$$

Along the lines of the derivation of (6.35), this can be recast as

$$\omega = \int \frac{dp}{2\pi i} \frac{p \phi'(p)}{\phi(p)} = -\frac{1}{\pi} \int_0^\infty dq \ln(1 - \tilde{\rho}(q)), \quad (6.37)$$

which is (2.21).

6.7 Moments of the first positive position

The following analysis of the moments of H and E parallels that exposed in section 4. Within the present setting, using (2.60), with

$$\langle H \rangle = \sqrt{D} = \frac{\prod'_b z_b}{\prod_a p_a}, \quad (6.38)$$

(see (6.26)), the expression (6.28) yields

$$\hat{f}_E(p) = \frac{\prod'_b (1 + p/z_b)}{\prod_a (1 + p/p_a)}. \quad (6.39)$$

The generating function $K(p)$ of the cumulants c_k of E , introduced in (4.2), therefore reads

$$K(p) = \ln \hat{f}_E(p) = \sum'_b \ln(1 + p/z_b) - \sum_a \ln(1 + p/p_a). \quad (6.40)$$

Expanding this result as a power series in p , we obtain the following expression for the cumulants:

$$c_k = (k-1)! \left(\sum_a \frac{1}{p_a^k} - \sum'_b \frac{1}{z_b^k} \right). \quad (6.41)$$

Along the lines of the derivation of (6.35), this can be recast as

$$c_k = (k-1)! \int \frac{dp}{2\pi i} \frac{\phi'(p)}{p^k \phi(p)}. \quad (6.42)$$

This expression provides an alternative starting point to derive (4.22).

In what follows we investigate several examples of step distributions of the form (6.1) where the distribution of H can be investigated in some more detail.

6.8 Symmetric exponential distribution

The symmetric exponential step distribution,

$$\rho(x) = \frac{e^{-|x|}}{2}, \quad (6.43)$$

also known as the Laplace distribution, corresponds to the case $M = 1$ in (6.1), with $p_1 = 1$, and hence $D = 1$. We have

$$\phi(p) = 1 - \hat{\rho}(p) = -\frac{p^2}{1-p^2}, \quad \hat{f}(p) = \frac{1}{p+1}, \quad \hat{g}(p) = \frac{p+1}{p}, \quad (6.44)$$

thus

$$f(x) = f_E(x) = e^{-x}, \quad g_{\text{reg}}(x) = 1. \quad (6.45)$$

The exponential distribution of H has been known for long [3, 6] (see also [22–24]). In this very simple case, we have $\mathbf{A} = \mathbf{B} = 1$, as well as $\langle H^k \rangle = \langle E^k \rangle = k!$ and $c_k = (k-1)!$.

The symmetric exponential step distribution is the single example where n -dependent quantities will be worked out explicitly. We have

$$\phi(s, p) = \frac{1 - s - p^2}{1 - p^2}, \quad (6.46)$$

hence $z_1 = \sqrt{1-s}$. The expressions (6.7) and (6.8) read

$$\hat{g}(s, p) = \frac{p+1}{p+\sqrt{1-s}}, \quad \hat{f}(s, p) = \frac{1-\sqrt{1-s}}{p+1}, \quad (6.47)$$

so

$$g_{\text{reg}}(s, x) = (1 - \sqrt{1-s})e^{-\sqrt{1-s}x}, \quad (6.48)$$

$$f(s, x) = (1 - \sqrt{1-s})e^{-x}. \quad (6.49)$$

The expression (6.49) implies that the density $f_n(x)$ assumes a factorised form:

$$f_n(x) = f_n e^{-x}. \quad (6.50)$$

In other words, the random variables N and H are statistically independent. The first-passage probability f_n is given by (6.17), whereas the distribution of H is exponential (see (6.45)).

The expression (6.48) implies that the density $g_n(x)$ is of the form

$$g_n(x) = P_n(x) e^{-x} \quad (n \geq 1), \quad (6.51)$$

where $P_n(x)$ is a polynomial of degree $n-1$. We have

$$\begin{aligned} P_1(x) &= \frac{1}{2}, & P_2(x) &= \frac{1}{8}(2x+1), & P_3(x) &= \frac{1}{16}(x+1)^2, \\ P_4(x) &= \frac{1}{384}(4x^3 + 18x^2 + 30x + 15), \end{aligned} \quad (6.52)$$

and so forth. Omitting details, simple algebra yields the general form

$$P_n(x) = \frac{1}{2^{2n-1}} \sum_{m=0}^{n-1} T_{n-1, n-1-m} \frac{(2x)^m}{m!}, \quad (6.53)$$

where the integers

$$T_{m,k} = \frac{m+1-k}{m+1} \binom{m+k}{m} \quad (k = 0, \dots, m) \quad (6.54)$$

are known as the ballot numbers, and listed as sequence number A009766 in the OEIS [55]. We have in particular

$$P_n(0) = \frac{T_{n-1, n-1}}{2^{2n-1}} = f_n, \quad (6.55)$$

in agreement with (2.19) and (6.50). Finally, setting $s = 1$ in (6.48) yields

$$g_{\text{reg}}(x) = \sum_{n \geq 1} g_n(x) = 1. \quad (6.56)$$

This identity implies the sum rule

$$\sum_{n \geq 1} P_n(x) = e^x. \quad (6.57)$$

6.9 Double symmetric exponential distribution

The next example of distributions of the form (6.1) corresponds to $M = 2$. In full generality, this double symmetric exponential step distribution reads

$$\rho(x) = \frac{1}{2}(w_1 p_1 e^{-p_1|x|} + w_2 p_2 e^{-p_2|x|}) \quad (6.58)$$

and involves three parameters. The decay rates are assumed to be distinct and ordered as $0 < p_1 < p_2$. We have

$$\phi(p) = 1 - \hat{\rho}(p) = -\frac{p^2(z^2 - p^2)}{(p_1^2 - p^2)(p_2^2 - p^2)}, \quad (6.59)$$

where the zero z is real and given by

$$z = \sqrt{w_1 p_2^2 + w_2 p_1^2}. \quad (6.60)$$

It is advantageous to use z as the third independent parameter, besides p_1 and p_2 . In terms of z , we have

$$w_1 = \frac{z^2 - p_1^2}{p_2^2 - p_1^2}, \quad w_2 = \frac{p_2^2 - z^2}{p_2^2 - p_1^2}, \quad \sqrt{D} = \frac{z}{p_1 p_2}. \quad (6.61)$$

Figure 8 shows the domain over which z can be varied. For $p_1 < z < p_2$ (red region), both weights w_1 and w_2 are positive. For $p_2 < z < z_{\text{max}}$ (blue region), with

$$z_{\text{max}} = \sqrt{p_1^2 + p_1 p_2 + p_2^2}, \quad (6.62)$$

the step distribution is still positive, even though w_2 is negative. For $z = z_{\max}$, the step distribution reads

$$\rho(x) = \frac{p_1 p_2}{2(p_2 - p_1)} (e^{-p_1|x|} - e^{-p_2|x|}) \quad (6.63)$$

and vanishes for $x = 0$.

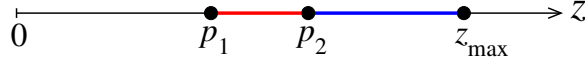


Fig. 8 Domain of allowed values of the real zero z . Red region ($p_1 < z < p_2$): both w_1 and w_2 are positive. Blue region ($p_2 < z < z_{\max}$): $\rho(x)$ is positive, even though w_2 is negative. At $z = z_{\max}$ (see (6.62)), $\rho(0)$ vanishes.

The general result (6.28) reads

$$\hat{f}(p) = 1 - \frac{p(p+z)}{(p+p_1)(p+p_2)}, \quad (6.64)$$

yielding the following explicit expression for the distribution of H :

$$f(x) = \frac{p_1(z-p_1)e^{-p_1x} - p_2(z-p_2)e^{-p_2x}}{p_2 - p_1}. \quad (6.65)$$

Figures 9 and 10 show the distributions $\rho(x)$ and $f(x)$ for $p_1 = 1$, $p_2 = 3/2$ and several z (see legend), including $z_{\max} = \sqrt{19}/2 \approx 2.179449$, for which $\rho(0) = 0$. The expressions (6.58) and (6.65) become independent of z for $p_1 e^{-p_1x} = p_2 e^{-p_2x}$, i.e.,

$$x_* = \frac{\ln(p_2/p_1)}{p_2 - p_1}, \quad f(x_*) = 2\rho(x_*) = \left(\frac{p_1^{p_2}}{p_2^{p_1}}\right)^{1/(p_2-p_1)}. \quad (6.66)$$

We have

$$\ell = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{z}, \quad \omega = p_1 + p_2 - z, \quad (6.67)$$

hence (see (2.22))

$$\mathbf{A} = 1 + \frac{(z-p_1)(p_2-z)}{z^2}, \quad \mathbf{B} = 1 + \frac{(z-p_1)(p_2-z)}{p_1 p_2}. \quad (6.68)$$

Both quantities equal unity for either $z = p_1$ or $z = p_2$, where (6.58) and (6.65) reduce to single exponentials. They are larger than unity in the red region ($p_1 < z < p_2$) and smaller than unity in the blue region ($p_2 < z < z_{\max}$).

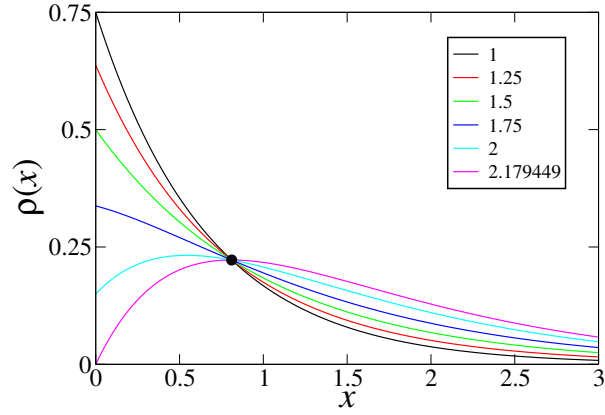


Fig. 9 Positive part of the step distribution $\rho(x)$, as given by (6.58), for $p_1 = 1$, $p_2 = 3/2$ and several z (see legend), including $z_{\max} = \sqrt{19}/2 \approx 2.179449$, for which $\rho(0) = 0$. Black symbol: point $x_* = 2\ln(3/2) \approx 0.810930$ where all curves cross at $\rho(x_*) = 2/9$ (see (6.66)).

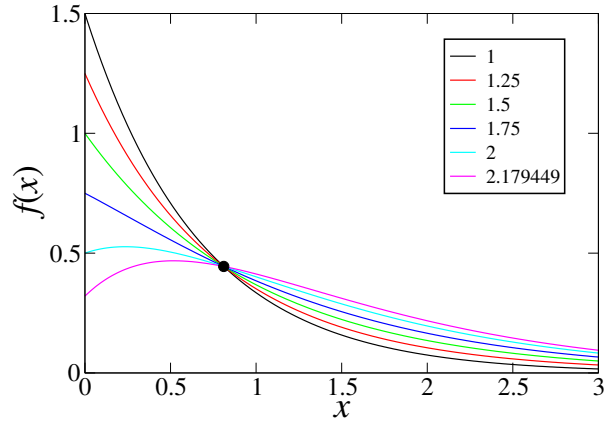


Fig. 10 Distribution $f(x)$ of the first positive position, as given by (6.65), for the same parameter values as figure 9, yielding $f(x_*) = 4/9$.

The dimensionless quantity \mathbf{A} reaches its minimum,

$$\mathbf{A} = \frac{2\sqrt{3} - 1}{3} \approx 0.821367, \quad (6.69)$$

in the limit $p_2 \rightarrow p_1$ and for $z = z_{\max} = p_1\sqrt{3}$. In this situation, setting $p_1 = 1$, the step distribution becomes the symmetric linear-times-exponential distribution

$$\rho(x) = \frac{|x|e^{-|x|}}{2}, \quad (6.70)$$

for which (6.65) becomes

$$f(x) = \left(2 - \sqrt{3} + (\sqrt{3} - 1)x\right) e^{-x}, \quad (6.71)$$

in agreement with [22–24].

To close, we mention that the formula (6.27) leads to

$$\hat{g}(p) = \frac{(p + p_1)(p + p_2)}{p(p + z)}, \quad (6.72)$$

so that

$$g_{\text{reg}}(x) = \frac{p_1 p_2}{z} + \frac{(z - p_1)(p_2 - z)}{z} e^{-zx}, \quad (6.73)$$

whereas the distribution of the equilibrium excess length E is given by (see (2.60))

$$\hat{f}_E(p) = \frac{p_1 p_2}{z} \frac{p + z}{(p + p_1)(p + p_2)}, \quad (6.74)$$

hence

$$f_E(x) = \frac{p_1 p_2}{z} \frac{(z - p_1)e^{-p_1 x} - (z - p_2)e^{-p_2 x}}{p_2 - p_1}. \quad (6.75)$$

In the above formulas, the prefactor $p_1 p_2 / z$ equals $1/\sqrt{D}$, as should be. One can also verify from (6.75) that $\langle E \rangle = \ell$, consistent with (6.67).

6.10 Symmetric Erlang distributions

Consider the family of symmetric Erlang distributions

$$\rho(x) = \frac{|x|^{M-1} e^{-|x|}}{2(M-1)!}, \quad (6.76)$$

parametrised by the integer $M \geq 1$ ⁹. The symmetric exponential distribution (6.43) is recovered for $M = 1$, and the linear-times-exponential distribution (6.70) for $M = 2$. The $M \rightarrow \infty$ limit is singular, as the limiting step density consists of two symmetric delta peaks at ± 1 in the reduced variable x/M .

The distribution (6.76) represents a peculiar, maximally degenerate form of (6.1), obtained by letting the M decay rates p_a simultaneously approach unity, while the corresponding weights w_a diverge appropriately, thereby generalising the construction of the distribution (6.70) for $M = 2$. We have

$$\phi(p) = 1 - \frac{1}{2} \left(\frac{1}{(1+p)^M} + \frac{1}{(1-p)^M} \right), \quad (6.77)$$

and so

$$D = \frac{M(M+1)}{2}. \quad (6.78)$$

⁹For usual (one-sided) Erlang distributions, the parameter M is called the shape parameter.

Expanding (6.77), we find that the $M - 1$ zeros z_b of $\phi(p)$ and their opposites satisfy a polynomial equation of degree $2M - 2$, namely

$$\sum_{k=1}^M \left((-1)^k \binom{M}{k} - \binom{M}{2k} \right) p^{2k-2} = 0. \quad (6.79)$$

Figure 11 shows a plot of these zeros in the complex p -plane for $M = 10$. Even for the modest value $M = 10$, the zeros sit near the circles with unit radii centered at $p = \pm 1$. More details will be given below. These zeros have appeared in several earlier works involving the family of symmetric Erlang distributions (6.76). They are studied in detail in [56], which focuses on the thermodynamics of random-field Ising chains. Additionally, a recent work [57] explores the statistics of gaps in random walks with the same family of step distributions.

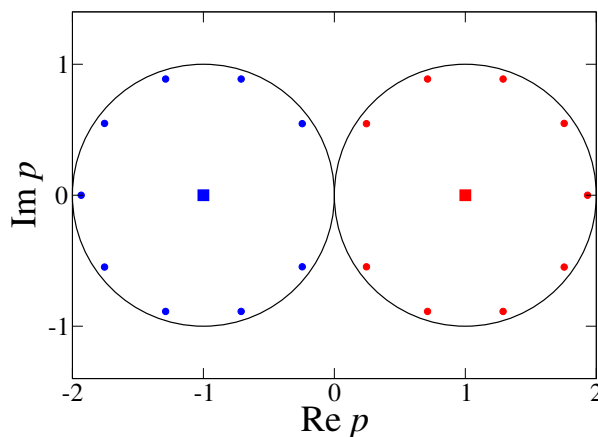


Fig. 11 The 9 zeros z_b (red) and their opposites (blue) in the complex p -plane for $M = 10$. The circles have unit radii and are centered at $p = \pm 1$, with $p = 1$ (red square) being the common value of the decay rates, and $p = -1$ (blue square) its opposite.

The distribution of the first positive position H reads, using (6.28),

$$f(x) = \int \frac{dp}{2\pi i} e^{px} \left(1 - \frac{p}{(p+1)^M} \prod_b' (p + z_b) \right). \quad (6.80)$$

Setting $p = \varepsilon - 1$, this becomes

$$f(x) = e^{-x} \int \frac{d\varepsilon}{2\pi i} e^{\varepsilon x} \left(1 + \frac{1-\varepsilon}{\varepsilon^M} \prod_b' (\varepsilon + z_b - 1) \right). \quad (6.81)$$

The integrand has a multiple pole of order M at $\varepsilon = 0$. Evaluating the corresponding residue yields

$$f(x) = e^{-x} \sum_{k=0}^{M-1} (S_k - S_{k+1}) \frac{x^k}{k!}. \quad (6.82)$$

This expression for $f(x)$ has the same structure as the step density (6.76). It is the product of the decaying exponential e^{-x} by a polynomial in x of degree $M - 1$. The coefficients of the latter polynomial involve the elementary symmetric polynomials S_k of the variables $z_b - 1$. These are defined by expanding the product

$$\prod'_b (\varepsilon + z_b - 1) = \sum_{k=0}^{M-1} S_k \varepsilon^{M-1-k}. \quad (6.83)$$

We have

$$S_0 = 1, \quad S_1 = \sum'_b (z_b - 1), \quad \dots, \quad S_{M-1} = \prod'_b (z_b - 1), \quad S_M = 0. \quad (6.84)$$

Setting $x = 0$ in (6.82), we obtain

$$f(0) = S_0 - S_1 = M - \sum'_b z_b, \quad (6.85)$$

in agreement with the expression (6.36) of ω . The asymptotic decay of $f(x)$ follows the proportionality predicted in (3.8), with

$$K = 2S_{M-1} = 2 \prod'_b (z_b - 1). \quad (6.86)$$

The distribution $f_E(x)$ of the recurrence length E assumes a form similar to the expression (6.82) of $f(x)$. The formula (2.54) yields

$$f_E(x) = \frac{e^{-x}}{\sqrt{D}} \sum_{k=0}^{M-1} S_k \frac{x^k}{k!}, \quad (6.87)$$

where D is given by (6.78).

For the first three values of the integer M , the distribution of H can be made explicit:

1. For $M = 1$, the symmetric exponential distribution (6.43) is recovered. The set of zeros is empty, so that (6.82) reduces to (6.45).
2. For $M = 2$, the symmetric linear-times-exponential distribution (6.70) is recovered. There is a real zero at $z_1 = \sqrt{3}$, and so $S_1 = \sqrt{3} - 1$, so that (6.82) becomes (6.71).

3. For $M = 3$, there is a pair of conjugate complex zeros, $z_1 = a + ib$ and $z_2 = a - ib$, with

$$a = \frac{\sqrt{2\sqrt{6} + 3}}{2} \approx 1.405256, \quad b = \frac{\sqrt{2\sqrt{6} - 3}}{2} \approx 0.689017. \quad (6.88)$$

The expression (6.82) then reads

$$f(x) = \left(3 - 2a + (4a - \sqrt{6} - 3)x + (\sqrt{6} + 1 - 2a)\frac{x^2}{2} \right) e^{-x}. \quad (6.89)$$

Figure 12 shows the distribution of H for the first six values of M . The product $Mf(x)$ is plotted against the ratio x/M . This rescaling ensures that the plotted curves have a mild dependence on M .

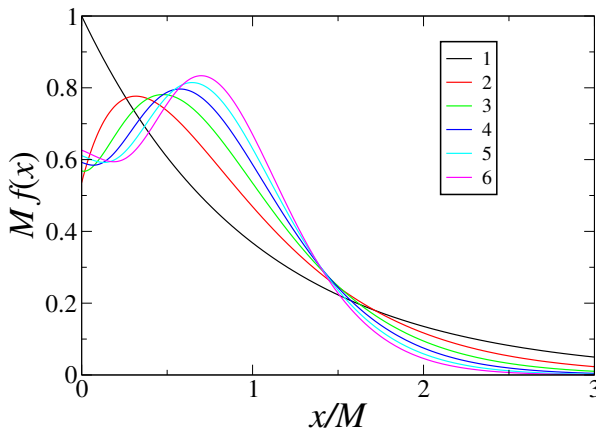


Fig. 12 Distribution of H for the first six values of the integer M . The product $Mf(x)$ is plotted against the ratio x/M .

The remainder of this section is devoted to an analysis of the crossover to the symmetric binary distribution which takes place at large M . Figure 13 shows a comparison between the step distribution $\rho(x)$ and the distribution $f(x)$ of the first positive position for $M = 100$. The products $M\rho(x)$ and $Mf(x)$ are again plotted against the ratio x/M . As expected, $\rho(x)$ exhibits a peak around $x/M = 1$ (and a symmetric one around $x/M = -1$, but only positive values of x are shown). More surprisingly, the distribution $f(x)$ exhibits a bimodal structure, with a first peak in the region where $x \ll M$ and a second one around $x/M = 1$, closely resembling that of $\rho(x)$. Both peaks are separated by a pronounced dip.

The above features are corroborated by the following analysis of the first few moments of H (see (4.9)) at large M . The diffusion coefficient is given by (6.78), whereas (4.14) yields

$$c_2 = \frac{(M+2)(M+3)}{12}. \quad (6.90)$$

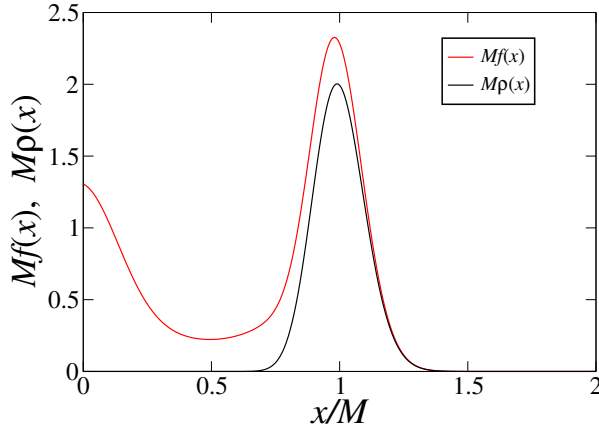


Fig. 13 Comparison between the distribution $f(x)$ of the first positive position (red curve) and the positive part of the step distribution $\rho(x)$ (black curve) for $M = 100$. The products $Mf(x)$ and $M\rho(x)$ are plotted against the ratio x/M .

The extrapolation length (see (6.34))

$$\ell = M - \sum'_b \frac{1}{z_b} \quad (6.91)$$

deserves some more care. The asymptotic behaviour of this quantity, and of related sums and products over the zeros z_b , has been thoroughly studied in [56]. To leading order, as M becomes large, the zeros are uniformly distributed on the circle centered at $p = 1$ with a radius close to unity (see figure 11), and more precisely given by $2^{-1/M}$. We choose to number the zeros so as to have

$$z_b \approx 1 - 2^{-1/M} e^{-2\pi i b/M} \quad (b = 1, \dots, M-1). \quad (6.92)$$

For large but finite M , the correction to the uniform circular distribution (6.92) is most significant for the zeros located near the origin, i.e., either $b \ll M$ or $M-b \ll M$. The leading correction assumes the following scaling form [56, Sec. 6.2]. Setting

$$\xi = \frac{2\pi b}{\sqrt{M}}, \quad (6.93)$$

we have

$$z_b \approx \frac{2\pi i b + Y(\xi)}{M}, \quad Y(\xi) = \frac{\xi^2}{2} + \ln \left(1 + \sqrt{1 - e^{-\xi^2}} \right). \quad (6.94)$$

The scaling function $Y(\xi)$ is real, and therefore only affects the real parts of the zeros. Inserting (6.94) into (6.91), we obtain after some algebra

$$\ell \approx \frac{M}{2} \left(1 - \frac{1}{\pi\sqrt{M}} \ln \frac{M}{M_0} \right), \quad (6.95)$$

with

$$M_0 = 4\pi^2 \exp\left(-2 \int_0^\infty \frac{d\xi}{\xi^2} \left(Y(\xi) - \frac{\xi^2}{2} - \xi e^{-\xi}\right)\right) \approx 8.166752. \quad (6.96)$$

The difference $M - \ell$ is denoted by $\Omega_{(-1)}$ in [56], and the expression (6.95) corresponds to equation (6.37) therein.

Inserting the expressions (6.78), (6.95) and (6.90) of D , $c_1 = \ell$ and c_2 into (4.9), we obtain the following asymptotic formulas for the first three moments of H at large M , where relative corrections of order $1/M$ are neglected:

$$\begin{aligned} \langle H \rangle &\approx \frac{M}{\sqrt{2}}, \\ \langle H^2 \rangle &\approx \frac{M^2}{\sqrt{2}} \left(1 - \frac{1}{\pi\sqrt{M}} \ln \frac{M}{M_0}\right), \\ \langle H^3 \rangle &\approx \frac{M^3}{\sqrt{2}} \left(1 - \frac{3}{2\pi\sqrt{M}} \ln \frac{M}{M_0}\right). \end{aligned} \quad (6.97)$$

The leading-order estimates imply that the peaks observed in figure 13 for $M = 100$ indeed become asymptotically well separated, and that the weights W_1 and W_2 of the first and second peak respectively read

$$W_1 = 1 - \frac{1}{\sqrt{2}} \approx 0.292893, \quad W_2 = \frac{1}{\sqrt{2}} \approx 0.707107. \quad (6.98)$$

The negative corrections in $(\ln M)/\sqrt{M}$ occurring in the expressions (6.97) of $\langle H^2 \rangle$ and $\langle H^3 \rangle$ most probably affect higher-order moments as well.

In terms of the dimensionless quantity \mathbf{A} defined in (2.22), the asymptotic expression (6.95) translates to

$$\mathbf{A} \approx \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\pi\sqrt{M}} \ln \frac{M}{M_0}\right). \quad (6.99)$$

Figure 14 shows that \mathbf{A} has a non-monotonic dependence on the integer M , dropping fast from $\mathbf{A} = 1$ at $M = 1$ (out of scale) to $\mathbf{A} \approx 0.821367$ for $M = 2$ (see (6.69)), reaching a minimum,

$$\mathbf{A} \approx 0.642960, \quad (6.100)$$

for $M = 40$ (arrow), and going up very slowly to its limit $\mathbf{A}_\infty = 1/\sqrt{2}$ (dashed line), according to (6.99).

The quantity \mathbf{A} has been shown to provide a measure of the dispersion of the distribution of H (see (4.10)) and to satisfy the inequality $\mathbf{A} > 1/2$. It seems quite plausible that \mathbf{A} reaches a non-trivial absolute minimum \mathbf{A}_{\min} for some well-defined step distribution. An exploratory numerical study supports this hypothesis and suggests that \mathbf{A}_{\min} is only slightly below the value given in (6.100). The exact determination of \mathbf{A}_{\min} and of the parent step distribution is left as a challenging open problem.

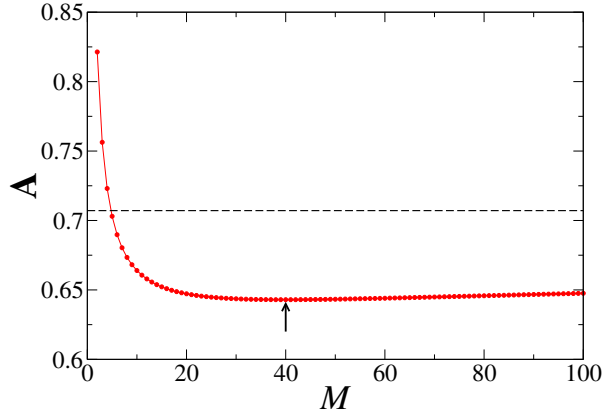


Fig. 14 Dimensionless quantity \mathbf{A} defined in (2.22) plotted against the integer M between 2 and 100. Dashed horizontal line: limit $\mathbf{A}_\infty = 1/\sqrt{2}$. Arrow: minimum $\mathbf{A} \approx 0.642960$ reached for $M = 40$.

Author contribution statement

Both authors contributed equally to the present work, were equally involved in the preparation of the manuscript, and have read and approved the final manuscript.

Data availability statement

Data sharing not applicable to this article.

Conflict of interest

The authors declare no conflict of interest.

A Details of derivations

A.1 Derivation of equation (2.52)

Reasoning along the lines of [43], it is easy to obtain the probability density of Σ_{R_x+1} in Laplace space. The double Laplace transform of this density with respect to its two arguments (space x and value y of the random variable) reads

$$\mathcal{L}_{x,y} f_{\Sigma_{R_x+1}}(x, y) = \mathcal{L}_x \langle e^{-u \Sigma_{R_x+1}} \rangle = \frac{\hat{f}(u) - \hat{f}(p+u)}{p(1 - \hat{f}(p+u))}. \quad (\text{A.1})$$

Taking the derivative of minus this expression with respect to u at $u = 0$ gives

$$\mathcal{L}_x \langle \Sigma_{R_x+1} \rangle = \frac{\langle H \rangle}{p(1 - \hat{f}(p))}. \quad (\text{A.2})$$

The right-hand side of this equation equals the Laplace transform with respect to x of the right-hand side of (2.52) because (see, e.g., [43, Eq. (3.4)])

$$\mathcal{L}_x \langle R_x \rangle = \frac{\hat{f}(p)}{p(1 - \hat{f}(p))}. \quad (\text{A.3})$$

Note that $\langle \Sigma_{R_x} \rangle$ is not equal to $\langle H \rangle \langle R_x \rangle$.

A.2 Proof that (2.72) implies the Pollaczek-Spitzer formula

Recall (2.72)

$$\ln \frac{1}{1 - \tilde{f}(s, q)} = \sum_{n \geq 1} \frac{s^n}{n} \int_0^\infty dx e^{iqx} f_{x_n}(x) = J(q). \quad (\text{A.4})$$

Consider

$$I(p) = \int_0^\infty dx e^{-px} \sum_{n \geq 1} \frac{s^n}{n} f_{x_n}(x) \quad \text{Re}(p) > 0, \quad (\text{A.5})$$

which is such that $J(q) = I(-iq)$. The goal is to show that $I(p)$ is given by (A.9), which is the integral appearing in the Pollaczek-Spitzer formula (1.5) or (6.14) (see also (2.67)). The proof goes as follows. Rewrite

$$\begin{aligned} \sum_{n \geq 1} \frac{s^n}{n} f_{x_n}(x) &= \sum_{n \geq 1} \frac{s^n}{n} \int_{-\infty}^\infty \frac{dq}{2\pi} e^{iqx} \tilde{\rho}(q)^n \\ &= - \int_{-\infty}^\infty \frac{dq}{2\pi} e^{iqx} \ln(1 - s\tilde{\rho}(q)). \end{aligned} \quad (\text{A.6})$$

Thus

$$\begin{aligned} I(p) &= - \int_0^\infty dx e^{-px} \int_{-\infty}^\infty \frac{dq}{2\pi} e^{iqx} \ln(1 - s\tilde{\rho}(q)) \\ &= - \int_{-\infty}^\infty \frac{dq}{2\pi} \ln(1 - s\tilde{\rho}(q)) \int_0^\infty dx e^{-px} e^{iqx} \\ &= - \int_{-\infty}^\infty \frac{dq}{2\pi} \frac{\ln(1 - s\tilde{\rho}(q))}{p - iq}. \end{aligned} \quad (\text{A.7})$$

The integral is split into two parts

$$-I(p) = \int_{-\infty}^0 \frac{dq}{2\pi} \frac{\ln(1 - s\tilde{\rho}(q))}{p - iq} + \int_0^\infty \frac{dq}{2\pi} \frac{\ln(1 - s\tilde{\rho}(q))}{p - iq}, \quad (\text{A.8})$$

yielding finally

$$I(p) = -\frac{p}{\pi} \int_0^\infty dq \frac{\ln(1 - s\tilde{\rho}(q))}{p^2 + q^2}. \quad (\text{A.9})$$

A.3 Derivation of equation (4.22)

We start by simplifying notations by setting (see (4.20))

$$A(q) = \ln \frac{1 - \tilde{\rho}(q)}{Dq^2} = \sum_{n \geq 1} a_{2n} q^{2n}, \quad a_{2n} = 2(-1)^n \frac{c_{2n}}{(2n)!}. \quad (\text{A.10})$$

The Mellin transform

$$\mu_K(s) = \int_0^\infty dq q^{s-1} A(q) \quad (\text{A.11})$$

is convergent for $-2 < \text{Re } s < 0$. It has a meromorphic continuation in the whole left-hand half-plane ($\text{Re } s < 0$), with poles at $s = -2m$ for $m = 1, 2, \dots$. We are interested in the values $\mu_K(-2m - 1)$ for $m = 0, 1, \dots$, which enter (4.21). These quantities can be derived by splitting the definition (A.11) as

$$\mu_K(s) = \mu_{K,1}(s) + \mu_{K,2}(s), \quad (\text{A.12})$$

with

$$\mu_{K,1}(s) = \int_0^1 dq q^{s-1} A(q) \quad (\text{Re } s > -2), \quad (\text{A.13})$$

$$\mu_{K,2}(s) = \int_1^\infty dq q^{s-1} A(q) \quad (\text{Re } s < 0). \quad (\text{A.14})$$

We henceforth fix the value of the integer m . The expression (A.13) can be recast as

$$\begin{aligned} \mu_{K,1}(s) &= \int_0^1 dq q^{s-1} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right) + \int_0^1 dq q^{s-1} \sum_{n=1}^m a_{2n} q^{2n} \\ &= \int_0^1 dq q^{s-1} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right) + \sum_{n=1}^m \frac{a_{2n}}{s+2n}. \end{aligned} \quad (\text{A.15})$$

The integral entering the above expressions is convergent for $-2m-2 < \text{Re } s < -2m$. We have in particular

$$\mu_{K,1}(-2m-1) = \int_0^1 \frac{dq}{q^{2m+2}} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right) - \sum_{n=1}^m \frac{a_{2n}}{2m-2n+1}. \quad (\text{A.16})$$

The corresponding expression for $\mu_{K,2}(s)$ can be directly evaluated at $s = -2m-1$:

$$\begin{aligned} \mu_{K,2}(-2m-1) &= \int_1^\infty \frac{dq}{q^{2m+2}} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right) + \int_1^\infty \frac{dq}{q^{2m+2}} \sum_{n=1}^m a_{2n} q^{2n} \\ &= \int_1^\infty \frac{dq}{q^{2m+2}} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right) + \sum_{n=1}^m \frac{a_{2n}}{2m-2n+1}. \end{aligned} \quad (\text{A.17})$$

Summing up (A.16) and (A.17) yields

$$\mu_K(-2m-1) = \int_0^\infty \frac{dq}{q^{2m+2}} \left(A(q) - \sum_{n=1}^m a_{2n} q^{2n} \right). \quad (\text{A.18})$$

Inserting this expression into (4.21), using the notations (A.10), we obtain (4.22).

A.4 Derivation of equation (6.7)

1. In the definition (6.6) of the Laplace transform of $g(s, x)$,

$$\hat{g}(s, p) = \int_0^\infty dx g(s, x) e^{-px} \quad (\text{A.19})$$

for $\text{Re } p > 0$, replace $g(s, x)$ by the right-hand side of the linear equation (2.64), obtaining

$$\hat{g}(s, p) = 1 + s \int_0^\infty dx e^{-px} \int_0^\infty dy g(s, y) \rho(x-y). \quad (\text{A.20})$$

2. In (A.20) express $g(s, y)$ in terms of its Laplace transform,

$$g(s, y) = \int \frac{dq}{2\pi i} \hat{g}(s, q) e^{qy}, \quad (\text{A.21})$$

where the integration contour is vertical with $\text{Re } q > 0$, and similarly for $\rho(x-y)$, obtaining

$$\hat{g}(s, p) = 1 + s \int_0^\infty dx e^{-px} \int_0^\infty dy \int \frac{dq}{2\pi i} \hat{g}(s, q) e^{qy} \int \frac{dr}{2\pi i} \hat{\rho}(r) e^{r(x-y)}. \quad (\text{A.22})$$

3. In (A.22) perform the integrations over x and y , obtaining

$$\hat{g}(s, p) = 1 + s \int \frac{dq}{2\pi i} \hat{g}(s, q) \int \frac{dr}{2\pi i} \frac{\hat{\rho}(r)}{(p-r)(r-q)} \quad (\text{A.23})$$

for $0 < \text{Re } q < \text{Re } r < \text{Re } p$ and $\text{Re } r < p_1$, with p_1 being the smallest decay rate entering (6.1).

4. In (A.23) shift the q -contour to the right. The contribution of the pole at $q = r$ yields

$$\hat{g}(s, p) = 1 + s \int \frac{dr}{2\pi i} \frac{\hat{g}(s, r)\hat{\rho}(r)}{p-r} \quad (\text{A.24})$$

for $0 < \text{Re } r < \text{Re } p$ and $\text{Re } r < p_1$.

5. In (A.24) shift the r -contour to the right of the pole at $r = p$, obtaining

$$\hat{g}(s, p) = 1 + s\hat{g}(s, p)\hat{\rho}(p) + s \int \frac{dr}{2\pi i} \frac{\hat{g}(s, r)\hat{\rho}(r)}{p-r}, \quad (\text{A.25})$$

i.e., using the definition (6.4) of $\phi(s, p)$,

$$\phi(s, p)\hat{g}(s, p) = 1 + s \int \frac{dr}{2\pi i} \frac{\hat{g}(s, r)\hat{\rho}(r)}{p-r}, \quad (\text{A.26})$$

for $0 < \text{Re } p < \text{Re } r < p_1$.

6. In (A.26) shift the r -contour to the right. The contributions of the poles at $r = p_a$ yield an expression of the form

$$\phi(s, p)\hat{g}(s, p) = 1 + \sum_a \frac{C_a}{p - p_a}, \quad (\text{A.27})$$

where the C_a are constants.

7. This last stage is the gist of the factorisation technique, which works as follows in the present setting. The Laplace transform $\hat{g}(s, p)$ is analytic for $\text{Re } p > 0$. The zeros of $\phi(s, p)$ at $p = z_b$ must therefore be zeros of the right-hand side of (A.27). Moreover, this right-hand side goes to unity as $p \rightarrow +\infty$. We are thus left with the product formula

$$\phi(s, p)\hat{g}(s, p) = \frac{\prod_b (p - z_b)}{\prod_a (p - p_a)}, \quad (\text{A.28})$$

or equivalently

$$\hat{g}(s, p) = \frac{\prod_a (p + p_a)}{\prod_b (p + z_b)}, \quad (\text{A.29})$$

which is (6.7).

A.5 Derivation of equation (6.8)

1. In the expression of the Laplace transform of $f(s, x)$,

$$\hat{f}(s, p) = \int_0^\infty dx f(s, x) e^{-px} \quad (\text{A.30})$$

for $\text{Re } p > 0$, replace $f(s, x)$ by the right-hand side of its expression (2.65), obtaining

$$\hat{f}(s, p) = s \int_0^\infty dx e^{-px} \int_0^\infty dy g(s, y) \rho(x + y). \quad (\text{A.31})$$

2. In (A.31) express $g(s, y)$ and $\rho(x + y)$ in terms of their Laplace transforms, obtaining

$$\hat{f}(s, p) = s \int_0^\infty dx e^{-px} \int_0^\infty dy \int \frac{dq}{2\pi i} \hat{g}(s, q) e^{qy} \int \frac{dr}{2\pi i} \hat{\rho}(r) e^{-r(x+y)}. \quad (\text{A.32})$$

3. In (A.32) perform the integrations over x and y , obtaining

$$\hat{f}(s, p) = s \int \frac{dq}{2\pi i} \hat{g}(s, q) \int \frac{dr}{2\pi i} \frac{\hat{\rho}(r)}{(p+r)(r-q)} \quad (\text{A.33})$$

for $0 < \text{Re } q < \text{Re } r < p_1$, with p_1 being the smallest decay rate entering (6.1).

4. In (A.33) shift the q -contour to the right of the pole at $q = r$, obtaining

$$\hat{f}(s, p) = s \int \frac{dr}{2\pi i} \frac{\hat{g}(s, r) \hat{\rho}(r)}{p+r} \quad (\text{A.34})$$

for $0 < \text{Re } r < p_1$. Using (6.4) and (6.9), the integrand can be recast as

$$\begin{aligned} s\hat{g}(s, r)\hat{\rho}(r) &= (1 - \phi(s, r))\hat{g}(s, r) \\ &= \hat{g}(s, r) - \frac{1}{\hat{g}(s, -r)} = (\hat{g}(s, r) - 1) + \left(1 - \frac{1}{\hat{g}(s, -r)}\right). \end{aligned} \quad (\text{A.35})$$

5. In (A.34) replace the integrand by the last line of (A.35), obtaining

$$\hat{f}(s, p) = \int \frac{dr}{2\pi i} \frac{\hat{g}(s, r) - 1}{p+r} + \int \frac{dr}{2\pi i} \left(1 - \frac{1}{\hat{g}(s, -r)}\right) \frac{1}{p+r}. \quad (\text{A.36})$$

In the first integral, shift the r -contour to the right: the integral vanishes. In the second integral, shift the r -contour to the left. The contribution of the pole at $r = -p$ yields

$$\hat{f}(s, p) = 1 - \frac{1}{\hat{g}(s, p)}, \quad (\text{A.37})$$

which is (6.8).

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